# Several inequalities used for construction of the $\bar{X}$ and $S$ control charts with unequal sample sizes

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# Overview

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## Shewhart-type $\bar{X}$ and $S$ charts

In general, the Shewhart-type control charts are made up of

- **UCL**: the upper control limit
- **CL**: the center line
- **LCL**: the lower control limit

This control charts are widely used and well developed when sample sizes are **equal**.

- However, when sample sizes are **not equal**, there exist only *ad hoc* methods.
- For the location estimator, two estimators are used. Then when one is better?
- Especially for the scale estimator, they are all **biased**.

# Basics on the Shewhart-type charts

## Assumptions

- $m$ samples but each subgroup has a **different sample size**.
- Let $X_{ij}$ denote the $i$th subgroup of size $n_i$, where $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n_i$.
- Assume that $X_{ij}$ are IID normal normal with mean $\mu$ and variance $\sigma^2$.

## $\bar{X}$ chart

For the $i$th ($i = 1, 2, \ldots, m$) subgroup, we have

$$\frac{\bar{X}_i - E(\bar{X}_i)}{\mathrm{SE}(\bar{X}_i)} \overset{\bullet}{\sim} N(0, 1),$$

where

$$\bar{X}_i = \frac{1}{n_i} \sum_{j=1}^{n_i} X_{ij}.$$

### $\bar{X}$ chart — continued

Since $E(\bar{X}_i) = \mu$ and $\mathrm{SE}(\bar{X}_i) = \sigma/n_i$, the $\mathrm{CL} \pm 3 \cdot \mathrm{SE}$ control limits are obtained by setting

$$\frac{\bar{X}_i - \mu}{\sigma/n_i} = \pm 3$$

and solving for $\bar{X}_i$.

Note that we can obtain the CI if we solve the above for $\mu$.

# Basics on the Shewhart-type charts

## $\bar{X}$ chart — continued

Solving for $\bar{X}_i$, the $\bar{X}$ control chart is constructed as follows

$$\begin{aligned}\textbf{UCL} &= \mu + \frac{3\sigma}{\sqrt{n_i}}\\ \textbf{CL} &= \mu\\ \textbf{LCL} &= \mu - \frac{3\sigma}{\sqrt{n_i}}.\end{aligned}$$

## Issues

- The population parameters $\mu$ and $\sigma$ are **not** known in general.
- Especially, conventional *ad hoc* scale estimators for $\sigma$ are **biased** when the sample sizes are **not** equal.

# Basics on the Shewhart-type charts

## $S$ chart

Similarly, we also have

$$\frac{S_i - E(S_i)}{\mathrm{SE}(S_i)} \overset{\bullet}{\sim} N(0,1),$$

where $S_i$ is the sample standard deviation with a sample of size $n_i$. For the $\mathrm{CL} \pm 3 \cdot \mathrm{SE}$ control limits, we can set up

$$\frac{S_i - E(S_i)}{\mathrm{SE}(S_i)} = \pm 3,$$

where $E(S_i) = c_4(n_i)\sigma$, $\mathrm{Var}(S_i) = (1 - c_4(n_i)^2)\sigma^2$, and

$$c_4(n_i) = \sqrt{\frac{2}{n_i - 1}} \frac{\Gamma(n_i/2)}{\Gamma((n_i - 1)/2)}$$

## $S$ chart — continued

Using $E(S_i) = c_4(n_i)\sigma$ and $\mathrm{Var}(S_i) = (1 - c_4(n_i)^2)\sigma^2$, we have

$$\frac{S_i - c_4(n_i)\sigma}{\sqrt{1 - c_4(n_i)^2}\,\sigma} = \pm 3.$$

Solving for $S_i$,

$$\begin{aligned} \textbf{UCL} &= c_4(n_i)\sigma + 3\sqrt{1 - c_4(n_i)^2}\sigma \\ \textbf{CL} &= c_4(n_i)\sigma \\ \textbf{LCL} &= c_4(n_i)\sigma - 3\sqrt{1 - c_4(n_i)^2}\sigma. \end{aligned}$$

## Issue again

- Especially, conventional *ad hoc* scale estimators for $\sigma$ are **biased**.

