Lidskii's Theorem

A beatiful result for finite dimensions...

Theorem

Let X be a Hibert space and $A \in L(X)$ be a trace-class operator with eigenvalues $\lambda_1, \lambda_2, \dots$. Then

$$\operatorname{Tr} A = \sum_{n=1}^{\infty} \lambda_n .$$

... extended to infinite dimensions.

Reference

For $\dim X < \infty$ i.e. $A \in \mathbb{C}^{n \times n}$ with eigenvalues $\lambda_1, \dots, \lambda_n$ we know

$$\det(\lambda I - A) = \lambda^n - \operatorname{tr} A \cdot \lambda^{n-1} + \dots + (-1)^n \det A = (\lambda - \lambda_1)(\lambda - \lambda_2) \cdot \dots \cdot (\lambda - \lambda_n)$$

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Generalized Eigenspaces

Theorem (c.f. Werner)

Let X be a Banach space, $A \in K(X)$, $0 \neq \lambda \in \sigma(A)$.
Then $\exists\, N = N(\lambda)$, $R = R(\lambda)$ subspaces of X with

a) $\dim N < \infty$
b) $X = N \oplus R$
c) $A(N) \subset N$, $A(R) \subset R$
d) $\lambda \in \sigma(A|N)$, $\lambda \notin \sigma(A|R)$.

Idea

Use a concept from linear algebra (Jordan canonical form):

$$S = \lambda I - A$$

$$N_0 = \{0\}, \quad N_n = \ker S^n$$

$$R_0 = X, \quad R_n = \operatorname{ran} S^n$$

$S = \lambda I - A$

$N_0 = \{0\}, \quad N_n = \ker S^n$

$R_0 = X, \quad R_n = \operatorname{ran} S^n$

Lemma

$\dim N_n < \infty \quad \forall n$

proof

$$S^n = (\lambda I - A)^n$$

$$= \lambda^n \left(I - \tfrac{1}{\lambda} A\right)^n$$

$$= \lambda^n \Big(I - \underbrace{\sum_{k=1}^{n} \binom{n}{k} \left(\tfrac{-1}{\lambda}\right)^k A^k}_{\text{compact}}\Big)$$

Riesz-Schauder $\Rightarrow$ $\dim N_n < \infty$ and N_n, R_n closed.

Lemma

$\exists\, p \in \mathbb{N} : \quad N_p = N_{p+1} = N_{p+2} = \dots$

$\exists\, q \in \mathbb{N} : \quad R_q = R_{q+1} = R_{q+2} = \dots$

$S = \lambda I - A$

$N_0 = \{0\}, \quad N_m = \ker S^m$

$R_0 = X, \quad R_m = \operatorname{ran} S^m$

proof

For contradiction, assume $\quad N_0 \subsetneq N_1 \subsetneq N_2 \subsetneq \dots .$

$R_0 \supsetneq R_1 \supsetneq R_2 \supsetneq \dots$

Fischer-Riesz gives us

$$(x_n)_n : \|x_n\| = 1, \quad d(x_n, N_m) > \tfrac{1}{2} \quad \forall n > m .$$

$$(x_n)_n : \|x_n\| = 1, \quad d(x_n, R_m) > \tfrac{1}{2} \quad \forall n < m .$$

But then

$m < n \Rightarrow N_m \subset N_{n-1}$

$$\|Ax_m - Ax_n\| = \|\underbrace{Ax_m - \lambda x_m}_{Sx_m \in N_m} + \underbrace{\lambda x_m}_{N_m} \underbrace{- Ax_n + \lambda x_n}_{-Sx_n \in N_{n-1}} \underbrace{- \lambda x_n}_{N_n}\| > \tfrac{1}{2} .$$

$$\|Ax_m - Ax_n\| = \|\underbrace{Ax_m - \lambda x_m}_{Sx_m \in R_{m+1}} + \underbrace{\lambda x_m}_{R_m} \underbrace{- Ax_n + \lambda x_n}_{-Sx_n \in R_{n+1}} \underbrace{- \lambda x_n}_{R_n}\| > \tfrac{1}{2} .$$

