Statistical Simulation

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Overview

1. Definition of Statistical Simulation

2. Problems can be solved by statistical simulations
   - Estimation of the probability when tossing a die or dice
   - Ratio of the circumference of a circle
   - Efficiency of estimators
   - Breakdown point
   - Correction factors (d_2 and c_4)
   - Hypothesis testing
   - Development of `rt.test` R package
   - Development of `weibullness` R package

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Definition of Statistical Simulation

Definition (wikipedia)

Monte Carlo methods (or Monte Carlo experiments) are a broad class of computational algorithms that rely on **repeated random sampling** to obtain numerical results. Their essential idea is using randomness to solve problems that might be **deterministic in principle**. They are often used in physical and mathematical problems and are most useful when it is difficult or impossible to use other approaches. *Monte Carlo methods are mainly used in three(~~actually much more~~?) problem classes*: optimization, numerical integration, and generating draws from a probability distribution.

- Monte Carlo methods $\Longrightarrow$ statistical or stochastic simulation.
- It can solve **stochastic** and **deterministic** problems.
- More applications in statistics: power, distribution, efficiency, bias-correction, etc.

Tips for Simulation

From my teaching experience, many students are confused with:

- sample size n and
- the iteration (replication) number I.

- Sample size n is in general the number of random variable(s).
- The iteration number I is the number of repeated experiments.

Problems can be solved by statistical simulations

In this talk, we will solve various problems by statistical simulations.

1. Estimation of the probability when tossing a die or dice.
2. The ratio of the circumference of a circle, $\pi = 3.14\ldots$.
3. Efficiency of estimators.
4. Breakdown point.
5. Bias correction factors (d_2 and c_4) in Quality Control Chart.
6. Hypothesis testing.
7. R package (`rt.test`)
8. R package (`weibullness`)

(1) Estimation of the probability

We are interested in the probability when tossing a single die.

R Code for tossing a fair die

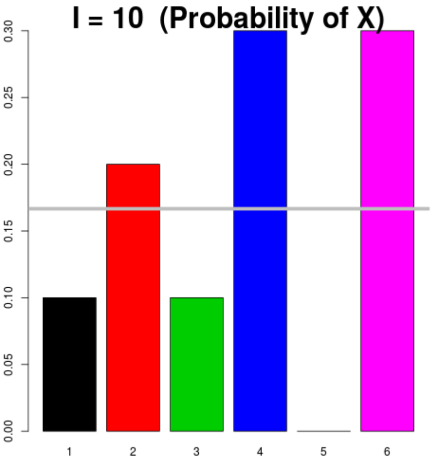
```
set.seed(1)
ITER = 10
x = sample(1:6, size=ITER, replace=TRUE)
table(x) / ITER
table( factor(x, levels=1:6)) / ITER  # cosmetic
```

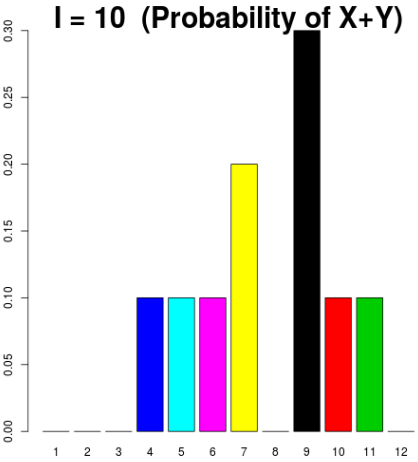
What is the prob. of sum of two outcomes when tossing two dice.

R Code for tossing two dice (sum of two)

```
set.seed(1)
ITER = 10
x = sample(1:6, size=ITER, replace=TRUE)
y = sample(1:6, size=ITER, replace=TRUE)
table(x+y) / ITER
table( factor(x+y, levels=1:12)) / ITER  # cosmetic
```

(1) Estimation of the probability





~~Homework~~: 1. Find the probability of the difference of two dice.
2. What if a die is not fair?

