

Miscellaneous

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Overview

- 1 A measure of heaviness of a tail
- 2 Different meanings of robustness in engineering
- 3 Cauchy and Normal distribution
- 4 Smooth empirical distribution and KDE
- 5 Revisit: robustness issues (pdf and KDE)
- 6 Skewed distribution

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A measure of heaviness of a tail

Definition of kurtosis

$$\text{kurtosis} = E \left[\left(\frac{X - \mu}{\sigma} \right)^4 \right]. \quad \text{Note: the kurtosis of normal is } \mathbf{3}.$$

NIST: kurtosis

Kurtosis is a measure of whether the data are heavy-tailed or light-tailed relative to a normal distribution. That is, data sets with **high kurtosis tend to have heavy tails, or outliers**. See NIST (2018) at [▶ NIST](#).

Interpretation of kurtosis (Wikipedia)

... The exact interpretation of the measure of kurtosis used to be **disputed**, but is *now settled(?)* (Westfall, 2014). ... Therefore, kurtosis measures outliers only; it measures **nothing about the peak(?)**.

See [▶ wiki\(Kurtosis\)](#).

尖度 (일본)

峯度 (중국)

鈍尖度 (my suggestion)

A measure of heaviness of a tail

Heavy distribution

- Rigorously, a heavy-tailed distribution means any distribution whose tail has heavier tail(s) than the **exponential**.

See [wiki\(heavy tail\)](#).

- **Occasionally**, a heavy-tailed (fat-tailed) distribution means any distribution that has **heavier tail(s) than the normal**. (kurtosis greater than 3).

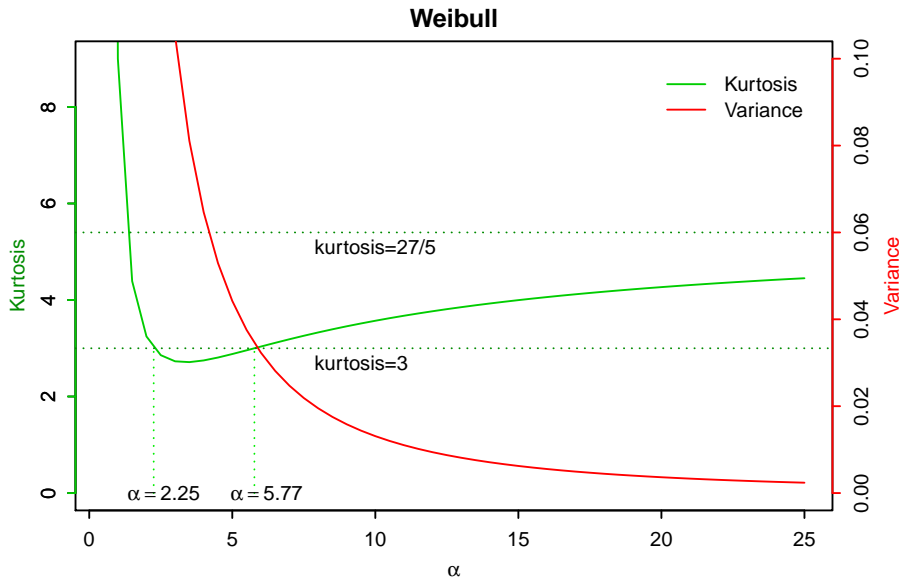
Danger of using kurtosis (Example: Weibull distribution)

$$\text{kurtosis} = \frac{\Gamma(1 + \frac{4}{\alpha}) - 4\Gamma(1 + \frac{3}{\alpha})\Gamma(1 + \frac{1}{\alpha}) + 6\Gamma(1 + \frac{2}{\alpha})\Gamma(1 + \frac{1}{\alpha})^2 - 3\Gamma(1 + \frac{1}{\alpha})^4}{(\Gamma(1 + \frac{2}{\alpha}) - \Gamma(1 + \frac{1}{\alpha})^2)^2}$$

