Solution 1 (April Herwig) *4.4*

a) Let $Y = \prod_{n\in\mathbb{N}} X_i$. *For each* $n \in \mathbb{N}$ *define* $\delta_n(x,y) = \rho \circ d_{X_n}(x,y)$ *where* d_{X_n} *is the metric on* X_n *and* ρ *is a monotone increasing homeomorphism* $\rho : [0,\infty) \to [0,1)$, *eg.* $x \mapsto \frac{x}{1+x}$. *Define the metric on* Y *as*

$$d : Y \times Y \to [0,\infty),\ (x,y) \mapsto \sum_{n\in\mathbb{N}} \frac{1}{2^n}\,\delta_n(x,y)\,.$$

This clearly satisfies the conditions for a metric since each d_{X_n} *is a metric and* ρ *is monotone increasing. We will show that this metric induces the same topology as the product topology.*

Let $U \subseteq Y$ *be open in the product topology, ie.*

$$U = \prod_{n\in\mathbb{N}} U_n, \quad U_n \text{ open}, \quad \exists\, N \in \mathbb{N} : U_m = X_m\ \forall m > N\,.$$

For $x \in U$, *each space* X_n, $n \in \mathbb{N}$ *has an* $\epsilon_n > 0$ *with* $B_{\epsilon_n}(x_n) \subseteq U_n$. *Define*

$$\epsilon = min\left\{\frac{1}{2^n}\rho(\epsilon_n) | n \leq N\right\}\,.$$

Then ϵ *satisfies* $x \in B_\epsilon(x) \subseteq U$ *since for* $y \neq x$ *we have*

$$d_{X_n}(y_n, x_n) > \epsilon_n \;\Leftrightarrow\; \frac{1}{2^n}\delta_n(y_n,x_n) > \frac{1}{2^n}\rho(\epsilon_n)\,.$$

Hence U *is open in the metric topology.*

Now let $x \in Y$, $U = B_\epsilon(x) \subseteq Y$, $y \in U$. *Set* $N = \lceil \log(\frac{4}{2+\epsilon}) \rceil$ *and consider*

$$V = \prod_{n\leq N} B_{\epsilon/2}(y_n) \;\times\; \prod_{m>N} X_m\,.$$

V *is open in the product topology and* $y \in V$. *Finally, we will show* $V \subseteq U$ *and hence* U *is a neighborhood of all its points, so open.*

$$\underbrace{\sum_{n=1}^{N} \frac{1}{2^n}\frac{\epsilon}{2}}_{\leq \frac{\epsilon}{2}} + \underbrace{\sum_{m=N+1}^{\infty} \frac{1}{2^n}}_{=1-2(1-2^{-N})} \overset{!}{<} \epsilon$$

$$1 - 2(1-2^{-N}) < \frac{\epsilon}{2} \quad\Leftrightarrow\quad N > \frac{\log(\frac{4}{2+\epsilon})}{\log(2)}\,.$$

b) Let $X = \{0,1\}$ *equipped with the discrete topology.* X *is trivially metrizable. Consider*

$$Y = X^{\mathbb{R}} = \prod_{x\in\mathbb{R}} \{0,1\} = \{f : \mathbb{R} \to X | \}\,.$$

Let $f_0 \equiv 0$ *the zero function. A neighborhood basis of* f_0 *is given by*

$$\mathcal{B}_{f_0} = \bigcup_{\substack{K \subseteq \mathbb{R} \\ \textit{finite}}} \underbrace{\{f \in Y \mid f(x) = 0 \ \forall x \in \mathbb{R} \setminus K\}}_{=: \, U_K} .$$

Assume Y is metrizable. Then it is homeomorphic to a metric space, and since metric spaces are first countable, Y is first countable. It follows that there exists a countable subset $\{U_{K_i}\}_{i \in \mathbb{N}} \subseteq \mathcal{B}_{f_0}$ which is still a neighborhood basis of f_0. In particular, $\bigcup_{i \in \mathbb{N}} K_i$ is countable.

$$\Rightarrow \ \exists \, \bar{x} \in \mathbb{R} \setminus \bigcup_{i \in \mathbb{N}} K_i \neq \emptyset \, .$$

Finally, consider

$$U_{\{\bar{x}\}} = \{f \in Y \mid f(\bar{x}) = 0\} \in \mathcal{N}_{f_0} \, ,$$

$$\bigcap_{i \in \mathbb{N}} U_{K_i} = \left\{ f \in Y \mid f(x) = 0 \ \forall x \in \bigcup_{i \in \mathbb{N}} K_i \right\} .$$

Then $U_{\{\bar{x}\}}$ is an open neighborhood of of f_0, but

$$\bigcap_{i \in \mathbb{N}} U_{K_i} \not\subseteq U_{\{\bar{x}\}}$$

which implies no U_{K_i} is a subset of $U_{\{\bar{x}\}}$, a contradiction to $\{U_{K_i}\}_{i \in \mathbb{N}}$ being a neighborhood basis.