

# 오염 데이터와 그 대책

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## 1 Introduction

- Illustrative Example
- View from physics (mean vs. median)
- View from distance (mean vs. median)
- Cocktail of mean and median

## 2 How to measure the performance of estimators?

- Performance
- Breakdown point
- Efficiency
- Unbiasedness (Fisher-consistency and Finite-sample)

## 3 Robustness in what sense

## 4 Miscellaneous

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# 1. Introduction: Illustrative Example

## Sample mean and variance

$$\bar{x} = \frac{1}{n} \sum_{i=1}^n x_i \text{ (mean)} \quad \text{and} \quad S^2 = \frac{1}{n-1} \sum_{i=1}^n (x_i - \bar{x})^2 \text{ with } S = \sqrt{S^2} \text{ (SD).}$$

## Illustrative Example

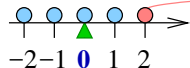
	Original data (-2, -1, 0, 1, <b>2</b> )	Contaminated data (-2, -1, 0, 1, <b>102</b> )
Mean	0	<b>20</b>
Median	0	0
SD	1.58	<b>45.9</b>
IQR	2	2

# 1. Introduction: View from physics (mean vs. median)

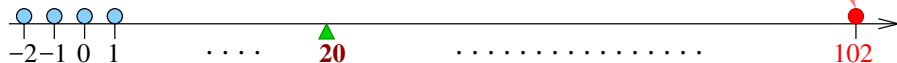
Why the mean is **not** robust? Recall mean:  $\bar{X} = \frac{1}{n}X_1 + \frac{1}{n}X_2 + \cdots + \frac{1}{n}X_n$

- Data:  $Y = (-2, -1, 0, 1, 2)$ : mean = 0 and median = 0
- Data:  $Y = (-2, -1, 0, 1, 102)$ : mean = 20 and **median** = 0

No contamination

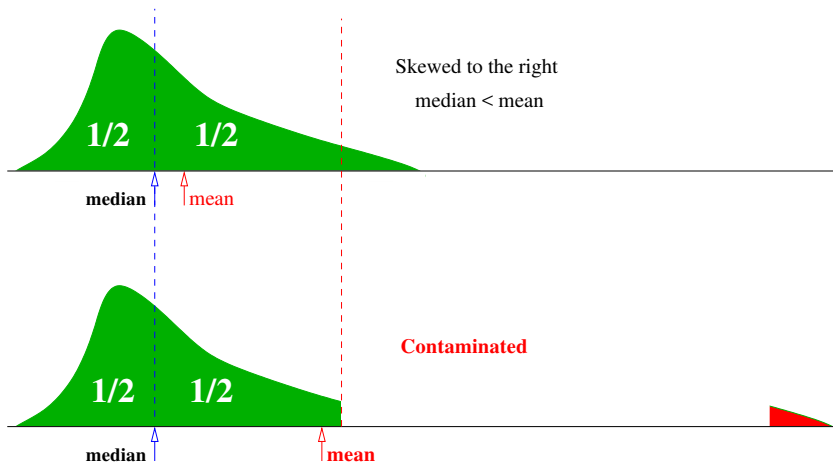


**Contamination**



The mean is the center of **gravity** while the median is just the middle one.  
The mean is influenced by the **gravity** (leverage) while the median is NOT.

# 1. Introduction: View from physics (mean vs. median)



- The mean is the center of **gravity** of pdf pizza.
- The median is the center of **area** (half-half area) of pdf pizza.



