Applications (사례 연구)

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Overview

1. Robust statistics와 응용 사례
   - Robust statistics (Basic applications)
   - Robust statistics (t-test)
   - Robust statistics (RD with contaminated data)
2. Missing · Incomplete Data와 응용 사례
   - Missing · Incomplete (RD with unbalanced samples)
   - Missing · Incomplete (Competing risks with censoring, masking, etc.)
   - Missing · Incomplete (Load-sharing)
   - Missing · Incomplete (Grouped Data)
3. Future work (missing with contamination)

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Basic applications (estimating μ and σ)

Consider observations from $X_i \sim N(\mu, \sigma^2)$. We need to estimate μ and σ^2.

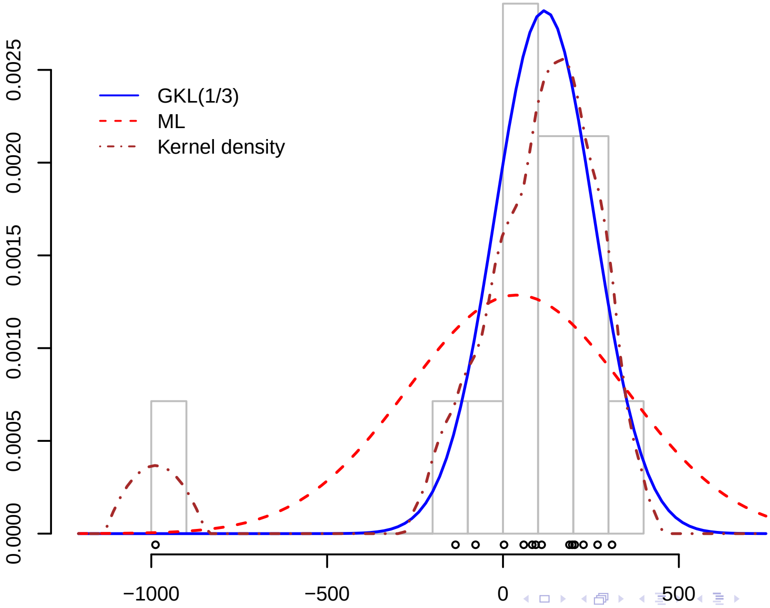
- MLE (maximum likelihood estimator)

$$\hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} X_i \quad \text{and} \quad \hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \hat{\mu})^2$$

- BUE (best unbiased estimator) or UMVUE

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \quad \text{and} \quad S^2 = \frac{1}{\boldsymbol{n-1}}\sum_{i=1}^{n}(X_i - \bar{X})^2$$

- MDE (minimum distance estimator): KL, GKL, etc.
  MLE is a special case of GKL (Basu et al., 2011) which can have robustness. MDE is asymptotically fully efficient, but its calculation is quite complex. Thus, HL and Shamos are recommended (Talk-2).


GKL(1/3)
ML
Kernel density
0.0000
0.0005
0.0010
0.0015
0.0020
0.0025
−1000
−500
0
500

Robust statistics와 응용 사례 (t-test)

Darwin (1876) collected the data: the growth of pairs of corn (especially Zea May) seedings, one produced by self-fertilization and the other produced by cross-fertilization. For the data set, see Friendly et al. (2018).

Cross	23.500	12.000	21	22	19.125	21.500	22.125	20.375
	18.25	21.625	23.25	21	22.125	23.0	12	
Self	17.375	20.375	20	20	18.375	18.625	18.625	15.250
	16.50	18.000	16.25	18	12.750	15.5	**18**	
Difference	6.125	−8.375	1.000	2.000	0.750	2.875	3.500	5.125
	1.750	3.625	7.000	3.000	9.375	7.500	−6.000	

We can test $H_0 : \mu_x = \mu_y$ and $H_1 : \mu_x \neq \mu_y$, equivalently, $H_0 : \mu_d = 0$ and $H_1 : \mu_d \neq 0$, where $\mu_d = \mu_x - \mu_y$. A typical paired sample t-test with

$$T = \frac{\bar{D} - 0}{S_D/\sqrt{n}},$$

where $\bar{D}$ and S_D are the sample mean and standard deviation, which this becomes a one-sample t-test.

