# Applications (사례 연구)

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# Overview

1. Robust statistics와 응용 사례
   - Robust statistics (Basic applications)
   - Robust $t$-test
   - Robust design with contaminated data
2. Missing · Incomplete Data와 응용 사례
   - RD with unbalanced samples
   - RD with incomplete data
   - Competing risks with censoring, masking, etc.
   - Load-sharing
   - Grouped Data
3. Future work
   - Robust control chart with unbalanced samples
   - Competing risks

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## Basic applications (estimating $\mu$ and $\sigma$)

Consider observations from $X_i \sim N(\mu, \sigma^2)$. We need to estimate $\mu$ and $\sigma^2$.

- MLE (maximum likelihood estimator)

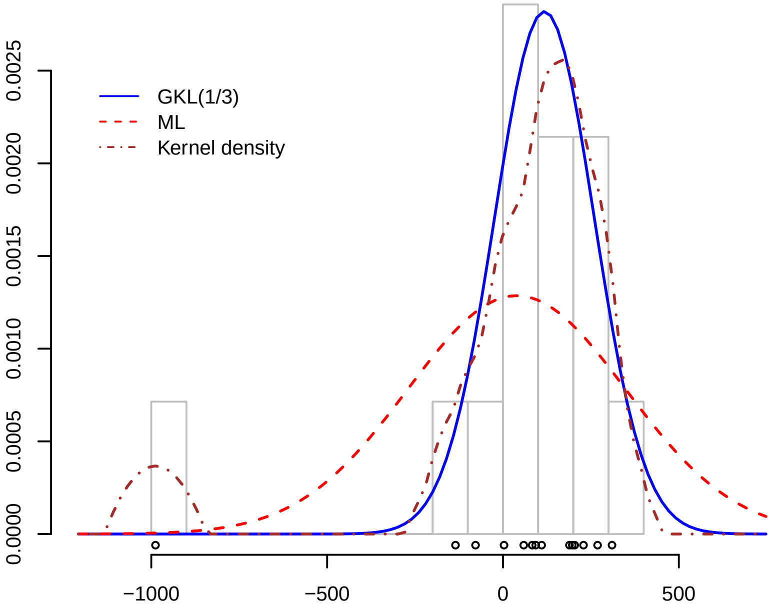
$$\hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} X_i \quad \text{and} \quad \hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \hat{\mu})^2$$

- BUE (best unbiased estimator) or UMVUE

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \quad \text{and} \quad S^2 = \frac{1}{\boldsymbol{n-1}}\sum_{i=1}^{n}(X_i - \bar{X})^2$$

- MDE (minimum distance estimator): KL, GKL, etc.
  MLE is a special case of GKL (Basu et al., 2011) which can have robustness. MDE is asymptotically fully efficient, but its calculation is quite complex. Thus, HL and Shamos are recommended (Talk-2).

# 1. Robust statistics와 응용 사례: Basic applications



Darwin (1876) collected the data: the growth of pairs of corn (especially Zea May) seedings, one produced by self-fertilization and the other produced by cross-fertilization. For the data set, see Friendly et al. (2018).

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| Cross | 23.500 | 12.000 | 21 | 22 | 19.125 | 21.500 | 22.125 | 20.375 |
| | 18.25 | 21.625 | 23.25 | 21 | 22.125 | 23.0 | 12 | |
| Self | 17.375 | 20.375 | 20 | 20 | 18.375 | 18.625 | 18.625 | 15.250 |
| | 16.50 | 18.000 | 16.25 | 18 | 12.750 | 15.5 | **18** | |
| Difference | 6.125 | −8.375 | 1.000 | 2.000 | 0.750 | 2.875 | 3.500 | 5.125 |
| | 1.750 | 3.625 | 7.000 | 3.000 | 9.375 | 7.500 | −6.000 | |

We can test $H_0 : \mu_x = \mu_y$ and $H_1 : \mu_x \neq \mu_y$, equivalently, $H_0 : \mu_d = 0$ and $H_1 : \mu_d \neq 0$, where $\mu_d = \mu_x - \mu_y$. A typical paired sample $t$-test with

$$T = \frac{\bar{D} - 0}{S_D/\sqrt{n}},$$

where $\bar{D}$ and $S_D$ are the sample mean and standard deviation, which this becomes a one-sample $t$-test.

## Theorem 1 (Park, 2018a)

*Let $X_1, X_2, \ldots, X_n$ be a random sample from a normal distribution with mean $\mu$ and variance $\sigma^2$. Then we have*

$$T_A = \sqrt{\frac{2n}{\pi}}\Phi^{-1}\left(\frac{3}{4}\right)\frac{\operatorname*{median}_{1\le i\le n} X_i - \mu}{\operatorname*{median}_{1\le i\le n}\left|X_i - \operatorname*{median}_{1\le i\le n} X_i\right|} \xrightarrow{d} N(0,1). \tag{1}$$