# Location parameter estimation

## Two conventional estimators for $\mu$

Montgomery (2013) provided the following location estimators for $\mu$ in Equations (6.2) and (6.30) of his book

- $\bar{\bar{X}}_A = \dfrac{\bar{X}_1 + \bar{X}_2 + \cdots + \bar{X}_m}{m} = \dfrac{1}{m}\sum_{i=1}^{m} \bar{X}_i$
- $\bar{\bar{X}}_B = \dfrac{n_1\bar{X}_1 + n_2\bar{X}_2 + \cdots + n_m\bar{X}_m}{N} = \dfrac{1}{N}\sum_{i=1}^{m} n_i\bar{X}_i.$

where $N = \sum_{i=1}^{m} n_i$.

Note that these estimators for $\mu$ are unbiased
while the *ad hoc* estimates for $\sigma$ are **biased**.

## Issue

Which one of $\bar{\bar{X}}_A$ or $\bar{\bar{X}}_B$ is selected?
**The one with the smaller variance!**

# Location parameter estimation

## Theorem 1

*If $X_{ij}$ are normal with mean $\mu$ and variance $\sigma^2$, then we have*

$$\mathrm{Var}(\bar{\bar{X}}_A) \geq \mathrm{Var}(\bar{\bar{X}}_B).$$

## Proof.

It is immediate from $\mathrm{Var}(\bar{X}_i) = \sigma^2/n_i$ that we have

- $\mathrm{Var}(\bar{\bar{X}}_A) = \frac{1}{m^2}\sum_{i=1}^m \mathrm{Var}(\bar{X}_i) = \dfrac{\sigma^2}{m^2}\displaystyle\sum_{i=1}^m \frac{1}{n_i} = \dfrac{\sigma^2}{m}\dfrac{1}{\bar{n}_h}$
- $\mathrm{Var}(\bar{\bar{X}}_B) = \frac{1}{N^2}\sum_{i=1}^m n_i^2 \mathrm{Var}(\bar{X}_i) = \dfrac{\sigma^2}{N^2}\displaystyle\sum_{i=1}^m n_i = \dfrac{\sigma^2}{N} = \dfrac{\sigma^2}{m}\dfrac{1}{\bar{n}_a}$

where

$$\bar{n}_h = \left( \frac{1}{m} \sum_{i=0}^{m} n_i^{-1} \right)^{-1} \quad \text{and} \quad \bar{n}_a = \frac{1}{m} \sum_{i=1}^{m} n_i.$$

Then we have

$$\frac{\mathrm{Var}(\bar{\bar{X}}_A)}{\mathrm{Var}(\bar{\bar{X}}_B)} = \frac{\bar{n}_a}{\bar{n}_h}.$$

- $\bar{n}_h$ and $\bar{n}_a$ are harmonic mean and arithmetic mean, respectively.
- $\bar{n}_a \geq \bar{n}_h$.
- Thus, $\mathrm{Var}(\bar{\bar{X}}_A)/\mathrm{Var}(\bar{\bar{X}}_B) \geq 1$ which results in $\mathrm{Var}(\bar{\bar{X}}_A) \geq \mathrm{Var}(\bar{\bar{X}}_B)$.

$\square$

Which one of $\bar{\bar{X}}_A$ or $\bar{\bar{X}}_B$ is selected? **$\bar{\bar{X}}_B$!**

# Scale parameter estimation

## $\bar{X}$ chart — Recalled

Substituting $\mu = \bar{\bar{X}}_B$, we have

$$\begin{aligned} \textbf{UCL} &= \bar{\bar{X}}_B + \frac{3\sigma}{\sqrt{n_i}} \\ \textbf{CL} &= \bar{\bar{X}}_B \\ \textbf{LCL} &= \bar{\bar{X}}_B - \frac{3\sigma}{\sqrt{n_i}}. \end{aligned}$$

Thus, we need to estimate $\sigma$.

# Scale parameter estimation

Section 6.3.2 of Montgomery (2013) recommended the followings to estimate $\sigma$:

- $S_p$
  where $S_p^2 = \sum_{i=1}^{m}(n_i - 1)S_i^2/(N - m)$ (pooled sample variance).
- $\bar{S}^*$
  where $\bar{S}^* = \bar{S}/c_4(\bar{n})$, $\bar{S} = \sum_{i=1}^{m} S_i/m$ and $\bar{n} = \sum_{i=1}^{m} n_i/m$.

## Issues

- Are they unbiased?
- Which one is better?

# Scale parameter estimation

Note that $S_p^2$ is unbiased for $\sigma^2$ so $S_p$ should underestimate $\sigma$. This is easily proved using the Jensen's inequality as follows.