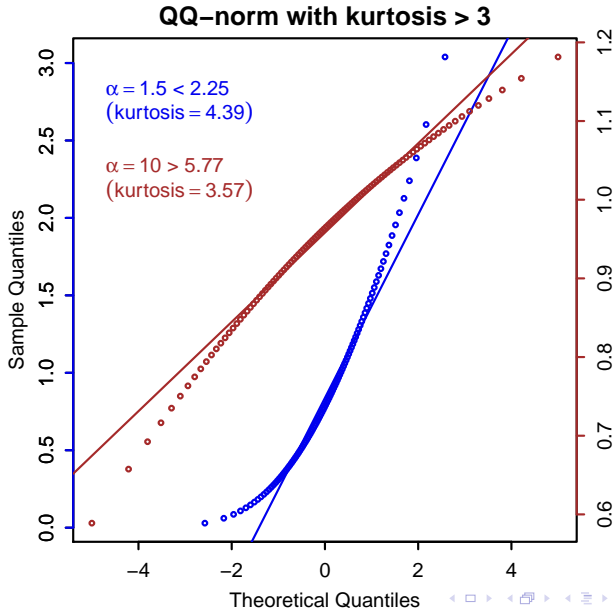
$$\text{variance} = \Gamma(1 + \frac{2}{\alpha}) - \Gamma(1 + \frac{1}{\alpha})^2$$

◇NB: as $\alpha \rightarrow \infty$, the variance goes to 0 and the kurtosis goes to 27/5.

A measure of heaviness of a tail (danger of using kurtosis)



A measure of heaviness of a tail (danger of using kurtosis)



A measure of heaviness of a tail (alternative to kurtosis)

- In some cases, the kurtosis does not explain the heaviness of a distribution properly.
 - ⇒ Use the QQ-norm if one wants to compare with a normal distribution. (**baseline**: normal distribution)
 - ⇒ Use the definition of heavy-tailed dist. provided in Foss et al. (2013). (**baseline**: exponential distribution)
According to this definition, Weibull has a heavy tail **only if** α (shape) less than one.
 - Also, for many distributions including the Cauchy, t -distributions with 1,2,3,4 degrees of freedom, and so on, the value of the kurtosis is infinity (∞).
 - ⇒ The definition in Foss et al. (2013) can be used, but it does not provide the strength of heaviness.
 - ⇒ L -kurtosis by Hosking (1990) can be a good alternative.
- ◇ **Suggestion**: **Kurtosis** + **QQ-norm** + **L -kurtosis**.
- ◇ Not a fan of Foss et al. (2013): too complex and no strength.

Different meanings of robustness in engineering

- In practice, robust estimators are widely used when dealing with **heavy-tailed** distributions and **contaminated** data, etc.
- We will look at various interpretations of robustness.

Definition of Robustness in **statistics literature**

- Robustness implies **stability** of parameter estimation under departure from the true model (contamination, model departure, etc.)
In statistics, it is also called **outlier-resistance**.
- Outliers can be from contamination or from a nature of **heavy** distribution (surprising observation).
- ◇ The meaning of robustness in engineering is slightly different.

Different meanings of robustness in engineering

Robust to noise. Noise in what sense?

- Robust to **contamination** (outlier / influential).
Robust to something **wrong**.
- Robust to **model departure** (usually departure from the normality).
Robust to something **different** from normal.
For example, the t -test is robust; see Remark 8.3.1 of (Hogg et al., 2013). (Roughly speaking, it is due to CLT).
- Robust to **surprising observation**
Robust to something **surprising**.
An outlier (from a **heavy** distribution). Thus, not a contamination.
Not necessarily influential.
No influential observation to the MLE of Laplace (median).
For **Cauchy**, we can easily meet surprising outlying observations.
- Robust to **uncontrollable noise** (Robust Design)
Robust to something **uncontrollable**.

Different meanings of robustness in engineering

Measures of robustness property (review)

- Robust to **contamination** \implies **Breakdown point**.

The percentage of contaminated observations before the estimator becomes non-sensical and incorrect.

Refer to Talk-1 at [▶ Seminar/2018](#).

- Robust to **model departure** \implies **Relative efficiency**.

$$\text{RE}(\hat{\theta}_2, \hat{\theta}_1) = \frac{\text{Var}(\hat{\theta}_1)}{\text{Var}(\hat{\theta}_2)} \times 100\%$$

where $\hat{\theta}_1$ is often a reference or baseline estimator (usually, the MLE under a normal distribution).

