

Lidskii's Theorem

A beautiful result for finite dimensions ...

Theorem

Let X be a Hilbert space and $A \in K(X)$ be a trace-class operator with eigenvalues $\lambda_1, \lambda_2, \dots$. Then

$$\text{Tr } A = \sum_{n=1}^{\infty} \lambda_n.$$

... extended to infinite dimensions.

Reference

For $\dim X < \infty$ i.e. $A \in \mathbb{C}^{n \times n}$ with eigenvalues $\lambda_1, \dots, \lambda_n$ we know

$$\det(\lambda I - A) = \lambda^n - \text{tr}A \cdot \lambda^{n-1} + \dots + (-1)^n \det A = (\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_n)$$

Contents

- Generalized eigenspaces
- (Antisymmetric) tensor products
- Operator determinants
- Properties of the determinant
- A one-line proof of Lidskii's Theorem
- Bonus: Tr and \det of integral operators

Generalized Eigenspaces

Theorem (c.f. Werner) compact

Let X be a Banach space, $A \in K(X)$, $0 \neq \lambda \in \sigma(A)$.

Then $\exists N = N(\lambda), R = R(\lambda)$ subspaces of X with

- a) $\dim N < \infty$ finiteness
- b) $X = N \oplus R$ splitting
- c) $A(N) \subset N, A(R) \subset R$ invariance
- d) $\lambda \in \sigma(A|N), \lambda \notin \sigma(A|R)$. isolation

Idea

Use a concept from linear algebra (Jordan canonical form).

$$S = \lambda I - A$$

$$N_0 = \{0\}, N_m = \ker S^m$$

$$R_0 = X, R_m = \text{ran } S^m$$

Lemma

$$\exists p \in \mathbb{N} : N_p = N_{p+1} = N_{p+2} = \dots$$

$$\exists q \in \mathbb{N} : R_q = R_{q+1} = R_{q+2} = \dots \text{ analogous.}$$

proof

For contradiction, assume $N_0 \subsetneq N_1 \subsetneq N_2 \subsetneq \dots$.

Riesz's lemma gives $(x_n)_n : x_n \in N_n, d(x_n, N_m) > \frac{1}{2} \forall n > m$.

But then

$$\|Ax_m - Ax_n\| = \left\| \underbrace{Ax_m - Ax_n}_{Sx_m \in N_m} + \underbrace{Ax_n - Ax_n}_{N_m} - \underbrace{Ax_n - Sx_n}_{-Sx_n \in N_{m-1}} + \underbrace{Sx_n}_{N_{m-1} \text{ since } m \leq n-1} \right\| > \frac{1}{2}.$$

\Rightarrow No convergent subseq. of $(Ax_n)_n$ $\xrightarrow{\text{Schauder}}$ to compactness

Hence $\exists p : N_p = N_{p+1}$. For $x \in N_{p+2}$,

$$Sx \in N_{p+1} = N_p \Rightarrow 0 = S^p(Sx) = S^{p+1}x \Rightarrow x \in N_{p+1}$$

so $N_{p+2} = N_{p+1}$. Analogously $N_{p+2} = N_{p+3} = \dots$.

$$S = \lambda I - A$$

$$N_0 = \{0\}, \quad N_m = \ker S^m$$

$$R_0 = X, \quad R_m = \text{ran } S^m$$

$$\begin{aligned} S^m &= (\lambda I - A)^m \\ &= \lambda^m (I - \lambda^{-1}A)^m \\ &= \lambda^m \left(I - \sum_{k=1}^m \binom{m}{k} (-\lambda^{-1}A)^k \right) \end{aligned}$$

compact

Riesz-Schauder
 $\Rightarrow \dim N_m < \infty$ and N_m, R_m closed

We can now write

$$N = N_p, \quad R = R_p.$$

Then a) follows from
 Riesz-Schauder and b), c), d)
 are (essentially) finite-dimensional
 linear algebra.