## Theorem 2 (Jeong et al., 2018)

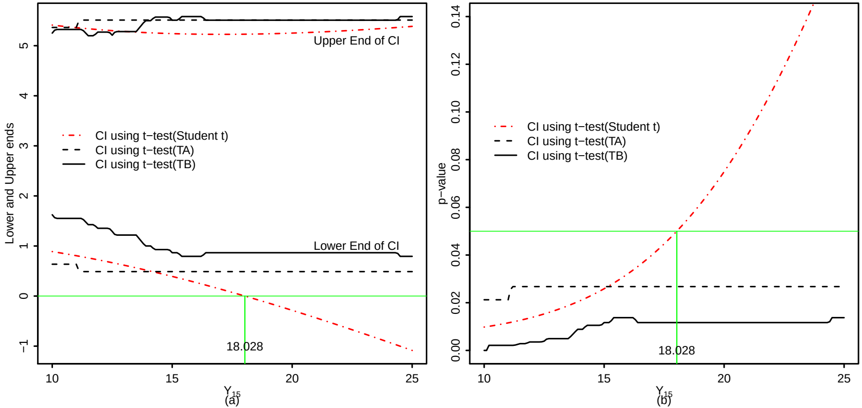
*Let $X_1, X_2, \ldots, X_n$ be a random sample from a normal distribution with mean $\mu$ and variance $\sigma^2$. Then we have*

$$T_B = \sqrt{\frac{3n}{2\pi}}\Phi^{-1}\left(\frac{3}{4}\right)\frac{\operatorname*{median}_{i\le j}\left(X_i + X_j\right) - 2\mu}{\operatorname*{median}_{i\le j}\left(|X_i - X_j|\right)} \xrightarrow{d} N(0,1). \tag{2}$$

- The above Theorems work well with a large sample size because these are based on asymptotic standard normal distribution.
- Recently, Park and Wang (2018) developed the `rt.test` R package used the empirical distributions instead of the asymptotic standard normal distribution. Using the `rt.test`, we can carry out robustified $t$-test easily.
- Also, we can obtain the confidence intervals using the above robustified test statistics. By checking if zero is included inside each interval, we can test the hypothesis

$$H_0 : \mu_d = 0 \quad \text{and} \quad H_1 : \mu_d \neq 0.$$

# 1. Robust statistics와 응용 사례: Robust $t$-test



This idea can be easily applied to control charting which is similar to a confidence interval.

- Phase I: use robustified control chart.
- Phase II: use conventional control chart.

This work is partially done (balanced case). See `rcc` function in `rQCC`.

```
> library("rQCC")
> help(rcc)
> tmp = c(
72, 84, 79, 49, 56, 87, 33, 42, 55, 73, 22, 60, 44, 80, 54, 74,
97, 26, 48, 58, 83, 89, 91, 62, 47, 66, 53, 58, 88, 50, 84, 69,
57, 47, 41, 46, 13, 10, 30, 32, 26, 39, 52, 48, 46, 27, 63, 34,
49, 62, 78, 87, 71, 63, 82, 55, 71, 58, 69, 70, 67, 69, 70, 94,
55, 63, 72, 49, 49, 51, 55, 76, 72, 80, 61, 59, 61, 74, 62, 57 )
> data2 = matrix(tmp, ncol=4, byrow=TRUE)
> rcc(data2, loc="HL2", scale="shamos")
     LCL       CL      UCL
36.99703 59.26250 81.52797
```

## Robust Design (Dual Response)

- The process mean response function.

$$\hat{M}(\mathbf{x}) = \hat{\beta}_0 + \sum_{i=1}^{k} \hat{\beta}_i x_i + \sum_{i=1}^{k} \hat{\beta}_{ii} x_i^2 + \sum_{i<j}^{k} \hat{\beta}_{ij} x_i x_j.$$

- The process variance response function.

$$\hat{V}(\mathbf{x}) = \hat{\eta}_0 + \sum_{i=1}^{k} \hat{\eta}_i x_i + \sum_{i=1}^{k} \hat{\eta}_{ii} x_i^2 + \sum_{i<j}^{k} \hat{\eta}_{ij} x_i x_j.$$

## In the above, we need to estimate $M(\mathbf{x})$, $V(\mathbf{x})$, $\beta$ and $\eta$

- The $M(\mathbf{x})$ and $V(\mathbf{x})$ can be estimated using robust estimators such as HL and Shamos estimators. For more details (HL and other estimators), see Talk-2 at ▸ 2018/Seminar
- The $\beta$ and $\eta$ can be estimated using the regression method.

Refer to Park and Leeds (2016) and Talk-2 at 2018/Seminar

**Method A**: $\hat{M}(\mathbf{x})$ using the sample **mean** and
$\hat{V}(\mathbf{x})$ using the sample **variance**.
(BASELINE – without contamination!)