Since $\sqrt{T}$ is concave, we have

$$E(\sqrt{T}) < \sqrt{E(T)}.$$

With $T = S_p^2$, we have

$$E(\sqrt{S_p^2}) = \boldsymbol{E(S_p)} < \sqrt{E(S_p^2)} = \sqrt{\sigma^2} = \boldsymbol{\sigma} \implies E(S_p) < \sigma.$$

- $S_p$ always **underestimate** $\sigma$,
- so it is also **biased**.

Note: whenever $S^2$ is unbiased for $\sigma^2$, $S$ underestimates $\sigma$ regardless of an underlying distribution.

# Scale parameter estimation

Then how about $\bar{S}^*$?

**Recall $\bar{S}^*$**

$$\bar{S}^* = \frac{\bar{S}}{c_4(\bar{n})},$$

where $\bar{S} = \sum_{i=1}^{m} S_i/m$ and $\bar{n} = \sum_{i=1}^{m} n_i/m$.

It **underestimates** $\boldsymbol{\sigma}$ since

$$E[\bar{S}^*] = \frac{\frac{1}{m}\sum_{i=1}^{m} E[S_i]}{c_4(\bar{n})\sigma} = \frac{\frac{1}{m}\sum_{i=1}^{m} c_4(n_i)}{c_4(\bar{n})} \cdot \sigma < \sigma.$$

How to prove the above? The claim is to show (the proof is tricky)

$$\frac{1}{m}\sum_{i=1}^{m} c_4(n_i) < c_4(\bar{n}) \qquad \text{(the proof will be provided later)}$$

# Scale parameter estimation

As aforementioned, both $S_p$ and $\bar{S}^*$ underestimate $\sigma$ — **biased!**

## We propose

- $\bar{S}_A = \dfrac{S_1/c_4(n_1) + S_2/c_4(n_2) + \cdots + S_m/c_4(n_m)}{m} = \dfrac{1}{m}\displaystyle\sum_{i=1}^{m} \dfrac{S_i}{c_4(n_i)}$
- $\bar{S}_B = \dfrac{S_1 + S_2 + \cdots + S_m}{c_4(n_1) + c_4(n_2) + \cdots + c_4(n_m)} = \dfrac{\sum_{i=1}^{m} S_i}{\sum_{i=1}^{m} c_4(n_i)}$

Using $E(S_i) = c_4(n_i)\sigma$ under the normality assumption, we see that these are **unbiased** for $\sigma$.

## Issue

Which one is used?
That is, which one has a smaller variance?

**Answer**: $\bar{S}_B$ (the proof will be provided later).

# Scale parameter estimation

It is immediate upon using

$$\mathrm{Var}(S_i) = E(S_i^2) - E(S_i)^2 = \sigma^2\{1 - c_4(n_i)^2\}$$

that we have

$$\mathrm{Var}(\bar{S}_A) = \frac{\sigma^2}{m^2}\sum_{i=1}^{m}\left\{\frac{1}{c_4(n_i)^2} - 1\right\}$$

and

$$\mathrm{Var}(\bar{S}_B) = \sigma^2 \cdot \frac{\sum_{i=1}^{m}\{1 - c_4(n_i)^2\}}{\{\sum_{i=1}^{m} c_4(n_i)\}^2}.$$

In the following section, we will provide several inequalities which are needed for proving it.

# Inequalities and their proofs

In the previous sections, the followings are left to prove

- $\mathrm{Var}(\bar{S}_A) \geq \mathrm{Var}(\bar{S}_B)$
- $E[\bar{S}^*] = \dfrac{\frac{1}{m}\sum_{i=1}^{m} E[S_i]}{c_4(\bar{n})\sigma} = \dfrac{\frac{1}{m}\sum_{i=1}^{m} c_4(n_i)}{c_4(\bar{n})} \cdot \sigma < \sigma$

  Note: $\bar{S}^*$ is recommended for the estimator of $\sigma$ in Section 6.3.2 of Montgomery (2013).

- To prove the above, it is essential to figure out the behaviors of the $c_4$ function which is again given by

$$c_4(x) = \sqrt{\frac{2}{(x-1)}}\frac{\Gamma(x/2)}{\Gamma((x-1)/2)}$$

- To this end, we will use the Wallis (1656) and Watson (1959) representations.