(2) Ratio of the circumference of a circle, $\pi = 3.14\ldots$

The idea is very simple — consider a dart game.
The probability that a pin drops in a circle when a pin was thrown at random is given by:

$$\frac{\pi}{4} = \frac{\text{Area of unit circle}}{\text{Area of square}} \approx \frac{\#\text{ of pins in a circle}}{\#\text{ of all the pins}} = \hat{p}.$$

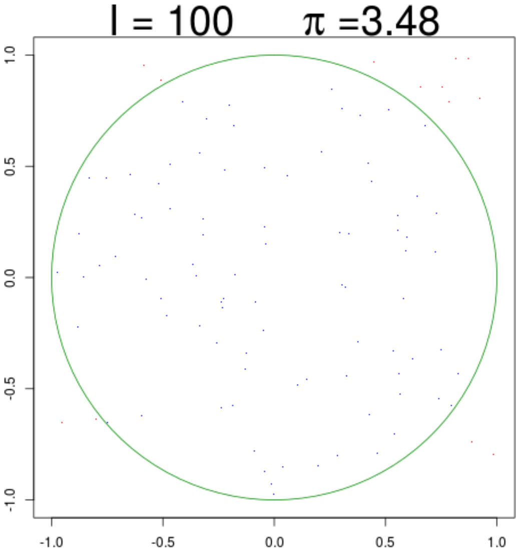
Thus, we have $\pi \approx 4 \times \hat{p}$.
♢ NB: Avoid using `for()` loop in R.

R Code for finding $\pi = 3.14\ldots$

```
set.seed(1)
ITER = 100
x = runif(ITER, min=-1, max=1)
y = runif(ITER, min=-1, max=1)
4 * sum (x^2 + y^2 < 1) / ITER
```

Also, refer to Talk-1 at seminar.

(2) Ratio of the circumference of a circle, $\pi = 3.14\ldots$



(3) Efficiency of Estimators (mean, median, HL)

The relative efficiency of $\hat{\theta}_2$ given $\hat{\theta}_1$ is defined as

$$\mathrm{RE}(\hat{\theta}_2, \hat{\theta}_1) = \frac{\mathrm{Var}(\hat{\theta}_1)}{\mathrm{Var}(\hat{\theta}_2)} \times 100\% \tag{1}$$

where $\hat{\theta}_1$ is often a reference or baseline estimator (usually MLE).

Breakdown Points and AREs of the estimators under consideration.

	Location			Scale			
	Mean	Median	HL	SD	IQR	MAD	Shamos
ARE	100%	64%	96%	100%	38%	37%	86%
Breakdown	0%	50%	29%	0%	25%	50%	29%

- Refer to §2.2 of Lehmann Lehmann (1999), Rousseeuw and Croux (1993) and Serfling (2011); Staudte and Sheather (1990).
- $\mathrm{HL} = \mathrm{median}(X_i + X_j)/2$ for all $i < j$ or $i \leq j$.

Refer to Talk-2 at seminar.

(3) Efficiency of Estimators (mean, median, HL)

Here, we will focus on RE or ARE of median and HL with mean as a baseline.

R Code for Relative Efficiency

```
# Load Hodges-Lehmann and Shamos estimators.
source("https://raw.githubusercontent.com/AppliedStat/seminar/master/R/Rsec.R")
ITER = 1000
n = 50
means = medians = HLs = numeric(ITER)

for ( i in 1:ITER ) {
    x = rnorm(n)
    means[i]   = mean(x)
    medians[i] = median(x)
    HLs[i]     = HL(x)
}
c( var(means) / var(medians),  var(means) / var(HLs) )
```

◊NB: As $n \to \infty$ and $I \to \infty$, RE converges to ARE.

(4) Breakdown point

Definition (Breakdown point)

The finite-sample breakdown point is the maximum proportion of incorrect observations (i.e. arbitrarily large observations) that an estimator can handle without leading to an egregiously incorrect estimate.