$n < m \Rightarrow R_m \subset R_n$

$\Rightarrow$ No convergent subseq. of $(Ax_n)_n$ ↯ ← to compactness

$S = \lambda I - A$

$N_0 = \{0\}$, $N_m = \ker S^m$

$R_0 = X$, $R_m = \operatorname{ran} S^m$

Hence

$\exists p : N_p = N_{p+1}$. For $x \in N_{p+2}$,

$\exists q : R_q = R_{q+1}$. For $x \in R_{q+2}$,

$Sx \in N_{p+1} = N_p \Rightarrow 0 = S^p(Sx) = S^{p+1}x$

$\Rightarrow x \in N_{p+1}$ ← $N_{p+1} = \ker S^{p+1}$

$\exists y \in X : S^{q+2}y = x \Rightarrow x = S^{q+1}Sy \in R_{q+1} = R_q$

so $N_{p+2} = N_{p+1}$. Analogously $N_{p+2} = N_{p+3} = \cdots$.

so $R_{q+2} = R_{q+1}$. Analogously $R_{q+2} = R_{q+3} = \cdots$.

We can now write $r = \max\{p, q\}$,

$$N = N_r, \quad R = R_r.$$

Then a) follows from Riesz-Schauder and b), c), d) are (essentially) finite-dimensional linear algebra.

For X as a Hilbert space we can define

$$P_\lambda : X \longrightarrow N$$

the orthogonal projection onto N.

$$S = \lambda I - A$$

$$N_0 = \{0\}, \quad N_m = \ker S^m$$

$$R_0 = X, \quad R_m = \operatorname{ran} S^m$$

Lemma

$N \cap R = \{0\}$

proof

Let $u \in N \cap R$.

$\Rightarrow u = S^r y$ for some $y \in X$ $\quad u \in R_r$

$\Rightarrow \underbrace{0 = S^r u}_{\in N_r} = S^{2r} y$

$\Rightarrow y \in N_{2r} = N_r$ $\quad N_r = N_{r+1} = N_{r+2} = \dots$

$\Rightarrow \underbrace{0 = S^r y}_{\in N_r} = u.$

Lemma

$X = N \oplus R$

proof

For $x \in X$,

$S^r x \in R_r = R_{2r}$ $\quad R_r = R_{r+1} = R_{r+2} = \dots$ $\quad R_r = \operatorname{ran} S^r$

$\Rightarrow \exists y \in X: S^r x = S^{2r} y.$

Then

$x = (x - S^r y) + S^r y \in N \oplus R.$ $\quad S^r(x - S^r y) = 0$

Lemma

$A(N) \subset N \quad A(R) \subset R$

proof

$S(N_r) = (\lambda I - A)(N_r) \subset N_{r-1} \subset N_r \iff A(N_r) \subset N_r$ $\quad N_0 \subset N_1 \subset \dots$

$S(R_r) = (\lambda I - A)(R_r) \subset R_{r+1} = R_r \iff A(R_r) \subset R_r$

λI keeps all subspaces invariant

Lemma

$\lambda \in \sigma(A|N), \quad \lambda \notin \sigma(A|R)$

proof

By construction $E_\lambda = \bigcup_{m \geq 0} \ker S^m \subset N$ and N, R are invariant.

A Singular Value Inequality

We will always assume X is a Hilbert space with inner product $(\cdot, \cdot)$, and $A \in L(X)$ is trace-class.

$\lambda_1, \lambda_2, \dots$ (resp. $s_1, s_2, \dots$) are the eigenvalues (resp. singular values) of A in descending order

$$A v_j = \lambda_j v_j, \qquad |\lambda_1| \geq |\lambda_2| \geq \dots, \qquad s_1 \geq s_2 \geq \dots .$$

A can be decomposed as $A = U|A|$ where U is unitary and $|A|$ is positive and self-adjoint.

Spectral theory $\Rightarrow$ ONB of $\overline{\operatorname{ran} A}$ $\{z_j\}_j$ with $|A| z_j = s_j z_j$.