# 1. Introduction: View from distance (mean vs. median)

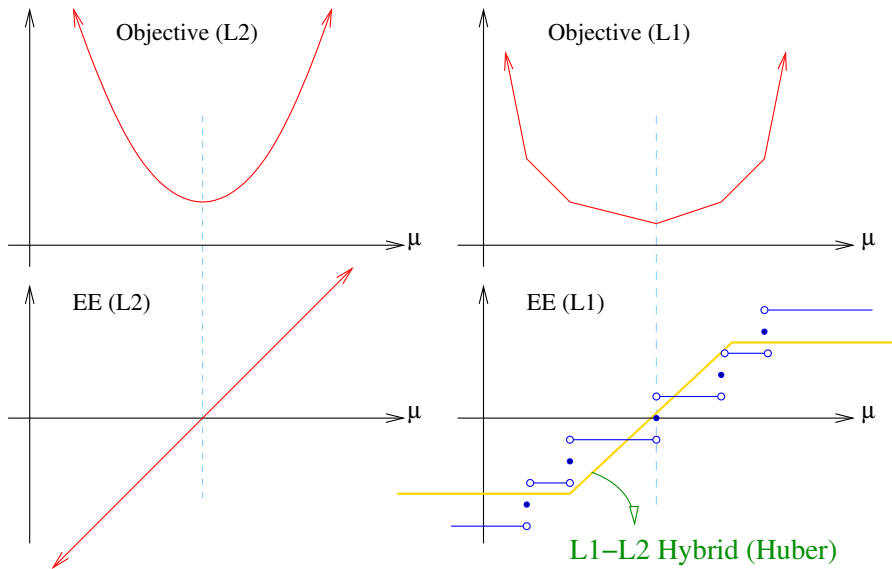
## View from distance (mean and median)

	Mean (minimizer of $L_2$ )	Median (minimizer of $L_1$ )
Objective	$\arg \min_{\mu} \sum_{i=1}^n (x_i - \mu)^2$	$\arg \min_{\mu} \sum_{i=1}^n  x_i - \mu $
EE	$\sum_{i=1}^n (x_i - \mu)(-1) = 0$	$\sum_{i=1}^n (x_i - \mu)(-1) = 0$
EE	$g_{L_2}(\mu) = \mu - \bar{x} = 0$	$g_{L_1}(\mu) = \frac{1}{n} \sum_{i=1}^n (\mu - x_i) = 0$
Problem(?)	Too sensitive	Too dull

Solution to Problem:

Hybrid ( $L_1$  and  $L_2$ ): Winsorization or Huber estimation (filtering).

# 1. Introduction: View from distance (mean vs. median)



# 1. Introduction: Cocktail of mean and median

## median of pairwise averages

Mean	Median
$\bar{x} = \frac{1}{n}(x_1 + x_2 + \cdots + x_n)$	$\tilde{x} = \text{median}_{1 \leq i \leq n} x_i$

Hodges-Lehmann (HL)

$$\text{HL} = \text{median} \left( \frac{x_i + x_j}{2} \right)$$

Note: Hodges-Lehmann (HL) has three different versions

$$\text{HL1} = \text{median}_{i < j} \left( \frac{x_i + x_j}{2} \right)$$

$$\text{HL2} = \text{median}_{i \leq j} \left( \frac{x_i + x_j}{2} \right)$$

$$\text{HL3} = \text{median}_{\forall (i,j)} \left( \frac{x_i + x_j}{2} \right)$$

# 1. Introduction: Cocktail of mean and median

What is the benefit of cocktail? How to measure their performance?

	Mean	Median	HL*	Huber*
Breakdown	0%	50%	29%	50%
ARE	100%	64%	96%	95%

Huber is **not** in closed form and its ARE **depends** on a threshold.

For more details (HL and other estimators), see Talk-2 at [▶ 2018/Seminar](#)

## 2. How to measure the performance of estimators?

### Asymptotic property

- **Breakdown point**: the proportion of incorrect observations (e.g. arbitrarily large observations) an estimator can handle as the sample size  $n$  goes to infinity.
- **ARE** (asymptotic relative efficiency): the ratio of variance of MLE to variance of the corresponding estimator as the sample size  $n$  goes to infinity.
- **Fisher-consistency**: roughly unbiasedness as the sample size  $n$  goes to infinity. (Most of location estimators are Fisher-consistent, but scale estimators are not).

### Finite-sample property (Park et al., 2020)

- Finite-sample Breakdown point
- ~~Finite-sample relative efficiency~~  $\implies$  Relative Efficiency
- ~~Finite-sample Fisher-consistency~~  $\implies$  Unbiasedness with finite sample

## 2. Performance: Breakdown point

### Mean with a sample of size $n = 10$

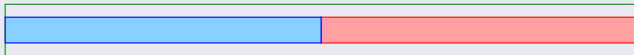
It breaks down even with a single extreme value (say,  $Y_{10} = \infty$ ).

$$\text{Mean} = \frac{1}{10} Y_1 + \frac{1}{10} Y_2 + \cdots + \frac{1}{10} Y_{10} \quad (0\% \text{ finite-sample breakdown})$$



### Median with a sample of size $n = 10$

OK up to **4** extremes out of  $n = 10$ : Median =  $(Y_{(5)} + Y_{(6)})/2$ .  
That is, 40% finite-sample breakdown and 50% breakdown points.



