오염 데이터와 그 대책

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August 5, 2020

Hosted by SEC

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Overview

Overview

Overview

Sample mean and variance

$$\overline{x} = \frac{1}{n}\sum_{i=1}^{n} x_i \text{ (mean)} \quad \text{and} \quad S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \overline{x})^2 \text{ with } S = \sqrt{S^2} \text{ (SD)}.$$

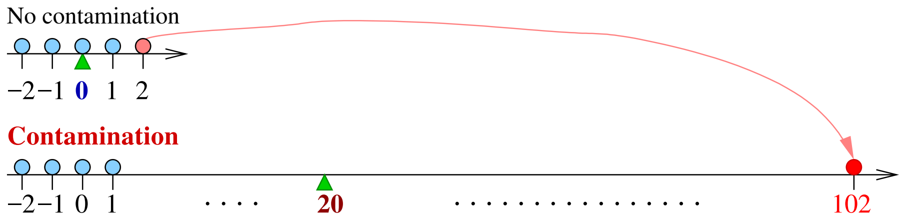
Example

	Original data $(-2, -1, 0, 1, \mathbf{2})$	Contaminated data $(-2, -1, 0, 1, \mathbf{102})$
Mean	0	**20**
Median	0	0
SD	1.58	**45.9**
IQR	2	2

Introduction: View from physics (mean vs. median)

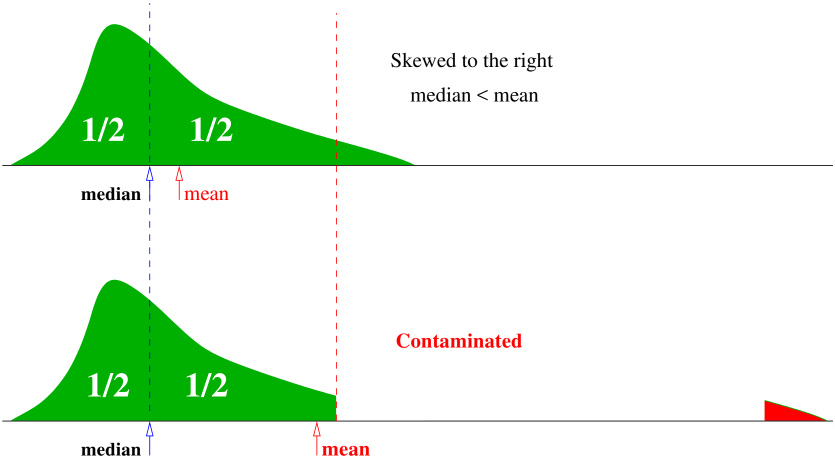
Why the mean is not robust? Recall mean: $\bar{X} = \frac{1}{n}X_1 + \frac{1}{n}X_2 + \cdots + \frac{1}{n}X_n$

- Data: $Y = (-2, -1, 0, 1, 2)$: mean $= 0$ and median $= 0$
- Data: $Y = (-2, -1, 0, 1, 102)$: mean $= 20$ and **median $= 0$**



The mean is the center of gravity while the median is just the middle one.
The mean is influenced by the gravity (leverage) while the median is NOT.

Introduction: View from physics (mean vs. median)



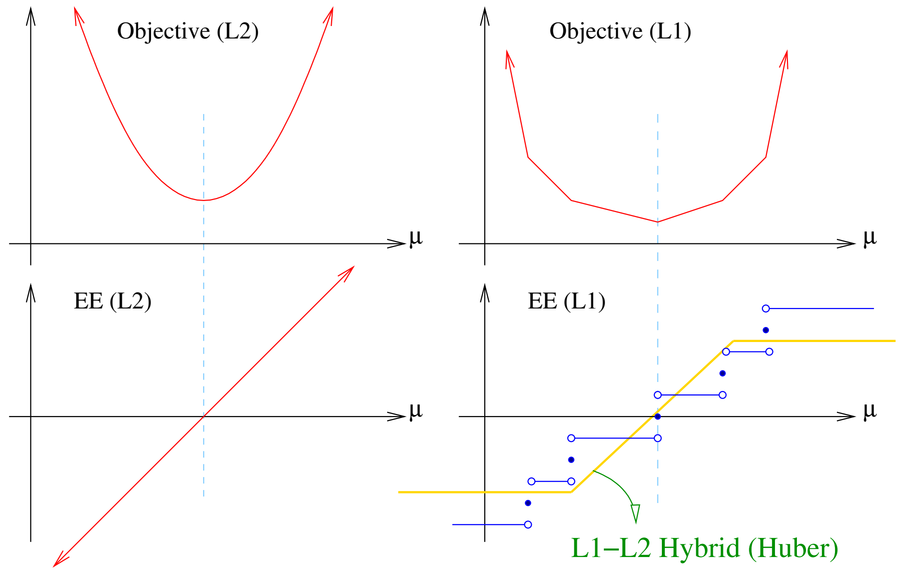
- The mean is the center of **gravity** of pdf ~~pizza~~.
- The median is the center of **area** (half-half area) of pdf ~~pizza~~.

