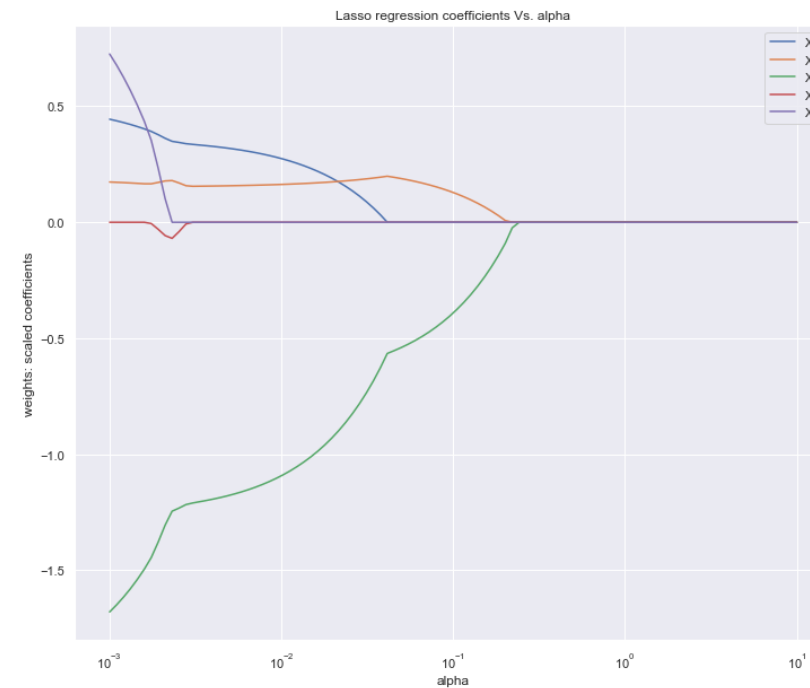
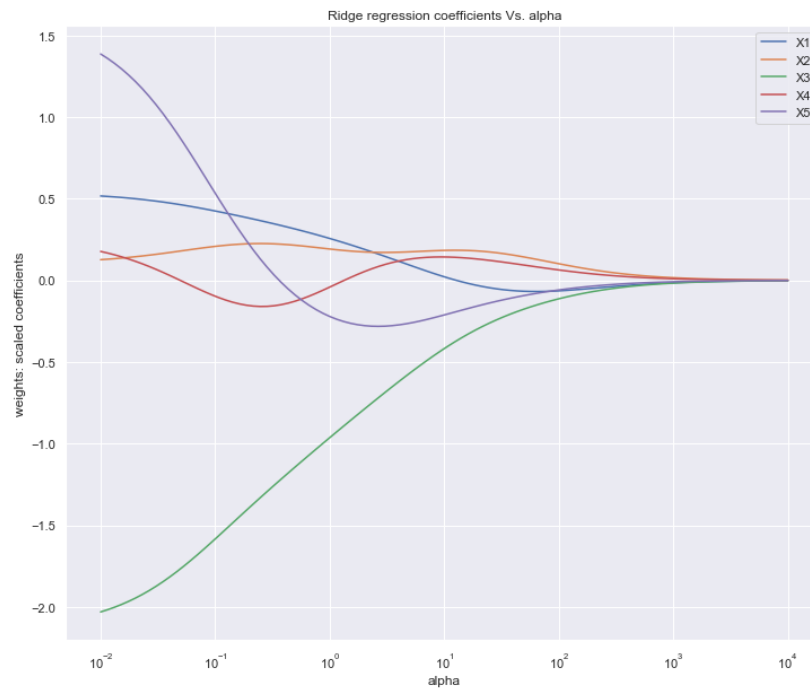


# Class 8- Regularization (Ridge, Lasso and Elastic Net)



Prof. Pedram Jahangiry



# ➔ Norms

- In mathematics, the **norm** of a vector is its **length**.
- In regression analysis, to fit our linear model, we need a measure of **mismatch!**
- Our vector is error at each training data. **We want to measure the length of error!**