**Method B**: $\hat{M}(\mathbf{x})$: **median** and $\hat{V}(\mathbf{x})$: the squared **MAD**

**Method C**: $\hat{M}(\mathbf{x})$: **median** and $\hat{V}(\mathbf{x})$: the squared **IQR**

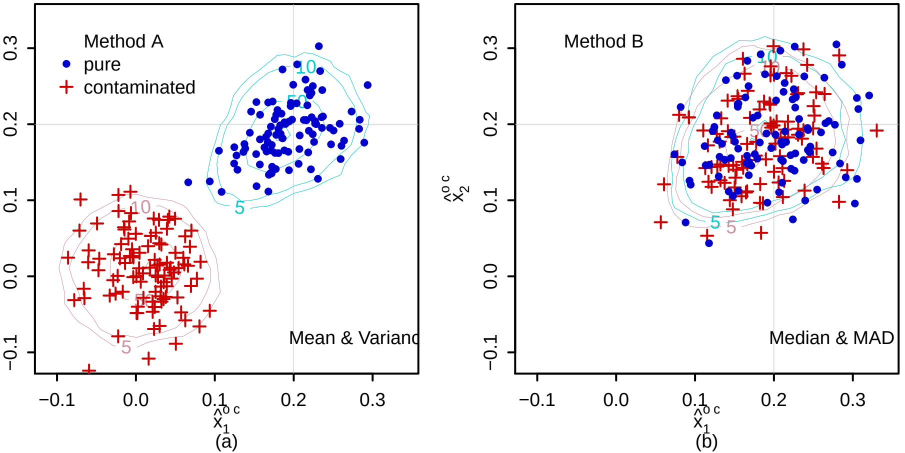
**Method D**: $\hat{M}(\mathbf{x})$: **HL (Hodges-Lehmann)** and
$\hat{V}(\mathbf{x})$: the squared **Shamos**

**Method E**: $\hat{M}(\mathbf{x})$: **median** and $\hat{V}(\mathbf{x})$: the squared **Shamos**

**Method F**: $\hat{M}(\mathbf{x})$: **HL** and $\hat{V}(\mathbf{x})$: the squared **MAD**.

**Method G**: $\hat{M}(\mathbf{x})$: **HL** and $\hat{V}(\mathbf{x})$: the squared **IQR**.

# 1. Robust statistics와 응용 사례: Robust Design/강건설계



# 2. Missing · Incomplete: RD with unbalanced samples

## Recall: Robust Design (Dual Response)

$$\text{Mean response}: \hat{M}(\mathbf{x}) = \hat{\beta}_0 + \sum_{i=1}^{k} \hat{\beta}_i x_i + \sum_{i=1}^{k} \hat{\beta}_{ii} x_i^2 + \sum_{i<j}^{k} \hat{\beta}_{ij} x_i x_j$$

$$\text{Variance response}: \hat{V}(\mathbf{x}) = \hat{\eta}_0 + \sum_{i=1}^{k} \hat{\eta}_i x_i + \sum_{i=1}^{k} \hat{\eta}_{ii} x_i^2 + \sum_{i<j}^{k} \hat{\eta}_{ij} x_i x_j$$

## Unbalanced Data Set

| $i$ | $x_{i1}$ | $x_{i2}$ | $Y_{ir_i}$ | | | | | | | $\overline{Y}_i$ | $S_i^2$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | -1 | -1 | 84.3 | 57.0 | 56.5 | | | | | 65.93 | 253.06 |
| 2 | 0 | -1 | 75.7 | 87.1 | 71.8 | 43.8 | 51.6 | | | 66.00 | 318.28 |
| 3 | 1 | -1 | 65.9 | 47.9 | 63.3 | | | | | 59.03 | 94.65 |
| 4 | -1 | 0 | 51.0 | 60.1 | 69.7 | 84.8 | 74.7 | | | 68.06 | 170.35 |
| 5 | 0 | 0 | 53.1 | 36.2 | 61.8 | 68.6 | 63.4 | 48.6 | 42.5 | 53.46 | 139.89 |
| 6 | 1 | 0 | 46.5 | 65.9 | 51.8 | 48.4 | 64.4 | | | 55.40 | 83.11 |
| 7 | -1 | 1 | 65.7 | 79.8 | 79.1 | | | | | 74.87 | 63.14 |
| 8 | 0 | 1 | 54.4 | 63.8 | 56.2 | 48.0 | 64.5 | | | 57.38 | 47.54 |
| 9 | 1 | 1 | 50.7 | 68.3 | 62.9 | | | | | 60.63 | 81.29 |

### Theorem 3

*Let $Y_1, \ldots, Y_r$ be a random sample of size $r$ from the probability density function $f(y)$ with a finite fourth moment and let $\mu = E(Y)$ and $\theta_k = E(Y-\mu)^k$, $k = 2, 3, 4$. Then we have*

$$\mathrm{Var}(\overline{Y}) = \frac{1}{r}\mu \quad \text{and} \quad \mathrm{Var}(S^2) = \frac{1}{r}(\theta_4 - \frac{r-3}{r-1}\theta_2^2).$$

*Especially, if $Y_i$'s have independent and identical normal distribution, then* $\mathrm{Var}(S^2) = 2\sigma^4/(r-1)$.

### Proof.

See Casella and Berger (2002). □

- $\mathrm{Var}(\overline{Y}) \propto 1/r$.
- $\mathrm{Var}(\overline{S^2}) \propto$ ?. Under the normality, $\mathrm{Var}(\overline{S^2}) \propto 1/(r-1)$.

# 2. Missing · Incomplete: RD with unbalanced samples

Under the normality assumption, we can solve this problem by using the weighted least squares (WLS) regression instead of the ordinary least squares (OLS) regression (Cho and Park, 2005).