View from distance (mean and median)

	Mean (minimizer of L_2)	Median (minimizer of L_1)
Objective	$\arg\min_{\mu} \sum_{i=1}^{n} (x_i - \mu)^2$	$\arg\min_{\mu} \sum_{i=1}^{n} \lvert x_i - \mu \rvert$
EE	$\sum_{i=1}^{n} (x_i - \mu)(-1) = 0$	$\sum_{i=1}^{n} (x_i - \mu)(-1) = 0$
EE	$g_{L_2}(\mu) = \mu - \bar{x} = 0$	$g_{L_1}(\mu) = \frac{1}{n} \sum_{i=1}^{n} (\mu - x_i) = 0$
Problem(?)	Too sensitive	Too dull

Solution to Problem:
Hybrid (L_1 and L_2): Winsorization or Huber estimation (filtering).

Introduction: View from distance (mean vs. median)



Introduction: Cocktail of mean and median

median of pairwise averages

Mean	Median
$\bar{x} = \dfrac{1}{n}(x_1 + x_2 + \cdots + x_n)$	$\tilde{x} = \operatorname*{median}_{1\le i\le n} x_i$

Hodges-Lehmann (HL)

$$\mathrm{HL} = \mathrm{median}\left(\frac{x_i + x_j}{2}\right)$$

For more details (HL and other estimators), see Talk-2 at ▸ Seminar/2018

What is the benefit of cocktail? How to measure their performance?

	Mean	Median	HL*	Huber*
Breakdown	0%	50%	29%	50%
ARE	100%	64%	96%	95%

Huber is **not** in closed form and its ARE **depends** on a threshold.

2. How to measure the performance of estimators?

Asymptotic property

- **Breakdown point**: the proportion of incorrect observations (e.g. arbitrarily large observations) an estimator can handle as the sample size n goes to infinity.
- **ARE** (asymptotic relative efficiency): the ratio of variance of MLE to variance of the corresponding estimator as the sample size n goes to infinity.
- **Fisher-consistency**: roughly unbiasedness as the sample size n goes to infinity. (Most of location estimators are Fisher-consistent, but scale estimators are not).

Finite-sample property (Park et al., 2020)

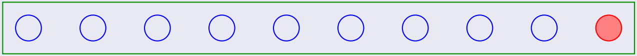
- Finite-sample Breakdown point
- ~~Finite-sample relative efficiency~~ $\Longrightarrow$ Relative Efficiency.
- ~~Finite-sample Fisher-consistency~~ $\Longrightarrow$ Unbiasedness with finite sample.

Performance: Breakdown point

Mean with a sample of size $n = 10$

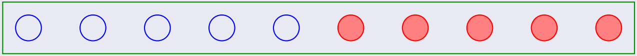
It breaks down even with a single extreme value (say, $Y_{10} = \infty$).

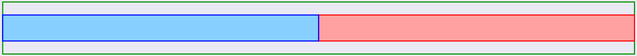
$$\mathrm{Mean} = \frac{1}{10}Y_1 + \frac{1}{10}Y_2 + \cdots + \frac{1}{10}Y_{10} \quad \text{(0\% finite-sample breakdown)}$$



Median with a sample of size $n = 10$

OK up to **4** extremes out of $n = \mathbf{10}$: $\mathrm{Median} = (Y_{(5)} + Y_{(6)})/2$.
That is, 40% finite-sample breakdown and 50% breakdown points.