If X is seperable, write $w_j = U z_j$

$$\Rightarrow \quad A = \sum_{j=1}^{\infty} s_j (\cdot, z_j) w_j .$$

"singular value decomposition"

Theorem (Lalesco-Schur-Weyl)

For any $N \in \mathbb{N}$,
$$\prod_{j=1}^{N} |\lambda_j| \le \prod_{j=1}^{N} s_j .$$

Corollary

$$\sum_{j=1}^{\infty} |\lambda_j| \le \sum_{j=1}^{\infty} s_j$$

proof

Consider
$$E_N = \operatorname{span}\{v_1, \dots, v_N\},$$
$$P_N : X \twoheadrightarrow E_N \quad \text{ON projection},$$
$$A_N := A|_{E_N} : E_N \to E_N .$$

finite-dimensional matrix!

Then $A_N = u_N |A_N|$.

u_N unitary

$$\Rightarrow \det A_N = \det |A_N| \overset{\text{finite dims}}{\Rightarrow} \prod_{j=1}^{N} \lambda_j(A_N) = \prod_{j=1}^{N} s_j(A_N).$$

$\lambda_j(|B|) = s_j(B)$

We have $AP_N|_{E_N} = A_N, \quad AP_N|_{E_N^\perp} = 0$.

$$\Rightarrow \lambda_j(A_N) = \lambda_j(AP_N), \quad s_j(A_N) = s_j(AP_N) \quad \forall j = 1, \dots, N.$$

But clearly $s_j(AP_N) \le s_j(A)$.

Operator Determinants (Preview)

Many definitions:

Fredholm

$$\det(I + zA) = \prod_{j=1}^{\infty} (1 + z\lambda_j)$$

Grothendieck - Krein

$$\det(I + zA) = \sum_{k=0}^{\infty} z^k \cdot \operatorname{Tr} \Lambda^k(A)$$

Dunford - Schwarz

$$\det(I + \mu A) = \exp \circ \operatorname{Tr} \circ \operatorname{Log}(I + \mu A)$$

for $|\mu| \ll 1$ + analytic continuation

Lidskii's theorem will show they are all equivalent!

Antisymmetric Tensor Products

Definition

For $x_1, x_2, \dots, x_n \in X$, let $(x_1 \otimes x_2 \otimes \dots \otimes x_n) : X \times X \times \dots \times X \longrightarrow \mathbb{C}$ be the multilinear map

$$x_1 \otimes x_2 \otimes \dots \otimes x_n : (y_1, \dots, y_n) \longmapsto \prod_{i=1}^{n} (x_i, y_i).$$

In general: If $U = \operatorname{span} B_U$, $W = \operatorname{span} B_W$, $U \otimes W$ is spanned by the maps $B_U \times B_W \to \mathbb{R}$ which have finitely many nonzero values.
- But that is precisely $\{ (x,y) \mapsto \langle x,u \rangle \cdot \langle y,w \rangle \mid u \in B_U, w \in B_W \}$

The span of such $(x_1 \otimes x_2 \otimes \dots \otimes x_n)$'s can be given an inner product

$$\langle x_1 \otimes \dots \otimes x_n, y_1 \otimes \dots \otimes y_n \rangle = \prod_{i=1}^{n} (x_i, y_i).$$

The (metric) completion of this span is a Hilbert space, denoted $X \otimes \dots \otimes X$.

Example

$X = L^2(I)$, $I = [-1, 1]$. What is $X \otimes X$?

$$f \otimes g : (\phi, \psi) \longmapsto \langle f, \phi \rangle_{L^2(I)} \cdot \langle g, \psi \rangle_{L^2(I)}$$

$$= \int_I \overline{f(x)}\, \phi(x)\, dx \int_I \overline{g(y)}\, \psi(y)\, dy$$

$$= \int_{I^2} \underbrace{\overline{f(x) g(y)}}_{=: \overline{u(x,y)}} \ \underbrace{\phi(x) \psi(y)}_{=: v(x,y)} \ d(x,y)$$

$$= \langle u, v \rangle_{L^2(I^2)}$$

$\Rightarrow$ $f \otimes g \longmapsto u$ induces an isometric isomorphism $X \otimes X \cong L^2(I^2)$.

only "induces" an iso since product-type functions do not form all of $L^2(I^2)$: only the closure!

