Several inequalities used for construction of the $\bar{X}$ and S control charts with unequal sample sizes

Chanseok PARK (朴燦奭)

Applied Statistics Laboratory
Department of Industrial Engineering
Pusan National University

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Overview

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2. Location parameter estimation
3. Scale parameter estimation
4. Inequalities and their proofs
5. Summary

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Shewhart-type $\bar{X}$ and S charts

Shewhart-type $\bar{X}$ and S charts

In general, the Shewhart-type control charts are made up of

- **UCL**: the upper control limit
- **CL**: the center line
- **LCL**: the lower control limit

This control charts are widely used and well developed when sample sizes are **equal**.

- However, when sample sizes are **not equal**, there exist only *ad hoc* methods.
- For the location estimator, two estimators are used. Then when one is better?
- Especially for the scale estimator, they are all **biased**.

Basics on the Shewhart-type charts

Assumptions

- m samples but each subgroup has a **different sample size**.
- Let X_{ij} denote the ith subgroup of size n_i, where $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n_i$.
- Assume that X_{ij} are IID normal normal with mean μ and variance σ^2.

$\bar{X}$ chart

For the ith $(i = 1, 2, \ldots, m)$ subgroup, we have

$$\frac{\bar{X}_i - E(\bar{X}_i)}{\mathrm{SE}(\bar{X}_i)} \overset{\bullet}{\sim} N(0, 1),$$

where

$$\bar{X}_i = \frac{1}{n_i} \sum_{j=1}^{n_i} X_{ij}.$$

Basics on the Shewhart-type charts

$\bar{X}$ chart — continued

Since $E(\bar{X}_i) = \mu$ and $\mathrm{SE}(\bar{X}_i) = \sigma/n_i$, the $\mathrm{CL} \pm 3 \cdot \mathrm{SE}$ control limits are obtained by setting

$$\frac{\bar{X}_i - \mu}{\sigma/n_i} = \pm 3$$

and solving for $\bar{X}_i$.

Note that we can obtain the CI if we solve the above for μ.

Basics on the Shewhart-type charts

$\bar{X}$ chart — continued

Solving for $\bar{X}_i$, the $\bar{X}$ control chart is constructed as follows

$$\begin{aligned}\textbf{UCL} &= \mu + \frac{3\sigma}{\sqrt{n_i}}\\ \textbf{CL} &= \mu\\ \textbf{LCL} &= \mu - \frac{3\sigma}{\sqrt{n_i}}.\end{aligned}$$

Issues

- The population parameters μ and σ are **not** known in general.
- Especially, conventional *ad hoc* scale estimators for σ are **biased** when the sample sizes are **not** equal.

Basics on the Shewhart-type charts

S chart

Similarly, we also have

$$\frac{S_i - E(S_i)}{\mathrm{SE}(S_i)} \overset{\bullet}{\sim} N(0,1),$$

where S_i is the sample standard deviation with a sample of size n_i.
For the $\mathrm{CL} \pm 3 \cdot \mathrm{SE}$ control limits, we can set up

$$\frac{S_i - E(S_i)}{\mathrm{SE}(S_i)} = \pm 3,$$

where $E(S_i) = c_4(n_i)\sigma$, $\mathrm{Var}(S_i) = (1 - c_4(n_i)^2)\sigma^2$, and

$$c_4(n_i) = \sqrt{\frac{2}{n_i - 1}} \frac{\Gamma(n_i/2)}{\Gamma((n_i-1)/2)}$$

Basics on the Shewhart-type charts

S chart — continued

Using $E(S_i) = c_4(n_i)\sigma$ and $\mathrm{Var}(S_i) = (1 - c_4(n_i)^2)\sigma^2$, we have

$$\frac{S_i - c_4(n_i)\sigma}{\sqrt{1 - c_4(n_i)^2}\,\sigma} = \pm 3.$$

Solving for S_i,

$$\begin{aligned}
\mathbf{UCL} &= c_4(n_i)\sigma + 3\sqrt{1 - c_4(n_i)^2}\sigma \\
\mathbf{CL} &= c_4(n_i)\sigma \\
\mathbf{LCL} &= c_4(n_i)\sigma - 3\sqrt{1 - c_4(n_i)^2}\sigma.
\end{aligned}$$

Issue again

- Especially, conventional *ad hoc* scale estimators for σ are **biased**.