For X as a Hilbert space we can define

$$\rho_1 : x \longrightarrow N$$

the orthogonal projection into N .

$$S = \lambda I - A$$

$$N_0 = \{0\}, \quad N_m = \ker S^m$$

$$R_3 = X, \quad R_m = \text{rank } S^m$$

Lemna

$$N \cap R = \{0\}$$

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Let $a \in N \cap R$. $\rightarrow a \in R$

$\Rightarrow u = s^r y$ for some $y \in X$

$$\Rightarrow O = S^2 u = S^2 r_y$$

$$\Rightarrow y \in N_{z,r} = N_r$$

$$\sigma = \text{sgn } \zeta_N$$

Levine

$$A(N) \subset N \quad A(R) \subset R$$

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$$S(N_r) = (\lambda I - A)(N_r) \subset N_{r-1} \subset N_r \iff$$

$$S(R_r) = (\lambda I - A)(R_r) \subset R_{r+1} = R_r \iff A(R_r) \subset R_r$$

λI keeps all subspace invariant

Lemur

$$\lambda \in \sigma(A|N), \quad \lambda \notin \sigma(A|R)$$

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By construction $E_1 = \cup_{i=0}^{\infty} h_i S \cap cN$ and N, R are invariant.

Recap + Notation

We will always assume X is a separable Hilbert space with inner product (\cdot, \cdot) , and $A \in L(X)$ is trace-class.

$\lambda_1, \lambda_2, \dots$ (resp. s_1, s_2, \dots) are the eigenvalues (resp. singular values) of A in descending order

$$Av_j = \lambda_j v_j, \quad |\lambda_1| \geq |\lambda_2| \geq \dots, \quad s_1 \geq s_2 \geq \dots .$$

A can be decomposed as $A = U|A|$ where U is unitary and $|A|$ is positive and self-adjoint.

Spectral theory \Rightarrow ONB of $\overline{\text{ran}|A|}$ $\{z_j\}_j$ with $|A|z_j = s_j z_j$.

Write $w_j = Uz_j$

$$\Rightarrow A = \sum_{j=1}^{\infty} s_j (\cdot, z_j) w_j .$$

"Canonical decomposition"

Operator Determinants (Preview)

Many definitions:

Gohberg - Krein

$$\det(I + zA) = \prod_{j=1}^{\infty} (1 + z\lambda_j)$$

Fredholm

$$\det(I + zA) = \sum_{k=0}^{\infty} z^k \cdot \text{Tr } \Lambda^k(A)$$

Dunford - Schwarz

$$\det(I + \mu A) = \exp \circ \text{Tr} \circ \log (I + \mu A)$$

for $|\mu| \ll 1$ + analytic continuation

Lidskii's theorem will show they are all equivalent!

Antisymmetric Tensor Products

Definition

For $x_1, x_2, \dots, x_n \in X$, let $(x_1 \otimes x_2 \otimes \dots \otimes x_n) : X \times X \times \dots \times X \rightarrow \mathbb{C}$ be the multilinear map

$$x_1 \otimes x_2 \otimes \dots \otimes x_n : (y_1, \dots, y_n) \mapsto \prod_{i=1}^n (x_i, y_i).$$

The (metric) completion of span of such $(x_1 \otimes x_2 \otimes \dots \otimes x_n)$'s is a Hilbert space, denoted $X \otimes \dots \otimes X$.

Operators $A_1, A_2, \dots, A_n \in L(X)$ induce a map $X \otimes \dots \otimes X \rightarrow \mathbb{C}$

$$(A_1 \otimes \dots \otimes A_n)(l) : (y_1, \dots, y_n) \mapsto l(A_1^* y_1, \dots, A_n^* y_n)$$

$\in X \otimes \dots \otimes X$

adjoint ↗

In general: $A_1 \otimes A_2$ is the unique linear map that satisfies $(A_1 \otimes A_2)(x \otimes y) = A_1 x \otimes A_2 y$. The universal property of the tensor product yields uniqueness.

write $\Lambda^n(A) = A \otimes \dots \otimes A$. If $\{e_i\}_i$ is an orthonormal basis of X , then $\{e_1 \otimes \dots \otimes e_n\}_{1 \leq i_1 < \dots < i_n}$ is an ONS of $X \otimes \dots \otimes X$.