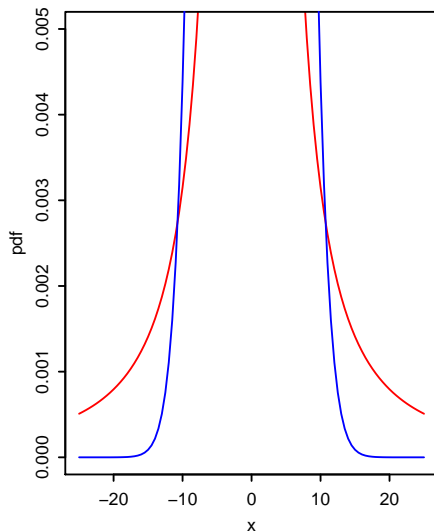
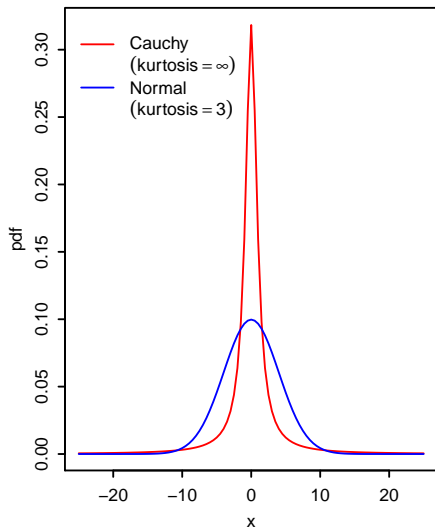
Refer to Talk-5 at [▶ Seminar/2018](#).

- Robust to **uncontrollable noise** (Robust Design) \implies **SN ratio**.

Cauchy and Normal distribution

Cauchy: $f(x) = \frac{\beta}{\pi} \frac{1}{\beta^2 + (x - \alpha)^2}$

Normal: $f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$



Cauchy and Normal distribution

R Code to compare Cauchy and Normal

```
set.seed(14)  # Change this seed.
n = 10;       # n = 10000
x = rcauchy(n)
y = rnorm(n)
OUT = rbind( c(mean(x),sd(x)), c(mean(y),sd(y)) )
colnames(OUT) = c("mean", "SD")
rownames(OUT) = c("Cauchy(x)", "Normal(y)")
OUT
```

- Cauchy is very dangerous due to **gravity with leverage**.
- Robust estimators are needed for most of heavy-tailed distributions.
- MLE under heavy distributions are robust in general.
(MLE of t -distribution, Laplace, Cauchy, etc).

Smooth empirical distribution and KDE

Suppose that there are n observations: x_1, x_2, \dots, x_n , and their order statistics are denoted by $y_1 < y_2 < \dots < y_n$. The value of the empirical distribution at a given point, y_k , is given by

$$\hat{F}_n(y_k) = \frac{k}{n}.$$

In general, the empirical distribution function of t (**without sorting**) can be written by

$$\hat{F}_n(t) = \frac{1}{n} \sum_{i=1}^n \mathbb{I}(x_i \leq t),$$

where $\mathbb{I}(\cdot)$ is an indicator function.

◇NB: $\hat{F}_n(t)$ also works for tied observations.

We can re-write $\hat{F}_n(t)$ as

$$\hat{F}_n(t) = \frac{1}{n} \sum_{i=1}^n \mathbb{P}(X \leq t | \mu_i = x_i),$$

where X is a random variable which degenerates at $\mu_i = x_i$.

Instead of **degenerating** \mathbb{P} , one can use any **smooth** distribution function.