## OLS versus WLS

- OLS: $\mathbf{Y} \sim N(\mathbf{X}\beta, \sigma^2\mathbf{I})$ and $\hat{\beta} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y}$.
- WLS: $\mathbf{Y} \sim N(\mathbf{X}\beta, \sigma^2\mathbf{W}^{-1})$ and $\hat{\beta} = (\mathbf{X}'\mathbf{W}\mathbf{X})^{-1}\mathbf{X}'\mathbf{W}\mathbf{Y}$.

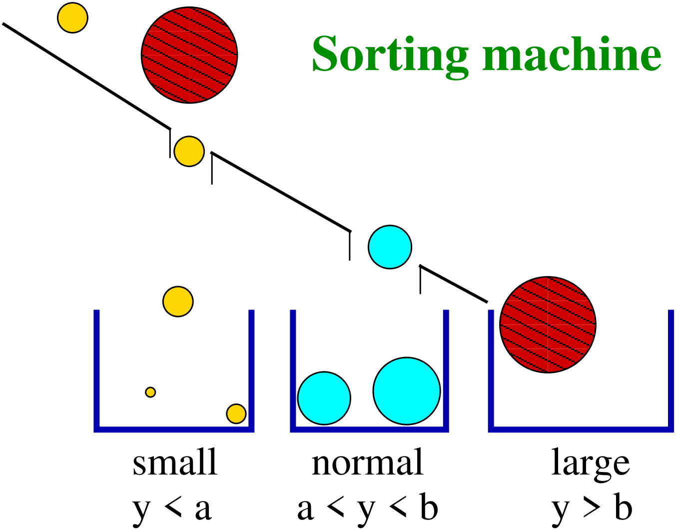
## Mean and Variance responses

- Mean response: $\mathbf{W} = \mathrm{diag}(r_1, r_2, \ldots, r_n)$
- Variance response: $\mathbf{W} = \mathrm{diag}(r_1 - 1, r_2 - 1, \ldots, r_n - 1)$

where $n$ is the number of the design points.

## What if normality is **not** satisfied

$\mathrm{Var}(\overline{S^2})$ is **not** proportional to $1/(r-1) \Rightarrow$ **multiple imputation**.


Sorting machine
small
y < a
normal
a < y < b
large
y > b

## Incomplete Data with grouping

| | | | Full observations | | | | | Interval observations | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $i$ | $x_{i1}$ | $x_{i2}$ | $y_{i1}$ | $y_{i2}$ | $y_{i3}$ | $y_{i4}$ | $y_{i5}$ | $(-\infty, 45)$ | $[45, 55]$ | $(55, \infty)$ |
| 1 | −1 | −1 | 55.1 | 61.4 | 53.5 | 72.4 | 62.6 | 2 | 17 | 81 |
| 2 | 0 | −1 | 65.5 | 59.2 | 60.4 | 57.3 | 65.0 | 2 | 58 | 40 |
| 3 | 1 | −1 | 58.7 | 63.3 | 56.9 | 49.4 | 67.7 | 4 | 18 | 78 |
| 4 | −1 | 0 | 61.3 | 52.1 | 54.3 | 47.3 | 57.9 | 6 | 46 | 48 |
| 5 | 0 | 0 | 54.5 | 47.8 | 49.8 | 44.4 | 51.8 | 8 | 84 | 8 |
| 6 | 1 | 0 | 45.0 | 54.1 | 62.0 | 59.0 | 55.8 | 6 | 47 | 47 |
| 7 | −1 | 1 | 50.5 | 56.0 | 54.3 | 60.2 | 47.8 | 5 | 33 | 62 |
| 8 | 0 | 1 | 52.3 | 60.5 | 53.8 | 62.1 | 57.9 | 7 | 43 | 50 |
| 9 | 1 | 1 | 75.5 | 44.0 | 83.7 | 58.0 | 56.5 | 4 | 23 | 73 |

- All observations: Clearly the best
- Full observations only: (measurement cost is expensive).
- Interval observations only: (measurement cost is cheap or free).

Which of full or interval is better? (It depends on the sample size).

## Recall: Robust Design (Dual Response)

$$\text{Mean response}: \hat{M}(\mathbf{x}) = \hat{\beta}_0 + \sum_{i=1}^{k} \hat{\beta}_i x_i + \sum_{i=1}^{k} \hat{\beta}_{ii} x_i^2 + \sum_{i<j}^{k} \hat{\beta}_{ij} x_i x_j$$

$$\text{Variance response}: \hat{V}(\mathbf{x}) = \hat{\eta}_0 + \sum_{i=1}^{k} \hat{\eta}_i x_i + \sum_{i=1}^{k} \hat{\eta}_{ii} x_i^2 + \sum_{i<j}^{k} \hat{\eta}_{ij} x_i x_j$$

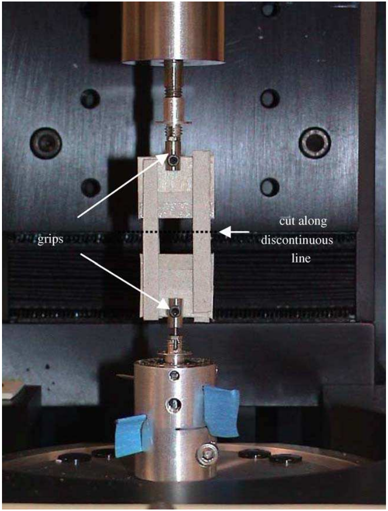
For the application of incomplete/grouped data to the robust design, see Lee and Park (2006). (Below, $n = 5$ full obs. and 100 interval obs.)