R Code for Breakdown point

```
n = 100   # Number of observations
k =  20   # Number of contaminated obs.
x = c(runif(n-k), rep(Inf,k))
c(median(x), HL(x) )
k / n
```

- Idea: increase k until an estimator breaks down.
- Try $k = 28, 29, 30, \ldots, 49, 50, 51$.

(4) Breakdown point

R Code for Breakdown point (a better code)

```
n = 100   # Number of observations
for ( k in 1: (n/2) ) {     # repeat with a different k
     x = c(runif(n-k), rep(Inf,k))
     print( c(k/n,median(x),HL(x)) )
}
```

- We obtained the finite-sample breakdown point.
- As increase n along with k, we can find the ~~finite-sample~~ breakdown point.

◇ The breakdown point of the median is $1/2$ and that of the Hodges-Lehmann is $1 - 1/\sqrt{2} = 0.29289$.
Also, refer to $\boxed{\text{Talk-2}}$ at ▸ seminar.

(5-1) Bias correction factor (d_2)

The range, $R = X_{(n)} - X_{(1)}$, is not unbiased under the normality assumption. To make it unbiased, we adjust R with the correction factor d_2 so that R/d_2 is unbiased, that is, $E[R/d_2] = \sigma$. Then we have

$$d_2 = \frac{1}{\sigma} E[R].$$

Note that we have $X = \sigma Z + \mu$, where Z is from $N(0,1)$. Thus, we have

$$d_2 = E[Z_{(n)} - Z_{(1)}] = 2 \int_0^\infty \left\{ 1 - \left[\Phi(z) \right]^n - \left[1 - \Phi(z) \right]^n \right\} dz,$$

where $\Phi(\cdot)$ is the CDF of $N(0,1)$.
Refer to Section 1 of $\boxed{\text{Talk-5-extra.pdf}}$ at ▸ seminar.

- To prove the above, it takes almost ten pages.
- d_2 depends on the sample size n.

(5-1) Bias correction factor (d_2)

Idea for estimating $E[Y]$.

The theoretical expectation $E[Y]$ is approximated by

$$\frac{1}{I}\sum_{i=1}^{I} Y_i.$$

Since $d_2 = E[Z_{(n)} - Z_{(1)}] = E[Z_{(n)}] - E[Z_{(1)}]$, we have

$$E[Z_{(n)}] \approx \frac{1}{I}\sum_{i=1}^{I} \max_i \quad \text{and} \quad E[Z_{(1)}] \approx \frac{1}{I}\sum_{i=1}^{I} \min_i,$$

where $\max_i$ and $\min_i$ are the maximum and minimum from a sample of size $\boxed{n}$ at the $\boxed{i}$th iteration.

(5-1) Bias correction factor (d_2)

R Code for d_2

```
set.seed(1)
n = 10          # sample size
ITER = 1E4      # The larger, the more accurate d2
MAX = MIN = numeric(ITER)
for ( i in 1:ITER ) {
    Z = rnorm(n)
    MAX[i] = max(Z)
    MIN[i] = min(Z)
}
mean(MAX) - mean(MIN)
```

- Better R codes are provided in Section 1 of Talk-5-extra.pdf at seminar.
- For $n = 10$, we have $d_2 = 3.078$ from the quality control textbook.
- The value through the simulation is given by $d_2 = 3.092639$.

(5-2) Bias correction factor (c_4)

It is well know that the sample variance S^2 is **unbiased** under the normal distribution assumption. That is

$$E[S^2] = \sigma^2,$$

where $S^2 = \sum_{i=1}^{n}(X_i - \bar{X})^2/(n-1)$. However, the sample standard deviation S is not unbiased so that

$$E[S] \neq \sigma.$$

Most quality control textbooks use the correction factor, c_4, so that $E[S/c_4] = \sigma$. The correction factor c_4 is theoretically derived as

$$c_4 = \sqrt{\frac{2}{n-1}} \cdot \frac{\Gamma(n/2)}{\Gamma(n/2 - 1/2)}.$$

Note that c_4 is an unbiasing factor which is a function of sample size and it was originally used in ASTM E-11 (1976).