**Wallis' production formula (Wallis, 1656)**

$$\frac{2}{1}\cdot\frac{4}{3}\cdot\frac{6}{5}\cdot\frac{8}{7}\cdot\frac{10}{9}\cdot\frac{12}{11}\cdots\to\infty \quad \text{(diverges to } \boldsymbol{\infty}\text{)}$$

$$\times\ \frac{2}{3}\cdot\frac{4}{5}\cdot\frac{6}{7}\cdot\frac{8}{9}\cdot\frac{10}{11}\cdot\frac{12}{13}\cdots\to 0 \quad \text{(trivially goes to } \mathbf{0}\text{)}$$

$$=\frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdot\frac{8}{7}\cdot\frac{8}{9}\cdot\frac{10}{9}\cdot\frac{10}{11}\cdots\to\frac{\boldsymbol{\pi}}{\mathbf{2}}$$

Another version is also given by

$$\frac{1}{\sqrt{\pi(n+\frac{1}{2})}} < \boxed{\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\cdots\cdot\frac{2n-1}{2n}} < \frac{1}{\sqrt{\pi n}}$$

# Inequalities and their proofs

The Wallis' production formula can also be rewritten as

$$\boxed{\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\cdots\cdot\frac{2n-1}{2n}} = \frac{1}{\sqrt{\pi(n+\theta)}},$$

where and $\boxed{n \geq 1}$ and $0 < \theta < \frac{1}{2}$.
This formula will play a pivotal role in proving the inequalities in this talk.

Note: it is provided in his book, *Arithmetica Infinitorum* in the year of 1656, written in Latin. Recently, Stedall (2004) translated into English.

- Extension of the Wallis' production formula by Watson (1959)
- $c_4(x)$ function written by the Watson's $\theta(x)$ function
- Using the Watson's $\theta(x)$, we can provide several useful results

Recall

$$\boxed{\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\cdots\cdot\frac{2n-1}{2n}} = \frac{1}{\sqrt{\pi(n+\theta)}},$$

where and $\boxed{n \geq 1}$ and $0 < \theta < \frac{1}{2}$. We have

$$\begin{aligned}\frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)} &= \frac{\sqrt{\pi}\cdot\frac{1}{2}}{1}\cdot\frac{1+\frac{1}{2}}{2}\cdot\frac{2+\frac{1}{2}}{3}\cdot\cdots\cdot\frac{n-\frac{1}{2}}{n} \\ &= \sqrt{\pi}\cdot\boxed{\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\cdots\cdot\frac{2n-1}{2n}} = \frac{1}{\sqrt{n+\theta}}\end{aligned}$$

Motivated by the above, Watson (1959) extended to

$$\frac{\Gamma(x+\frac{1}{2})}{\Gamma(x+1)} = \frac{1}{\sqrt{x+\theta(x)}}, \tag{1}$$

where $\boxed{x > -\frac{1}{2}}$ and $0 < \theta(x) < \frac{1}{2}$.

# Inequalities and their proofs

Solving (1) for $\theta(x)$, we have

$$\theta(x) = -x + \frac{\Gamma(x+1)^2}{\Gamma(x+\frac{1}{2})^2} = -x + x\frac{\Gamma(x)\Gamma(x+1)}{\Gamma(x+\frac{1}{2})^2}. \tag{2}$$

Since

$$c_4(x)^2 = \frac{1}{(x-1)/2}\frac{\Gamma(x/2)^2}{\Gamma((x-1)/2)^2} = \frac{\Gamma(x/2)^2}{\Gamma((x-1)/2)\Gamma((x+1)/2)},$$

we have

$$c_4(2x+1)^2 = \frac{\Gamma(x+\frac{1}{2})^2}{\Gamma(x)\Gamma(x+1)}. \tag{3}$$

Substituting (3) into (2), we have

$$\boxed{\theta(x) = -x + x \cdot \frac{1}{c_4(2x+1)^2}} \quad \text{or} \quad \boxed{c_4(2x+1) = \sqrt{\frac{x}{\theta(x)+x}}}$$

- We succeeded in rewriting $c_4$ function using the Watson's $\theta(x)$ function (extended version of Wallis product).
- Using the nice properties of $\theta(x)$, we can get several useful results easily.

**Properties of $\theta(x)$**

- $0 < \theta < \frac{1}{2}$.
- $\theta(x)$ is monotonically decreasing.

## Lemma 1

- $\sqrt{\dfrac{x-1}{x}} < c_4(x) < 1$

## Proof.

Recall $\theta(x) = -x + x/c_4(2x+1)^2$ and $0 < \theta(x) < \frac{1}{2}$. Let $x = (t-1)/2$. Then we have

$$0 < -x + \frac{x}{c_4(2x+1)^2} < \frac{1}{2} \implies 0 < -\left(\frac{t-1}{2}\right) + \frac{(t-1)/2}{c_4(t)^2} < \frac{1}{2}.$$

Note: $c_4(t)$ is defined on $t > 1$. Solving the above for $c_4(t)^2$, we have

$$\frac{t-1}{t} < c_4(t)^2 < 1.$$

$\square$

# Inequalities and their proofs

## Lemma 2

- $c_4(x)$ is monotonically increasing
- The function

$$\frac{1}{c_4(x)^2} - 1$$

is monotonically decreasing.