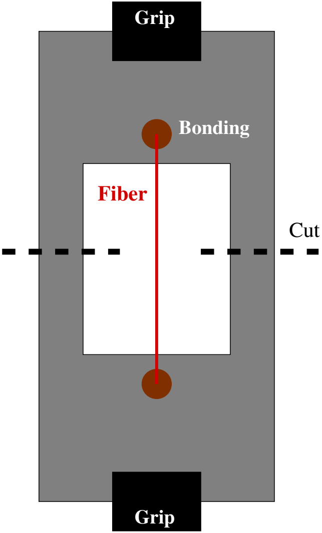
## Empirical bias and MSE of the mean and variance under considered method

| | Full obs. only | | All observations | | Interval obs. only | |
|---|---|---|---|---|---|---|
| Estimate | $\hat{\mu}_1$ | $\hat{\sigma}_1^2$ | $\hat{\mu}_2$ | $\hat{\sigma}_2^2$ | $\hat{\mu}_3$ | $\hat{\sigma}_3^2$ |
| Bias | 0.0956 | −0.9653 | 0.0059 | 2.8018 | 0.0009 | 4.5426 |
| Variance | 20.2007 | 4777.45 | 1.2125 | 762.0400 | 1.3103 | 989.9277 |
| MSE | 20.2098 | 4778.38 | 1.2125 | 769.8901 | 1.3103 | 1010.5630 |

- Full observations are costly.
  Interval observations are cheap or free.
- Curvature of profile likelihood can be used for precision of estimators, which can be used for equivalent sample sizes with the same precision (Lee and Park, 2006).
- The parameter estimation with incomplete/grouped data is tricky.
  The EM, MCEM, QEM can be used (Park, 2018b).
  The MI can also be used but this is also tricky.
  In general, the model is known, the MLE (EM) is better than the MI.
- This idea can be applied to various applications with parameter estimates.

## Illustrative Example: Tensile Testing Equipment



Most multi-modal strength analyses of materials have been studied based on the so-called weakest link theory which requires two assumptions (Beetz, 1982; Goda and Fukunaga, 1986):

**Assumptions**

**A1** The material contains inherently many strength-limiting defects, and its strength depends on the weakest defect of all of them.

**A2** There are no interactions among the defects.

**What if the above assumptions are not satisfied**

**multiple imputation** can be considered.

**Strength data with three fracture causes (modes)**

| Strength | Mode | Strength | Mode | Str | Mode | Str | Mode |
|---|---|---|---|---|---|---|---|
| 54 | $\{3\}$ | 7 | $\{1,2,3\}$ | 86 | $\{2\}$ | 104 | $\{1\}$ |
| 143 | $\{2\}$ | 81 | $\{3\}$ | 141 | $\{1\}$ | 89 | $\{3\}$ |
| 97 | $\{3\}$ | 52 | $\{3\}$ | 79 | $\{3\}$ | 9 | $\{3\}$ |
| 104 | $\{3\}$ | 40 | $\{3\}$ | 23 | $\{3\}$ | 111 | $\{1,2,3\}$ |
| 71 | $\{1,2\}$ | 82 | $\{2\}$ | 8 | $\{3\}$ | 150 | 0 |
| 98 | $\{1\}$ | 3 | $\{3\}$ | 17 | $\{3\}$ | 79 | $\{2\}$ |
| 24 | $\{2\}$ | 130 | $\{2\}$ | 41 | $\{2\}$ | 94 | $\{2\}$ |
| 138 | $\{3\}$ | 5 | $\{3\}$ | 43 | $\{2,3\}$ | 150 | 0 |
| 38 | $\{3\}$ | 32 | $\{2\}$ | 9 | $\{3\}$ | 77 | $\{2\}$ |
| 78 | $\{3\}$ | 16 | $\{3\}$ | 92 | $\{2\}$ | 76 | $\{3\}$ |
| 150 | 0 | 33 | $\{3\}$ | 80 | $\{2\}$ | 100 | $\{2\}$ |
| 46 | $\{3\}$ | 137 | $\{1,2\}$ | 92 | $\{3\}$ | 108 | $\{2\}$ |
| 109 | $\{1\}$ | 71 | $\{1\}$ | 60 | $\{2\}$ | 88 | $\{1\}$ |
| 7 | $\{3\}$ | 11 | $\{3\}$ | 150 | 0 | 150 | 0 |
| 42 | $\{2\}$ | 6 | $\{3\}$ | 43 | $\{3\}$ | 124 | $\{1,2\}$ |

**Specimens in tensile strength experiments are broken down due to**

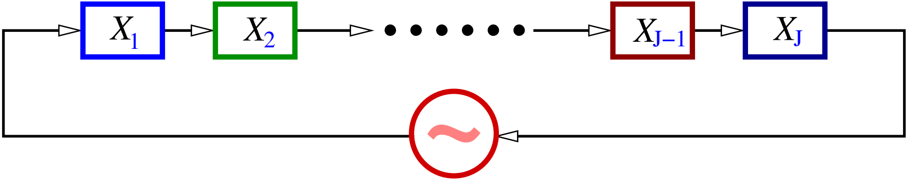
- several causes (**competing risks**)
- with the cause of fracture not properly identified (**missing**)
- along with censoring due to time and cost considerations on experiments.

