

A Categorical Approach to Dynamical Systems and Chaos

- 2.11. Monomorphisms, epimorphisms, opposite category, commutative diagrams and initial/final objects, functors, natural transformations, category of functors, equivalence of categories [Bra, 2.4–3.6,6.7], [Lei, 1.2–1.3]
- 2.12. Representable functors, Yoneda Lemma and simple applications, tensor product [Bla, 4.1–4.2], [Bla, 5.1–5.2]

What is a dynamical system?

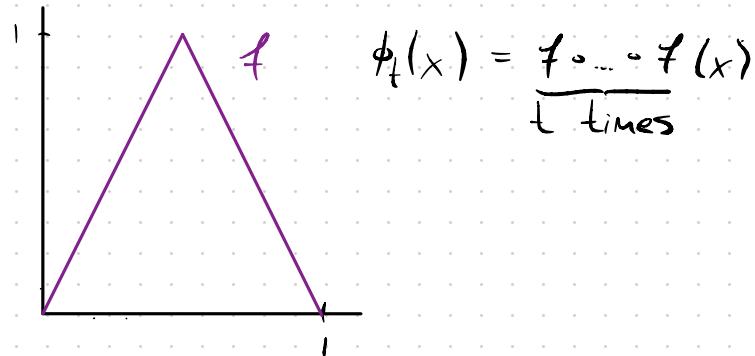
Flows "Lorenz63"

$$\begin{aligned}\dot{x} &= \sigma \cdot (y - x) \\ \dot{y} &= x \cdot (\rho - z) - y \\ \dot{z} &= x \cdot y - \beta \cdot z\end{aligned}$$

$\phi_t(x)$ "flow map"

[https://youtu.be/
B4ftJl77PTw?
si=LXj4AHLhl8W
7oOb](https://youtu.be/B4ftJl77PTw?si=LXj4AHLhl8W7oOb)

Iterated maps "Tent map"



here: metric space

Unified language! ↗ "time domain", often $\mathbb{N}_0, \mathbb{Z}, [0, \infty), \mathbb{R}$

Let X be a space, Γ be an additive semigroup. A dynamical system is a family of maps $\{\phi_t\}_{t \in \Gamma}$ with the properties

- $\phi_0 = \text{id}_X$
- $\phi_t \circ \phi_s = \phi_{t+s} \quad \forall t, s \in \Gamma$

Functors In the spirit of unifying language...

Definition

A functor F between two categories \mathcal{U} and \mathcal{B} maps objects and morphisms in \mathcal{U} to objects and morphisms in \mathcal{B} , in the following structure-preserving way:

- $F(\text{id}_A) = \text{id}_{F(A)} \quad \forall A \in \text{ob}(\mathcal{U})$
- $\forall A_1, A_2, A_3 \in \text{ob}(\mathcal{U})$ the diagram commutes:

$$\begin{array}{ccc} F(A_1) & \xrightarrow{F(g)} & F(A_2) \\ & F(fog) \swarrow & \downarrow F(f) \\ & & F(A_3) \end{array}$$

Viewing \mathcal{T} as a single-object category,
we notice a dynamical system is
just a functor into MetTop ,
the category of metrizable
spaces and continuous maps.

$$\mathcal{T}: \begin{array}{c} * \\ \curvearrowleft \\ t \\ \curvearrowright \\ s \end{array}$$

$$\begin{array}{ccc} \phi(*) = X & \xrightarrow{\phi_t} & X \\ \dashrightarrow & & \downarrow \phi_s \\ \phi_{s+t} & \searrow & X \end{array}$$

Chaos

$$\exists r > 0 : (\forall x \neq y \in X \ \exists t \in T : d(\phi_t x, \phi_t y) > r)$$

A dynamical system is chaotic if it has sensitive dependence on initial conditions, is topologically transitive, has dense periodic orbits.

$$\exists x \in X : \overline{\bigcup_{t \in T} \{\phi_t x\}} = X \quad \forall x \in X, \delta > 0 \ \exists y \in X, T \in T : d(x, y) < \delta \wedge \phi_T y = y$$

Left shift map

$$X = \Sigma_2, \quad T = \mathbb{N}_0, \quad \sigma_1(x_0 x_1 x_2 \dots) = (x_1 x_2 x_3 \dots)$$

{ space of binary representations of numbers in $[0, 1]$

This system is chaotic.

sensitive dependence: left shift "doubles all differences"

topological transitivity: concatenate all finite sequences

periodic orbits are dense: repeated sequences $\underbrace{00\dots 0}_{n} \underbrace{100\dots 0}_{n} \dots$

That's great and all, but how can we verify chaos in "real" systems?

