

Determinants of Trace Class Operators - Lidskii's Theorem

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1 Lidskii's Theorem

Theorem 1.1 (Lidskii). Let X be a Hilbert space and $A \in L(X)$ be a trace-class operator with eigenvalues $\lambda_1, \lambda_2, \dots$. Then

$$\operatorname{Tr} A = \sum_{n=1}^{\infty} \lambda_n. \quad (1)$$

2 Tools: Generalized Eigenspaces and a Product Form

Definition 2.1 (Notation). We consider a separable Hilbert space X with inner product (\cdot, \cdot) . $A \in K(X)$ is a trace-class operator.

Let $\lambda_1, \lambda_2, \dots$ (resp. s_1, s_2, \dots) be the eigenvalues (resp. singular values) of A in descending order:

$$A v_j = \lambda_j v_j, \quad |\lambda_1| \geq |\lambda_2| \geq \dots, \quad s_1 \geq s_2 \geq \dots. \quad (2)$$

A can be decomposed as $A = U|A|$ where U is unitary and $|A|$ is positive and self-adjoint. $\{z_j\}_j$ is an orthonormal basis of $\overline{\operatorname{Range}|A|}$. Writing $w_j = U z_j$, we have

$$A = \sum_{j=1}^{\infty} s_j (\cdot, z_j) w_j. \quad (3)$$

Theorem 2.2 (Generalized eigenspaces). Let $0 \neq \lambda \in \sigma(A)$. Then there exist subspaces $N = N(\lambda)$, $R = R(\lambda)$ of X with

$$\text{a) (finiteness)} \quad \dim N < \infty, \quad (4)$$

$$\text{b) (splitting)} \quad X = N \oplus R, \quad (5)$$

$$\text{c) (invariance)} \quad A(N) \subset N, \quad A(R) \subset R, \quad (6)$$

$$\text{d) (isolation)} \quad \lambda \in \sigma(A|N), \quad \lambda \in \sigma(A|R). \quad (7)$$

Write $P_\lambda : X \rightarrow N$ for the orthogonal projection of X into N .

Lemma 2.3 (Hadamard factorization). Let $f : \mathbb{C} \rightarrow \mathbb{C}$ be an entire function with $f(0) = 1$. Let $\{z_j\}_j$ be the zeros of f and assume $\sum_{j=1}^{\infty} |z_j|^{-1} < \infty$. Finally suppose the boundedness condition

$$\forall \epsilon > 0 \quad \exists C > 0 : |f(z)| \leq C \cdot e^{\epsilon|z|} \quad \forall z \in \mathbb{C}. \quad (8)$$

Then

$$f(z) = \prod_{j=1}^{\infty} \left(1 - \frac{z}{z_j}\right). \quad (9)$$

Lemma 2.4 (Weyl-Horn). We have the singular value inequality

$$\sum_{j=1}^{\infty} |\lambda_j| \leq \sum_{j=1}^{\infty} s_j. \quad (10)$$

3 (Antisymmetric) Tensor Products in Hilbert Spaces

Definition 3.1. For $x_1, x_2, \dots, x_n \in X$, let $x_1 \otimes x_2 \otimes \dots \otimes x_n$ be the multilinear map

$$(x_1 \otimes x_2 \otimes \dots \otimes x_n) : X \times X \times \dots \times X \rightarrow \mathbb{C}, \quad (y_1, y_2, \dots, y_n) \mapsto \prod_{j=1}^n (x_j, y_j). \quad (11)$$

The (Hilbert) span of such $x_1 \otimes \dots \otimes x_n$ (w.r.t. the natural inner product)

$$\langle x_1 \otimes x_2 \otimes \dots \otimes x_n, y_1 \otimes y_2 \otimes \dots \otimes y_n \rangle = \prod_{j=1}^n (x_j, y_j). \quad (12)$$

forms a Hilbert space, denoted $X \otimes X \otimes \dots \otimes X$.

If $\{e_j\}_j$ is an orthonormal basis of X , then $\{e_{\iota_1} \otimes e_{\iota_2} \otimes \dots \otimes e_{\iota_n}\}_{\iota_1 < \iota_2 < \dots < \iota_n}$ is an orthonormal basis of $X \otimes X \otimes \dots \otimes X$.

Operators $A_1, A_2, \dots, A_n \in L(X)$ induce a map $X \otimes \dots \otimes X \odot$:

$$(A_1 \otimes A_2 \otimes \dots \otimes A_n)(\ell) : (y_1, y_2, \dots, y_n) \mapsto \ell(A_1^* y_1, A_2^* y_2, \dots, A_n^* y_n) \quad (13)$$

Denote the antisymmetrization of $x_1 \otimes \dots \otimes x_n$ as

$$x_1 \wedge x_2 \wedge \dots \wedge x_n = \frac{1}{\sqrt{n!}} \sum_{\pi \in S_n} (-1)^\pi x_{\pi(1)} \otimes x_{\pi(2)} \otimes \dots \otimes x_{\pi(n)} \quad (14)$$

where S_n is the symmetric group with n elements, $(-1)^\pi$ is the sign of a permutation.

Denote by $\Lambda^n(X)$ the (Hilbert) span of such $x_1 \wedge x_2 \wedge \dots \wedge x_n$. $\Lambda^0(X) = \mathbb{C}$, $\Lambda^1(X) = X$. The same statement about orthonormal bases can be made for $\Lambda^n(X)$.

Proposition 3.2 (Properties of the tensor products).

$$\text{a) } (A_1 \otimes A_2 \otimes \dots \otimes A_n)(x_1 \otimes x_2 \otimes \dots \otimes x_n) = (A_1 x_1) \otimes (A_2 x_2) \otimes \dots \otimes (A_n x_n), \quad (15)$$

$$\text{b) } |\Lambda^n(A)| = \Lambda^n(|A|), \quad (16)$$

$$\text{c) } \Lambda^n(A) \text{ has singular values } s_{\iota_1} \cdot s_{\iota_2} \cdot \dots \cdot s_{\iota_n} \quad \forall \iota_1 < \iota_2 < \dots < \iota_n, \quad (17)$$

$$\text{d) } \text{If } \dim X = d < \infty, \text{ then } \text{Tr } \Lambda^d(A) = \det A, \quad \text{Tr } \Lambda^{d+m}(A) = 0 \quad \forall m > 0. \quad (18)$$

4 Operator Determinants

Definition 4.1 (Fredholm's determinant).

$$\det(I + zA) = \sum_{k=0}^{\infty} z^k \cdot \text{Tr } \Lambda^k(A). \quad (19)$$

Lemma 4.2. $A \mapsto \det(I + zA)$ is continuous in the trace-class. In particular the determinant satisfies

$$\det((I + A)(I + B)) = \det(I + A) \cdot \det(I + B). \quad (20)$$

The technique to prove Lidskii's Theorem 1.1 is to show that Lemma 2.3 applies. This shows that the preceding definition is equivalent to the following:

Definition 4.3 (Groh'berg-Krein's determinant).

$$\det(I + zA) = \prod_{j=1}^{\infty} (1 + \lambda_j z). \quad (21)$$