- **L1** norm: Least absolute errors

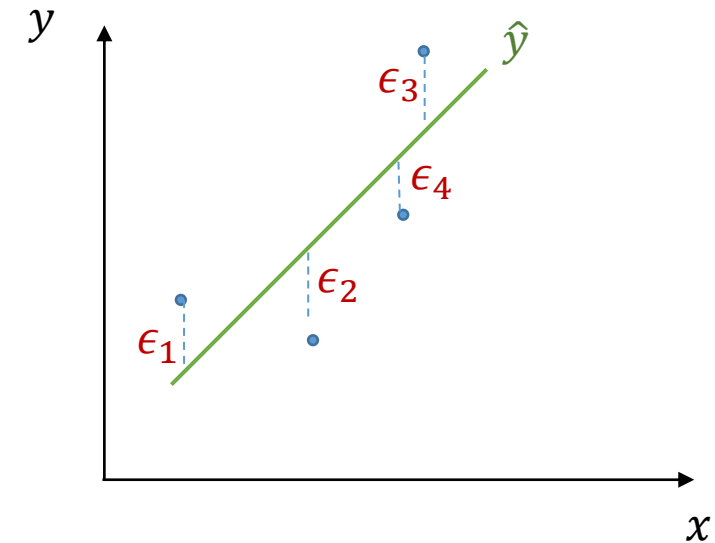
Manhattan norm

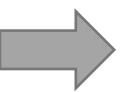
$$L^1 = \sum_i |\epsilon_i|$$

- **L2** norm: Least squares

Euclidean norm

$$L^2 = \sum_i (\epsilon_i)^2$$

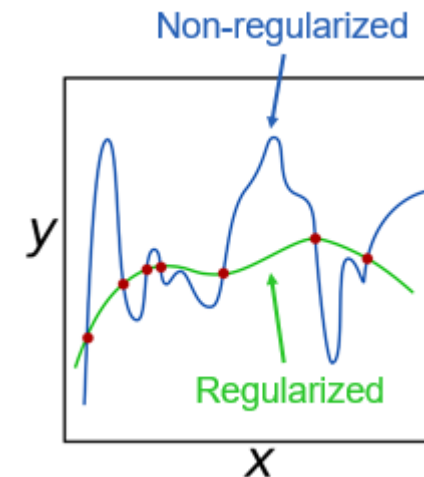




# Regularization

- ❑ In machine learning there are often **many features** (usually **correlated** with each other). This can lead to **overfitting** and models that are **unnecessarily complex**.
- ❑ **Regularization** force the learning algorithm to build a **less complex model**. In practice, that often leads to **slightly** higher bias but **significantly** reduces the variance.
- ✓ The two most widely used types of regularization are called **L1** and **L2** regularization. The idea is quite simple. To create a regularized model, we modify the loss function by adding a penalizing term whose value is higher when the model is more complex.

$$\text{Min}_{w,b} (\text{MSE} + \text{penalty}) = \text{Min} \left[ \frac{1}{N} \sum_{i=1}^N \left( y_i - f_{w,b}(X_i) \right)^2 + \text{penalty}(w) \right]$$



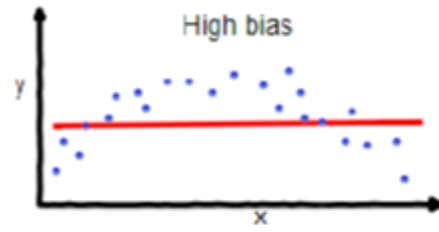
# → Penalized regression

$$\text{Min}_{w,b} (\text{MSE} + \text{penalty}) = \text{Min} \left[ \frac{1}{N} \sum_{i=1}^N \left( y_i - f_{w,b}(X_i) \right)^2 + \text{penalty}(w) \right]$$