Performance: Breakdown point

Other estimators (Breakdown point)

- For more details (HL and other estimators), see Talk-2 at Seminar/2018
- Refer to rQCC R Package (Park and Wang, 2020) at https://cran.r-project.org/web/packages/rQCC/

  Location: mean, median, Hodges-Lehmann(HL1, HL2, HL3)
  Scale: variance, Std. dev., range, MAD, Shamos

```
> install.packages("rQCC")  # if rQCC is not installed
> library("rQCC")
> help(package="rQCC")        # For help page
> finite.breakdown (n=10, method="median")
  0.4
> RE (n=10, method="median")
  0.7229247
```

Note: rQCC R Package is developed for robust quality control chart.

Performance: Finite-sample Breakdown point

Table 1: Finite-sample breakdown points (%).

n	median/MAD	HL1/Shamos	HL2	HL3
2	00.000	00.000	00.000	00.000
3	33.333	00.000	00.000	00.000
4	25.000	00.000	25.000	25.000
5	40.000	20.000	20.000	20.000
6	33.333	16.667	16.667	16.667
7	42.857	14.286	28.571	28.571
8	37.500	25.000	25.000	25.000
9	44.444	22.222	22.222	22.222
10	40.000	20.000	30.000	20.000
	$\cdots$	$\cdots$	$\cdots$	$\cdots$
50	48.000	28.000	28.000	28.000
	$\cdots$	$\cdots$	$\cdots$	$\cdots$
∞	50	$100(1-\sqrt{1/2})$	$100(1-\sqrt{1/2})$	$100(1-\sqrt{1/2})$

Performance: Efficiency, RE and ARE

The RE (relative efficiency) and ARE (asymptotic relative efficiency)

$$\mathrm{RE}(\hat{\theta}_1|\hat{\theta}_0) = \frac{\mathrm{Var}(\hat{\theta}_0)}{\mathrm{Var}(\hat{\theta}_1)} \times 100\%$$

$$\mathrm{ARE}(\hat{\theta}_1|\hat{\theta}_0) = \frac{\mathrm{AVar}(\hat{\theta}_0)}{\mathrm{AVar}(\hat{\theta}_1)} \times 100\%, \quad \text{as } n \to \infty$$

where $\hat{\theta}_0$ is a reference or baseline estimator (say, MLE without contamination).

- The larger RE or ARE, the better its performance.
- It is quite difficult to obtain the RE and ARE theoretically.
- See Park et al. (2020) for RE and Serfling (2011) for ARE.

Performance: Asymptotic Relative Efficiency

ARE of Location and Scale Estimators along with breakdown points

Location	Mean	Median	**HL**	**Huber**
Breakdown	0%	**50%**	29%	**50%**
ARE	**100%**	64%	**96%**	**95%**

Scale	SD	IQR	MAD	**Shamos**
Breakdown	0%	25%	**50%**	29%
ARE	**100%**	38%	37%	**86%**

Note: the above results are based on $n \to \infty$.

Performance: Relative Efficiency

Table 2: RE (%) of the median and Hodges-Lehmann estimators to the sample mean and those of the Fisher-consistent MAD and Shamos estimators to the sample standard deviation under the normal distribution.

n	median	HL1	HL2	HL3	MAD	Shamos
2	100.0	100.0	100.0	100.0	90.91	45.45
3	74.27	91.99	97.84	91.99	69.58	41.99
4	83.82	00.00	91.33	91.33	85.62	58.84
5	69.74	94.19	92.99	92.99	50.48	53.84
6	77.63	94.17	92.95	94.32	59.32	55.92
7	67.86	94.07	92.48	92.97	45.20	61.80
8	74.30	94.09	93.22	93.42	51.32	63.20
9	66.86	94.45	92.97	93.65	42.87	66.18
10	72.29	94.26	93.08	93.98	47.46	67.32
	...	...	...	...	...	...
50	65.50	95.25	94.95	95.11	38.44	82.08

Note: for $n = 2$, breakdown points of median, HL1, HL2, HL3 have zero.

Performance: Unbiasedness

Finite-sample unbiasedness and Fisher-consistency

As an illustration, the sample variance $S_n^2 = \frac{1}{n-1}\sum(X_i - \bar{X})^2$ is unbiased for σ^2 under $N(\mu, \sigma^2)$, but the standard deviation S_n is not unbiased. However, as $n \to \infty$, $S_n \to \sigma$. That is,

Estimator	Unbiased?	Fisher-consistent?[a]
S_n^2 for σ^2	$E(S_n^2) = \sigma^2$ (Yes)	$S_n^2 \to \sigma^2$ (Yes)
S_n for σ	$E(S_n) \neq \sigma$ (No)	$S_n \to \sigma$ (Yes)

With $c_4 = \sqrt{2/(n-1)} \cdot \Gamma(n/2)/\Gamma(n/2 - 1/2)$, S_n/c_4 is unbiased.