Robust statistics와 응용 사례 (t-test)

Theorem 1 (Park, 2018)

Let $X_1, X_2, \ldots, X_n$ be a random sample from a normal distribution with mean μ and variance σ^2. Then we have

$$T_A = \sqrt{\frac{2n}{\pi}}\Phi^{-1}\left(\frac{3}{4}\right)\frac{\operatorname*{median}_{1\le i\le n} X_i - \mu}{\operatorname*{median}_{1\le i\le n}\left|X_i - \operatorname*{median}_{1\le i\le n} X_i\right|} \xrightarrow{d} N(0,1). \tag{1}$$

Theorem 2 (Jeong et al., 2018)

Let $X_1, X_2, \ldots, X_n$ be a random sample from a normal distribution with mean μ and variance σ^2. Then we have

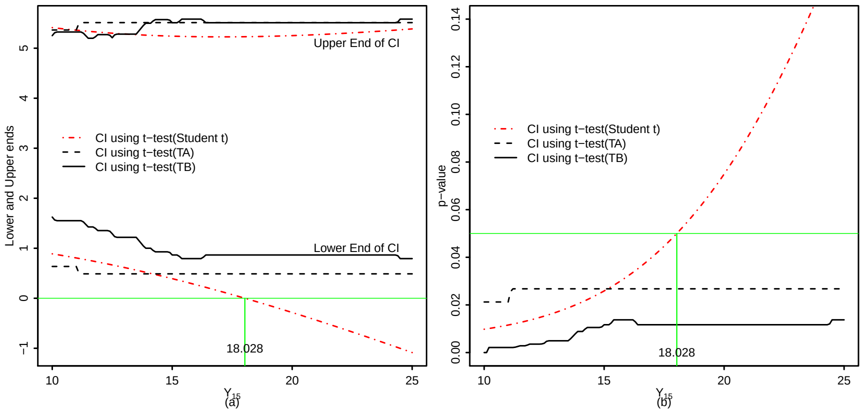
$$T_B = \sqrt{\frac{3n}{2\pi}}\Phi^{-1}\left(\frac{3}{4}\right)\frac{\operatorname*{median}_{i\le j}\left(X_i + X_j\right) - 2\mu}{\operatorname*{median}_{i\le j}\left(|X_i - X_j|\right)} \xrightarrow{d} N(0,1). \tag{2}$$

Robust statistics와 응용 사례 (t-test)

- The above Theorems work well with a large sample size because these are based on asymptotic standard normal distribution.
- Recently, Park and Wang (2018) developed the `rt.test` R package used the empirical distributions instead of the asymptotic standard normal distribution. Using the `rt.test`, we can carry out robustified t-test easily.
- Also, we can obtain the confidence intervals using the above robustified test statistics. By checking if zero is included inside each interval, we can test the hypothesis

$$H_0 : \mu_d = 0 \quad \text{and} \quad H_1 : \mu_d \neq 0.$$

Robust statistics와 응용 사례 (t-test)



Robust statistics와 응용 사례 (t-test)

This idea can be easily applied to control charting which is similar to a confidence interval.

- Phase I: use robustified control chart.
- Phase II: use conventional control chart.

This work is partially done (balanced case). See `rcc` function in `rQCC`.

```
> library("rQCC")
> help(rcc)
> tmp = c(
72, 84, 79, 49, 56, 87, 33, 42, 55, 73, 22, 60, 44, 80, 54, 74,
97, 26, 48, 58, 83, 89, 91, 62, 47, 66, 53, 58, 88, 50, 84, 69,
57, 47, 41, 46, 13, 10, 30, 32, 26, 39, 52, 48, 46, 27, 63, 34,
49, 62, 78, 87, 71, 63, 82, 55, 71, 58, 69, 70, 67, 69, 70, 94,
55, 63, 72, 49, 49, 51, 55, 76, 72, 80, 61, 59, 61, 74, 62, 57 )
> data2 = matrix(tmp, ncol=4, byrow=TRUE)
> rcc(data2, loc="HL2", scale="shamos")
     LCL       CL      UCL
36.99703 59.26250 81.52797
```