Isomorphic systems

Conjugacy

We can study the dynamics of one system using a simpler "conjugate" system, that is, a system (Y, τ, ψ) such that the diagram commutes for some homeomorphism h .

$$\begin{array}{ccc} X & \xrightarrow{\phi_t} & X \\ h \downarrow & & \uparrow h^{-1} \\ Y & \xrightarrow{\psi_t} & Y \end{array}$$

The tent map is chaotic.

"Itinerary map" technique: Let $x \in [0, 1]$, define

$$h(x) = y_0 y_1 y_2 \dots \text{ where } y_i = \begin{cases} 1, & \phi_i x \geq \frac{1}{2}, \\ 0, & \text{else.} \end{cases}$$

With some work, h is a homeomorphism. And by definition,

$$\begin{array}{ccc} x & \mapsto & \phi_i x \\ h \downarrow & & \downarrow h \\ y_0 y_1 \dots & \mapsto & y_1 y_2 \dots \end{array}$$

Natural transformations

What did we just do?

Definition

Given categories \mathcal{A} and \mathcal{B} , a natural transformation β is a morphism in the category of functors from \mathcal{A} to \mathcal{B} . — denoted $[\mathcal{A}, \mathcal{B}]$

We write $\gamma: F \Rightarrow G$. If the functors

$F: \mathcal{A} \rightarrow \mathcal{B}$ and $G: \mathcal{B} \rightarrow \mathcal{A}$ satisfy

$$G \circ F \cong \text{id}_{\mathcal{A}}, \quad F \circ G \cong \text{id}_{\mathcal{B}}$$

then we say \mathcal{A} is equivalent to \mathcal{B} , write $\mathcal{A} \cong \mathcal{B}$.

Specifically, a natural transformation β is a collection of morphisms

$$\gamma = \{\gamma_A\}_{A \in \text{ob}(\mathcal{A})}$$

such that $\forall f: A_1 \xrightarrow{\text{ob}(\mathcal{A})} A_2$, the diagram commutes.

$$\begin{array}{ccc} F(A_1) & \xrightarrow{F(f)} & F(A_2) \\ \gamma_{A_1} \downarrow & & \downarrow \gamma_{A_2} \end{array}$$

$$\begin{array}{ccc} G(A_1) & \xrightarrow{G(f)} & G(A_2) \end{array}$$

Historical Note

RESEARCH ANNOUNCEMENT

APPEARED IN BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 32, Number 1, January 1995, Pages 66-72

CHAOS IN THE LORENZ EQUATIONS: A COMPUTER-ASSISTED PROOF

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ABSTRACT. A new technique for obtaining rigorous results concerning the global dynamics of nonlinear systems is described. The technique combines abstract existence results based on the Conley index theory with computer-assisted computations. As an application of these methods it is proven that for an explicit parameter value the Lorenz equations exhibit chaotic dynamics.

The proof of Theorem 1 has five distinct components:

1. algebraic invariants based on the Conley index theory which guarantee the structure of the global dynamics, in this case the semi-conjugacy to the full two shift;
2. an extension of these invariants to multivalued maps;
3. a theory of finite representable multivalued maps which, when combined with the above-mentioned algebraic invariants, serves to bridge the gap between the continuous dynamics (in this case the Lorenz equations) and the finite dynamics of the computer;
4. the numerical computations of the finite multivalued map of interest;
5. the combinatorial computations of the Conley index for the multivalued map.

The rough idea of the general scheme of the proof is as follows. Choose a potential isolating neighborhood N for the Poincaré map g . Select a finite representable multivalued map \mathcal{G} (definitions follow) such that \mathcal{G} is an *extension* of g , i.e. $g(x) \in \mathcal{G}(x)$ for all $x \in \text{dom } \mathcal{G}(x)$. Perform a computer calculation both to determine \mathcal{G} and to check whether N is an isolating neighborhood for \mathcal{G} . Theorem

Postscript Equivalent equivalence

Previously mentioned

If the functors $F: \mathcal{A} \rightarrow \mathcal{B}$ and $G: \mathcal{B} \rightarrow \mathcal{A}$ satisfy

$$G \circ F \cong \text{id}_{\mathcal{A}}, \quad F \circ G \cong \text{id}_{\mathcal{B}}$$

then we say \mathcal{A} is equivalent to \mathcal{B} , write $\mathcal{A} \simeq \mathcal{B}$.

Theorem

F induces an equivalence $\mathcal{A} \simeq \mathcal{B}$ iff F is

- fully faithful: $f \mapsto F(f)$ is a bijection

$$\text{Hom}_{\mathcal{A}}(A_1, A_2) \rightarrow \text{Hom}_{\mathcal{B}}(F(A_1), F(A_2))$$

for any $A_1, A_2 \in \text{ob}(\mathcal{A})$,

- essentially surjective: every $B \in \mathcal{B}$ is isomorphic to some $F(A)$, $A \in \mathcal{A}$.