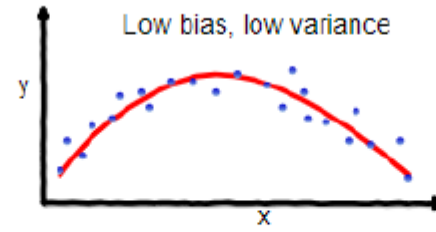
- **Penalized regression** is useful for reducing a large number of features to a manageable set and for making good predictions especially where features are correlated (i.e., when classical linear regression breaks down).
- **Penalized regression** can be used to avoid **overfitting**.
- To use the penalized regression, we need to first **standardize the features**. This will allow us to compare the magnitudes of regression coefficients for the feature variables.

- 1) Ridge regression
- 2) LASSO regression
- 3) Elastic Net regression

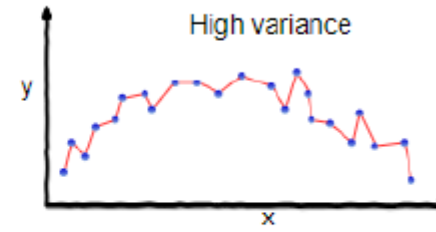
The only  
difference is  
in the penalty  
term



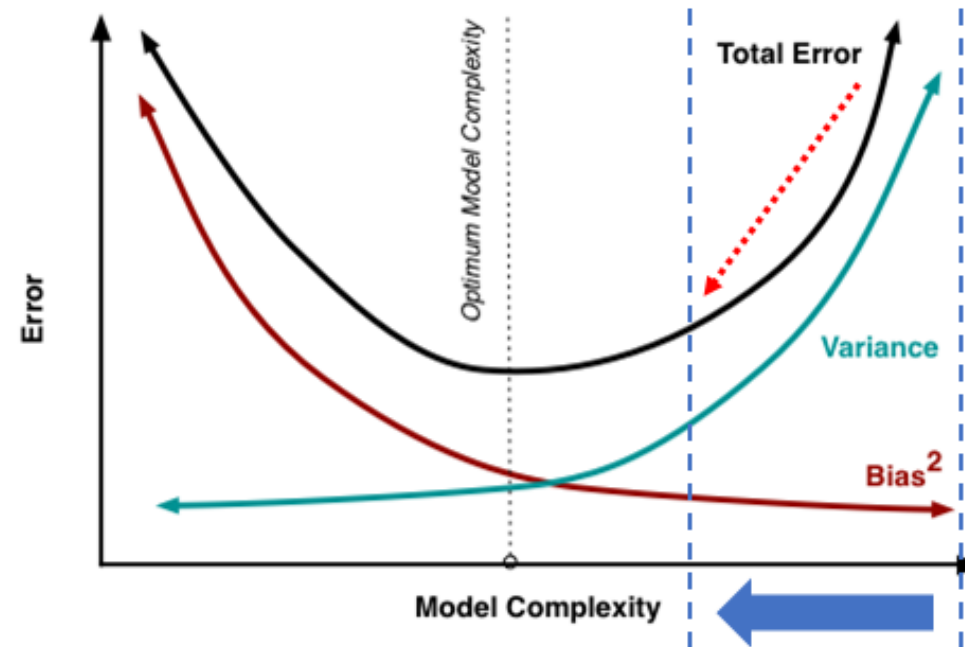
underfitting



Good balance



overfitting



Penalized  
regression

# Part I

## Ridge Regression

# → 1) Ridge regression

$$\begin{aligned} \text{Min}_{w,b} (\text{MSE} + \text{penalty}) &= \text{Min} \left[ \frac{1}{N} \sum_{i=1}^N \left( y_i - f_{w,b}(X_i) \right)^2 + \text{penalty}(w) \right] \\ &= \text{Min} \left[ \frac{1}{N} \sum_{i=1}^N \left( y_i - f_{w,b}(X_i) \right)^2 + \lambda \sum_{j=1}^D w_j^2 \right] \end{aligned}$$