Location parameter estimation

Two conventional estimators for μ

Montgomery (2013) provided the following location estimators for μ in Equations (6.2) and (6.30) of his book

- $\bar{\bar{X}}_A = \dfrac{\bar{X}_1 + \bar{X}_2 + \cdots + \bar{X}_m}{m} = \dfrac{1}{m}\sum_{i=1}^{m} \bar{X}_i$
- $\bar{\bar{X}}_B = \dfrac{n_1\bar{X}_1 + n_2\bar{X}_2 + \cdots + n_m\bar{X}_m}{N} = \dfrac{1}{N}\sum_{i=1}^{m} n_i\bar{X}_i.$

where $N = \sum_{i=1}^{m} n_i$.

Note that these estimators for μ are unbiased
while the *ad hoc* estimates for σ are **biased**.

Issue

Which one of $\bar{\bar{X}}_A$ or $\bar{\bar{X}}_B$ is selected?
The one with the smaller variance!

Location parameter estimation

Theorem 1

If X_{ij} are normal with mean μ and variance σ^2, then we have

$$\mathrm{Var}(\bar{\bar{X}}_A) \geq \mathrm{Var}(\bar{\bar{X}}_B).$$

Proof.

It is immediate from $\mathrm{Var}(\bar{X}_i) = \sigma^2/n_i$ that we have

- $\mathrm{Var}(\bar{\bar{X}}_A) = \frac{1}{m^2} \sum_{i=1}^{m} \mathrm{Var}(\bar{X}_i) = \dfrac{\sigma^2}{m^2} \displaystyle\sum_{i=1}^{m} \frac{1}{n_i} = \frac{\sigma^2}{m} \frac{1}{\bar{n}_h}$
- $\mathrm{Var}(\bar{\bar{X}}_B) = \frac{1}{N^2} \sum_{i=1}^{m} n_i^2 \mathrm{Var}(\bar{X}_i) = \dfrac{\sigma^2}{N^2} \displaystyle\sum_{i=1}^{m} n_i = \frac{\sigma^2}{N} = \frac{\sigma^2}{m} \frac{1}{\bar{n}_a}$

where

$$\bar{n}_h = \left(\frac{1}{m}\sum_{i=0}^{m} n_i^{-1}\right)^{-1} \quad \text{and} \quad \bar{n}_a = \frac{1}{m}\sum_{i=1}^{m} n_i.$$

Then we have

$$\frac{\mathrm{Var}(\bar{\bar{X}}_A)}{\mathrm{Var}(\bar{\bar{X}}_B)} = \frac{\bar{n}_a}{\bar{n}_h}.$$

- $\bar{n}_h$ and $\bar{n}_a$ are harmonic mean and arithmetic mean, respectively.
- $\bar{n}_a \geq \bar{n}_h$.
- Thus, $\mathrm{Var}(\bar{\bar{X}}_A)/\mathrm{Var}(\bar{\bar{X}}_B) \geq 1$ which results in $\mathrm{Var}(\bar{\bar{X}}_A) \geq \mathrm{Var}(\bar{\bar{X}}_B)$.

$\square$

Which one of $\bar{\bar{X}}_A$ or $\bar{\bar{X}}_B$ is selected? **$\bar{\bar{X}}_B$!**

Scale parameter estimation

$\bar{X}$ chart — Recalled

Substituting $\mu = \bar{\bar{X}}_B$, we have

$$\begin{aligned}\textbf{UCL} &= \bar{\bar{X}}_B + \frac{3\sigma}{\sqrt{n_i}}\\ \textbf{CL} &= \bar{\bar{X}}_B\\ \textbf{LCL} &= \bar{\bar{X}}_B - \frac{3\sigma}{\sqrt{n_i}}.\end{aligned}$$

Thus, we need to estimate σ.

Scale parameter estimation

Section 6.3.2 of Montgomery (2013) recommended the followings to estimate σ:

- S_p
  where $S_p^2 = \sum_{i=1}^m (n_i - 1)S_i^2/(N - m)$ (pooled sample variance).

- $\bar{S}^*$
  where $\bar{S}^* = \bar{S}/c_4(\bar{n})$, $\bar{S} = \sum_{i=1}^m S_i/m$ and $\bar{n} = \sum_{i=1}^m n_i/m$.

Issues

- Are they unbiased?
- Which one is better?

Scale parameter estimation

Note that S_p^2 is unbiased for σ^2 so S_p should underestimate σ. This is easily proved using the Jensen's inequality as follows.