Refer to Section 2 of Talk-5-extra.pdf at seminar.

(5-2) Bias correction factor (c_4)

Since $E[S/c_4] = \sigma$, we have $c_4 = \frac{1}{\sigma}E[S] = E[S/\sigma]$. Note that S/σ is a **pivot**. Similar to the case of the range R, we have

$$c_4 = E[S_Z],$$

where S_Z is the sample standard deviation with a sample of size n from $N(0, 1)$.

R Code for d_2

```
set.seed(1)
n = 10          # sample size
ITER = 1E4      # The larger, the more accurate d2
S = numeric(ITER)
for ( i in 1:ITER ) {
    Z = rnorm(n)
    S[i] = sd(Z)
}
mean(S)
```

(5-3) Comparing R, R/d_2, S, S/c_4

We have studied R, R/d_2, S, and S/c_4.
Then, are R/d_2 and S/c_4 better?
We usually check the bias and variance (or relative efficiency).
Let $\hat{\theta}_i$ be the ith estimate from the simulation. Then the empirical bias and variance are given by

- Empirical bias:

$$\frac{1}{I}\sum_{i=1}^{I}\hat{\theta}_i - \theta.$$

- Empirical variance:

$$\frac{1}{I-1}\sum_{i=1}^{I}(\hat{\theta}_i - \bar{\hat{\theta}})^2.$$

♢NB: Based on the above variance, we can estimate the relative efficiency.

(5-3) Comparing R, R/d_2, S, S/c_4

R Code for empirical bias and variance

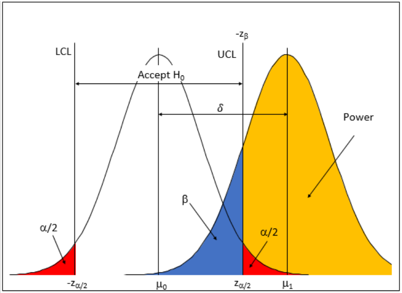
```
# Load Hodges-Lehmann and Shamos estimators.
source("https://raw.githubusercontent.com/AppliedStat/seminar/master/R/Rsec.R")
set.seed(1)
ITER = 1E5
R = Rd2 = SD = SDc4 = numeric(ITER)
for ( i in 1:ITER ) {
    X = rnorm(n)
    R[i]  = diff(range(X))
    SD[i] = sd(X)
}
Rd2 = R / d2(n)
SDc4 = SD / c4(n)

# Bias of estimators
c( mean(R)-1, mean(Rd2)-1, mean(SD)-1, mean(SDc4)-1 )
# Var. of estimators
c( var(R), var(Rd2), var(SD), var(SDc4) )
```

(6) Hypothesis testing

Consider the statistical hypothesis testing:

$$H_0 : \mu = \mu_0 \quad \text{versus} \quad H_1 : \mu \neq \mu_0.$$

		Decision H_0	Decision H_1
Truth	H_0	**OK** (coverage) $(1-\alpha)$	Type-I error (α)
	H_1	Type-II error (β)	**OK** (power) $(K(\cdot) = 1 - \beta)$



◊NB: Power is usually a function of μ or $\delta = \mu_1 - \mu_0$.

(6-1) Hypothesis testing (Example)

Example 8.5-2 from Hogg et al. (2015)

Let $X_1, X_2, \ldots, X_{25}$ be a random sample from $N(\mu, 100)$. We want to test $H_0 : \mu = 60$ versus $H_1 : \mu > 60$.
When we decided to reject H_0 if $\bar{X} > 62$ (critical/rejection region), find the power function of μ.