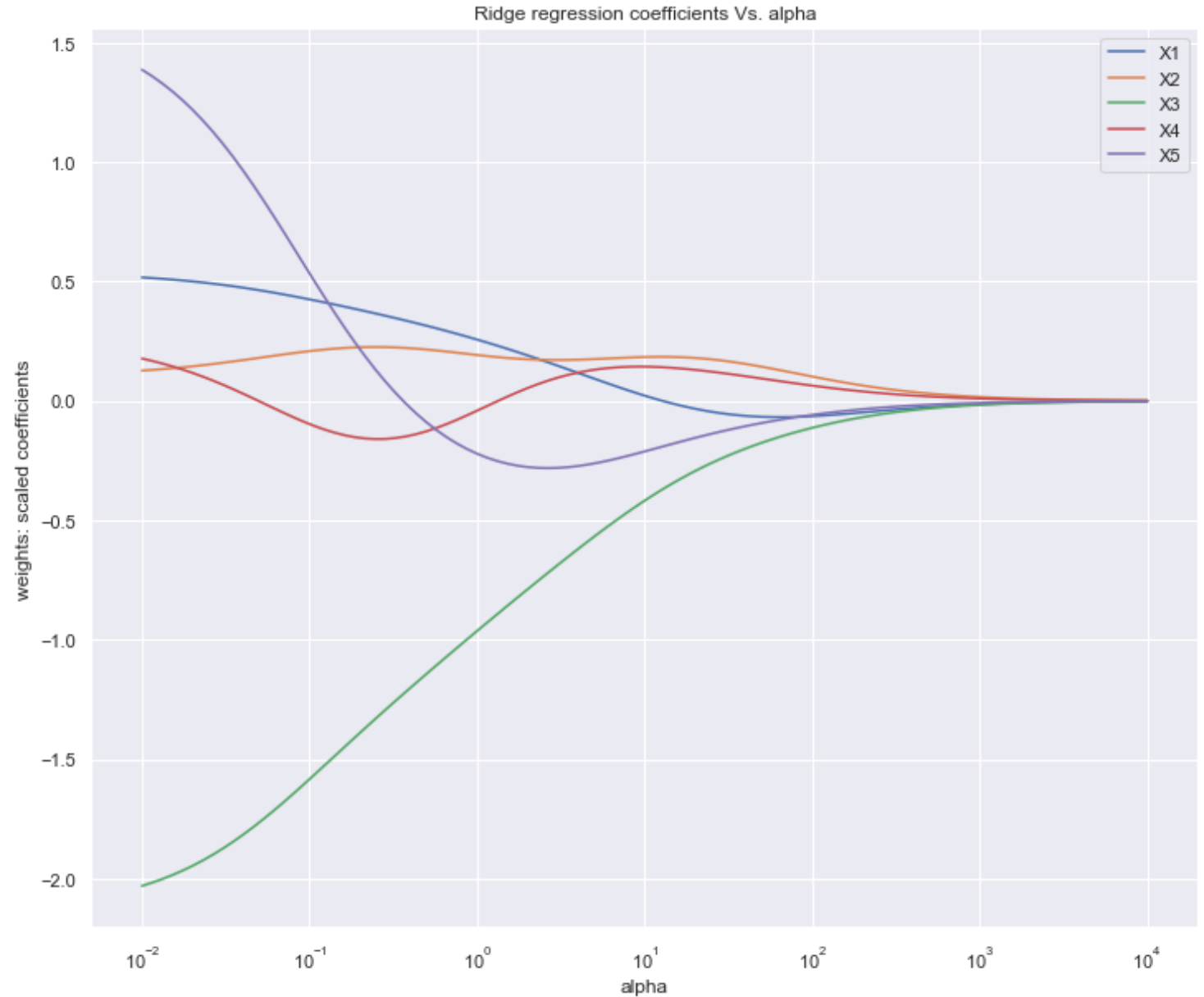
- Ridge regression uses **L2** norm.
- The shrinkage penalty has the effect of shrinking the estimates of  $w_j$  towards zero.
- The tuning parameter  $\lambda$  serves to control the relative impact of the penalty term on the regression coefficient estimates.
- Selecting a good value for  $\lambda$  is critical; cross-validation is used for this.
- It is best to apply ridge regression after variable **standardization**.