Since $\sqrt{T}$ is concave, we have

$$E(\sqrt{T}) < \sqrt{E(T)}.$$

With $T = S_p^2$, we have

$$E(\sqrt{S_p^2}) = \boldsymbol{E(S_p)} < \sqrt{E(S_p^2)} = \sqrt{\sigma^2} = \boldsymbol{\sigma} \implies E(S_p) < \sigma.$$

- S_p always **underestimate** σ,
- so it is also **biased**.

Note: whenever S^2 is unbiased for σ^2, S underestimates σ regardless of an underlying distribution.

Scale parameter estimation

Then how about $\bar{S}^*$?

Recall $\bar{S}^*$

$$\bar{S}^* = \frac{\bar{S}}{c_4(\bar{n})},$$

where $\bar{S} = \sum_{i=1}^{m} S_i/m$ and $\bar{n} = \sum_{i=1}^{m} n_i/m$.

It **underestimates** $\boldsymbol{\sigma}$ since

$$E[\bar{S}^*] = \frac{\frac{1}{m}\sum_{i=1}^{m} E[S_i]}{c_4(\bar{n})\sigma} = \frac{\frac{1}{m}\sum_{i=1}^{m} c_4(n_i)}{c_4(\bar{n})} \cdot \sigma < \sigma.$$

How to prove the above? The claim is to show (the proof is tricky)

$$\frac{1}{m}\sum_{i=1}^{m} c_4(n_i) < c_4(\bar{n}) \qquad \text{(the proof will be provided later)}$$

Scale parameter estimation

As aforementioned, both S_p and $\bar{S}^*$ underestimate σ — **biased!**

We propose

- $$\bar{S}_A = \frac{S_1/c_4(n_1) + S_2/c_4(n_2) + \cdots + S_m/c_4(n_m)}{m} = \frac{1}{m}\sum_{i=1}^{m}\frac{S_i}{c_4(n_i)}$$
- $$\bar{S}_B = \frac{S_1 + S_2 + \cdots + S_m}{c_4(n_1) + c_4(n_2) + \cdots + c_4(n_m)} = \frac{\sum_{i=1}^{m} S_i}{\sum_{i=1}^{m} c_4(n_i)}$$

Using $E(S_i) = c_4(n_i)\sigma$ under the normality assumption, we see that these are **unbiased** for σ.

Issue

Which one is used?
That is, which one has a smaller variance?

Answer: $\bar{S}_B$ (the proof will be provided later).

Scale parameter estimation

It is immediate upon using

$$\mathrm{Var}(S_i) = E(S_i^2) - E(S_i)^2 = \sigma^2\{1 - c_4(n_i)^2\}$$

that we have

$$\mathrm{Var}(\bar{S}_A) = \frac{\sigma^2}{m^2}\sum_{i=1}^{m}\left\{\frac{1}{c_4(n_i)^2} - 1\right\}$$

and

$$\mathrm{Var}(\bar{S}_B) = \sigma^2 \cdot \frac{\sum_{i=1}^{m}\{1 - c_4(n_i)^2\}}{\left\{\sum_{i=1}^{m} c_4(n_i)\right\}^2}.$$

In the following section, we will provide several inequalities which are needed for proving it.

Inequalities and their proofs

In the previous sections, the followings are left to prove

- $\mathrm{Var}(\bar{S}_A) \geq \mathrm{Var}(\bar{S}_B)$
- $E[\bar{S}^*] = \dfrac{\frac{1}{m}\sum_{i=1}^{m} E[S_i]}{c_4(\bar{n})\sigma} = \dfrac{\frac{1}{m}\sum_{i=1}^{m} c_4(n_i)}{c_4(\bar{n})} \cdot \sigma < \sigma$

  Note: $\bar{S}^*$ is recommended for the estimator of σ in Section 6.3.2 of Montgomery (2013).

- To prove the above, it is essential to figure out the behaviors of the c_4 function which is again given by

$$c_4(x) = \sqrt{\frac{2}{(x-1)}}\,\frac{\Gamma(x/2)}{\Gamma((x-1)/2)}$$

- To this end, we will use the Wallis (1656) and Watson (1959) representations.