Robust statistics와 응용 사례 (Robust Design/강건설계)

Robust Design (Dual Response)

- The process mean response function.

$$\hat{M}(\mathbf{x}) = \hat{\beta}_0 + \sum_{i=1}^{k} \hat{\beta}_i x_i + \sum_{i=1}^{k} \hat{\beta}_{ii} x_i^2 + \sum_{i<j}^{k} \hat{\beta}_{ij} x_i x_j.$$

- The process variance response function.

$$\hat{V}(\mathbf{x}) = \hat{\eta}_0 + \sum_{i=1}^{k} \hat{\eta}_i x_i + \sum_{i=1}^{k} \hat{\eta}_{ii} x_i^2 + \sum_{i<j}^{k} \hat{\eta}_{ij} x_i x_j.$$

Note: original x are centered and re-scaled to $x \in [-1, 1]$.
We need to estimate, $M(\mathbf{x})$, $V(\mathbf{x})$, β and η.
The β and η can be estimated using the least squares method, etc.
For more details (HL and other estimators), see Talk-2 at ▸ Seminar/2018

Robust statistics와 응용 사례 (Robust Design/강건설계)

Refer to Park and Leeds (2016).

Method A: $\hat{M}(\mathbf{x})$ using the sample **mean** and
$\hat{V}(\mathbf{x})$ using the sample **variance**.
(BASELINE – without contamination!)

Method B: $\hat{M}(\mathbf{x})$: **median** and $\hat{V}(\mathbf{x})$: the squared **MAD**

Method C: $\hat{M}(\mathbf{x})$: **median** and $\hat{V}(\mathbf{x})$: the squared **IQR**

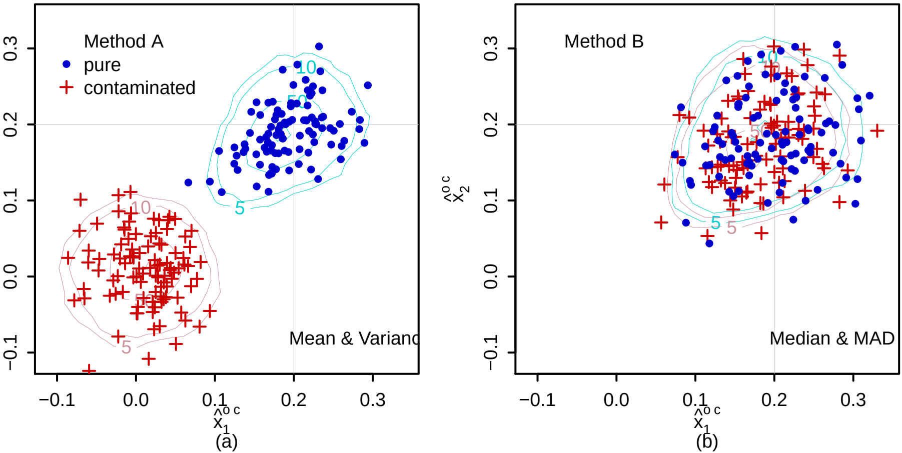
Method D: $\hat{M}(\mathbf{x})$: **HL (Hodges-Lehmann)** and
$\hat{V}(\mathbf{x})$: the squared **Shamos**

Method E: $\hat{M}(\mathbf{x})$: **median** and $\hat{V}(\mathbf{x})$: the squared **Shamos**

Method F: $\hat{M}(\mathbf{x})$: **HL** and $\hat{V}(\mathbf{x})$: the squared **MAD**.

Method G: $\hat{M}(\mathbf{x})$: **HL** and $\hat{V}(\mathbf{x})$: the squared **IQR**.