## Proof.

Proofs are omitted. Please refer to the manuscript. □

# Inequalities and their proofs

- Recall: we claim $\mathrm{Var}(\bar{S}_A) \geq \mathrm{Var}(\bar{S}_B)$.
- To prove this, we need Lemma 3 below.

**Lemma 3 (Chebyshev's sum inequality)**

If $a_1 \geq a_2 \geq \cdots \geq a_m$ and $b_1 \geq b_2 \geq \cdots \geq b_m$, then

$$m \sum_{i=1}^{m} a_i b_i \geq \sum_{i=1}^{m} a_i \cdot \sum_{i=1}^{m} b_i.$$

Similarly, if $a_1 \leq a_2 \leq \cdots \leq a_m$ and $b_1 \geq b_2 \geq \cdots \geq b_m$, then

$$m \sum_{i=1}^{m} a_i b_i \leq \sum_{i=1}^{m} a_i \cdot \sum_{i=1}^{m} b_i.$$

- The original formula is in an integral form (Chebyshev, 1882).
- Its proof can be found in Chebyshev (1883).

## Proof of Chebyshev's sum inequality.

When $a_i$ and $b_i$ are both decreasing, $(a_i - a_j)$ and $(b_i - b_j)$ have the **same sign** for any $i$ and $j$. Thus, we have

$$\sum_{i=1}^{m}\sum_{j=1}^{m}(a_i - a_j)(b_i - b_j) \geq 0,$$

which results in

$$2m\sum_{i=1}^{m} a_i b_i - 2\sum_{i=1}^{m} a_i \sum_{j=1}^{m} b_j \geq 0.$$

Next, when $a_i$ are increasing and $b_i$ are decreasing, $(a_i - a_j)$ and $(b_i - b_j)$ have **different signs** or zero for any $i$ and $j$, Thus, we have $\sum_{i=1}^{m}\sum_{j=1}^{m}(a_i - a_j)(b_i - b_j) \leq 0$. □

# Inequalities and their proofs

## Recall: we propose

- $$\bar{S}_A = \frac{S_1/c_4(n_1) + S_2/c_4(n_2) + \cdots + S_m/c_4(n_m)}{m} = \frac{1}{m}\sum_{i=1}^{m}\frac{S_i}{c_4(n_i)}$$
- $$\bar{S}_B = \frac{S_1 + S_2 + \cdots + S_m}{c_4(n_1) + c_4(n_2) + \cdots + c_4(n_m)} = \frac{\sum_{i=1}^m S_i}{\sum_{i=1}^m c_4(n_i)}$$

## Theorem 2

*We have* $\mathrm{Var}(\bar{S}_A) \geq \mathrm{Var}(\bar{S}_B)$.

## Recall: we showed

$\mathrm{Var}(\bar{S}_A) = \frac{\sigma^2}{m^2}\sum_{i=1}^m \left\{\frac{1}{c_4(n_i)^2} - 1\right\}$ and $\mathrm{Var}(\bar{S}_B) = \sigma^2 \cdot \frac{\sum_{i=1}^m \left\{1 - c_4(n_i)^2\right\}}{\left\{\sum_{i=1}^m c_4(n_i)\right\}^2}$

# Inequalities and their proofs

## Sketch proof.

It suffices to show

$$\frac{1}{m^2}\sum_{i=1}^{m}\left\{\frac{1}{c_4(n_i)^2}-1\right\} \geq \frac{\sum_{i=1}^{m}\left\{1-c_4(n_i)^2\right\}}{\left\{\sum_{i=1}^{m} c_4(n_i)\right\}^2}$$

- First, let $a_i = c_4(n_i)$ (increasing) and $b_i = 1/c_4(n_i)^2 - 1$ (decreasing)
- Apply the Chebyshev's sum inequality with $a_i$ and $b_i$. Then we have an intermediate result.

- Next, let $a_i = c_4(n_i)$ (increasing) and $b_i = 1/c_4(n_i) - c_4(n_i)$ (decreasing)
- Apply the Chebyshev's sum inequality again to the above intermediate result with $a_i$ and $b_i$. Then we have a result.