Wallis' production formula (Wallis, 1656)

$$\frac{2}{1}\cdot\frac{4}{3}\cdot\frac{6}{5}\cdot\frac{8}{7}\cdot\frac{10}{9}\cdot\frac{12}{11}\cdots\to\infty \quad \text{(diverges to } \infty\text{)}$$

$$\times\ \frac{2}{3}\cdot\frac{4}{5}\cdot\frac{6}{7}\cdot\frac{8}{9}\cdot\frac{10}{11}\cdot\frac{12}{13}\cdots\to 0 \quad \text{(trivially goes to } \mathbf{0}\text{)}$$

$$= \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdot\frac{8}{7}\cdot\frac{8}{9}\cdot\frac{10}{9}\cdot\frac{10}{11}\cdots\to\frac{\pi}{2}$$

Another version is also given by

$$\frac{1}{\sqrt{\pi(n+\frac{1}{2})}} < \boxed{\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\cdots\cdot\frac{2n-1}{2n}} < \frac{1}{\sqrt{\pi n}}$$

The Wallis' production formula can also be rewritten as

$$\boxed{\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \cdots \cdot \frac{2n-1}{2n}} = \frac{1}{\sqrt{\pi(n+\theta)}},$$

where and $\boxed{n \geq 1}$ and $0 < \theta < \frac{1}{2}$.
This formula will play a pivotal role in proving the inequalities in this talk.

Note: it is provided in his book, *Arithmetica Infinitorum* in the year of 1656, written in Latin. Recently, Stedall (2004) translated into English.

- Extension of the Wallis' production formula by Watson (1959)
- $c_4(x)$ function written by the Watson's $\theta(x)$ function
- Using the Watson's $\theta(x)$, we can provide several useful results

Recall

$$\boxed{\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\cdots\cdot\frac{2n-1}{2n}} = \frac{1}{\sqrt{\pi(n+\theta)}},$$

where and $\boxed{n \geq 1}$ and $0 < \theta < \frac{1}{2}$. We have

$$\frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)} = \frac{\sqrt{\pi}\cdot\frac{1}{2}}{1}\cdot\frac{1+\frac{1}{2}}{2}\cdot\frac{2+\frac{1}{2}}{3}\cdot\cdots\cdot\frac{n-\frac{1}{2}}{n}$$
$$= \sqrt{\pi}\cdot\boxed{\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\cdots\cdot\frac{2n-1}{2n}} = \frac{1}{\sqrt{n+\theta}}$$

Motivated by the above, Watson (1959) extended to

$$\frac{\Gamma(x+\frac{1}{2})}{\Gamma(x+1)} = \frac{1}{\sqrt{x+\theta(x)}}, \tag{1}$$

where $\boxed{x > -\frac{1}{2}}$ and $0 < \theta(x) < \frac{1}{2}$.

Solving (1) for $\theta(x)$, we have

$$\theta(x) = -x + \frac{\Gamma(x+1)^2}{\Gamma(x+\frac{1}{2})^2} = -x + x\frac{\Gamma(x)\Gamma(x+1)}{\Gamma(x+\frac{1}{2})^2}. \qquad (2)$$

Since

$$c_4(x)^2 = \frac{1}{(x-1)/2}\frac{\Gamma(x/2)^2}{\Gamma((x-1)/2)^2} = \frac{\Gamma(x/2)^2}{\Gamma((x-1)/2)\Gamma((x+1)/2)},$$

we have

$$c_4(2x+1)^2 = \frac{\Gamma(x+\frac{1}{2})^2}{\Gamma(x)\Gamma(x+1)}. \qquad (3)$$

Substituting (3) into (2), we have

$$\boxed{\theta(x) = -x + x\cdot\frac{1}{c_4(2x+1)^2}} \quad \text{or} \quad \boxed{c_4(2x+1) = \sqrt{\frac{x}{\theta(x)+x}}}$$

Inequalities and their proofs

- We succeeded in rewriting c_4 function using the Watson's $\theta(x)$ function (extended version of Wallis product).
- Using the nice properties of $\theta(x)$, we can get several useful results easily.

Properties of $\theta(x)$

- $0 < \theta < \frac{1}{2}$.
- $\theta(x)$ is monotonically decreasing.

Lemma 1

- $\sqrt{\dfrac{x-1}{x}} < c_4(x) < 1$

Proof.

