Lidskii's Theorem

A beatiful result for finite dimensions ...

Theorem

Let X be a Hilbert space and $A \in K(X)$ be a trace-class operator with eigenvalues $\lambda_1, \lambda_2, \dots$. Then

$$\operatorname{Tr} A = \sum_{n=1}^{\infty} \lambda_n .$$

... extended to infinite dimensions.

Reference

For $\dim X < \infty$ i.e. $A \in \mathbb{C}^{n \times n}$ with eigenvalues $\lambda_1, \dots, \lambda_n$ we know

$$\det(\lambda I - A) = \lambda^n - \operatorname{tr} A \cdot \lambda^{n-1} + \dots + (-1)^n \det A = (\lambda - \lambda_1)(\lambda - \lambda_2) \cdot \ldots \cdot (\lambda - \lambda_n)$$

Contents

Generalized Eigenspaces

Theorem (c.f. Werner)

compact

Let X be a Banach space, $A \in K(X)$, $0 \neq \lambda \in \sigma(A)$.
Then $\exists\, N = N(\lambda)$, $R = R(\lambda)$ subspaces of X with

a) $\dim N < \infty$ — finiteness

b) $X = N \oplus R$ — splitting

c) $A(N) \subset N$, $A(R) \subset R$ — invariance

d) $\lambda \in \sigma(A|N)$, $\lambda \notin \sigma(A|R)$. — isolation

Idea

Use a concept from linear algebra (Jordan canonical form):

$$S = \lambda I - A$$

$$N_0 = \{0\}, \quad N_m = \ker S^m$$

$$R_0 = X, \quad R_m = \operatorname{ran} S^m$$

Lemma

$\exists\, p \in \mathbb{N}: \quad N_p = N_{p+1} = N_{p+2} = \dots .$

$\exists\, q \in \mathbb{N}: \quad R_q = R_{q+1} = R_{q+2} = \dots$ analogous.

$S = \lambda I - A.$

$N_0 = \{0\}, \quad N_m = \ker S^m$

$R_0 = X, \quad R_m = \operatorname{ran} S^m$

proof

For contradiction, assume $N_0 \subsetneq N_1 \subsetneq N_2 \subsetneq \dots .$

Riesz's lemma gives $(x_n)_n$: $x_n \in N_n$, $\|x_n\| = 1$, $d(x_n, N_m) > \frac{1}{2}$ $\forall n > m$,

$$S^m = (\lambda I - A)^m = \lambda^m (I - \lambda^{-1} A)^m = \lambda^m \Big(I - \underbrace{\sum_{k=1}^{m} \binom{m}{k} (-\lambda^{-1} A)^k}_{\text{compact}}\Big)$$

Riesz-Schauder $\Rightarrow \dim N_m < \infty$ and N_m, R_m closed

But then

$$\|Ax_m - Ax_n\| = \| \underbrace{\underbrace{Ax_m - \lambda x_m}_{Sx_m \in N_{m-1}} + \underbrace{\lambda x_m}_{N_m} \underbrace{- Ax_n + \lambda x_n}_{-Sx_n \in N_{n-1}}}_{N_{n-1} \text{ since } m \le n-1} - \underbrace{\lambda x_n}_{N_n} \| > \frac{1}{2}.$$

$\Rightarrow$ No convergent subseq. of $(Ax_n)_n$ $\lightning$ to compactness

Hence $\exists\, p$: $N_p = N_{p+1}$. A calculation then shows $N_{p+1} = N_{p+2} = \dots .$

We can now write

$$N = N_p, \quad R = R_p.$$

Then a) follows from Riesz-Schauder and b), c), d) are (essentially) finite-dimensional linear algebra.

For X as a Hilbert space we can define

$$P_\lambda : X \longrightarrow N$$

the orthogonal projection onto N.

Lemma

$N \cap R = \{0\}$

proof

Let $u \in N \cap R$.