Since $n = 25$ and $X_i \sim N(\mu, 100)$, we have $\bar{X} \sim N(\mu, 100/25)$. Then $Z = (\bar{X} - \mu)/\sqrt{100/25} = (\bar{X} - \mu)/2 \sim N(0, 1)$. Thus, under $H_1 : \mu$, the power function is given by

$$
\begin{aligned}
K(\mu) &= P[\bar{X} > 62 \mid \mu] \\
&= P\left[\frac{\bar{X} - \mu}{2} > \frac{62 - \mu}{2}\right] = P\left[Z > \frac{62 - \mu}{2}\right] \\
&= 1 - \Phi\left[\frac{62 - \mu}{2}\right].
\end{aligned}
$$

(6-1) Hypothesis testing (Example: Power)

Example 8.5-2 from Hogg et al. (2015)

The example state that the power at $\mu = 65$ is $K(65) = 0.9332$ ...

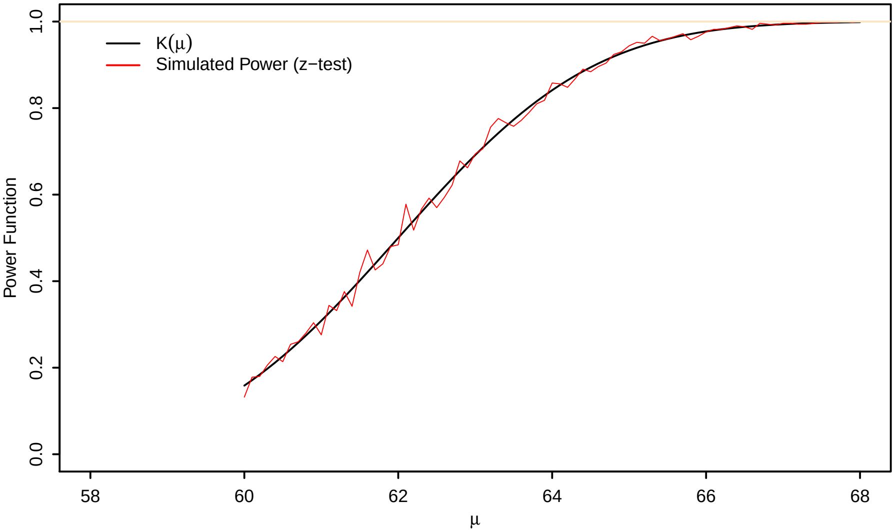
R Code for Example 8.5-2

```
# Theoretical power
K = function(mu) { 1-pnorm( (62-mu)/2 ) }
# Empirical power (through simulation)
set.seed(1);
ITER = 10000
n=25; sigma=10; mu=65
count = 0
for ( i in 1:ITER ) {
    X = rnorm(n=n, mean=mu, sd=sigma)
    # count if it is in a critical region.
    if (mean(X) > 62) count = count+1
}
c( count / ITER, K(65) )
```

(6-1) Hypothesis testing (Power function)

R Code for Figure 8.5-2: Power function $K(\mu)$

```
# Empirical power (through simulation)
set.seed(1)
ITER = 1000
n=25; sigma=10; MU=seq(60, 68, length=81)
power = numeric(length(MU))
for ( j in 1:length(MU) ) {
    mu = MU[j]
    for ( i in 1:ITER ) {
        X = rnorm(n=n, mean=mu, sd=sigma)
        if ( mean(X) >= 62 ) power[j] = power[j] + 1/ITER
   }
}
# Compare the empirical power with the theoretical power
plot(MU, K(MU), xlim=c(58,68), ylim=c(0,1),  type="l" )
lines(MU, power, col="red")
```


K(μ)
Simulated Power (z−test)
Power Function
1.0
0.8
0.6
0.4
0.2
0.0
58
60
62
64
66
68
μ

(6-2) Testing $H_0 : \mu = \mu_0$ vs. $H_0 : \mu \neq \mu_0$

- σ is known (z-test): reject H_0 if

$$|z| = \frac{|\bar{x} - \mu_0|}{\sigma/\sqrt{n}} > z_{\alpha/2}$$

$$K_z(\mu) = 1 - \Phi\left(z_{\alpha/2} + \frac{\mu - \mu_0}{\sigma/\sqrt{n}}\right) + \Phi\left(-z_{\alpha/2} + \frac{\mu - \mu_0}{\sigma/\sqrt{n}}\right)$$