Recall $\theta(x) = -x + x/c_4(2x+1)^2$ and $0 < \theta(x) < \frac{1}{2}$. Let $x = (t-1)/2$. Then we have

$$0 < -x + \frac{x}{c_4(2x+1)^2} < \frac{1}{2} \implies 0 < -\left(\frac{t-1}{2}\right) + \frac{(t-1)/2}{c_4(t)^2} < \frac{1}{2}.$$

Note: $c_4(t)$ is defined on $t > 1$. Solving the above for $c_4(t)^2$, we have

$$\frac{t-1}{t} < c_4(t)^2 < 1.$$

$\square$

Lemma 2

- $c_4(x)$ is monotonically increasing
- The function

$$\frac{1}{c_4(x)^2} - 1$$

is monotonically decreasing.

Proof.

Proofs are omitted. Please refer to the manuscript. □

Inequalities and their proofs

- Recall: we claim $\mathrm{Var}(\bar{S}_A) \geq \mathrm{Var}(\bar{S}_B)$.
- To prove this, we need Lemma 3 below.

Lemma 3 (Chebyshev's sum inequality)

If $a_1 \geq a_2 \geq \cdots \geq a_m$ and $b_1 \geq b_2 \geq \cdots \geq b_m$, then

$$m \sum_{i=1}^{m} a_i b_i \geq \sum_{i=1}^{m} a_i \cdot \sum_{i=1}^{m} b_i .$$

Similarly, if $a_1 \leq a_2 \leq \cdots \leq a_m$ and $b_1 \geq b_2 \geq \cdots \geq b_m$, then

$$m \sum_{i=1}^{m} a_i b_i \leq \sum_{i=1}^{m} a_i \cdot \sum_{i=1}^{m} b_i .$$

- The original formula is in an integral form (Chebyshev, 1882).
- Its proof can be found in Chebyshev (1883).

Proof of Chebyshev's sum inequality.

When a_i and b_i are both decreasing, $(a_i - a_j)$ and $(b_i - b_j)$ have the **same sign** for any i and j. Thus, we have

$$\sum_{i=1}^{m}\sum_{j=1}^{m}(a_i - a_j)(b_i - b_j) \geq 0,$$

which results in

$$2m\sum_{i=1}^{m} a_i b_i - 2\sum_{i=1}^{m} a_i \sum_{j=1}^{m} b_j \geq 0.$$

Next, when a_i are increasing and b_i are decreasing, $(a_i - a_j)$ and $(b_i - b_j)$ have **different signs** or zero for any i and j, Thus, we have $\sum_{i=1}^{m}\sum_{j=1}^{m}(a_i - a_j)(b_i - b_j) \leq 0$. $\square$

Inequalities and their proofs

Recall: we propose

- $\bar{S}_A = \dfrac{S_1/c_4(n_1) + S_2/c_4(n_2) + \cdots + S_m/c_4(n_m)}{m} = \dfrac{1}{m}\displaystyle\sum_{i=1}^{m} \dfrac{S_i}{c_4(n_i)}$
- $\bar{S}_B = \dfrac{S_1 + S_2 + \cdots + S_m}{c_4(n_1) + c_4(n_2) + \cdots + c_4(n_m)} = \dfrac{\sum_{i=1}^{m} S_i}{\sum_{i=1}^{m} c_4(n_i)}$

Theorem 2

We have $\mathrm{Var}(\bar{S}_A) \geq \mathrm{Var}(\bar{S}_B)$.

Recall: we showed

$$\mathrm{Var}(\bar{S}_A) = \frac{\sigma^2}{m^2}\sum_{i=1}^{m}\left\{\frac{1}{c_4(n_i)^2} - 1\right\} \quad \text{and} \quad \mathrm{Var}(\bar{S}_B) = \sigma^2 \cdot \frac{\sum_{i=1}^{m}\left\{1 - c_4(n_i)^2\right\}}{\left\{\sum_{i=1}^{m} c_4(n_i)\right\}^2}$$

Sketch proof.

It suffices to show

$$\frac{1}{m^2}\sum_{i=1}^{m}\left\{\frac{1}{c_4(n_i)^2}-1\right\} \geq \frac{\sum_{i=1}^{m}\left\{1-c_4(n_i)^2\right\}}{\left\{\sum_{i=1}^{m} c_4(n_i)\right\}^2}$$

- First, let $a_i = c_4(n_i)$ (increasing) and $b_i = 1/c_4(n_i)^2 - 1$ (decreasing)
- Apply the Chebyshev's sum inequality with a_i and b_i. Then we have an intermediate result.