$u \in R_r$

$\Rightarrow u = S^r y$ for some $y \in X$

$\Rightarrow \underbrace{0}_{\in N_r} = S^r u = S^{2r} y$

$N_r = N_{r+1} = N_{r+2} = \dots$

$\Rightarrow y \in N_{2r} = N_r$

$\Rightarrow \underbrace{0 = S^r y}_{\in N_r} = u.$

Lemma

$X = N \oplus R$

proof

For $x \in X$,

$R_r = R_{r+1} = R_{r+2} = \dots$

$R_r = \operatorname{ran} S^r$

$S^r x \in R_r = R_{2r}$

$\Rightarrow \exists y \in X : S^r x = S^{2r} y.$

Then

$S^r(x - S^r y) = 0$

$x = (x - S^r y) + S^r y \in N \oplus R.$

Lemma

$A(N) \subset N \quad A(R) \subset R$

proof

$N_0 \subset N_1 \subset \dots$

$S(N_r) = (\lambda I - A)(N_r) \subset N_{r-1} \subset N_r \iff A(N_r) \subset N_r$

$S(R_r) = (\lambda I - A)(R_r) \subset R_{r+1} = R_r \iff A(R_r) \subset R_r$

λI keeps all subspaces invariant

Lemma

$\lambda \in \sigma(A|N), \quad \lambda \notin \sigma(A|R)$

proof

By construction $E_\lambda = \bigcup_{m \geq 0} \ker S^m \subset N$ and N, R are invariant.

Recap + Notation

We will always assume X is a seperable Hilbert space with inner product $(\cdot,\cdot)$, and $A \in K(X)$ is trace-class.

$\lambda_1, \lambda_2, \dots$ (resp. $s_1, s_2, \dots$) are the eigenvalues (resp. singular values) of A in descending order

$$A v_j = \lambda_j v_j, \qquad |\lambda_1| \geq |\lambda_2| \geq \dots, \qquad s_1 \geq s_2 \geq \dots .$$

A can be decomposed as $A = U|A|$ where U is unitary and $|A|$ is positive and self-adjoint.

Spectral theory $\Rightarrow$ ONB of $\overline{\operatorname{ran}|A|}$ $\{z_j\}_j$ with $|A| z_j = s_j z_j$.

Write $w_j = U z_j$

$$\Rightarrow \quad A = \sum_{j=1}^{\infty} s_j (\cdot, z_j) w_j .$$

"Canonical decomposition"

Operator Determinants (Preview)

Many definitions:

Goh'berg - Krein

$$\det(I+zA) = \prod_{j=1}^{\infty}(1+z\lambda_j)$$

Fredholm

$$\det(I+zA) = \sum_{k=0}^{\infty} z^k \cdot \operatorname{Tr} \Lambda^k(A)$$

Dunford - Schwarz

$$\det(I+\mu A) = \exp \circ \operatorname{Tr} \circ \operatorname{Log}(I+\mu A)$$

for $|\mu| \ll 1$ + analytic continuation

Lidskii's theorem will show they are all equivalent!

Antisymmetric Tensor Products

Definition

For $x_1, x_2, \dots, x_n \in X$, let $(x_1 \otimes x_2 \otimes \dots \otimes x_n) : X \times X \times \dots \times X \longrightarrow \mathbb{C}$ be the multilinear map

$$\boxed{x_1 \otimes x_2 \otimes \dots \otimes x_n} : (y_1, \dots, y_n) \longmapsto \prod_{i=1}^{n} (x_i, y_i).$$

In general: If
$U = \operatorname{span} B_U$, $W = \operatorname{span} B_W$,
$U \otimes W$ is spanned by the maps
$B_U \times B_W \to \mathbb{R}$ which have
finitely many nonzero values.
- But that is precisely
$\{ (x,y) \mapsto \langle x,u \rangle \cdot \langle y,w \rangle \mid u \in B_U, w \in B_W \}$

The (metric) completion of span of such $(x_1 \otimes x_2 \otimes \dots \otimes x_n)$'s is a Hilbert space, denoted $X \otimes \dots \otimes X$.

Operators $A_1, A_2, \dots, A_n \in L(X)$ induce a map $X \otimes \dots \otimes X \circlearrowleft$

adjoint

$$\boxed{(A_1 \otimes \dots \otimes A_n)}(\ell) : (y_1, \dots, y_n) \longmapsto \ell(A_1^* y_1, \dots, A_n^* y_n).$$

$\in X \otimes \dots \otimes X$

In general: $A_1 \otimes A_2$
is the unique linear map
that satisfies
$(A_1 \otimes A_2)(x \otimes y) = A_1 x \otimes A_2 y$.
The universal property of the
tensor product yields uniqueness.