Example

$X = L^2(I)$, $I = [-1, 1]$. What is $X \otimes X$?

$$f \otimes g : (\phi, \psi) \mapsto \langle f, \phi \rangle_{L^2(I)} \langle g, \psi \rangle_{L^2(I)}$$

$$= \int_I \overline{f(x)} \phi(x) dx \int_I \overline{g(y)} \psi(y) dy$$

$$= \int_{I^2} \underbrace{\overline{f(x)g(y)}}_{=: u(x,y)} \underbrace{\phi(x)\psi(y)}_{=: v(x,y)} d(x,y)$$

$$= \langle u, v \rangle_{L^2(I^2)}$$

$\Rightarrow f \otimes g \mapsto u$ induces an isometric isomorphism $X \otimes X \cong L^2(I^2)$.

{ only "induces" an iso since product-type functions do not form all of $L^2(I^2)$: only the closure! }

Definition

$$x \wedge \dots \wedge x_n = \frac{1}{n!} \sum_{\pi \in S_n} (-1)^\pi x_{\pi(1)} \otimes \dots \otimes x_{\pi(n)}.$$

"antisymmetrization"

sign of the permutation

\hookrightarrow symmetric group on $\{1, \dots, n\}$

Again, the completion of the span of such $x_1 \dots x_n$'s is a Hilbert space, denoted $\Lambda^n(X)$. An ONB $\{e_i\}_i$ induces an ONB $\{e_{i_1} \wedge \dots \wedge e_{i_n}\}_{i_1 < \dots < i_n}$ of $\Lambda^n(X)$. $\Lambda^0(X) := \mathbb{C}$, $\Lambda^1(X) = X$.

Proposition

a) $(A_1 \otimes \dots \otimes A_n)(x_1 \otimes \dots \otimes x_n) = (A_1 x_1) \otimes \dots \otimes (A_n x_n)$

universal property of
the tensor product

b) $|\Lambda^n(A)| = \Lambda^n(|A|)$ \leftarrow prove by showing
 $\Lambda^n(A)^* \Lambda^n(A) = \Lambda^n(|A|) \Lambda^n(|A|)$

c) $\Lambda^n(A)$ has singular values s_1, \dots, s_{2n} , $2_1 < \dots < 2_n$

d) If $\dim X = d < \infty$ then $\{\text{eigenvalues of } \Lambda^n(|A|)\}$

$\dim \Lambda^d(X) = \binom{d}{n}$ $\rightarrow \Lambda^d(X) = \mathbb{C}$, $\text{Tr } \Lambda^{dn}(A) = \det A$, $\text{Tr } \Lambda^{d+n}(A) = 0 \quad \forall n > 0$.

$$\begin{aligned} \text{RTP: } |\Lambda^n(A)| &= \Lambda^n(|A|). \\ (\Delta^n(A))^* \Delta^n(A) &+ \omega_- \otimes \omega_+ + \omega_+ \otimes \omega_- \\ &= (\Delta^n(|A|))^* + \omega_- \otimes \omega_+ \cdot \Delta^n(|A|) + \omega_+ \otimes \omega_- \\ &= ((A(\psi_1) \otimes \dots \otimes (A(\psi_n)) \cdot (A(\psi_1) \otimes \dots \otimes (A(\psi_n)))^* \\ &= \prod_{i=1}^n (\psi_i, |A|^* |A| \psi_i) \\ &= \prod_{i=1}^n (\psi_i, A^* A \psi_i) \\ &= (\Delta^n(A))^* \Delta^n(A) + \omega_- \otimes \omega_+ + \omega_+ \otimes \omega_- \end{aligned}$$

Operator Determinants

Motivation

For $\dim X = d < \infty$,

$$\sigma(A) = \{\lambda_1, \dots, \lambda_d\} \Rightarrow \sigma(\Lambda^k(A)) = \{\lambda_{i_1} \cdot \dots \cdot \lambda_{i_k}\}_{i_1 < \dots < i_k}.$$

$$\Rightarrow \sum_{k=0}^{\infty} \text{Tr} \Lambda^k(A) = \sum_{k=0}^d \sum_{i_1 < \dots < i_k} \lambda_{i_1} \cdot \dots \cdot \lambda_{i_k} = \prod_{n=1}^N (1 + \lambda_n)$$