Application I (Robust Design with Contaminated Data)



Missing · Incomplete (RD with unbalanced samples)

Recall: Robust Design (Dual Response)

$$\text{Mean response}: \hat{M}(\mathbf{x}) = \hat{\beta}_0 + \sum_{i=1}^{k} \hat{\beta}_i x_i + \sum_{i=1}^{k} \hat{\beta}_{ii} x_i^2 + \sum_{i<j}^{k} \hat{\beta}_{ij} x_i x_j$$

$$\text{Variance response}: \hat{V}(\mathbf{x}) = \hat{\eta}_0 + \sum_{i=1}^{k} \hat{\eta}_i x_i + \sum_{i=1}^{k} \hat{\eta}_{ii} x_i^2 + \sum_{i<j}^{k} \hat{\eta}_{ij} x_i x_j$$

Unbalanced Data Set

i	x_{i1}	x_{i2}	Y_{ir_i}							$\overline{Y}_i$	S_i^2
1	-1	-1	84.3	57.0	56.5					65.93	253.06
2	0	-1	75.7	87.1	71.8	43.8	51.6			66.00	318.28
3	1	-1	65.9	47.9	63.3					59.03	94.65
4	-1	0	51.0	60.1	69.7	84.8	74.7			68.06	170.35
5	0	0	53.1	36.2	61.8	68.6	63.4	48.6	42.5	53.46	139.89
6	1	0	46.5	65.9	51.8	48.4	64.4			55.40	83.11
7	-1	1	65.7	79.8	79.1					74.87	63.14
8	0	1	54.4	63.8	56.2	48.0	64.5			57.38	47.54
9	1	1	50.7	68.3	62.9					60.63	81.29

Missing · Incomplete (RD with unbalanced samples)

Theorem 3

Let $Y_1, \dots, Y_r$ be a random sample of size r from the probability density function $f(y)$ with a finite fourth moment and let $\mu = E(Y)$ and $\theta_k = E(Y-\mu)^k$, $k = 2, 3, 4$. Then we have

$$\mathrm{Var}(\overline{Y}) = \frac{1}{r}\mu \quad \text{and} \quad \mathrm{Var}(S^2) = \frac{1}{r}(\theta_4 - \frac{r-3}{r-1}\theta_2^2).$$

Especially, if Y_i's have independent and identical normal distribution, then $\mathrm{Var}(S^2) = 2\sigma^4/(r-1)$.

Proof.

See Casella and Berger (2002). □

- $\mathrm{Var}(\overline{Y}) \propto 1/r$.
- $\mathrm{Var}(\overline{S^2}) \propto$?. Under the normality, $\mathrm{Var}(\overline{S^2}) \propto 1/(r-1)$.

Missing · Incomplete (RD with unbalanced samples)

Under the normality assumption, we can solve this problem by using the weighted least squares (WLS) regression instead of the ordinary least squares (OLS) regression (Cho and Park, 2005).

OLS versus WLS

- OLS: $\mathbf{Y} \sim N(\mathbf{X}\beta, \sigma^2\mathbf{I})$ and $\hat{\beta} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y}$.
- WLS: $\mathbf{Y} \sim N(\mathbf{X}\beta, \sigma^2\mathbf{W}^{-1})$ and $\hat{\beta} = (\mathbf{X}'\mathbf{W}\mathbf{X})^{-1}\mathbf{X}'\mathbf{W}\mathbf{Y}$.

Mean and Variance responses

- Mean response: $\mathbf{W} = \mathrm{diag}(r_1, r_2, \ldots, r_n)$
- Variance response: $\mathbf{W} = \mathrm{diag}(r_1 - 1, r_2 - 1, \ldots, r_n - 1)$

where n is the number of the design points.

What if normality is **not** satisfied

$\mathrm{Var}(\overline{S^2})$ is **not** proportional to $1/(r-1) \Rightarrow$ **multiple imputation**.

Missing · Incomplete (Competing risks)

....

Will be added

....

Missing · Incomplete (Load-sharing)

....

Will be added

....

Missing · Incomplete (Grouped Data)

....

Will be added

....

Future work (missing with contamination)

XXXX

- XXX
- XXX

References

Basu, A., Shioya, H., and Park, C. (2011). Statistical Inference: The Minimum Distance Approach. Monographs on Statistics and Applied Probability. Chapman & Hall.

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