Determining Box-Joint Finger Widths of Variable Size

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1 Introduction

In this document, we outline methods for computing box-joint finger widths with the following properties:

1. The center finger is the widest and is centered on the midpoint of the joint.

2. Each finger away from the center finger is a width-d less than its adjoining finger towards the centerline.

Note that the finger widths are such that the pattern is symmetric about the centerline of the joint. Figure [] shows the example pattern.

Be warned, this document is written using a lot of mathematics notation and contains a lot of algebra. I've tried to simplify the notation as much as I can. If you're uncomfortable with the math, that's fine. You're probably also a much better woodworker than I am.

The nomenclature used in this document is as follows:

- B = The router bit (or dado) width. This may also be viewed as the minimum width of a interior cut. I should note here that this document can be used regardless of your preferred unit of measure; it holds for the English system just as well as for metric.

- W = total width of the joint. We assume in this document that W is an exact multiple of B.

- N = total number of fingers, both on one board and the adjoining board. Because of the joint symmetry about the center, we know that N is an odd number, with $(N-1)/2$ fingers on each side of the center finger.

- $M = (N-1)/2$. On the edge with the centered cut, this is the number of cuts to one side.

- d = the change in width from a finger to its adjoining finger, from the adjoining board. $d = 0$ gives a traditional equally-spaced, box joint (with a caveat to be explained below).

- f_i = width of finger i, where $i = 0, 1, 2, \ldots, M$. The indexing is defined so that the center finger has width f_0, it has two adjoining fingers of width f_1, and we strive to have $d = f_i - f_{i+1}$.

2 Equally-spaced case

As a special case, we want to be able to handle the equally-spaced case; that is, when $d = 0$. Let

$$W_b = \frac{W}{B} \,. \tag{1}$$

Note that with the assumptions above, W_b is an exact integer. There then are two cases:

1. W_b is odd. Then $N = W_b$ and all fingers are width B.

2. W_b is even. Then $N = W_b + 1$ and all fingers are width B, except for two fingers at the edge of the joint, which are width $B/2$. See Fig. [] for an example.

For the second case, another possibility is to offset the joints over by $B/2$. Then the pattern would no longer be symmetric about the center of the edge. It would be anti-symmetric. That's fine for standard box joints, but when we consider $d > 0$, we feel we need to maintain symmetry about the centerline. We'll discuss this issue more in the next below.

3 Derivation of Finger Widths for the General Case

The sum of the finger widths must equal the total width of the joint, so that

$$W = f_0 + 2(f_1 + f_2 + \ldots + f_M)\,, \tag{2}$$

$$= f_0 + 2\sum_{i=1}^{M} f_i\,. \tag{3}$$

To within the resolution of our measuring (say 1/32 inch or 1 mm), we want $d = f_i - f_{i+1}$, which gives

$$f_i = f_0 - id\,. \tag{4}$$

The only exception is for even-W_b, in which case for $i = M$ (the fingers at the edge) we set $f_M = (f_{M-1} - d)/2$, which can be written as

$$f_M = \frac{f_0 - Md}{2}\,. \tag{5}$$

For W_b either even or odd, we can write

$$f_M = \frac{f_0 - Md}{1 + \alpha}\,, \tag{6}$$

where $\alpha = 0$ for odd-W_b, and $\alpha = 1$ when even-W_b, so

$$\alpha = \mathrm{mod}(W_b + 1, 2)\,. \tag{7}$$

The width may then be written as

$$W = f_0 + 2\sum_{i=1}^{M}(f_0 - id) - \alpha(f_0 - Md)\,. \tag{8}$$

Next use the relations

$$2\sum_{i=1}^{M} 1 = N - 1, \qquad 2\sum_{i=1}^{M} i = M(M+1), \tag{9}$$

so that we have

$$W = Nf_0 - dM(M+1) - \alpha(f_0 - Md)\,. \tag{10}$$

Note that if $d = 0$, then the finger widths are all the same and we get back the familiar relationship $W = W_b f_0$.

Alternative forms for (10) are also useful. Let

- $C = f_0$, the center finger width.
- $E = (\alpha + 1) f_M$, the finger width at the edge of the joint for odd-W_b, or twice that for even-W_b.

We can solve eq. (6) for d to obtain

$$d = \frac{C - E}{M} \,. \tag{11}$$

Eq. (10) may then be written as

$$W = MC + (M + 1 - \alpha)E \,. \tag{12}$$

Therefore,

$$E = \frac{W - MC}{M + 1 - \alpha} \,. \tag{13}$$

Another option is to ensure that the minimum interior finger, f_{M-1}, is the bit (or dado) width B. Then (4) gives

$$d = \frac{C - B}{M - 1} \,, \tag{14}$$

and

$$W = \frac{1}{M - 1} \left[C(M^2 - 2M - 1) + BM(1 + M) + \alpha(C - BM) \right] \tag{15}$$

There are several alternative forms of these relations, discussed below.