Definition

For a trace-class $A \in K(X)$ in a Hilbert space X ,

$$\det(I + zA) = \sum_{k=0}^{\infty} z^k \cdot \text{Tr} \Lambda^k(A)$$

$$\det(I + zA) = \sum_{k=0}^{\infty} z^k \cdot \text{Tr } A^k (A)$$

Theorem

$A^k(A)$ is trace-class in $\mathcal{L}^k(X)$ with

$$\|A^k(A)\|_1 \leq \frac{\|A\|_1^k}{k!}.$$

In particular, $\det(I + zA)$ is well-defined and dominated by $\exp(|z| \|A\|_1)$.

proof

$$\|A\|_T^k = \left(\sum_{j=1}^{\infty} s_j \right)^k = k! \underbrace{\sum_{1 \leq s_1 < s_2 \dots < s_{k+1}} s_1 \dots s_{k+1}}_{k! \|A^k(A)\|_T} + [\dots].$$

rest terms

Cauchy product

Properties of the Determinant

Ideally we would want

$$\det(I + zA) = \prod_{j=1}^{\infty} (1 + \lambda_j z).$$

How best to show this?

Lemma

Let f be an entire function with $f(0) = 1$.

Write $f^{-1}(0) = \{z_1, z_2, \dots\}$.

Suppose $\sum_{n=1}^{\infty} |z_n|^{-1} < \infty$ and

zeros "run away fast"

bounded by exponentials

$$\forall \varepsilon > 0 \exists C > 0 : |f(z)| \leq C \exp(\varepsilon |z|) \quad \forall z.$$

Then

$$f(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right).$$

Lemma

$$\sum_{j=1}^{\infty} |\lambda_j(A)| \leq \text{Tr}(A)$$

If A has some $\lambda < 0$ eigenvalues / singular values, we extend by adding zeros.

proof

Consider $E_n = \{v_j(A)\}_{j=1}^n$, $P_n : X \rightarrow E_n$. \leftarrow orthogonal projection onto the span of the first n eigenvectors

Then $A|_{E_n}$ acts as a finite-dimensional matrix, for which we know

finite-dim determinant

$$\left| \prod_{j=1}^n \lambda_j(AP_n) \right| = \left| \prod_{j=1}^n \lambda_j(A|_{E_n}) \right| = \prod_{j=1}^n \lambda_j(|A|_{E_n}|) = \prod_{j=1}^n \lambda_j(|AP_n|)$$

But $\lambda_j(|AP_n|) = s_j(AP_n) \stackrel{\text{last talk}}{\leq} s_j(A)$

λ monotone, convex, $\lambda_j \geq 0$

$$\Rightarrow \sum_{j=1}^n \log |\lambda_j| \leq \sum_{j=1}^n \log s_j \quad \forall n \Rightarrow \sum_{j=1}^n |\lambda_j| \leq \sum_{j=1}^n s_j \quad \forall n$$

Theorem

The determinant satisfies the boundedness condition

$$\forall \varepsilon > 0 \exists C > 0 : |\det(I + zA)| \leq C \exp(|z|) \quad \forall z.$$

proof

Let $\varepsilon > 0$. choose $N \in \mathbb{N}$ with $\sum_{j=N+1}^{\infty} s_j \leq \frac{\varepsilon}{2}$. Then

$$|\det(I + zA)| \leq \sum_{k=0}^{\infty} |z|^k \|A^k(A)\|_{\text{Tr}}$$

$$(*) \leq \prod_{j=N+1}^{\infty} \exp(|z|s_j)$$

definition
of $\|\cdot\|_{\text{Tr}}$ $\rightarrow = \sum_{k=0}^{\infty} \sum_{1 \leq i_1 < i_2 < \dots < i_k} |z|^k s_{i_1} \dots s_{i_k}$

$$= \exp(|z| \sum_{j=N+1}^{\infty} s_j)$$

sum converges
uniformly
 \rightarrow factorizable $\rightarrow = \prod_{j=1}^{\infty} (1 + |z|s_j)$

$$< \exp(|z| \frac{\varepsilon}{2})$$

$$\begin{aligned} &= \underbrace{\prod_{j=1}^N (1 + |z|s_j)}_{\leq C \cdot \exp(|z| \frac{\varepsilon}{2})} \underbrace{\prod_{j=N+1}^{\infty} (1 + |z|s_j)}_{(*)} \end{aligned}$$

Lemma

$A \mapsto \det(I+A)$ is continuous in the trace-class.

proof

Let A, B be trace-class.