Example

$X = L^2(I)$, $\quad I = [-1, 1]$. What is $X \otimes X$?

$$f \otimes g : (\phi, \psi) \longmapsto \langle f, \phi \rangle_{L^2(I)} \cdot \langle g, \psi \rangle_{L^2(I)}$$

$$= \int_I \overline{f(x)}\, \phi(x)\, dx \int_I \overline{g(y)}\, \psi(y)\, dy$$

$$= \int_{I^2} \underbrace{\overline{f(x) g(y)}}_{=: \overline{u(x,y)}} \; \underbrace{\phi(x) \psi(y)}_{=: v(x,y)} \; d(x,y)$$

$$= \langle u, v \rangle_{L^2(I^2)}$$

$\Rightarrow$ $f \otimes g \longmapsto u$ induces an isometric isomorphism $X \otimes X \cong L^2(I^2)$.

only "induces" an iso since product-type functions do not form all of $L^2(I^2)$: only the closure!

Definition

sign of the permutation

$$\boxed{x_1 \wedge \dots \wedge x_n} = \frac{1}{n!} \sum_{\pi \in S_n} (-1)^\pi \, x_{\pi(1)} \otimes \dots \otimes x_{\pi(n)} .$$

"antisymmetrization"

symmetric group on $\{1, \dots, n\}$

The completion of the span of such $x_1 \wedge \dots x_n$'s is a Hilbert space w.r.t. the inner product

$$\langle x_1 \wedge \dots \wedge x_n , y_1 \wedge \dots \wedge y_n \rangle = \det \left((x_i, y_j)_{1 \le i,j \le n} \right)$$

This space is denoted $\boxed{\Lambda^n(X)}$. An ONB $\{e_i\}_i$ induces an ONB $\{e_{i_1} \wedge \dots \wedge e_{i_n}\}_{i_1 < \dots < i_n}$ of $\Lambda^n(X)$. $\Lambda^0(X) := \mathbb{C}$, $\Lambda^1(X) = X$.

$A \otimes \dots \otimes A$ leaves $\Lambda^n(X)$ invariant and is denoted $\boxed{\Lambda^n(A)}$ for simplicity.

Proposition

a) $(A_1 \wedge \dots \wedge A_n)(x_1 \wedge \dots \wedge x_n) = (A_1 x_1) \wedge \dots \wedge (A_n x_n)$

b) $|\Lambda^n(A)| = \Lambda^n(|A|)$

c) $\Lambda^n(A)$ has singular values $s_{i_1} \cdot \ldots \cdot s_{i_n}$, $i_1 < \dots < i_n$

d) If $\dim X = d < \infty$ then

$$\Lambda^d(X) = \mathbb{C}, \qquad \operatorname{Tr} \Lambda^d(A) = \det A, \qquad \operatorname{Tr} \Lambda^{d+m}(A) = 0 \ \forall m > 0.$$

proof

Each point is a direct calculation, using previous points.

a) is known as the universal property of the tensor product.

b) proven by showing $\Lambda^n(A)^* \Lambda^n(A) = \Lambda^n(|A|)^2$.

c) construct eigenvectors of $\Lambda^n(|A|)$ using a).

d) $\dim \Lambda^k(X) = \binom{d}{k}$.

RTP: $|\Lambda^n(A)| = \Lambda^n(|A|)$

$$\left(\Lambda^n(|A|)^* \Lambda^n(|A|)\, \phi_1 \otimes \dots \otimes \phi_n ,\ \psi_1 \otimes \dots \otimes \psi_n\right)$$

$$= \left(\Lambda^n(|A|)\, \phi_1 \otimes \dots \otimes \phi_n ,\ \Lambda^n(|A|)\, \psi_1 \otimes \dots \otimes \psi_n\right)$$

$$= \left((|A|\phi_1) \otimes \dots \otimes (|A|\phi_n),\ (|A|\psi_1) \otimes \dots \otimes (|A|\psi_n)\right)$$

$$= \prod_{i=1}^{n} (\phi_i, |A|^*|A|\psi_i)$$

$$= \prod_{i=1}^{n} (\phi_i, A^*A\psi_i)$$

$$= \left(\Lambda^n(A)^* \Lambda^n(A)\, \phi_1 \otimes \dots \otimes \phi_n ,\ \psi_1 \otimes \dots \otimes \psi_n\right)$$