- σ is unknown (t-test): reject H_0 if

$$|t| = \frac{|\bar{x} - \mu_0|}{S/\sqrt{n}} > t_{\alpha/2}(n-1)$$

$$K_t(\mu) = 1 - \Phi_{\nu,\delta}(t_{\alpha/2}) + \Phi_{\nu,\delta}(-t_{\alpha/2}),$$

where $\Phi_{\nu,\delta}(\cdot)$ is the CDF of the non-central t-distribution with $\nu = n - 1$ degrees of freedom and non-centrality $\delta = (\mu - \mu_0)/(\sigma/\sqrt{n})$.

◇ Refer to Section 3 of $\boxed{\text{Talk-5-extra.pdf}}$ at seminar
$\Longrightarrow$ 4 pages!

(6-2) Hypothesis testing (Power function when σ is known)

R Code for empirical power of z-test (σ is known)

```
Kz = function(mu, alpha, mu0, sigma, n) {
    z.cut = qnorm(1-alpha/2)
    tmp = (mu-mu0)/(sigma/sqrt(n))
    pnorm(z.cut + tmp, lower.tail=FALSE) + pnorm(-z.cut + tmp)
}
# Empirical Power
set.seed(1); ITER = 1E4
mu=1.5; mu0=0.5; alpha=0.05; sigma=1; n=5
power = 0
zcut = qnorm(1-alpha/2)
for ( i in 1:ITER ) {
  X = rnorm(n, mean=mu, sd=1)
  if(abs(mean(X)-mu0)/(sigma/sqrt(n)) > zcut) power=power+1/ITER
}
# Compare
c( Kz(mu, alpha, mu0, sigma, n), power )
```

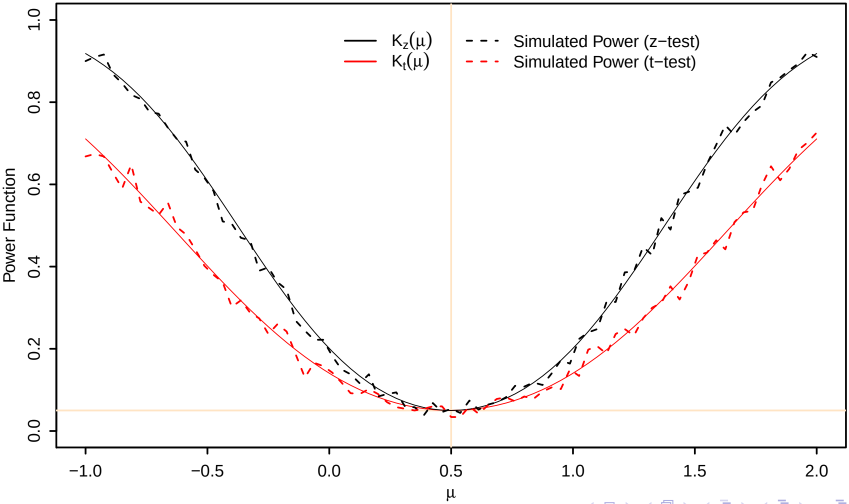
(6-3) Hypothesis testing (Power when σ is unknown)

R Code for empirical power of t-test (σ is unknown)

```
Kt = function(mu, alpha, mu0, sigma, n) {
   t.cut = qt(1-alpha/2,df=n-1)
   ncp = (mu-mu0)/(sigma/sqrt(n))
   pt(t.cut,df=n-1,ncp=ncp,lower.tail=F)+pt(-t.cut,df=n-1,ncp=ncp)
}
# Empirical Power
set.seed(1); ITER = 1E4
mu=1.5; mu0=0.5; alpha=0.05; sigma=1; n=5
power = 0
tcut = qt(1-alpha/2, df=n-1)  # tcut is used instead of zcut
for ( i in 1:ITER ) {
  X = rnorm(n, mean=mu, sd=1)
  # sd(X) is used instead of known sigma
  if(abs(mean(X)-mu0)/(sd(X)/sqrt(n)) > tcut) power=power+1/ITER
}
# Compare
c( Kt(mu, alpha, mu0, sigma, n), power )
```

