Question 1 Observe the following polyhedron $P^{\leq}(A,b)$ with

$$A = \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & 1 & 1 \\ \cdot & \cdot & \cdot & \cdot & 2 & 2 & 2 \\ \cdot & \cdot & \cdot & \cdot & 3 & 3 & 3 \end{pmatrix}, \quad b = \begin{pmatrix} 2 \\ 4 \\ 6 \\ 1 \\ 2 \\ 8 \\ 1 \end{pmatrix}.$$

a) Determine $lineal(P)$.

b) What is the affine dimension of a minimal face in P?

Solution

$$lineal(P) = ker(A) = \text{span}\{ \begin{pmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \\ 1 \\ -1 \\ \cdot \end{pmatrix}, \begin{pmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \\ 1 \\ \cdot \\ -1 \end{pmatrix} \}$$

For a minimal face F we have $F = x + lineal(P)$ for any $x \in F$. Since $lineal(P)$ has dimension 2, a minimal face will have affine dimension 2.

Question 2 A Hamilton cycle is a cycle in a graph G that visits every vertex.

Formulate an integer program for a given graph $G = (V, E)$ to determine if the graph contains a hamilton cycle.

Hint: A graph is connected if for any bipartition of the vertices, there must be an edge running between the bipartition.

Solution We wish to find a subset of edges that form a Hamilton cycle, so we define our variable $x \in \{0,1\}^E$. This is a viability problem, so the objective function is unimportant. A Hamilton cycle must satisfy two constraints:

- The induced graph must be a cycle. For a given vertex, exactly one edge must "enter" the vertex, and one must "exit" it. To model this:

$$\sum_{e\,=\,(v,w)} x_e = 2, \quad \forall\, v \in V,$$

- The induced graph must be connected. For any bipartition of the vertices, there must be edges that run between the bipartition. As with H4.2:

$$\sum_{\substack{e\,=\,(v,w)\\ v\in S\\ w\notin S}} x_e \geq 2, \quad \forall\, S \subset V,\ S \notin \{\emptyset, V\}.$$

This model does cause the same issue as in H4.2, which is that it can require an exponential number of constraints.