Definition

Operators $A_1, A_2, \dots, A_n \in L(X)$ induce a map $X \otimes \dots \otimes X \circlearrowright$

$$(A_1 \otimes \dots \otimes A_n)(\underset{\in X \otimes \dots \otimes X}{\ell}) : (y_1, \dots, y_n) \longmapsto \ell(A_1^* y_1, \dots, A_n^* y_n)$$

which obeys

$$(A_1 \otimes \dots \otimes A_n)(x_1 \otimes \dots \otimes x_n) = (A_1 x_1) \otimes \dots \otimes (A_n x_n).$$

In general: $A_1 \otimes A_2$ is the unique linear map that satisfies $(A_1 \otimes A_2)(x \otimes y) = A_1 x \otimes A_2 y$. The universal property of the tensor product yields uniqueness.

Write $\Lambda^n(A) = A \otimes \dots \otimes A$. Directly from the definition we have $\Lambda^n(AB) = \Lambda^n(A)\Lambda^n(B)$.

If $\{e_i\}_i$ is an orthonormal basis (ONB) of X, then $\{e_{i_1} \otimes \dots \otimes e_{i_n}\}_{i_1 < \dots < i_n}$ is an ONB of $X \otimes \dots \otimes X$.

Definition

sign of the permutation

$$x \wedge \dots \wedge x_n = n^{-\frac{1}{2}} \sum_{\pi \in S_n} (-1)^{\pi} \, x_{\pi(1)} \otimes \dots \otimes x_{\pi(n)} .$$

"antisymmetrization"

symmetric group on $\{1, \dots, n\}$

$$\Lambda^n(X) := \overline{\operatorname{span}\{ x_1 \wedge \dots \wedge x_n : x_1, \dots, x_n \in X \}} .$$

"exterior algebra"

Again, an ONB $\{e_i\}_i$ induces an ONB $\{e_{i_1} \wedge \dots \wedge e_{i_n}\}_{i_1 < \dots < i_n}$ of $\Lambda^n(X)$. $\Lambda^0(X) := \mathbb{C}$.

We have

$$\langle x_1 \wedge \dots \wedge x_n , y_1 \wedge \dots \wedge y_n \rangle = \det\big((x_i, y_j)_{i,j=1,\dots,n} \big) .$$

Example

If $\dim X = d < \infty$, then $\Lambda^d(X) = \mathbb{C}$:

$\{e_1, \dots, e_d\}$ ONB of X

$\Rightarrow$ $e_1 \wedge \dots \wedge e_d$ is the only element of the basis of $\Lambda^d(X)$.

Further,

$$\mathrm{Tr}(\Lambda^d(A))$$

$$= \langle \Lambda^d(A)(e_1 \wedge \dots \wedge e_d), (e_1 \wedge \dots \wedge e_d) \rangle$$

$$= \det\left((Ae_i, e_j)_{i,j=1,\dots,d} \right)$$

$$= \det A.$$

Combined with $\Lambda^d(AB) = \Lambda^d(A)\Lambda^d(B)$, this creates a simple proof that $\det AB = \det A \cdot \det B$.

Operator Determinants

Motivation

For $\dim X = d < \infty$,

$$\sigma(A) = \{\lambda_1, \dots, \lambda_N\} \Rightarrow \sigma(\Lambda^k(A)) = \{\lambda_{i_1} \cdot \ldots \cdot \lambda_{i_k}\}_{i_1 < \ldots < i_k}.$$

$$\Rightarrow \sum_{k=0}^{N} \operatorname{Tr} \Lambda^k(A) = \sum_{k=0}^{N} \sum_{i_1 < \ldots < i_k} \lambda_{i_1} \cdot \ldots \cdot \lambda_{i_k} = \prod_{n=1}^{N} (1 + \lambda_n)$$

Definition

For a trace-class $A \in K(X)$ in a Hilbert space X,

$$\boxed{\det(I + zA) = \sum_{k=0}^{\infty} z^k \cdot \operatorname{Tr} \Lambda^k(A)}$$

Theorem

$\Lambda^k(A)$ is trace-class in $\Lambda^k(X)$ with

$$\| \Lambda^k(A) \|_{Tr} \leq \frac{\|A\|_{Tr}^k}{k!}.$$

In particular, $\det(I + zA)$ is well-defined.