3.1 Normalized by C

We can divide eq. (12) by C to give

$$W_c = M + (M + 1)E_c \,, \tag{16}$$

where $W_c = W/C$ and $E_c = E/C$. We require that $0 < E_c \leq 1$, which gives

$$M < W_c \leq N \,. \tag{17}$$

A typical case is to specify E_c and N and compute W_c. The normalized eq. (4) becomes

$$\frac{f_i}{C} = 1 - i \frac{1 - E_c}{M} \,. \tag{18}$$

A choice I used for a project is $N = 11$ ($M = 5$) and $E_c = 0.25$, so that (16,18) reduce to

$$W_c = 6.5 \qquad \frac{f_i}{C} = 1 - 0.15i\,. \tag{19}$$

i	f_i/C
1	0.85
2	0.70
3	0.55
4	0.40
5	0.25

3.2 Constrained by the bit width

In this section, we consider the case where we want to ensure that the narrowest slot is the bit width, B. Let $C_b = C/B$ and $E_b = E/B$. Equations (15,4,6) become

$$W_b = \frac{1}{M-1}\left[C_b(M^2 - 2M - 1) + M(1+M) + \alpha(C_b - B)\right], \tag{20}$$

$$\frac{f_i}{B} = \beta_i\left(C_b - i\frac{C_b - 1}{M-1}\right), \tag{21}$$

where

$$\beta_i = \begin{cases} 1 & 0 \le i < M\,, \\ \frac{1}{1+\alpha} & i = M\,. \end{cases} \tag{22}$$

In addition, (13) becomes

$$E_b = \frac{W_b - MC_b}{M + 1 - \alpha}\,. \tag{23}$$

Now, the cut $i = M$ is at the edge of the board, which is allowed to be less than the bit width, because we allow the bit to extend over the ends of the joint. But the $i = M$ cut must be non-negative, which requires

$$C_b - M\frac{C_b - 1}{M-1} \ge 0\,, \tag{24}$$

or

$$C_b \le M\,. \tag{25}$$

Next, we may solve (20) for C_b to obtain

$$C_b = \frac{(M-1)W_b - (M+1-\alpha)M}{M^2 - 2M - 1 + \alpha} \,. \tag{26}$$

Now if we substitute this relation into (23), we obtain

$$E_b = \frac{M^2 - W_b}{(1+\alpha)(M^2 - 2M - 1 + \alpha)} \,. \tag{27}$$

We must have $E_b > 0$, which is satisfied if

$$M > \max(3-\alpha, \sqrt{W_b}) \,. \tag{28}$$

Equally-spaced fingers are when $W_b = N = 2M + 1 - \alpha$, so combined with the constraint above, we require that

$$M_{\min} \le M \le M_{\max} \tag{29}$$

where

$$M_{\min} = \max(3-\alpha, \sqrt{W_b}) \,, \qquad M_{\max} = \frac{W_b - 1 + \alpha}{2} \,. \tag{30}$$

Smaller values of M (but no smaller than $M_{\min}$) will increase the difference in width of the fingers. Of coarse M must be an integer, so we compute the constraints as

$$M_{\min} = \max\left(3-\alpha, \mathrm{ceil}\left(\sqrt{W_b}\right)\right) \,, \quad M_{\max} = \mathrm{floor}\left(\frac{W_b - 1 + \alpha}{2}\right) \,. \tag{31}$$

4 Summary

So let's summarize how we can generate variable-width fingers:

1. Say you're given an overall joint-width W and bit (or dado) width B.
2. Compute $W_b = W/B$.
3. Set $\alpha = 1$ for even-W_b and $\alpha = 0$ for odd-W_b.
4. Compute $M_{\min}$ and $M_{\max}$ from eq. (31).

5. Vary M (must be an integer) between $M_{\min}$ and $M_{\max}$ to generate different finger patterns. When $M = M_{\max}$, we get a standard box joint, with each finger the same width. When $M = M_{\min}$, we get the widest variation in finger sizes.

6. For a given choice of M, we can compute C_b from (26)

7. Then $C = f_0 = BC_b$. This is the finger width at the center of the joint.

8. For each i, $i = 1$ to $i = M$, the finger-widths f_i away from the center of the joint can be computed from (21).

9. One issue is accuracy. You often get a number such as $f_2 = 1.0438475$ inches. You obviously can't measure to that sort of accuracy.