## 2. Performance: Breakdown point

### Other estimators (Breakdown point)

- **Location:** mean, median, Hodges-Lehmann(HL1, HL2, HL3)
- **Scale:** variance, Std. dev., range, MAD, Shamos

For more details, see rQCC R Package (Park and Wang, 2020) at <https://cran.r-project.org/web/packages/rQCC/>

```
> install.packages("rQCC") # if rQCC is not installed
> library("rQCC")
> help(package="rQCC")      # For help page
> finite.breakdown (n=10, method="median")
0.4
> RE (n=10, method="median")
0.7229247
```

Note: rQCC R Package is developed for robust quality control chart.

## 2. Performance: Finite-sample Breakdown point

Table 1: Finite-sample breakdown points (%).

$n$	median/MAD	HL1/Shamos	HL2	HL3
2	00.000	00.000	00.000	00.000
3	33.333	00.000	00.000	00.000
4	25.000	00.000	25.000	25.000
5	40.000	20.000	20.000	20.000
6	33.333	16.667	16.667	16.667
7	42.857	14.286	28.571	28.571
8	37.500	25.000	25.000	25.000
9	44.444	22.222	22.222	22.222
10	40.000	20.000	30.000	20.000
...	...	...	...	...
50	48.000	28.000	28.000	28.000
...	...	...	...	...
$\infty$	50	$100(1 - \sqrt{1/2})$	$100(1 - \sqrt{1/2})$	$100(1 - \sqrt{1/2})$



## 2. Performance: Efficiency, RE and ARE

The RE (relative efficiency) and ARE (asymptotic relative efficiency)

$$\text{RE}(\hat{\theta}_1 | \hat{\theta}_0) = \frac{\text{Var}(\hat{\theta}_0)}{\text{Var}(\hat{\theta}_1)} \times 100\%$$

$$\text{ARE}(\hat{\theta}_1 | \hat{\theta}_0) = \frac{\text{AVar}(\hat{\theta}_0)}{\text{AVar}(\hat{\theta}_1)} \times 100\%, \quad \text{as } n \rightarrow \infty$$

where  $\hat{\theta}_0$  is a reference or baseline estimator (say, MLE without contamination).

- The larger RE or ARE, the better its performance.
- It is quite difficult to obtain the RE and ARE theoretically.
- See Park et al. (2020) for RE and Serfling (2011) for ARE.

## 2. Performance: Asymptotic Relative Efficiency

### ARE of Location and Scale Estimators along with breakdown points

<b>Location</b>	Mean	Median	<b>HL</b>	<b>Huber</b>
Breakdown	<b>0%</b>	<b>50%</b>	29%	<b>50%</b>
ARE	<b>100%</b>	64%	<b>96%</b>	<b>95%</b>
<b>Scale</b>	SD	IQR	MAD	<b>Shamos</b>
Breakdown	<b>0%</b>	25%	<b>50%</b>	29%
ARE	<b>100%</b>	38%	37%	<b>86%</b>

Note: the above results are based on  $n \rightarrow \infty$ .

## 2. Performance: Relative Efficiency

**Table 2:** RE (%) of the median and Hodges-Lehmann estimators to the sample mean and those of the **Fisher-consistent** MAD and Shamos estimators to the sample standard deviation under the normal distribution.