Say, we use normal CDF with $\mu_i = x_i$ and $\sigma_i = h_i$ (aka, bandwidth). That is, we have

$$\hat{F}_n(t) = \frac{1}{n} \sum_{i=1}^n P(X \leq t \mid \mu_i = x_i, \sigma_i = h_i),$$

where $X \sim N(x_i, h_i^2)$. Since $Z = (X - \mu_i)/\sigma_i$, we have

$$\begin{aligned} \hat{F}_n(t) &= \frac{1}{n} \sum_{i=1}^n P\left(Z \leq \frac{t - \mu_i}{\sigma_i} \mid \mu_i = x_i, \sigma_i = h_i\right) \\ &= \frac{1}{n} \sum_{i=1}^n P\left(Z \leq \frac{t - x_i}{h_i}\right) = \frac{1}{n} \sum_{i=1}^n \Phi\left(\frac{t - x_i}{h_i}\right), \end{aligned} \quad (1)$$

where Φ is a CDF of $N(0, 1)$.

Differentiating Eq. (1) with t , we have

$$\hat{f}_n(t) = \frac{1}{n} \sum_{i=1}^n \frac{1}{h_i} \phi\left(\frac{t - x_i}{h_i}\right),$$

Instead of $\phi(\cdot)$, one can use any kernel $k(\cdot)$ which is positive and its integration is one. Also, we can use any positive weights instead of 1. Then we have more general KDE

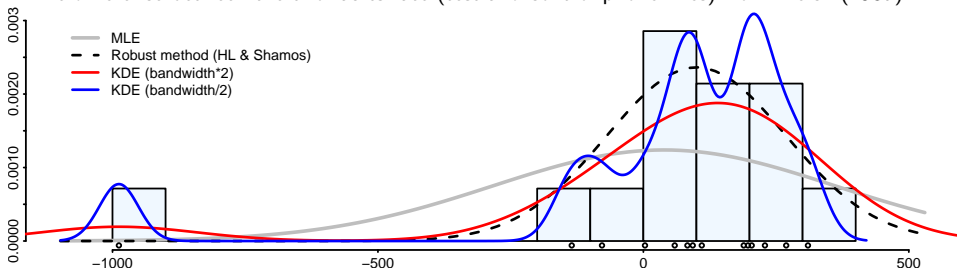
$$\hat{f}_n(t) = \frac{1}{\sum_{i=1}^n w_i} \sum_{i=1}^n w_i \frac{1}{h_i} k\left(\frac{t - x_i}{h_i}\right).$$

In many cases, we use a constant weight $w_i = 1$ and a fixed bandwidth $h_i = h$. Then we have

$$\hat{f}_n(t) = \frac{1}{n} \sum_{i=1}^n \frac{1}{h} k\left(\frac{t - x_i}{h}\right) = \frac{1}{nh} \sum_{i=1}^n k\left(\frac{t - x_i}{h}\right).$$

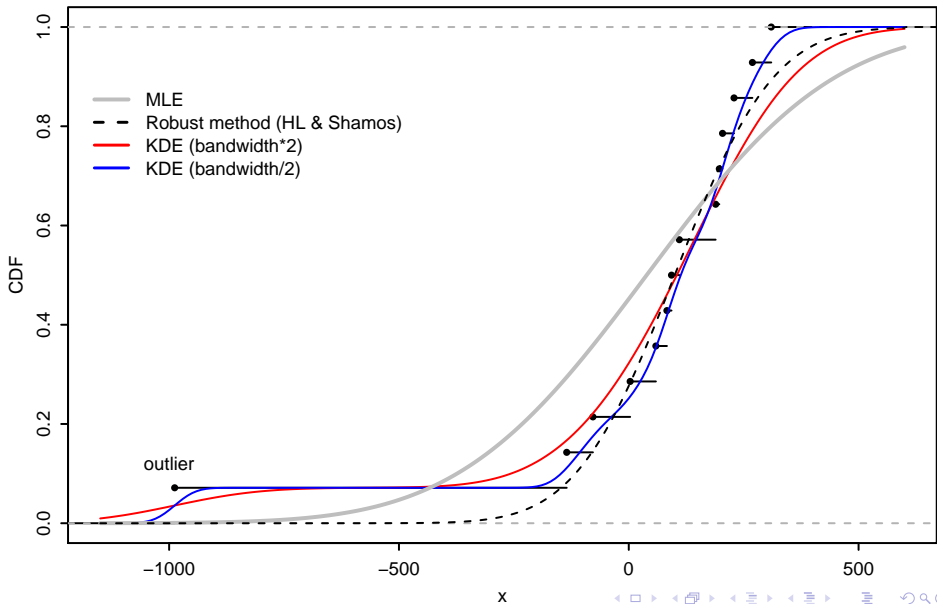
Revisit: robustness issues (pdf and KDE)

The difference between the two faults rate (test and control phone-lines) from Welch (1987).