For example, the fracture causes are due to:

- a surface defect (**mode 1**),
- an inner defect (**mode 2**), and
- an end effect at the clamp to hold the specimen (**mode 3**).
- The censored observations are denoted by **0**.
  (The observations were censored at **150** in the previous data set).

NOTE: In competing risks literature, **missing cause** is called **masking**.

The competing risks can be modeled as a system in series.




Let $X = \min(X_1, X_2, \ldots, X_J)$. Then the cdf of $X$ is easily obtained as

$$
\begin{aligned}
F(x|\boldsymbol{\Theta}) &= 1 - P[X > x] = 1 - P[\min(X_1, X_2, \ldots, X_J) > x] \\
&= 1 - P[X_1 > x, X_2 > x, \ldots, X_J > x] \\
&= 1 - P[X_1 > x] \cdot P[X_2 > x] \cdot \cdots \cdot P[X_J > x] \\
&= 1 - \prod_{j=1}^{J} \left\{ 1 - F_j(x|\boldsymbol{\theta}_j) \right\},
\end{aligned}
$$

where $\boldsymbol{\Theta} = (\boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \ldots, \boldsymbol{\theta}_J)$.

- Let $p$ be the dimension of $\boldsymbol{\theta}_j$. Then $\boldsymbol{\Theta} = (\boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \ldots, \boldsymbol{\theta}_J)$ has $J \times p$ parameters (curse of high dimensionality).
- Masking (missing cause) make the likelihood function more complex.
- EM method can help the likelihood function simple by **partitioning** the likelihood (low dimension).

### Brief Literature Review

- Cox (1959): Exponential with only two causes but no masking.
- Herman and Patell (1971): Exponential with multiple causes but no masking.
- Miyakawa (1984): Exponential with only two causes, only complete masking, but no censoring.
- Usher and Hodgson (1988), Usher and Guess (1989), Guess et al. (1991), and Reiser et al. (1995): general masking problem, but mainly on Exponential.

## Recent work

- Park and Kulasekera (2004): Exponential and Weibull with multiple causes, censoring, and masking (only complete masking). They provide the closed-form MLEs for Exponential. For Weibull, the closed-form MLE is available only when the common shape parameter is estimated by the likelihood function. They also show the bound for the Weibull shape estimate.
- Albert and Baxter (1995): EM for Exponential with partial masking.
- Park (2005) and Park and Padgett (2006): EM for various distributions including Exponential, Weibull, (log)normal and Wald with multiple causes, censoring,and general masking.
- Extension to Load-sharing problem: Park (2010, 2013).

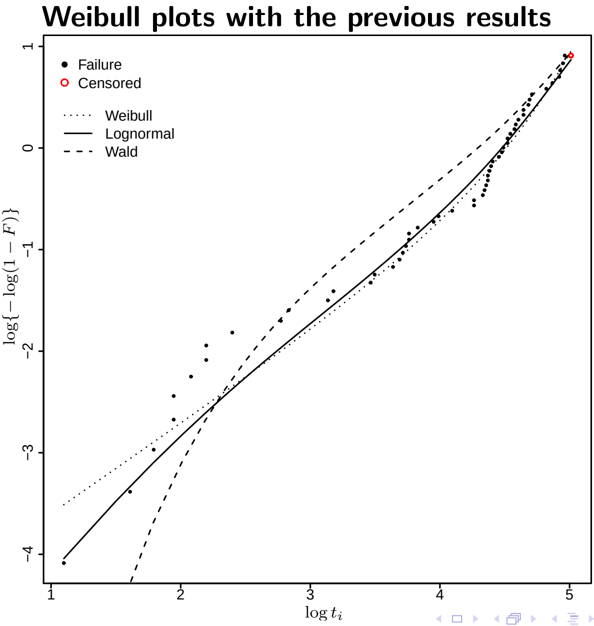
## Revisit: Strength data with three fracture causes (modes)

| Mode | Weibull $\lambda^{(j)}$ | Weibull $\alpha^{(j)}$ | Lognormal $\mu^{(j)}$ | Lognormal $\sigma^{(j)}$ | Wald $\mu^{(j)}$ | Wald $\lambda^{(j)}$ |
|---|---|---|---|---|---|---|
| Surface | $2.615 \times 10^{-10}$ | 4.308 | 5.0390 | 0.360 | 165.4 | 1172.5 |
| Inner defect | $8.916 \times 10^{-6}$ | 2.329 | 4.853 | 0.6732 | 169.4 | 261.6 |
| End effect | $1.131 \times 10^{-2}$ | 0.880 | 4.693 | 1.7034 | 7478.7 | 30.2 |

