Determinants of Trace Class Operators - Lidskii's Theorem

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1 Lidskii's Theorem

Theorem 1.1 (Lidksii)**.** Let X be a Hilbert space and $A \in L(X)$ be a trace-class operator with eigenvalues λ_1, λ_2, ... Then

$$\operatorname{Tr} A = \sum_{n=1}^{\infty} \lambda_n \ . \tag{1}$$

2 Tools: Generalized Eigenspaces and a Product Form

Definition 2.1 (Notation)**.** We consider a separable Hilbert space X with inner product $(\cdot, \cdot)$. $A \in K(X)$ is a trace-class operator.

Let λ_1, λ_2, ... (resp. s_1, s_2, ...) be the eigenvalues (resp. singular values) of A in descending order:

$$A\, v_j = \lambda_j v_j, \quad |\lambda_1| \geq |\lambda_2| \geq \dots, \quad s_1 \geq s_2 \geq \dots . \tag{2}$$

A can be decomposed as $A = U\,|A|$ where U is unitary and $|A|$ is positive and self-adjoint. $\{z_j\}_j$ is an orthonormal basis of $\overline{\operatorname{Range}|A|}$. Writing $w_j = U z_j$, we have

$$A = \sum_{j=1}^{\infty} s_j (\cdot, z_j) w_j \, . \tag{3}$$

Theorem 2.2 (Generalized eigenspaces)**.** Let $0 \neq \lambda \in \sigma(A)$. Then there exist subspaces $N = N(\lambda)$, $R = R(\lambda)$ of X with

$$\begin{aligned} &\text{a) (finiteness)} && \dim N < \infty, && (4)\\ &\text{b) (splitting)} && X = N \oplus R, && (5)\\ &\text{c) (invariance)} && A(N) \subset N, \;\; A(R) \subset R, && (6)\\ &\text{d) (isolation)} && \lambda \in \sigma(A\,|\,N), \;\; \lambda \in \sigma(A\,|\,R). && (7) \end{aligned}$$

Write $P_\lambda : X \twoheadrightarrow N$ for the orthogonal projection of X into N.

Lemma 2.3 (Hadamard factorization)**.** Let $f : \mathbb{C} \circlearrowleft$ be an entire function with $f(0) = 1$. Let $\{z_j\}_j$ be the zeros of f and assume $\sum_{j=1}^{\infty} |z_j|^{-1} < \infty$. Finally suppose the boundedness condition

$$\forall\, \epsilon > 0 \;\; \exists\, C > 0 : \;\; |f(z)| \leq C \cdot e^{\epsilon |z|} \;\; \forall\, z \in \mathbb{C}. \tag{8}$$

Then

$$f(z) = \prod_{j=1}^{\infty} (1 - \frac{z}{z_j}). \tag{9}$$

Lemma 2.4 (Weyl-Horn)**.** We have the singular value inequality

$$\sum_{j=1}^{\infty} |\lambda_j| \leq \sum_{j=1}^{\infty} s_j. \tag{10}$$

3 (Antisymmetric) Tensor Products in Hilbert Spaces

Definition 3.1. For $x_1, x_2, \ldots, x_n \in X$, let $x_1 \otimes x_2 \otimes \ldots \otimes x_n$ be the multilinear map

$$(x_1 \otimes x_2 \otimes \ldots \otimes x_n) : X \times X \times \ldots \times X \to \mathbb{C}, \quad (y_1, y_2, \ldots, y_n) \mapsto \prod_{j=1}^{n} (x_j,\, y_j)\,. \tag{11}$$

The (Hilbert) span of such $x_1 \otimes \ldots \otimes x_n$ (w.r.t. the natural inner product)

$$\langle\, x_1 \otimes x_2 \otimes \ldots \otimes x_n,\;\; y_1 \otimes y_2 \otimes \ldots \otimes y_n \,\rangle = \prod_{j=1}^{n} (x_j,\, y_j)\,. \tag{12}$$

forms a Hilbert space, denoted $X \otimes X \otimes \ldots \otimes X$.

Operators $A_1, A_2, \ldots, A_n \in L(X)$ induce a map $X \otimes \ldots \otimes X \circlearrowleft$:

$$(A_1 \otimes A_2 \otimes \ldots \otimes A_n)(\ell) : \;\; (y_1, y_2, \ldots, y_n) \mapsto \ell(A_1^* y_1,\, A_2^* y_2,\, \ldots,\, A_n^* y_n) \tag{13}$$

Denote the antisymmetrization of $x_1 \otimes \ldots \otimes x_n$ as

$$x_1 \wedge x_2 \wedge \ldots \wedge x_n = \frac{1}{n!} \sum_{\pi \in S_n} (-1)^{\pi}\, x_{\pi(1)} \otimes x_{\pi(2)} \otimes \ldots \otimes x_{\pi(n)} \tag{14}$$

where S_n is the symmetric group with n elements, $(-1)^{\pi}$ is the sign of a permutation.

Denote by $\Lambda^n(X)$ the (Hilbert) span of such $x_1 \wedge x_2 \wedge \ldots \wedge x_n$. $\Lambda^0(X) = \mathbb{C}$, $\Lambda^1(X) = X$.

If $\{e_j\}_j$ is an orthonormal basis of X, then $\{e_{\iota_1} \wedge e_{\iota_2} \wedge \ldots \wedge e_{\iota_n}\}_{\iota_1 < \iota_2 < \ldots < \iota_n}$ is an orthonormal basis of $\Lambda^n(X)$.

Proposition 3.2 (Properties of the tensor products)**.**

$$\text{a)} \quad (A_1 \otimes A_2 \otimes \ldots \otimes A_n)(x_1 \otimes x_2 \otimes \ldots \otimes x_n) = (A_1 x_1) \otimes (A_2 x_2) \otimes \ldots \otimes (A_n x_n), \tag{15}$$

$$\text{b)} \quad |\Lambda^n(A)| = \Lambda^n(|A|), \tag{16}$$

$$\text{c)} \quad \Lambda^n(A) \text{ has singular values } s_{\iota_1} \cdot s_{\iota_2} \cdot \ldots \cdot s_{\iota_n} \;\; \forall\, \iota_1 < \iota_2 < \ldots < \iota_n, \tag{17}$$

$$\text{d)} \quad \text{If } \dim X = d < \infty, \text{ then } \operatorname{Tr} \Lambda^d(A) = \det A, \;\; \operatorname{Tr} \Lambda^{d+m}(A) = 0 \;\; \forall\, m > 0. \tag{18}$$

4 Operator Determinants

Definition 4.1 (Fredholm's determinant)**.**

$$\det(I + zA) = \sum_{k=0}^{\infty} z^k \cdot \operatorname{Tr} \Lambda^k(A). \tag{19}$$

Lemma 4.2. $A \mapsto \det(I + zA)$ is continuous in the trace-class. In particular the determinant satisfies

$$\det\big(\; (I + A)(I + B) \;\big) = \det(I + A) \cdot \det(I + B). \tag{20}$$

The technique to prove Lidskii's Theorem 1.1 is to show that Lemma 2.3 applies. This shows that the preceeding definition is equivalent to the following:

Definition 4.3 (Groh'berg-Krein's determinant)**.**

$$\det(I + zA) = \prod_{j=1}^{\infty} (1 + \lambda_j z). \tag{21}$$