(6-3) Hypothesis testing: Power

Testing $H_0 : \mu = 1/2$ versus $H_1 : \mu \neq 1/2$ with $\alpha = 0.05$, $n = 5$, and $\sigma = 1$. Refer to Section 3.3 of Talk-5-extra.pdf at seminar.



t-test statistic

$$T = \frac{\bar{X} - \mu}{S_X/\sqrt{n}}, \tag{2}$$

where $\bar{X} = \sum X_i/n$, $S_X^2 = \sum (X_i - \bar{X})^2/(n-1)$ and X_i are from $N(\mu, \sigma^2)$. Substituting $X_i = \sigma Z_i + \mu$ into (2), we have

$$T = \frac{\bar{Z} - \mu}{S_Z/\sqrt{n}},$$

Note that T is a **pivot**.

- Assume that we know only the normal distribution.
- How to know the distribution of T above.

(7-1) Warming-up before `rt.test` R package

R Code for the critical value of the two-side t-test when $\alpha = 5\%$

```
set.seed(1)
ITER = 1.0E4; n=6;
Tstat = numeric(ITER)
for ( i in 1:ITER ) {
    Z = rnorm(n)
    Tstat[i] = mean(Z) / (sd(Z)/sqrt(n))
}

# Critical values
quantile(Tstat, prob=0.975)
qt(0.975, df=n-1)
qnorm(0.975)  # 1.96

# Plot
hist(Tstat,breaks=100,probability=T,xlim=c(-5,5),ylim=c(0,0.4))
lines( density(Tstat))
curve( dnorm(x), xlim=c(-5,5), add=T, col="red" )
```

(7-2) `rt.test` R package

Recall a basic idea

$$T = \frac{\boxed{\bar{X}} - \mu}{\boxed{S}/\sqrt{n}} \tag{3}$$

- $\boxed{\bar{X}}$ is replaced by median or Hodges and Lehmann (1963).
- $\boxed{S}$ is replaced by MAD or Shamos (1976).

- **Is it enough?** No.
- Then, **is it distributed as $N(0,1)$ or t-distribution?** Neither.
  T_A or $T_B \overset{\bullet}{\sim} N(0,1)$ (Refer to KIIE at Seoul, November 2018).
  Thus, T_A or T_B works for a large sample.
  What if a sample is small?

◇ NB: Refer to $\boxed{\text{Talk-2}}$ at ▸ seminar and `rt.test` R package at ▸ rt.test.

(7-2) `rt.test` R package

Recall T_A and T_B

Refer to Park and Wang (2018a) at ArXiv.

$$T_A = \sqrt{\frac{2n}{\pi}} \Phi^{-1}\left(\frac{3}{4}\right) \cdot \frac{\operatorname*{median}_{1\le i\le n} X_i - \mu}{\operatorname*{median}_{1\le i\le n} \left| X_i - \operatorname*{median}_{1\le i\le n} X_i \right|} \xrightarrow{d} N(0,1), \tag{4}$$

where $\Phi^{-1}(\cdot)$ is the inverse of the standard normal cumulative distribution function and $\xrightarrow{d}$ denotes convergence in distribution.

$$T_B = \sqrt{\frac{6n}{\pi}} \Phi^{-1}\left(\frac{3}{4}\right) \frac{\operatorname*{median}_{i\le j} \left(\frac{X_i + X_j}{2}\right) - \mu}{\operatorname*{median}_{i\le j} \left(|X_i - X_j|\right)} \xrightarrow{d} N(0,1). \tag{5}$$

$\Diamond$NB: T_A and T_B are **pivotal**. See Park (2018a) and Jeong et al. (2018).