## MSEs under the Competing Models

| Model | Weibull | Lognormal | Wald |
|---|---|---|---|
| MSE | 0.1322911 | **0.1314757** | 0.1396442 |

Note: For R program, refer to `Talk-R.r` at ▸ 2020/Talk-R


Weibull plots with the previous results
Failure
Censored
Weibull
Lognormal
Wald
$\log\{-\log(1-F)\}$
$\log t_i$
1
0
−1
−2
−3
−4
1
2
3
4
5

### Load-sharing models

- Consider a multi-component system connected in parallel, in which components fail one by one.
- Total load or traffic applied to the system is redistributed among the remaining surviving components as component fails.
- Kim and Kvam (2004) provided an important first step in drawing parametric inference on load-sharing properties. (high-dimensional, but no closed-form solution).

### Recent work

- Park (2010): Closed-form solution of Kim and Kvam (2004). But, only for exponential model.
- Park (2013): Extended to Weibull, etc.
  It may not be possible to obtain a closed-form solution.
  EM algorithm is used for partitioning (low-dimensional)

**Observed frequencies of intermittent inspection data (Nelson, 1982)**

| Inspection time | Observed failures |
|---|---|
| $0 \sim 6.12$ | 5 |
| $6.12 \sim 19.92$ | 16 |
| $19.92 \sim 29.64$ | 12 |
| $29.64 \sim 35.40$ | 18 |
| $35.40 \sim 39.72$ | 18 |
| $39.72 \sim 45.24$ | 2 |
| $45.24 \sim 52.32$ | 6 |
| $52.32 \sim 63.48$ | 17 |
| $63.48 \sim$ | 73 |

- These censored and grouped data can also be regarded as interval-censored data.
- Grouped data are popular in engineering experiments (Seo and Yum, 1993; Shapiro and Gulati, 1998; Xiong and Ji, 2004; Meeker, 1986; Nelson, 1990; Sun, 2006; Lee and Park, 2006; Park et al., 2017).

### Handling interval data is picky

- MLE is not in closed form.
- EM can be applied for normal, but not for Weibull.
- Thus, it is very picky to handle interval data.

### Recall EM algorithm

- **E-step:** $Q(\boldsymbol{\theta}|\boldsymbol{\theta}^{(t)}) = \int \ell^c(\boldsymbol{\theta}|\mathbf{y},\mathbf{z})p(\mathbf{z}|\mathbf{y},\boldsymbol{\theta}^{(t)})dz$
- **M-step:** $\boldsymbol{\theta}^{(t+1)} = \arg\max_{\boldsymbol{\theta}} Q(\boldsymbol{\theta}|\boldsymbol{\theta}^{(t)})$

**MC-EM** (Wei and Tanner, 1990a,b) and **Q-EM** (Park, 2018b) can ease this problem.

- **MC-EM**: uses stochastic Monte Carlo integration (more flexible).
- **Q-EM**: uses deterministic quantile integration (less flexible, but more accurate).

Note: For the analysis of the grouped data in (Nelson, 1982), refer to

## **Robust** control chart with **unbalanced** samples

- For balanced sample, see `rQCC` package (Park and Wang, 2020b). Due to missing, original sample can be unbalanced.
- For control chart with **unbalanced** samples, refer to Park and Wang (2020a).
  Note: this method is not robust.
- Thus, the future work would be:
  How to combine the methods of (Park and Wang, 2020b) (robust) and Park and Wang (2020a) (unbalanced) so that a new method is robust with unbalanced samples.
  $\Longrightarrow$ Robust control chart with unbalanced samples.

# 3. Future work: Competing risks

## Extension of Competing risks models

- Competing risks problems are solved with moderate tailed distribution. Thus, heavier tailed distributions can be considered, which may handle contaminated observations.
- Masking and censoring with competing risks are considered. Interval data with competing risks can be challenging.
- Load-sharing with Weibull distributions are solved. Various other distributions should be considered.
- EM works very well with competing risks model. Then, does MI work well? We need to compare these two methods. Since MI is less model-dependent (robust to miss-specifications), MI can be preferred if MI performs well.

# References


Albert, J. R. G. and Baxter, L. A. (1995). Applications of the EM algorithm to the analysis of life length data. Applied Statistics, 44:323–341.

Basu, A., Shioya, H., and Park, C. (2011). Statistical Inference: The Minimum Distance Approach. Monographs on Statistics and Applied Probability. Chapman & Hall.

Beetz, C. P. (1982). The analysis of carbon fibre strength distributions exhibiting multiple modes of failure. Fibre Science Technology, 16:45–59.

Casella, G. and Berger, R. L. (2002). Statistical Inference. Duxbury, Pacific Grove, CA, second edition.

Cho, B.-R. and Park, C. (2005). Robust design modeling and optimization with unbalanced data. Computers & Industrial Engineering, 48:173–180.

# References


Cox, D. R. (1959). The analysis of exponentially distributed lifetimes with two types of failures. Journal of the Royal Statistical Society B, 21:411–421.

Darwin, C. (1876). The Effect of Cross- and Self-fertilization in the Vegetable Kingdom. John Murry, London, 2nd edition.