$n$	median	HL1	HL2	HL3	MAD	Shamos
2	100.0	100.0	100.0	100.0	90.91	45.45
3	74.27	91.99	97.84	91.99	69.58	41.99
4	83.82	00.00	91.33	91.33	85.62	58.84
5	69.74	94.19	92.99	92.99	50.48	53.84
6	77.63	94.17	92.95	94.32	59.32	55.92
7	67.86	94.07	92.48	92.97	45.20	61.80
8	74.30	94.09	93.22	93.42	51.32	63.20
9	66.86	94.45	92.97	93.65	42.87	66.18
10	72.29	94.26	93.08	93.98	47.46	67.32
	...	...	...	...	...	...
50	65.50	95.25	94.95	95.11	38.44	82.08

Note: for  $n = 2$ , breakdown points of median, HL1, HL2, HL3 have zero.

## 2. Performance: Unbiasedness

### Finite-sample unbiasedness and Fisher-consistency

As an illustration, the sample variance  $S_n^2 = \frac{1}{n-1} \sum (X_i - \bar{X})^2$  is unbiased for  $\sigma^2$  under  $N(\mu, \sigma^2)$ , but the standard deviation  $S_n$  is not unbiased. However, as  $n \rightarrow \infty$ ,  $S_n \rightarrow \sigma$ . That is,

Estimator	Unbiased?	Fisher-consistent? <sup>a</sup>
$S_n^2$ for $\sigma^2$	$E(S_n^2) = \sigma^2$ (Yes)	$S_n^2 \rightarrow \sigma^2$ (Yes)
$S_n$ for $\sigma$	$E(S_n) \neq \sigma$ (No)	$S_n \rightarrow \sigma$ (Yes)

With  $c_4 = \sqrt{2/(n-1)} \cdot \Gamma(n/2)/\Gamma(n/2 - 1/2)$ ,  $S_n/c_4$  is unbiased.

Estimator	Unbiased?	Fisher-consistent?
$S_n^2$ for $\sigma^2$	$E(S_n^2) = \sigma^2$ (Yes)	$S_n^2 \rightarrow \sigma^2$ (Yes)
$S_n$ for $\sigma$	$E(S_n/c_4) = \sigma$ (Yes)	$S_n \rightarrow \sigma$ (Yes)

<sup>a</sup>For rigorous definition of Fisher-consistency, refer to Fisher (1922)

## 2. Performance: Unbiasedness

In general, location estimators are unbiased and Fisher-consistent as well. However, scale estimators are **neither** unbiased or Fisher-consistent.

Estimator	Original version	Fisher-consistent version
MAD	$\text{median} \{  Y_i - \text{median}(Y)  \}$	$\frac{\text{median} \{  Y_i - \text{median}(Y)  \}}{\Phi^{-1}(3/4)}$
IQR	$Y_{[3n/4]} - Y_{[n/4]}$	$\frac{Y_{[3n/4]} - Y_{[n/4]}}{\Phi^{-1}(3/4) - \Phi^{-1}(1/4)}$
Shamos	$\text{median}_{i < j} ( Y_i - Y_j )$	$\frac{\text{median}_{i < j} ( Y_i - Y_j )}{\sqrt{2}\Phi^{-1}(3/4)}$

For  $S_n$ , we have  $c_4$  (finite-sample unbiasing factor) in closed form. But, for **MAD** and **Shamos**, it may be **impossible** to obtain finite-sample unbiasing factors in **closed** form.  $\Rightarrow$  **Simulation-based** method.

Note: IQR is inferior to MAD or Shamos in a sense of both RE and breakdown.

Thus, we do not consider IQR. For more on simulation method, see Talk-5 at [2018/Seminar](#)

## 2. Performance: Unbiasedness

Refer to Section 3 of Park et al. (2020).

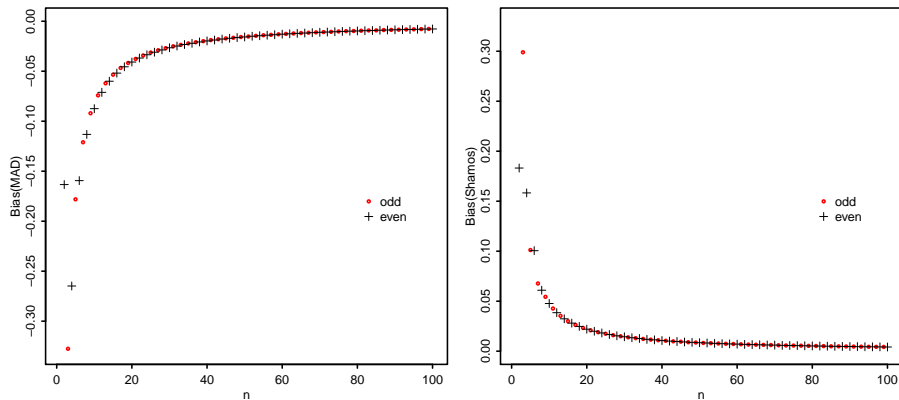


Figure 1: Empirical biases of the MAD and Shamos estimators with reference  $\sigma = 1$ .