The true model is:

$$y = f(x) = x + 2x^2 - 3x^3 + \epsilon$$

Imposed functional form:

$$\hat{y} = w_1x + w_2x^2 + w_3x^3 + w_4x^4 + w_5x^5$$





# Part II

## LASSO Regression

## → 2) LASSO regression

$$\begin{aligned} \text{Min}_{w,b} (\text{MSE} + \text{penalty}) &= \text{Min} \left[ \frac{1}{N} \sum_{i=1}^N \left( y_i - f_{w,b}(X_i) \right)^2 + \text{penalty}(w) \right] \\ &= \text{Min} \left[ \frac{1}{N} \sum_{i=1}^N \left( y_i - f_{w,b}(X_i) \right)^2 + \lambda \sum_{j=1}^D |w_j| \right] \end{aligned}$$

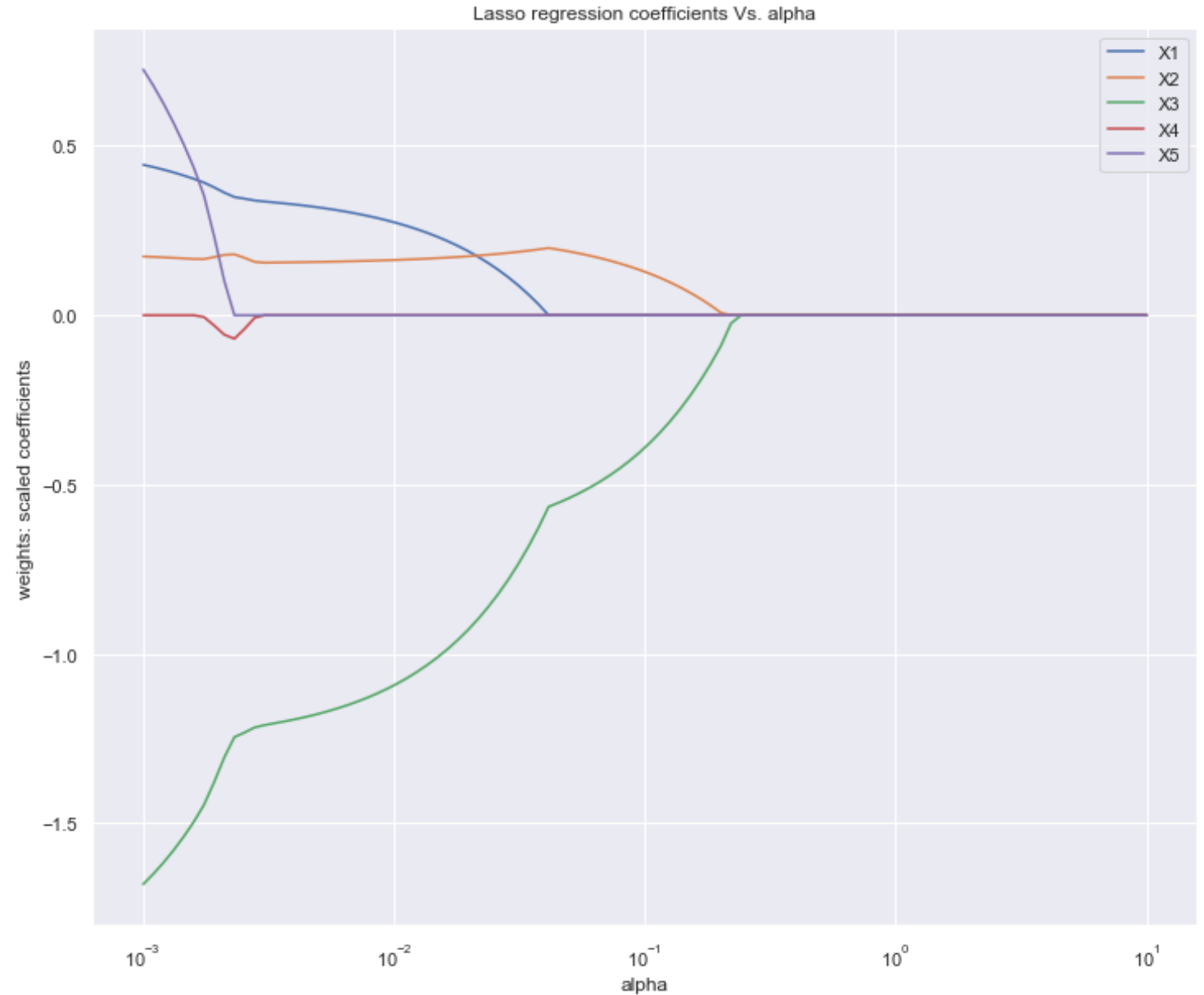
- LASSO stands for “Least Absolute Shrinkage and Selection Operator”
- LASSO regression uses **L1** norm.
- LASSO eliminates the least important features from the model, it automatically performs a type of **feature selection**.
- Selecting a good value for  $\lambda$  is critical; cross-validation is used for this.
- It is best to apply LASSO regression after variable **standardization**.