Friendly, M., Dray, S., Wickham, H., Hanley, J., Murphy, D., and Li, P. (2018). `HistData`: Data sets from the history of statistics and data visualization. `https://CRAN.R-project.org/package=HistData`. R package version 0.8-4.

Goda, K. and Fukunaga, H. (1986). The evaluation of the strength distribution of silicon carbide and alumina fibres by a multi-modal Weibull distribution. Journal of Materials Science, 21:4475–4480.

Guess, F. M., Usher, J. S., and Hodgson, T. J. (1991). Estimating system and component reliabilities under partial information of the cause of failure. Journal of Statistical Planning and Inference, 29:75–85.

# References


Herman, R. J. and Patell, R. K. N. (1971). Maximum likelihood estimation for multi-risk model. Technometrics, 13:385–396.

Jeong, R., Son, S. B., Lee, H. J., and Kim, H. (2018). On the robustification of the *z*-test statistic. Presented at KIIE Conference, Gyeongju, Korea. April 6, 2018.

Kim, H. and Kvam, P. H. (2004). Reliability estimation based on system data with an unknown load share rule. Lifetime Data Analysis, 10:83–94.

Lee, S. B. and Park, C. (2006). Development of robust design optimization using incomplete data. Computers & Industrial Engineering, 50:345–356.

Meeker, W. Q. (1986). Planning life tests in which units are inspected from failure. IEEE Transactions on Reliability, 35:571–578.

Miyakawa, M. (1984). Analysis of incomplete data in competing risks model. IEEE Transactions on Reliability, 33:293–296.

Nelson, W. (1982). Applied Life Data Analysis. John Wiley & Sons, New York.

# References


Nelson, W. (1990). Accelerated Testing: Statistical Models, Test Plans, and Data Analyses. John Wiley & Sons, New York.

Park, C. (2005). Parameter estimation of incomplete data in competing risks using the EM algorithm. IEEE Transactions on Reliability, 54:282–290.

Park, C. (2010). Parameter estimation for reliability of load sharing systems. IIE Transactions, 42:753–765.

Park, C. (2013). Parameter estimation from load-sharing system data using the expectation-maximization algorithm. IIE Transactions, 45:147–163.

Park, C. (2018a). Note on the robustification of the Student *t*-test statistic using the median and the median absolute deviation. `https://arxiv.org/abs/1805.12256`. ArXiv e-prints.

Park, C. (2018b). A quantile variant of the Expectation-Maximization algorithm and its application to parameter estimation with interval data. Journal of Algorithms & Computational Technology, 12:253–272.

Park, C. and Kulasekera, K. B. (2004). Parametric inference of incomplete data with competing risks among several groups. IEEE Transactions on Reliability, 53:11–21.

Park, C. and Leeds, M. (2016). A highly efficient robust design under data contamination. Computers & Industrial Engineering, 93:131–142.

Park, C. and Padgett, W. J. (2006). Analysis of strength distributions of multi-modal failures using the EM algorithm. Journal of Statistical Computation and Simulation, 76:619–636.

Park, C. and Wang, M. (2018). Empirical distributions of the robustified $t$-test statistics. `https://arxiv.org/abs/1807.02215`. ArXiv e-prints.

Park, C. and Wang, M. (2020a). A study on the X-bar and S control charts with unequal sample sizes. Mathematics, 8(5):698.

Park, C. and Wang, M. (2020b). `rQCC`: Robust quality control chart. `https://CRAN.R-project.org/package=rQCC`. R package version 1.20.7 (published on July 5, 2020).

Park, J. P., Park, C., Cho, J., and Bahn, C. B. (2017). Effects of cracking test conditions on estimation uncertainty for Weibull parameters considering time-dependent censoring interval. Materials, 10(1):3.

Reiser, B., Guttman, I., Lin, D. K. J., Guess, F. M., and Usher, H. S. (1995). Bayesian inference for masked system lifetime data. Applied Statistics, 44:79–90.

Seo, S.-K. and Yum, B.-J. (1993). Estimation methods for the mean of the exponential distribution based on grouped & censored data. IEEE Transactions on Reliability, 42:87–96.

Shapiro, S. S. and Gulati, S. (1998). Estimating the mean of an exponential distribution from grouped observations. Journal of Quality Technology, 30:107–118.

Sun, J. (2006). The Statistical Analysis of Interval-censored Failure Time Data. Springer, New York.

# References

Usher, J. S. and Guess, F. M. (1989). An iterative approach for estimating component reliability from masked system life data. Quality and Reliability Engineering International, 5:257–261.

Usher, J. S. and Hodgson, T. J. (1988). Maximum likelihood analysis of component reliability using masked system life-test data. IEEE Transactions on Reliability, 37:550–555.

Wei, G. C. G. and Tanner, M. A. (1990a). A Monte Carlo implementation of the EM algorithm and the poor man's data augmentation algorithm. Journal of the American Statistical Association, 85:699–704.

Wei, G. C. G. and Tanner, M. A. (1990b). Posterior computations for censored regression data. Journal of the American Statistical Association, 85:829–839.

Xiong, C. and Ji, M. (2004). Analysis of grouped and censored data from step-stress life test. IEEE Transactions on Reliability, 53:22–28.