## 2. Performance: Unbiased estimates with rQCC Package

A closed-form unbiasing factor  $c_4$  for  $S_n$ , but not a for MAD or Shamos. However, we can obtain the **unbiasing factors**  $c_5$  and  $c_6$  for MAD and Shamos thru Monte Carlo simulation.

$$\text{MAD}(\text{unbiased}) = \frac{1}{c_5(n)} \cdot \frac{\text{median} \{ |Y_i - \text{median}(Y)| \}}{\Phi^{-1}(3/4)}$$

$$\text{Shamos}(\text{unbiased}) = \frac{1}{c_6(n)} \cdot \frac{\text{median}_{i < j} (|Y_i - Y_j|)}{\sqrt{2}\Phi^{-1}(3/4)}$$

```
> install.packages("rQCC") # if rQCC is not installed
> library("rQCC")
> x = c(0:5, 50)
> mad(x) # Fisher-consistent MAD
> mad.unbiased(x) # unbiased MAD
> shamos(x) # Fisher-consistent Shamos
> shamos.unbiased(x) # unbiased Shamos
```

## 2. Performance: Summary

### Recall: Location and Scale Estimators

Location	Mean	Median	HL	Huber
Breakdown	0%	50%	29%	50%
ARE	100%	64%	96%	95%
Scale	SD	IQR	MAD	Shamos
Breakdown	0%	25%	50%	29%
ARE	100%	38%	37%	86%

Note: the above results are based on  $n \rightarrow \infty$  (**asymptotic results**).

- **Location:** HL (rQCC package) or Huber (MASS package)
- **Scale:** unbiased MAD, unbiased Shamos (rQCC package)  
Note: Rousseeuw and Croux (1993) estimator has high breakdown point (50%) with ARE 82%, but its **finite-sample** breakdown and RE are under development.
- AREs of mean and SD are under the ideal case (normal distribution without contamination). When contaminated or departed from normality, their AREs are really bad.



### 3. Robustness in what sense

#### Robust to what?

- Robust to **contamination**: **Wrong** observation (contamination). Influential observation (outlier and high leverage) in regression.
- Robust to **model departure (misspecification)** (usually departure from the normality): Robust to something **different** from normal.  
For example, the  $t$ -test is robust to model departure. See Remark 8.3.1 of (Hogg et al., 2013) (roughly due to CLT).  
But, the  $t$ -test is **not** robust to contamination. See `rt.test` in Talk-5 at [2018/Seminar](#). Also, multiple imputation (MI) is robust to misspecification.
- Robust to **surprising observation**: An outlier (from a **heavy-tailed** distribution).  
This is due to a **nature** of a heavy-tailed distribution (not contamination). For **Cauchy**, we can easily meet surprising outlying observations.
- Robust to **uncontrollable noise** (Robust Design): Robust to something **uncontrollable**.

## 4. Miscellaneous (symmetrically contaminated)

### Location Parameter

- Data:  $Y_0 = (-2, -1, 0, 1, 2)$ : mean = 0 and median = 0
- Data:  $Y_1 = (-102, -1, 0, 1, 102)$ : mean = 0 and median = 0

### Scale Parameter

Data	$S^2$	MAD	MAD(unbiased)	Shamos
$Y_0$	2.5	1.5	1.8	2.1
$Y_1$	5202.5	1.5	1.8	106.4

```
> install.packages("rQCC") # if rQCC is not installed
> library("rQCC")
> finite.breakdown (n=5, method="mad")
[1] 0.4
> finite.breakdown (n=5, method="shamos")
[1] 0.2
```

Note: Refer to Talk-R.r at [2020/Talk-R](#)

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