RTP: $|\Lambda^n(A)| = \Lambda^n(|A|)$

$|\Lambda^n(A)| \Rightarrow |\Lambda^n(A)|\,|\Lambda^n(A)| = \Lambda^n(A)^* \Lambda^n(A)$

$|\Lambda^n(A)| = |\Lambda^n(A)|^*$

$$\left(\Lambda^n(|A|)^* \Lambda^n(|A|)\, \phi_1 \otimes \dots \otimes \phi_n,\ \psi_1 \otimes \dots \otimes \psi_n\right)$$

$$= \left(\Lambda^n(|A|)\, \phi_1 \otimes \dots \otimes \phi_n,\ \Lambda^n(|A|)\, \psi_1 \otimes \dots \otimes \psi_n\right)$$

$$= \left((|A|\phi_1) \otimes \dots \otimes (|A|\phi_n),\ (|A|\psi_1) \otimes \dots \otimes (|A|\psi_n)\right)$$

$$= \prod_{i=1}^{n} (\phi_i, |A|^*|A|\psi_i)$$

$$= \prod_{i=1}^{n} (\phi_i, A^*A\psi_i)$$

$$= \left(\Lambda^n(A)^* \Lambda^n(A)\, \phi_1 \otimes \dots \otimes \phi_n,\ \psi_1 \otimes \dots \otimes \psi_n\right)$$

proof

A calculation shows $|\Lambda^k(A)| = \Lambda^k(|A|)$ so that $\Lambda^k(A)$ has singular values $s_{l_1} \cdot \dots \cdot s_{l_k}$ for $l_1 < \dots < l_k$.

$$\Rightarrow \quad \|\Lambda^k(A)\|_{Tr} = \sum_{l_1 < \dots < l_k} s_{l_1} \cdot \dots \cdot s_{l_k}.$$

But

rest terms

$$\|A\|_{Tr}^k = \left(\sum_{j=1}^{\infty} s_j\right)^k = k! \sum_{l_1 < \dots < l_k} s_{l_1} \cdot \dots \cdot s_{l_k} + [\dots].$$

Cauchy product

Properties of the Determinant

Ideally we would want

$$\boxed{\det(I + zA) = \prod_{j=1}^{\infty} (1 + \lambda_j z).}$$

How best to show this?

Lemma

Let f be an entire function with $f(0) = 1$.

Write $f^{-1}(0) = \{z_1, z_2, \dots\}$.

zeros "run away"

Suppose $\sum_{n=1}^{\infty} |z_n|^{-1} < \infty$ and

bounded by exponentials

$\forall \varepsilon > 0 \ \exists C > 0 : \ |f(z)| \leq C \exp(\varepsilon |z|) \ \forall z.$

Then

$$f(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right).$$

We need to show:

a) $\forall \varepsilon > 0 \; \exists C > 0 : |\det(I + zA)| \leq C \exp(\varepsilon |z|) \quad \forall z,$

b) Eigenvalues λ_j are precisely zeros $z_j = -\frac{1}{\lambda_j}$ of $\det(I + zA)$.

proof of a)

Let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ with $\sum_{j=N+1}^{\infty} s_j \leq \frac{\varepsilon}{2}$. Then

$$
\begin{aligned}
|\det(I+zA)| &\leq \sum_{k=0}^{\infty} |z|^k \, \|\Lambda^k(A)\|_{Tr} \\
&= \sum_{k=0}^{\infty} \sum_{l_1 < \dots < l_k} |z|^k s_{l_1} \cdot \ldots \cdot s_{l_k} \\
&\leq \prod_{j=1}^{\infty} (1 + |z| s_j) \\
&= \underbrace{\prod_{j=1}^{N} (1 + |z| s_j)}_{\leq \text{const} \cdot \exp(|z| \frac{\varepsilon}{2})} \underbrace{\prod_{j=N+1}^{\infty} (1 + |z| s_j)}_{(*)}
\end{aligned}
$$

$$
\begin{aligned}
(*) &\leq \prod_{j=N+1}^{\infty} \exp(|z| s_j) \\
&= \exp\Big(|z| \sum_{j=N+1}^{\infty} s_j\Big) \\
&< \exp\Big(|z| \frac{\varepsilon}{2}\Big)
\end{aligned}
$$

Lemma

$A \mapsto \det(I+A)$ is continuous in the trace-class.

proof

Let A, B be trace-class.

given by a power series

Consider $g(z) = \det\left(I + \frac{1}{2}(A+B) + z(A-B)\right)$. g is analytic.