Estimator	Unbiased?	Fisher-consistent?
S_n^2 for σ^2	$E(S_n^2) = \sigma^2$ (Yes)	$S_n^2 \to \sigma^2$ (Yes)
S_n for σ	$E(S_n/c_4) = \sigma$ (Yes)	$S_n \to \sigma$ (Yes)

[a] For rigorous definition of Fisher-consistency, refer to Fisher (1922)

Performance: Unbiasedness

In general, location estimators are unbiased and Fisher-consistent as well. However, scale estimators are **neither** unbiased or Fisher-consistent.

Estimator	Original version	Fisher-consistent version
MAD	$\mathrm{median}\left\{\lvert Y_i - \mathrm{median}(Y)\rvert\right\}$	$\dfrac{\mathrm{median}\left\{\lvert Y_i - \mathrm{median}(Y)\rvert\right\}}{\Phi^{-1}(3/4)}$
IQR	$Y_{[3n/4]} - Y_{[n/4]}$	$\dfrac{Y_{[3n/4]} - Y_{[n/4]}}{\Phi^{-1}(3/4) - \Phi^{-1}(1/4)}$
Shamos	$\underset{i<j}{\mathrm{median}}\left(\lvert Y_i - Y_j\rvert\right)$	$\dfrac{\mathrm{median}_{i<j}\left(\lvert Y_i - Y_j\rvert\right)}{\sqrt{2}\Phi^{-1}(3/4)}$

For S_n, we have c_4 (finite-sample unbiasing factor) in closed form. But, for MAD and Shamos, it may be impossible to obtain finite-sample unbiasing factors in **closed** form. $\Rightarrow$ Simulation-based method.

Note: IQR is inferior to MAD or Shamos in a sense of both RE and breakdown.

Thus, we do not consider IQR. For more on simulation method, see Talk-5 at ▸ Seminar/2018

Performance: Unbiasedness

Refer to Section 3 of Park et al. (2020).

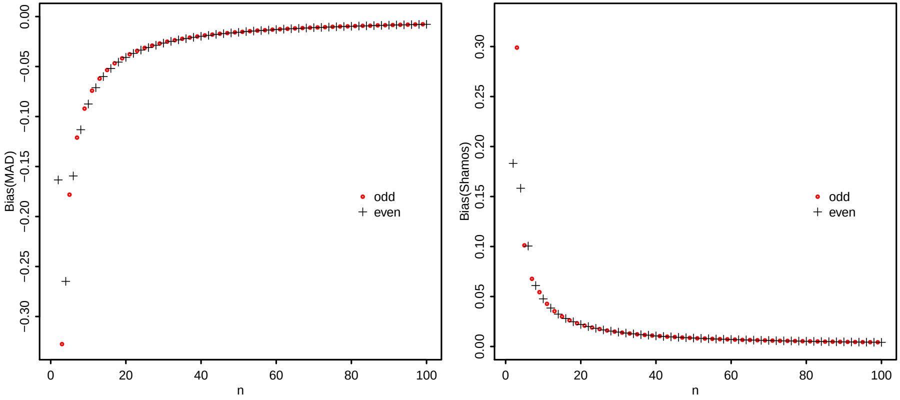


Figure 1: Empirical biases of the MAD and Shamos estimators with reference $\sigma = 1$.

Performance: Unbiased estimates with rQCC Package

A closed-form unbiasing factor c_4 for S_n, but not a for MAD or Shamos. However, we can obtain the unbiasing factors c_5 and c_6 for MAD and Shamos thru Monte Carlo simulation.

$$\mathrm{MAD(unbiased)} = \frac{1}{c_5(n)} \cdot \frac{\mathrm{median}\left\{\left|Y_i - \mathrm{median}(Y)\right|\right\}}{\Phi^{-1}(3/4)}$$

$$\mathrm{Shamos(unbiased)} = \frac{1}{c_6(n)} \cdot \frac{\mathrm{median}_{i<j}\left(\left|Y_i - Y_j\right|\right)}{\sqrt{2}\Phi^{-1}(3/4)}$$

```
> install.packages("rQCC")  # if rQCC is not installed
> library("rQCC")
> x = c(0:5, 50)
> mad(x)              # Fisher-consistent MAD
> mad.unbiased(x) # unbiased MAD
> shamos(x)             # Fisher-consistent Shamos
> shamos.unbiased(x) # unbiased Shamos
```