Operator Determinants

Motivation

For $\dim X = d < \infty$,

$$\sigma(A) = \{\lambda_1, \dots, \lambda_d\} \Rightarrow \sigma(\Lambda^k(A)) = \{\lambda_{i_1} \cdot \ldots \cdot \lambda_{i_k}\}_{i_1 < \dots < i_k}$$

$$\Rightarrow \sum_{k=0}^{\infty} \operatorname{Tr} \Lambda^k(A) = \sum_{k=0}^{d} \sum_{i_1 < \dots < i_k} \lambda_{i_1} \cdot \ldots \cdot \lambda_{i_k} = \prod_{n=1}^{N} (1 + \lambda_n) = \det(I + A)$$

Definition

For a trace-class $A \in K(X)$ in a Hilbert space X,

$$\boxed{\det(I + zA) = \sum_{k=0}^{\infty} z^k \cdot \operatorname{Tr} \Lambda^k(A)}$$

$$\det(I + zA) = \sum_{k=0}^{\infty} z^k \cdot \operatorname{Tr} \Lambda^k(A)$$

Theorem

$\Lambda^k(A)$ is trace-class in $\Lambda^k(X)$ with

$$\| \Lambda^k(A) \|_1 \leq \frac{\|A\|_1^k}{k!} .$$

In particular, $\det(I + zA)$ is well-defined and dominated by $\exp(|z| \cdot \|A\|_1)$.

proof

$$\|A\|_1^k = \Big(\sum_{j=1}^{\infty} s_j \Big)^k \overset{\text{Cauchy product}}{=} \underbrace{k! \sum_{i_1 < \dots < i_k} s_{i_1} \cdot \dots \cdot s_{i_k}}_{k! \, \|\Lambda^k(A)\|_1} + \underbrace{[\,\dots\,]}_{\text{rest terms}} .$$

Properties of the Determinant

Goal: $z_n = -\frac{1}{\lambda_j}$

Ideally we would want

$$\det(I + zA) = \prod_{j=1}^{\infty} (1 + \lambda_j z).$$

How best to show this?

Lemma

Let f be an entire function with $f(0) = 1$.

Write $f^{-1}(0) = \{z_1, z_2, \ldots\}$.

zeros "run away fast"

Suppose $\sum_{n=1}^{\infty} |z_n|^{-1} < \infty$ and

bounded by exponeitials

$$\forall \varepsilon > 0 \ \exists C > 0 : \ |f(z)| \leq C \exp(\varepsilon |z|) \ \forall z.$$

Then

$$f(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right).$$

Lemma

$A \longmapsto \det(I+A)$ is continuous in the trace-class.

proof

Let A, B be trace-class.

given by a power series

Consider $g(z) = \det\left(I + \frac{1}{2}(A+B) + z(A-B) \right)$. g is analytic.

$$|\det(I+A) - \det(I+B)|$$

$$= |g(\tfrac{1}{2}) - g(-\tfrac{1}{2})|$$

mean value theorem

$$\leq \sup_{-\frac{1}{2} \leq t \leq \frac{1}{2}} |g'(t)|$$

Cauchy integral theorem

$$\leq \frac{1}{R} \cdot \sup_{|z| \leq R + \frac{1}{2}} |g(z)|$$

choose $R = \frac{1}{\|A-B\|_1}$,

$|\det(I+A)| \leq \exp(\|A\|_1)$

$$\leq \|A - B\|_1 \cdot \exp(\|A\|_1 + \|B\|_1 + 1)$$

Corollary

$I+A+B+AB$

$$\det\big((I+A)(I+B)\big) = \det(I+A)\det(I+B) \quad (*)$$

proof

given by truncated canonical decomposition

Let $(A_n)_n \to A$, $(B_n)_n \to B$ in $\|\cdot\|_1$, $\dim \operatorname{ran} A_n = \dim \operatorname{ran} B_n < \infty$.
Then $(*)$ for A_n, B_n is just finite-dimensional determinants.
$\det(I+\cdot)$ is continuous.

Corollary

If $I+A$ is invertible, $\det(I+A) \neq 0$.

proof

Let $B = -A(I+A)^{-1}$. Then

$$I+A - A(I+A)^{-1} - AA(I+A)^{-1}$$
$$= I+A - A(I+A)(I+A)^{-1}$$
$$= I$$

$$\det(I+A)\det(I+B) = \det(I+A+B+AB) = \det I = 1.$$

Theorem

Let λ be an eigenvalue of A with (algebraic) multiplicity n. Then $\det(I+zA)$ has a root of order n at $z_0 = -\frac{1}{\lambda}$.

proof

Recall $P_\lambda : X \twoheadrightarrow N$ from theorem 1.