$$
\begin{aligned}
|\det(I+A) - \det(I+B)| &= \left| g\left(\tfrac{1}{2}\right) - g\left(-\tfrac{1}{2}\right) \right| \\
&\leq \sup_{-\frac{1}{2} \leq t \leq \frac{1}{2}} |g'(t)| \\
&\leq \frac{1}{R} \cdot \sup_{|z| \leq R + \frac{1}{2}} |g(z)| \\
&\leq \|A - B\|_{Tr} \cdot \text{const}
\end{aligned}
$$

mean value theorem

Cauchy integral theorem

choose $R = \frac{1}{\|A-B\|_{Tr}}$

Corollary

$$\det(I + A + B + AB) = \det(I+A)\det(I+B) \qquad (*)$$

$(I+A)(I+B)$

proof

given by truncated SVD

Let $(A_n)_n \to A$, $(B_n)_n \to B$ in $\|\cdot\|_{Tr}$, $\dim \operatorname{ran} A_n = \dim \operatorname{ran} B_n < \infty$.
Then $(*)$ for A_n, B_n is just finite-dimensional determinants.
$\det(I + \cdot)$ is continuous.

Corollary

If $I+A$ is invertible, $\det(I+A) \neq 0$.

proof

Let $B = -A(I+A)^{-1}$. Then

$$I+A - A(I+A)^{-1} - AA(I+A)^{-1}$$
$$= I+A - A(I+A)(I+A)^{-1}$$
$$= I$$

$$\det(I+A)\det(I+B) = \det(I+A+B+AB) = \det I = 1.$$

Theorem

Let λ be an eigenvalue of A with (algebraic) multiplicity n. Then $\det(I+zA)$ has a root of order n at $z_0 = -\frac{1}{\lambda}$.

proof

Recall $P_\lambda : X \twoheadrightarrow N$ from theorem 1. We have

$$\begin{tikzcd} X \arrow[r, "P_\lambda"] \arrow[d, dashed, "(I-P_\lambda)AP_\lambda"'] & N \arrow[d, "A"] \\ 0 & N \arrow[l, "I-P_\lambda"'] \end{tikzcd}$$

$$\begin{aligned} & I + zA \\ &= I + zA + zAP - zAP \\ &= I + zA(I-P) + zAP \\ &= I + zA(I-P) + zAP + \overbrace{zA(I-P)AP}^{=0} \end{aligned}$$

$$\Rightarrow \det(I+zA) = \underbrace{\det(I+zAP_\lambda)}_{\substack{=(1-z_0^{-1}z)^n \\ \text{finite-dimensional} \\ \text{determinant}}} \overbrace{\det(I+zA(I-P_\lambda))}^{\neq 0 \text{ since } \lambda \notin \sigma(A|R)}.$$

Now we can put it all together...

Theorem

$$\det(I+zA) = \prod_{j=1}^{\infty} (1+z\lambda_j)$$

proof

Let $f(z) = \det(I+zA)$. f has zeros $z_j = -\frac{1}{\lambda_j}$, $j=1,2,\ldots$.
Clearly, $f(0)=1$.

Lalesco-Schur-Weyl $\Rightarrow$ $\sum_{j=1}^{\infty} |z_j|^{-1} < \infty$.

checked earlier

$\forall \varepsilon > 0 \ \exists C > 0 : |\det(I+zA)| \leq C \exp(\varepsilon |z|) \quad \forall z$.

$\Rightarrow$ The expansion $f(z) = \prod_{j=1}^{\infty} \left(1 - \frac{z}{z_j}\right) = \prod_{j=1}^{\infty} (1+\lambda_j z)$ is valid.

A "One-line" Proof of Lidskii's Theorem

Corollary

$$\operatorname{Tr} A = \sum_{j=1}^{\infty} \lambda_j .$$

proof

$\Lambda^1(A) = A$

$$\det(I + zA) = \sum_{k=0}^{\infty} z^k \operatorname{Tr} \Lambda^k(A) = 1 + z \operatorname{Tr} A + \mathcal{O}(z^2)$$

$$\det(I + zA) = \prod_{j=1}^{\infty} (1 + \lambda_j z) = 1 + z\left(\sum_{j=1}^{\infty} \lambda_j\right) + \mathcal{O}(z^2)$$

Proof by comparing coefficients.