Performance: Summary

Recall: Location and Scale Estimators

Location	Mean	Median	**HL**	**Huber**
Breakdown	0%	**50%**	29%	**50%**
ARE	**100%**	64%	**96%**	**95%**
Scale	SD	IQR	MAD	**Shamos**
Breakdown	0%	25%	**50%**	29%
ARE	**100%**	38%	37%	**86%**

Note: the above results are based on $n \to \infty$.

- **Location**: HL (`rQCC` package) or Huber (`MASS` package)
- **Scale**: unbiased MAD, unbiased Shamos (`rQCC` package)
  Note: Rousseeuw and Croux (1993) estimator has 50% breakdown point with ARE 82%, but its finite-sample breakdown and RE are under development.
- ARE of mean and SD are under the ideal case (normal distribution without contamination). When contaminated or departed from normality, their AREs are really bad. After Winsorization, missing data occur.

3. Robustness in what sense

Robust to what?

- Robust to **contamination**: $\boxed{\textbf{Wrong}}$ observation (contamination). Influential observation (outlier and high leverage) in regression.
- Robust to **model departure (misspecification)** (usually departure from the normality): Robust to something $\boxed{\textbf{different}}$ from normal.
  For example, the *t*-test is robust to model departure. See Remark 8.3.1 of (Hogg et al., 2013) (roughly due to CLT).
  But, the *t*-test is not robust to contamination. See `rt.test` in Talk-5 at Seminar/2018. Also, MI (NORM) is robust to misspecification.
- Robust to **surprising observation**: An outlier (from a $\boxed{\textbf{heavy-taliled}}$ distribution).
  This is due to a nature of a heavy-tailed distribution (not contamination).
  For **Cauchy**, we can easily meet surprising outlying observations.
- Robust to **uncontrollable noise** (Robust Design): Robust to something $\boxed{\textbf{uncontrollable}}$.

4. Miscellaneous (symmetrically contaminated)

Location Parameter

- Data: $Y_0 = (-2, -1, 0, 1, 2)$: mean $= 0$ and median $= 0$
- Data: $Y_1 = (-102, -1, 0, 1, 102)$: mean $= 0$ and median $= 0$

Scale Parameter

Data	S^2	MAD	MAD(unbiased)	Shamos
Y_0	2.5	1.5	1.8	2.1
Y_1	5202.5	1.5	1.8	106.4

```
> install.packages("rQCC")  # if rQCC is not installed
> library("rQCC")
> finite.breakdown (n=5, method="mad")
[1] 0.4
> finite.breakdown (n=5, method="shamos")
[1] 0.2
```

Note: Refer to Talk-R.r at ▸ Talk-R

References

Fisher, R. A. (1922). On the mathematical foundations of theoretical statistics. Philosophical Transactions of the Royal Society of London. Series A, Containing Papers of a Mathematical or Physical Character, 222:309–368.

Hodges, J. L. and Lehmann, E. L. (1963). Estimates of location based on rank tests. Annals of Mathematical Statistics, 34:598–611.

Hogg, R. V., McKean, J. W., and Craig, A. T. (2013). Introduction to Mathematical Statistics. Pearson, Boston, MA, 7 edition.

Huber, P. J. (1964). Robust estimation of a location parameter. Annals of Mathematical Statistics, 35:73–101.

Huber, P. J. (1981). Robust Statistics. John Wiley & Sons, New York.

References

Park, C., Kim, H., and Wang, M. (2020). Investigation of finite-sample properties of robust location and scale estimators. Communication in Statistics – Simulation and Computation. doi:10.1080/03610918.2019.1699114.

Park, C. and Wang, M. (2020). `rQCC`: Robust quality control chart. `https://CRAN.R-project.org/package=rQCC`. R package version 1.20.7 (published on July 5, 2020).

Rousseeuw, P. J. and Croux, C. (1993). Alternatives to the median absolute deviation. Journal of the American Statistical Association, 88:1273–1283.

Serfling, R. J. (2011). Asymptotic relative efficiency in estimation. In Lovric, M., editor, Encyclopedia of Statistical Science, Part I, pages 68–82. Springer-Verlag, Berlin.