$$\Rightarrow \det(I+zA) = \det\big((I+zAP_\lambda)(I+zA(1-P_\lambda)) \big)$$

$$= \underbrace{\det(I+zAP_\lambda)}_{\substack{=(1-z_0^{-1}z)^n \\ \text{finite-dimensional} \\ \text{determinant}}} \cdot \underbrace{\det(I+zA(I-P_\lambda))}_{\substack{\neq 0 \text{ since } \lambda\notin\sigma(A|R) \\ \text{and } z\to z_0}} .$$

$$\begin{tikzcd}
X \arrow[r, "1-P_\lambda"] \arrow[d, dotted] & R \arrow[d, "A"] \\
0 & R \arrow[l, "P_\lambda"']
\end{tikzcd}$$

Lemma

$$\sum_{j=1}^{\infty} |\lambda_j(A)| \leq \|A\|_1$$

If A has some $d < \infty$ eigenvalues / singular values, we extend by adding zeros.

proof

Consider $E_n = \{v_j(A)\}_{j=1}^{n}$, $P_n : X \twoheadrightarrow E_n$.

orthogonal projection onto the span of the first n eigenvectors

Then $A|_{E_n}$ acts as a finite-dimensional matrix, for which we know

$$\left| \prod_{j=1}^{n} \lambda_j(AP_n) \right| = \left| \prod_{j=1}^{n} \lambda_j(A|_{E_n}) \right| \overset{\text{finite-dim determinant}}{=} \prod_{j=1}^{n} \lambda_j(\,|A|_{E_n}|\,) = \prod_{j=1}^{n} \lambda_j(|AP_n|)$$

But $\lambda_j(|AP_n|) = s_j(AP_n) \overset{\text{last talk}}{\leq} s_j(A)$

$$\Rightarrow \sum_{j=1}^{n} \log|\lambda_j| \leq \sum_{j=1}^{n} \log s_j \quad \forall n \quad \overset{e^x \text{ monotone, convex, } |\lambda_j| \geq 0}{\Longrightarrow} \quad \sum_{j=1}^{n} |\lambda_j| \leq \sum_{j=1}^{n} s_j \quad \forall n.$$

Theorem
The determinant satisfies the boundedness condition

$$\forall \varepsilon > 0 \ \exists C > 0 : \ |\det(I + zA)| \leq C \exp(\varepsilon |z|) \quad \forall z.$$

proof
Let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ with $\sum_{j=N+1}^{\infty} s_j \leq \frac{\varepsilon}{2}$. Then

$$|\det(I + zA)| \leq \sum_{k=0}^{\infty} |z|^k \|\Lambda^k(A)\|_1$$

definition of $\|\cdot\|_1$ $\longrightarrow$
$$= \sum_{k=0}^{\infty} \sum_{l_1 < \dots < l_k} |z|^k s_{l_1} \cdot \ldots \cdot s_{l_k}$$

sum converges uniformly => factorizable $\longrightarrow$
$$\leq \prod_{j=1}^{\infty} (1 + |z| s_j)$$

$$= \underbrace{\prod_{j=1}^{N} (1 + |z| s_j)}_{\leq C \cdot \exp(|z| \frac{\varepsilon}{2})} \underbrace{\prod_{j=N+1}^{\infty} (1 + |z| s_j)}_{(*)}$$

$$(*) \leq \prod_{j=N+1}^{\infty} \exp(|z| s_j)$$

$$= \exp\left(|z| \sum_{j=N+1}^{\infty} s_j\right)$$

$$\leq \exp\left(|z| \frac{\varepsilon}{2}\right)$$

Now we can put it all together...

Theorem

$$\det(I + zA) = \prod_{j=1}^{\infty} (1 + z\lambda_j)$$

proof

Let $f(z) = \det(I + zA)$. f has zeros $z_j = -\frac{1}{\lambda_j}$, $j = 1, 2, \ldots$.

Clearly $f(0) = 1$. Further $\sum_{j=1}^{\infty} |z_j|^{-1} < \infty$,

$\forall \varepsilon > 0 \; \exists C > 0 : |\det(I + zA)| \leq C \exp(\varepsilon |z|) \quad \forall z.$

$\Rightarrow$ The expansion $f(z) = \prod_{j=1}^{\infty} \left(1 - \frac{z}{z_j}\right) = \prod_{j=1}^{\infty} (1 + \lambda_j z)$ is valid.

