A Categorical Introduction to Dynamical Systems

Functors, Natural Transformations, Category of Functors, Equivalence of Categories

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1 Two Dynamical Systems

(a) Tent map (b) Left shift map on two symbols

2 Functors

Definition 2.1. A *functor* $F : \boldsymbol{\mathscr{A}} \to \boldsymbol{\mathscr{B}}$ between two categories $\boldsymbol{\mathscr{A}}$ and $\boldsymbol{\mathscr{B}}$ maps $\mathrm{ob}(\boldsymbol{\mathscr{A}}) \to \mathrm{ob}(\boldsymbol{\mathscr{B}})$, $\hom_{\boldsymbol{\mathscr{A}}} \to \hom_{\boldsymbol{\mathscr{B}}}$ in the following structure-preserving way:

- for all $A \in \mathrm{ob}(\boldsymbol{\mathscr{A}})$, $F(\mathrm{id}_A) = \mathrm{id}_{F(\boldsymbol{\mathscr{A}})}$;
- for all $A_1, A_2, A_3 \in \mathrm{ob}(\boldsymbol{\mathscr{A}})$, the diagram commutes

$$\begin{tikzcd}
F(A_1) \arrow[r, "F(g)"] \arrow[dr, "F(g\circ f)"'] & F(A_2) \arrow[d, "F(f)"] \\
& F(A_3)
\end{tikzcd}$$

Example 2.2. A *dynamical system* is a functor ϕ from an additive semigroup $\boldsymbol{\mathscr{T}}$ (viewed as a single-object category) to the category $\boldsymbol{\mathscr{MetTop}}$ of metrizable spaces with continuous functions.

3 Natural Transformations

Definition 3.1. A *natural transformation* is a morphism in the category $[\boldsymbol{\mathscr{A}}, \boldsymbol{\mathscr{B}}]$ of functors from $\boldsymbol{\mathscr{A}}$ to $\boldsymbol{\mathscr{B}}$. Specifically, a natural transformation (*natural morphism, natural isomorphism, etc.*) is a collection $\{\eta_A\}_{A \in \mathrm{ob}(\boldsymbol{\mathscr{A}})}$ of morphisms such that

- for all $A_1, A_2 \in \mathrm{ob}(\boldsymbol{\mathscr{A}})$ and $f : A_1 \to A_2$, the diagram commutes

$$\begin{tikzcd}
F(A_1) \arrow[r, "F(f)"] \arrow[d, "\eta_{A_1}"'] & F(A_2) \arrow[d, "\eta_{A_2}"] \\
G(A_1) \arrow[r, "G(f)"'] & G(A_2)
\end{tikzcd}$$

We write $\eta : F \implies G$, and write $\mathscr{A} \cong \mathscr{B}$ if there exists a natural isomorphism η.

Moreover, if two functors $F : \mathscr{A} \to \mathscr{B}$, $G : \mathscr{B} \to \mathscr{A}$ satisfy $G \circ F \cong \mathrm{id}_{\mathscr{A}}$, $F \circ G \cong \mathrm{id}_{\mathscr{B}}$, then we say $\mathscr{A}$ and $\mathscr{B}$ are *equivalent*, write $\mathscr{A} \simeq \mathscr{B}$.

Example 3.2. Given two dynamical systems, that is two functors $\phi, \psi : \mathscr{T} \to \mathscr{MetTop}$, a homeomorphism h which satisfies $\psi_1 = h^{-1} \circ \phi_1 \circ h$ is known in dynamical systems literature as a *topological conjugacy*. Such a homeomorphism is precisely a natural isomorphism when viewed in the categorical context.

Example 3.3. The tent map is topologically conjugate to the left shift on two symbols. The conjugacy is constructed using a so-called "itinerary map" technique. Systems which are conjugate share many properties, including *chaos*. This is to say, despite how simple the map looks - a piecewise linear map - after just a few iterations the dynamics become incalculably complicated!

4 Equivalence

Theorem 4.1. A functor $F : \mathscr{A} \to \mathscr{B}$ induces an equivalence of categories $\mathscr{A} \simeq \mathscr{B}$ if and only if F is

- *fully faithful*: for all $A_1, A_2 \in \mathrm{ob}(\mathscr{A})$, the map $f \mapsto F(f)$ is a bijection

$$\hom_{\mathscr{A}}(A_1, A_2) \to \hom_{F(\mathscr{A})}(F(A_1), F(A_2));$$

- *essentially surjective*: every $b \in \mathrm{ob}(\mathscr{B})$ is isomorphic to some $F(A)$, $A \in \mathrm{ob}(\mathscr{A})$.

References

[1] Martin Brandenburg. *Einführung in die Kategorientheorie*. Springer Spektrum Berlin, Heidelberg, 2016. DOI: https://doi.org/10.1007/978-3-662-47068-8.

[2] Tom Leinster. *Basic Category Theory*. 2016. arXiv: 1612.09375 [math.CT].