The true model is:

$$y = f(x) = x + 2x^2 - 3x^3 + \epsilon$$

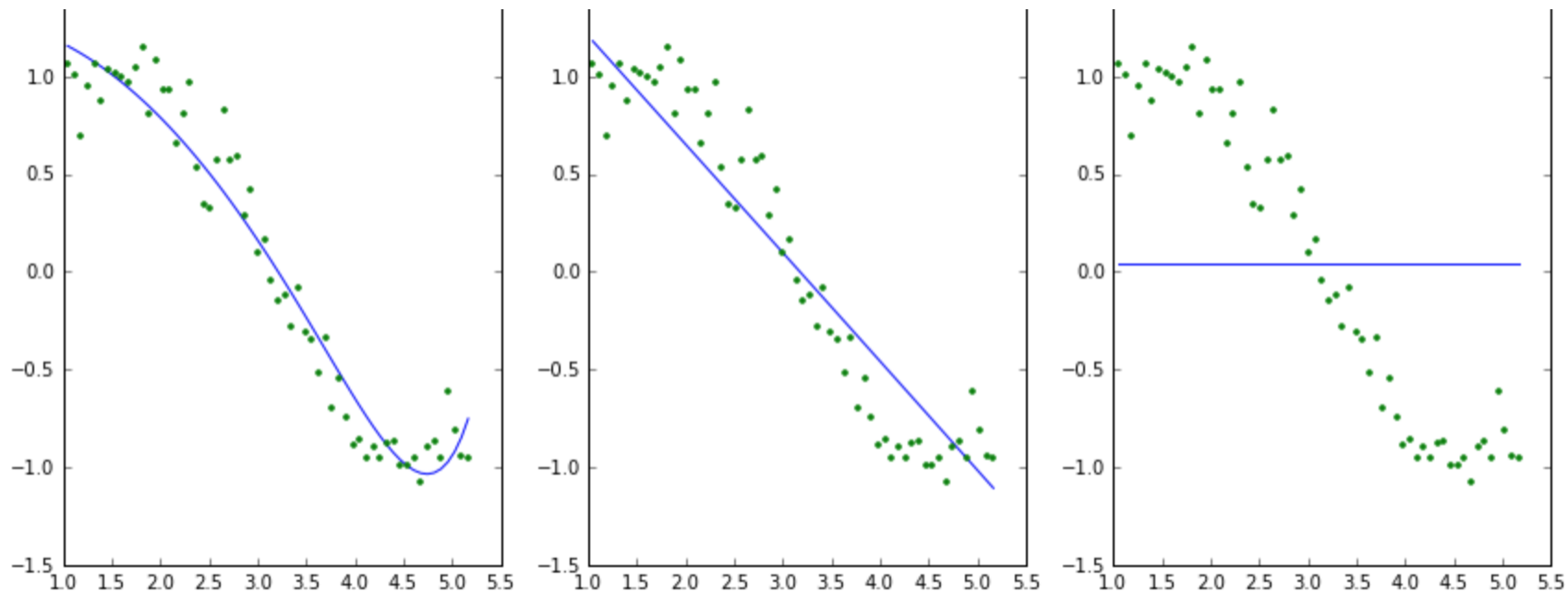
Imposed functional form:

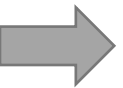
$$\hat{y} = w_1x + w_2x^2 + w_3x^3 + w_4x^4 + w_5x^5$$



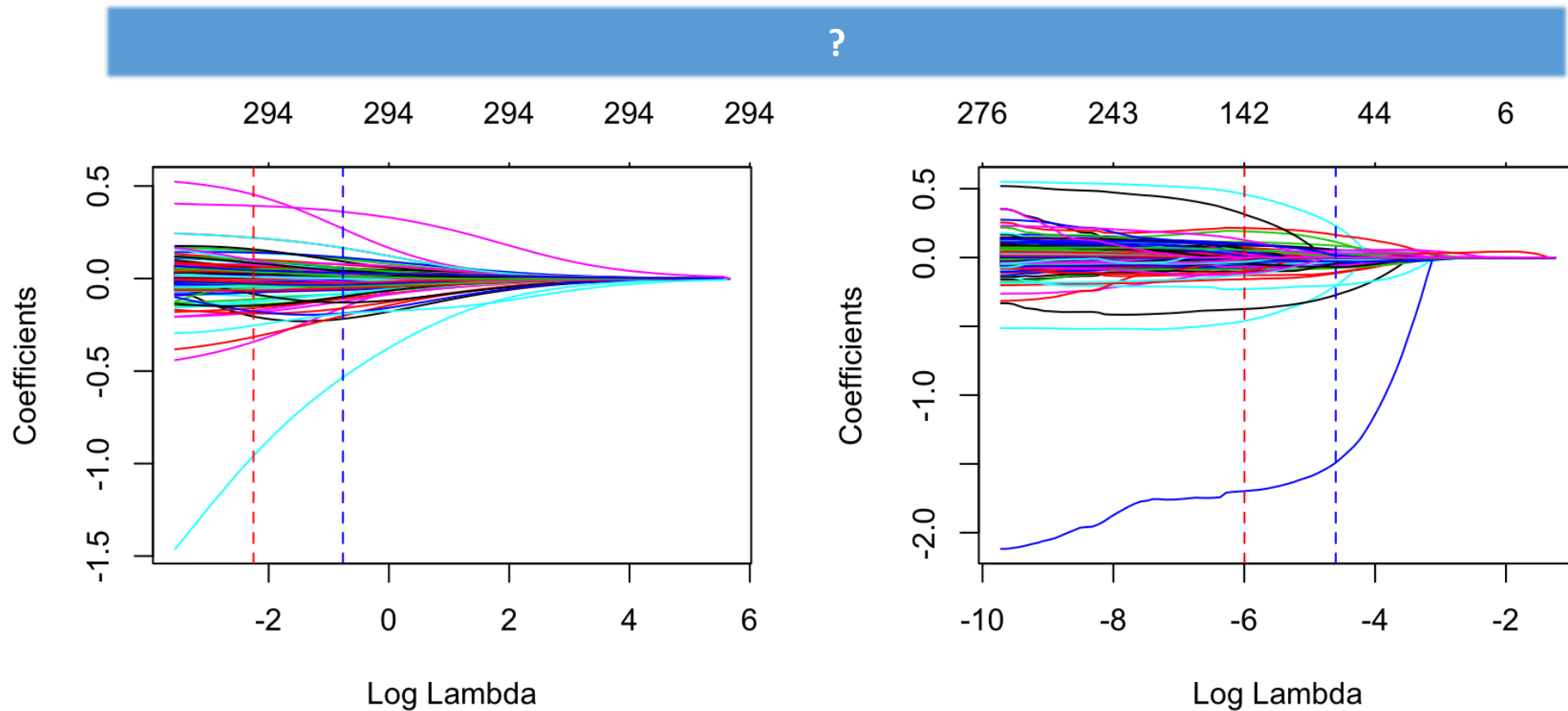
# ➔ Ridge and LASSO vs Lambda

As  $\lambda$  increases, the model becomes simpler





# Question of the day: Ridge vs LASSO?



# Part III

## Elastic Net Regression

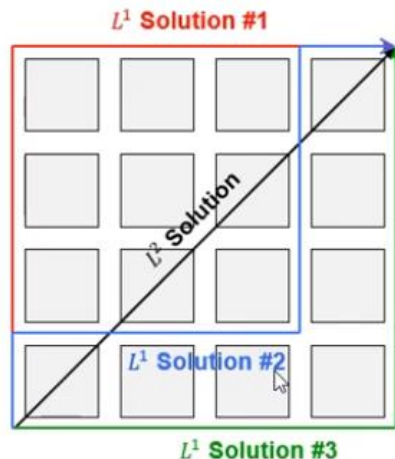
### → 3) Elastic Net Regression

$$\begin{aligned} \text{Min}_{w,b} (\text{MSE} + \text{penalty}) &= \text{Min} \left[ \frac{1}{N} \sum_{i=1}^N \left( y_i - f_{w,b}(X_i) \right)^2 + \text{penalty}(w) \right] \\ &= \text{Min} \left[ \frac{1}{N} \sum_{i=1}^N \left( y_i - f_{w,b}(X_i) \right)^2 + \lambda_1 \sum_{j=1}^D |w_j| + \lambda_2 \sum_{j=1}^D w_j^2 \right] \end{aligned}$$

- In LASSO some weights are reduced to zero, but others may be quite large. In Ridge, weights are small in magnitude, but they are not reduced to zero.
- In Elastic Net, we may be able to get the **best of both worlds** by making some weights zero while reducing the magnitude of the others.

# → Ridge vs LASSO vs Elastic Net

Property	Ridge	LASSO	Elastic Net