Consider $g(z) = \det(I + \frac{1}{2}(A+B) + z(A-B))$. g is analytic.

given by a
power series
↓

$$|\det(I+A) - \det(I+B)|$$

$$= |g(\frac{1}{2}) - g(-\frac{1}{2})| \quad) \text{ mean value theorem}$$

$$\leq \sup_{-\frac{1}{2} \leq t \leq \frac{1}{2}} |g'(t)| \quad) \text{ Cauchy integral theorem}$$

$$\leq \frac{1}{R} \cdot \sup_{|z| \leq R + \frac{1}{2}} |g(z)| \quad) \text{ choose } R = \frac{1}{\|A-B\|_1},$$

$$\leq \|A-B\|_1 \cdot \exp(\|A\|_1 + \|B\|_1 + 1)$$

$$|\det(I+A)| \leq \exp(\|A\|_1)$$

Corollary

$$\det((I+A)(I+B)) = \det(I+A) \det(I+B) \quad (*)$$

proof

Let $(A_n)_n \rightarrow A$, $(B_n)_n \rightarrow B$ in $\| \cdot \|_1$, $\dim \text{ran } A_n = \dim \text{ran } B_n < \infty$. Then $(*)$ for A_n, B_n is just finite-dimensional determinants. $\det(I + \cdot)$ is continuous.

Corollary

If $I+A$ is invertible, $\det(I+A) \neq 0$.

proof

Let $B = -A(I+A)^{-1}$. Then

$$\begin{aligned} I+A - A(I+A)^{-1} - AA(I+A)^{-1} \\ = I+A - A(I+A)(I+A)^{-1} \\ = I \end{aligned}$$

$$\det(I+A) \det(I+B) = \det(I+A+B+AB) = \det I = 1.$$

Theorem

Let λ be an eigenvalue of A with (algebraic) multiplicity n .
Then $\det(I + zA)$ has a root of order n at $z_0 = -\frac{1}{\lambda}$.

proof

Recall $P_\lambda : X \rightarrow N$ from theorem 1.

$$\Rightarrow \det(I + zA) = \det((I + zAP_\lambda)(I + zA(I - P_\lambda)))$$

$$\begin{array}{ccc} X & \xrightarrow{I-P_\lambda} & R \\ & \downarrow A & \\ 0 & \xleftarrow{P_\lambda} & R \end{array}$$

$$\begin{aligned} &= \underbrace{\det(I + zAP_\lambda)}_{= (1 - z_0^{-1}z)^n} \cdot \underbrace{\det(I + zA(I - P_\lambda))}_{\neq 0 \text{ since } I \notin \sigma(A|_R)}. \\ &\quad \text{finite-dimensional} \\ &\quad \text{determinant} \end{aligned}$$

Now we can put it all together...

Theorem

$$\det(I + zA) = \prod_{j=1}^{\infty} (1 + z\lambda_j)$$

proof

Let $f(z) = \det(I + zA)$. f has zeros $z_j = -\frac{1}{\lambda_j}$, $j=1, 2, \dots$.

Clearly, $f(0) = 1$.

Weierstrass-Hahn $\Rightarrow \sum_{j=1}^{\infty} |z_j|^{-1} < \infty$.

$\forall \epsilon > 0 \exists C > 0 : |\det(I + zA)| \leq C \exp(\epsilon |z|) \quad \forall z$.

\Rightarrow The expansion $f(z) = \prod_{j=1}^{\infty} (1 - \frac{z}{z_j}) = \prod_{j=1}^{\infty} (1 + \lambda_j z)$ is valid.