A "One-line" Proof of Lidskii's Theorem

Corollary

$$\operatorname{Tr} A = \sum_{j=1}^{\infty} \lambda_j .$$

proof

Proof by comparing coefficients.

$\Lambda^1(A) = A$

$$\det(I + zA) = \sum_{k=0}^{\infty} z^k \operatorname{Tr} \Lambda^k(A) = 1 + z \operatorname{Tr} A + \mathcal{O}(z^2)$$

$$\det(I + zA) = \prod_{j=1}^{\infty} (1 + \lambda_j z) = 1 + z\left(\sum_{j=1}^{\infty} \lambda_j\right) + \mathcal{O}(z^2)$$

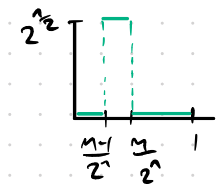
Bonus: Tr and det of Integral Operators

Question

For the operator

$$\mathbb{K} f = \int_0^1 K(\cdot, y)\, f(y)\, dy, \qquad f \in L^2(I), \quad K \in C^0(I, \mathbb{C})$$

what is $\operatorname{Tr} \mathbb{K}$, $\det(I + \mathbb{K})$?



Consider the system $(\phi_m)_n = 2^{n/2}\, \mathbb{1}_{[\frac{m-1}{2^n}, \frac{m}{2^n})}$,

$P_n : L^2 \twoheadrightarrow \{(\phi_m)_n\}_{m=1}^{n}$. Then

$$\operatorname{Tr}(P_n \mathbb{K} P_n) = \sum_{m=1}^{n} \langle (\phi_m)_n, \mathbb{K} (\phi_m)_n \rangle$$

K uniformly continuous

$$= 2^n \sum_{m=1}^{n} \iint_{\frac{m-1}{2^n}}^{m/2^n} K(x,y)\, dx\, dy \xrightarrow{n \to \infty} \int_0^1 K(x,x)\, dx$$

$$[2\Lambda^2(K)(f_1 \wedge f_2)](x_1, x_2) = [Kf_1 \otimes Kf_2 - Kf_2 \otimes Kf_1](x_1, x_2)$$

$$= \int_0^1 \int_0^1 K(x_1, y_1) f_1(y_1) K(x_2, y_2) f_2(y_2) \, dy_1 \, dy_2$$

$$- \int_0^1 \int_0^1 K(x_1, z_1) f_2(z_1) K(x_2, z_2) f_1(z_2) \, dz_1 \, dz_2$$

relabel $z_1 = y_2$, $z_2 = y_1$

$$= \int_0^1 \int_0^1 K(x_1, y_1) f_1(y_1) K(x_2, y_2) f_2(y_2) - K(x_1, y_2) f_1(y_1) K(x_2, y_1) f_2(y_2) \, dy_1 \, dy_2$$

$$= \int_0^1 \int_0^1 \underbrace{[K(x_1, y_1) K(x_2, y_2) - K(x_1, y_2) K(x_2, y_1)]}_{K_2(x_1, x_2, y_1, y_2)} f_1(y_1) f_2(y_2) \, dy_1 \, dy_2$$

$$= K_2(f_1 \otimes f_2)$$

Analogously,

$$\Lambda^n(\mathbb{K})(f_1 \wedge \dots \wedge f_2) = \frac{1}{n!} \sum_{\pi \in S_n} \mathbb{K} f_{\pi(1)} \otimes \dots \otimes \mathbb{K} f_{\pi(n)}$$

$$= \frac{1}{n!} \int_0^1 \cdots \int_0^1 \underbrace{\det\left(K(x_i, y_j)_{1 \le i,j \le n} \right)}_{K_n(x_1, \dots, x_n, y_1, \dots, y_n)} f_1(y_1) \cdots f_n(y_n) \, dy_1 \cdots dy_n$$

$$= \frac{1}{n!} \mathbb{K}_n (f_1 \otimes \dots \otimes f_n)$$

Hence,

$$n! \operatorname{Tr} \Lambda^n(\mathbb{K}) = \operatorname{Tr} \mathbb{K}_n = \int_0^1 \cdots \int_0^1 K_n(x_1, \dots, x_n, x_1, \dots, x_n) \, dx_1 \cdots dx_n \in \mathbb{C}$$

and so

$$\det(I + \mathbb{K}) = \sum_{n=0}^{\infty} \frac{\operatorname{Tr} \mathbb{K}_n}{n!}.$$