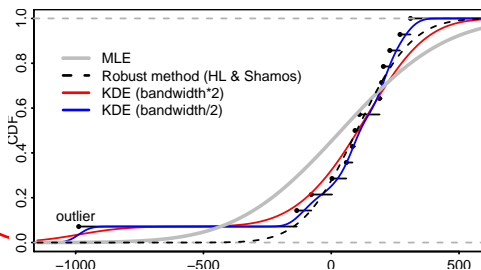
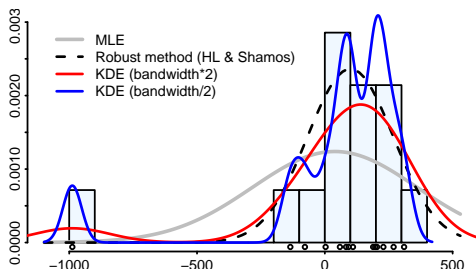


- -988, -135, -78, 3, 59, 83, 93, 110, 189, 197, 204, 229, 269, 310.
- Using the empirical CDF, $\hat{F}_n(-500) = 1/14 = 0.07142857$. ($n = 14$).
- Using the CDF with the MLE, $\hat{F}_{\hat{\theta}}(-500) = 0.04706641$.
- Using the CDF with robust estimation, $\hat{F}_{\hat{\theta}}(-500) = 0.0001891299$.
- Using the CDF with the KDE (large h), $\hat{F}_h(-500) = 0.07203472$.
- Using the CDF with the KDE (small h), $\hat{F}_h(-500) = 0.07142857$.

Revisit: robustness issues (CDF)



Revisit: robustness issues (pdf and KDE)



- The pdf (grey) with the MLE does not fit the histogram.
($\hat{\mu} = 38.92857$ and $\hat{\sigma} = 321.9428$). NB: **gravity and leverage**.
- The pdf (dashed) with the HL and Shamos fits the histogram well.
($\hat{\mu} = 100.0$ and $\hat{\sigma} = 168.7856$). NB: it is **robust**.
- The KDE (blue/red) is not robust.
But, the probability estimation is not so sensitive to an outlier.
- The empirical CDF (step function) is not robust either.
But, the probability estimation is not so sensitive to an outlier.
- The KDE depends on the bandwidth h .

Skewed distribution

Skewness

- Skewness is a measure of the **asymmetry** of a pdf.
(zero \implies symmetric. non-zero \implies asymmetric or skewed).
- A unimodal left-skewed distribution has a long left tail while a unimodal right-skewed distribution has a long right tail.
- A skewed distribution with a **light longer** tail is **not lethal**.
- A **long and heavy** tail may create potential problems.

Measure of skewness

- $E\left[\left(\frac{X-\mu}{\sigma}\right)^3\right]$ (Pearson). Most popular measure.
- $\frac{\mu - \text{median}}{\sigma}$ (old **nonparametric**).
◊NB: μ is just a center of gravity (so nonparametric).
- Others: quartile-based measure, L -moment by Hosking (1990), etc.

Skewed distribution

How to deal with skewed distributions

- One can use the empirical CDF or the KDE although they may not be robust.

Note: the estimation of a probability is **not so sensitive** to outliers.

- One can select parametric (asymmetric or skewed) distributions with robust estimation.

For example, Weibull, lognormal, Birnbaum-Saunders, gamma distribution, χ^2 -distribution, F -distribution, etc.

- One can normalize the data set (say, using Box-Cox Transformation). Then, use the existing methods developed under the normality.
- One can try data-driven weights for a general KDE to give smaller weights for outliers

$$\hat{f}_n(t) = \frac{1}{\sum_{i=1}^n w_i} \sum_{i=1}^n w_i \frac{1}{h} k\left(\frac{t - x_i}{h}\right).$$

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