A "One-line" Proof of Liouville's Theorem

Corollary

$$\operatorname{Tr} A = \sum_{j=1}^{\infty} \lambda_j.$$

proof

Proof by comparing coefficients.

$$\hookrightarrow \Lambda'(A) = A$$

$$\det(I + zA) = \sum_{k=0}^{\infty} z^k \operatorname{Tr} \Lambda^k(A) = 1 + z \operatorname{Tr} A + O(z^2)$$

$$\det(I + zA) = \prod_{j=1}^{\infty} (1 + \lambda_j z) = 1 + z \left(\sum_{j=1}^{\infty} \lambda_j \right) + O(z^2)$$

Bonus: Tr and det of Integral Operators

Question

For the operator

$$Kf = \int_0^1 K(\cdot, y) f(y) dy, \quad f \in L^2(I), \quad K \in C^0(I, \mathbb{C})$$

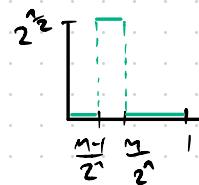
what is $\text{Tr } K$, $\det(I + K)$?

Consider the system $(\phi_m)_n = 2^{\frac{n}{2}} \mathbf{1}_{[\frac{m-1}{2^n}, \frac{m}{2^n})}$,

$P_n : L^2 \rightarrow \{(\phi_m)_n\}_{m=1}^{\infty}$. Then

$$\begin{aligned} \text{Tr}(P_n A P_n) &= \sum_{m=1}^{\infty} \langle (\phi_m)_n, K(\phi_m)_n \rangle \\ &= 2^n \sum_{m=1}^{\infty} \iint_{\frac{m-1}{2^n}}^{\frac{m}{2^n}} K(x, y) dx dy \xrightarrow{n \rightarrow \infty} \int_0^1 K(x, x) dx \end{aligned}$$

K uniformly
continuous



$$2\Lambda^2(\mathbb{K})(f_1, f_2) = (\mathbb{K}f_1 \otimes \mathbb{K}f_2 - \mathbb{K}f_2 \otimes \mathbb{K}f_1)(x_1, x_2)$$

$$= \int_0^1 \int_0^1 K(x_1, y_1) f_1(y_1) K(x_2, y_2) f_2(y_2) dy_1 dy_2$$

$$- \int_0^1 \int_0^1 K(x_1, z_1) f_2(z_1) K(x_2, z_2) f_1(z_2) dz_1 dz_2$$

relabel
 $z_1 = y_2, z_2 = y_1$

$$= \int_0^1 \int_0^1 K(x_1, y_1) f_1(y_1) K(x_2, y_2) f_2(y_2)$$

$$- K(x_1, y_2) f_1(y_1) K(x_2, y_1) f_2(y_2) dy_1 dy_2$$

$$= \int_0^1 \int_0^1 \underbrace{\left[K(x_1, y_1) K(x_2, y_2) - K(x_1, y_2) K(x_2, y_1) \right]}_{K_2(x_1, x_2, y_1, y_2)} f_1(y_1) f_2(y_2) dy_1 dy_2$$

$$= K_2(f_1 \otimes f_2)$$

Analogously,

$$\Lambda^n(\mathbb{K})(f_1, \dots, f_n) = \frac{1}{n!} \sum_{\pi \in S_n} \mathbb{K} f_{\pi(1)} \otimes \dots \otimes \mathbb{K} f_{\pi(n)}$$

$$\begin{aligned} &= \frac{1}{n!} \int_0^1 \dots \int_0^1 \frac{\det(K(x_i, y_j)_{1 \leq i, j \leq n})}{K_n(x_1, \dots, x_n, y_1, \dots, y_n)} f_1(y_1) \cdots f_n(y_n) dy_1 \cdots dy_n \\ &= \frac{1}{n!} K_n(f_1 \otimes \dots \otimes f_n) \end{aligned}$$

Hence,

$$n! \operatorname{Tr} \Lambda^n(\mathbb{K}) = \operatorname{Tr} K_n = \int_0^1 \dots \int_0^1 K_n(x_1, \dots, x_n, x_1, \dots, x_n) dx_1 \cdots dx_n \in \mathbb{C}$$

and so

$$\det(I + \mathbb{K}) = \sum_{n=0}^{\infty} \frac{\operatorname{Tr} K_n}{n!}.$$