- Next, let $a_i = c_4(n_i)$ (increasing) and $b_i = 1/c_4(n_i) - c_4(n_i)$ (decreasing)
- Apply the Chebyshev's sum inequality again to the above intermediate result with a_i and b_i. Then we have a result.

$\square$

Inequalities and their proofs

Recall

Section 6.3.2 of Montgomery (2013) uses

$$\bar{S}^* = \frac{\bar{S}}{c_4(\bar{n})},$$

where $\bar{S} = \sum_{i=1}^m S_i/m$ and $\bar{n} = \sum_{i=1}^m n_i/m$.

Claim: $\bar{S}^*$ **underestimates** $\boldsymbol{\sigma}$ since

$$E[\bar{S}^*] = \frac{\frac{1}{m}\sum_{i=1}^m E[S_i]}{c_4(\bar{n})\sigma} = \frac{\frac{1}{m}\sum_{i=1}^m c_4(n_i)}{c_4(\bar{n})} \cdot \sigma < \sigma.$$

It suffices to show

$$\frac{1}{m}\sum_{i=1}^m c_4(n_i) < c_4(\bar{n})$$

Lemma 4

The $c_4(x)$ function is log-concave.

Sketch proof.

Recall the c_4 function which is again given by

$$c_4(x) = \sqrt{\frac{2}{(x-1)}}\frac{\Gamma(x/2)}{\Gamma((x-1)/2)}$$

Thus, we have

$$c_4(x)^2 = \frac{1}{(x-1)/2}\frac{\Gamma(x/2)^2}{\Gamma((x-1)/2)^2} = \frac{\Gamma(x/2)^2}{\Gamma((x-1)/2)\Gamma((x+1)/2)}.$$

Sketch proof – continued.

- Taking the logarithm and dividing by two, we have

$$\log c_4(x) = \log\Gamma\left(\frac{x}{2}\right) - \frac{1}{2}\log\Gamma\left(\frac{x-1}{2}\right) - \frac{1}{2}\log\Gamma\left(\frac{x+1}{2}\right).$$

- From Section 11.14 (iv) of Schilling (2005) and Merkle (1996), the second derivative of $\log\Gamma(x)$ can be written as the sum of the series so that we have

$$\frac{d^2}{dx^2}\log\Gamma(x) = \sum_{k=0}^{\infty}\frac{1}{(x+k)^2}. \tag{4}$$

Sketch proof — continued.

After tedious algebra, we have

$$\frac{d^2}{dx^2}\log c_4(x) = \sum_{k=0}^{\infty}\frac{1}{(\frac{x}{2}+k)^2} - \frac{1}{2}\sum_{k=0}^{\infty}\frac{1}{(\frac{x-1}{2}+k)^2} - \frac{1}{2}\sum_{k=0}^{\infty}\frac{1}{(\frac{x+1}{2}+k)^2}$$
$$= \sum_{k=0}^{\infty}\frac{-12(x+2k)^2+4}{(x+2k)^2(x-1+2k)^2(x+1+2k)^2} < 0.$$

- Thus, we have $\frac{d^2}{dx^2}\big(-\log c_4(x)\big) > 0$ so $-\log c_4(x)$ is convex due to Theorem 6.4.6 of Bartle and Sherbert (2011).
- Then $\log c_4(x)$ is concave and $c_4(x)$ is thus log-concave, which completes the proof.

$\square$

Lemma 5

We have the following inequality

$$\frac{1}{m}\sum_{i=1}^{m} c_4(n_i) \le c_4(\bar{n}).$$

Proof.

The log-concavity of $c_4(x)$ from Lemma 4 guarantees that $c_4(x)$ is concave.
Thus, we can apply the Jensen's inequality to $c_4(x)$ and we have

$$\frac{1}{m}\sum_{i=1}^{m} c_4(n_i) \le c_4(\bar{n}),$$

where $\bar{n} = \sum_{i=1}^{m} n_i/m$. This completes the proof. $\square$

Summary

Summary

- The c_4 function plays a pivotal role in constructing the $\bar{X}$ and S charts.
- With the useful inequalities, we showed that the conventional *ad hoc* methods are **biased**, which are used for the $\bar{X}$ and S charts.
- We also proposed new **unbiased** estimators for σ which can be used for various control charts.
- Thus, we can construct **proper $\bar{X}$ and S charts** based on the new **unbiased** estimators.
- The behaviors of the c_4 function are much understood thanks to the connection with Wallis' product and Watson representation.
- Hopefully, these results on c_4 can produce more useful inequalities.

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