□

# Inequalities and their proofs

**Recall**

Section 6.3.2 of Montgomery (2013) uses

$$\bar{S}^* = \frac{\bar{S}}{c_4(\bar{n})},$$

where $\bar{S} = \sum_{i=1}^{m} S_i/m$ and $\bar{n} = \sum_{i=1}^{m} n_i/m$.

Claim: $\bar{S}^*$ **underestimates** $\boldsymbol{\sigma}$ since

$$E[\bar{S}^*] = \frac{\frac{1}{m}\sum_{i=1}^{m} E[S_i]}{c_4(\bar{n})\sigma} = \frac{\frac{1}{m}\sum_{i=1}^{m} c_4(n_i)}{c_4(\bar{n})} \cdot \sigma < \sigma.$$

It suffices to show

$$\frac{1}{m}\sum_{i=1}^{m} c_4(n_i) < c_4(\bar{n})$$

## Lemma 4

The $c_4(x)$ function is log-concave.

## Sketch proof.

Recall the $c_4$ function which is again given by

$$c_4(x) = \sqrt{\frac{2}{(x-1)}}\,\frac{\Gamma(x/2)}{\Gamma((x-1)/2)}$$

Thus, we have

$$c_4(x)^2 = \frac{1}{(x-1)/2}\,\frac{\Gamma(x/2)^2}{\Gamma((x-1)/2)^2} = \frac{\Gamma(x/2)^2}{\Gamma((x-1)/2)\Gamma((x+1)/2)}.$$

# Inequalities and their proofs

## Sketch proof – continued.

- Taking the logarithm and dividing by two, we have

$$\log c_4(x) = \log\Gamma\left(\frac{x}{2}\right) - \frac{1}{2}\log\Gamma\left(\frac{x-1}{2}\right) - \frac{1}{2}\log\Gamma\left(\frac{x+1}{2}\right).$$

- From Section 11.14 (iv) of Schilling (2005) and Merkle (1996), the second derivative of $\log\Gamma(x)$ can be written as the sum of the series so that we have

$$\frac{d^2}{dx^2}\log\Gamma(x) = \sum_{k=0}^{\infty}\frac{1}{(x+k)^2}. \tag{4}$$

## Sketch proof — continued.

After tedious algebra, we have

$$\frac{d^2}{dx^2}\log c_4(x) = \sum_{k=0}^{\infty}\frac{1}{(\frac{x}{2}+k)^2} - \frac{1}{2}\sum_{k=0}^{\infty}\frac{1}{(\frac{x-1}{2}+k)^2} - \frac{1}{2}\sum_{k=0}^{\infty}\frac{1}{(\frac{x+1}{2}+k)^2}$$
$$= \sum_{k=0}^{\infty}\frac{-12(x+2k)^2+4}{(x+2k)^2(x-1+2k)^2(x+1+2k)^2} < 0.$$

- Thus, we have $\frac{d^2}{dx^2}\big(-\log c_4(x)\big) > 0$ so $-\log c_4(x)$ is convex due to Theorem 6.4.6 of Bartle and Sherbert (2011).
- Then $\log c_4(x)$ is concave and $c_4(x)$ is thus log-concave, which completes the proof.

□

# Inequalities and their proofs

## Lemma 5

We have the following inequality

$$\frac{1}{m}\sum_{i=1}^{m} c_4(n_i) \leq c_4(\bar{n}).$$

## Proof.

The log-concavity of $c_4(x)$ from Lemma 4 guarantees that $c_4(x)$ is concave.
Thus, we can apply the Jensen's inequality to $c_4(x)$ and we have

$$\frac{1}{m}\sum_{i=1}^{m} c_4(n_i) \leq c_4(\bar{n}),$$

where $\bar{n} = \sum_{i=1}^{m} n_i/m$. This completes the proof. $\square$

# Summary

## Summary

- The $c_4$ function plays a pivotal role in constructing the $\bar{X}$ and $S$ charts.
- With the useful inequalities, we showed that the conventional *ad hoc* methods are **biased**, which are used for the $\bar{X}$ and $S$ charts.
- We also proposed new **unbiased** estimators for $\sigma$ which can be used for various control charts.
- Thus, we can construct **proper $\bar{X}$ and $S$ charts** based on the new **unbiased** estimators.
- The behaviors of the $c_4$ function are much understood thanks to the connection with Wallis' product and Watson representation.
- Hopefully, these results on $c_4$ can produce more useful inequalities.

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