Can shrink the coefficient estimate toward zero?	Yes	Yes	Yes
Can include all the features in the model even with large $\lambda$ ?	Yes	No	No
Can force some of the coefficient estimates to be exactly = 0? Hence, can be used for <u>feature selection</u> ? Or <u>sparse output</u> ? More explainable?	No	Yes	Yes
Is robust : resistant to <u>outliers</u> ?	No	Yes	Not very
No Analytical solution i.e., requires gradient descent?	No	Yes	Yes
Always unique solution?	Yes	No	Yes





# Appendix



## LASSO vs Ridge, behind the scene? (optional)

Why is it that the lasso, unlike ridge regression, results in coefficient estimates that are exactly equal to zero?

One can show that the lasso and ridge regression coefficient estimates solve the problems

$$\underset{\beta}{\text{minimize}} \sum_{i=1}^n \left( y_i - \beta_0 - \sum_{j=1}^p \beta_j x_{ij} \right)^2 \quad \text{subject to} \quad \sum_{j=1}^p |\beta_j| \leq s$$

and

$$\underset{\beta}{\text{minimize}} \sum_{i=1}^n \left( y_i - \beta_0 - \sum_{j=1}^p \beta_j x_{ij} \right)^2 \quad \text{subject to} \quad \sum_{j=1}^p \beta_j^2 \leq s,$$

# ➔ LASSO vs Ridge, behind the scene? (optional)