(7-2) rt.test R package

R Code for the critical value of rt.test R package

```
set.seed(1)
ITER = 1.0E4; n=6;
TA = numeric(ITER)
for ( i in 1:ITER ) {
    Z = rnorm(n)
    TA[i] = sqrt(2*n/pi)*qnorm(3/4)*median(Z)/mad(Z,constant=1)
}

# Install rt.test package
install.packages("rt.test")
library("rt.test")

# Critical values
quantile(TA, prob=0.975)
q.robustified.t(0.975, n=n) # This Package used ITER = 1.0E8

qnorm(0.975)  # 1.96 (compare it with the above)
```

(8) `weibullness` R package

Recall: Weibull distribution (reparametrized form)

$$F(x) = 1 - \exp\big(-\lambda x^{\alpha}\big) \text{ and } f(x) = \lambda\alpha x^{\alpha-1}\exp\big(-\lambda x^{\alpha}\big),$$

It is immediate from the Weibull CDF $F(x) = p$ that

$$\log(1-p) = -\lambda x_p^{\alpha}.$$

It follows that

$$\log\big\{-\log(1-p)\big\} = \log\lambda + \alpha\log x_p.$$

This implies that the plot of

$$\boxed{\log\big\{-\log(1-p)\big\}} \text{ versus } \boxed{\log x_p}$$

draws a straight line with the slope α and the intercept $\log\lambda$.
◊NB: Refer to Park (2017) and $\boxed{\text{Talk-3}}$ at ▸ seminar and `weibullness` R package at ▸ weibullness.

Recall: How to determine whether the data are from Weibull or not

$$H_0 : \text{Weibull} \quad \text{versus} \quad H_1 : \text{non-Weibull.}$$

- Recall: $\log\{-\log(1-p)\} = \log\lambda + \alpha \log x_p$.
- Thus $\boxed{\log\{-\log(1-\hat{p}_i)\}}$ versus $\boxed{\log x_{(i)}}$ draws a straight line, where $\hat{p}_i = (i-0.375)/(n+0.25)$. Thus, the linearity measure (sample correlation) can be used for Weibullness test.
- Idea: Large r implies Weibullness.

$$\boxed{H_0 : r \geq r_0 \quad \text{versus} \quad H_1 : r < r_0}$$

Then, how small is small enough for r to reject H_0?

- To this end, we should know the distribution of r and then find the critical value.

◊ NB: r is a **pivot**. Thus, for Weibull with any parameter values, the results are the same. $\Longrightarrow$ Use exponential distribution with rate one.

(8) `weibullness` R package

Algorithm

Step-1: Generate Weibull random observations of size n from any Weibull. Note Weibull(1,1) is an exponential distribution.

Step-2: Sort the data. Denote $x_{(i)}$.

Step-3: Calculate $\log\{-\log(1-\hat{p}_i)\}$, where $\hat{p}_i = (i-0.375)/(n+0.25)$ or $\hat{p}_i = (i-0.5)/n$.

Step-4: Calculate the sample correlation between
$$\log\{-\log(1-\hat{p}_i)\} \text{ and } \log x_{(i)}.$$

Step-5: Repeat the above (say, up to I iteration numbers).

Step-6: Find the empirical quantiles for critical values (say, 5%, 10%, etc.)

◊ NB: Compare the sample correlation with the empirical quantile (critical value) in order to make a decision.

(8) weibullness R package

R Code for the critical value of weibullness R package

```
set.seed(1)
ITER = 1.0E4; n=23;
cors = numeric(ITER)
for ( i in 1:ITER ) {
   x = sort( rexp(n) )      # Step 1-2
   p = ppoints(n)           # Step 3
   cors[i] = cor(log(-log(1-p)),log(x))     # Step 4
}  # Repeat (i=1,2,...,ITER)                 # Step 5

alphas = c(0.01,0.02,0.025,0.5,0.1,0.2)
quantile(cors, prob=alphas)                  # Step 6

# Install weibullness package
install.packages("weibullness")
library("weibullness")
wp.test.critical(alphas,n)   # This package used ITER=1E8
```

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