Technische Universität München

Department of Mathematics

Bachelor's Thesis

# GAIO.jl: Set-oriented Methods for Approximating Invariant Objects, and their Implementations in julia

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I assure the single handed composition of this bachelor's thesis only supported by declared resources.

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## Zusammenfassung

Bei einer in englischer Sprache verfassten Arbeit muss eine Zusammenfassung in deutscher Sprache vorangestellt werden. Dafür ist hier Platz.

# Contents

# 1 Dynamical Systems

## 1.1 Motivation

Our goal is to investigate the qualitative, long-term behavior of systems in which a given function describes the trajectory of a point in an ambient space. Such dynamical systems are used in modelling physcial phenomena, economic forecasting, differential equations, etc. We wish to construct topological *closed covers* of sets which describe the infinite dynamics of some portions of the system, as well as statistical *invariant measures* which describe much larger sets in the space, but with less information.

The basic technique of all the topological algorithms is to split a compact set $Q$ into a partition $\mathcal{P}$ of *boxes* - that is, generalized rectangles, each with center vector $c$ and componentwise radii $r$. The algorithms will begin with a set of boxes $\mathcal{B}$, and then repeatedly subdivide each box in $\mathcal{B}$ into two (or more) smaller boxes, examine the dynamics of the subdivided boxes, and refine the box set to include only the boxes we are interested in.

The algorithms described in the present paper have been previously implemented in the statistical programming language `matlab` [8], but is now being fully refactored and reimplemented in the open-source, composable language `julia` [1]. The reason for this change is julia's high-level abstraction capabilities, just-in-time compilation, and in-built set-theoretical functions, which create short, elegant code which is nonetheless more performant. Source code for `GAIO` in `matlab` and `julia` and be found in [7] and [11], respectively.

## 1.2 Definitions

This section should be treated as an index of definitions, to be referred back to as necessary during reading. In the following, we assume $M$ is a compact or a smooth manifold in $\mathbb{R}^d$, endowed with a metric $d$, and the map $f : M \to M$ is at least $\mathcal{C}^0$. Our setting is a *discrete, autonomous dynamical system*, that is, a system of the form:

$$x_{k+1} = f(x_k), \quad k = 0,\ 1,\ 2,\ \ldots \tag{1.1}$$

A continuous dynamical system $\dot{x} = F(x)$ can be *discretized* by, for example, considering the *Poincaré time-t map* over some $d-1$ dimensional hyperplane, or by setting one "step" of the system as integrating $F$ for a set time $t$.

We begin by giving a set of topological definitions of sets we wish to approximate.

**Definition 1.1** ((Forward-, Backward-) Invariant)**.** [5] A set $A$ is called *forward-invariant* if $f(A) \subset A$, *backward-invariant* if $f^{-1}(A) \subset A$, and *invariant* if it is both forward- and backward-invariant.

**Definition 1.2** (Attracting Set)**.** [3] An invariant set $A$ is called *attracting* with *fundamental neighborhood* $U$ if for every open set $V \supset A$ there is an $N \in \mathbb{N}$ such that the tail $\bigcup_{k \geq N} f^k(U)$ lies entirely within $A$. The attracting set is also called *global* if the *basin of attraction*

$$B(A) = \bigcap_{k \geq 0} f^{-k}(U) \tag{1.2}$$

is the whole of $\mathbb{R}^n$.

The basin of attraction acts in some sense as the set for which all points eventually arrive in $A$. Since the map $f$ is smooth, then the closure $\bar{A}$ is invariant too. With continuity it becomes clear that

$$A = \bigcap_{k \geq 0} f^k(U). \tag{1.3}$$

The global attractor is maximal in the sense that it contains all backward-invariant sets within the system. In particular, it contains local unstable manifolds.

**Definition 1.3** (Stable and Unstable Manifolds)**.** [9] Let $\bar{x}$ be a fixed point of the diffeomorphism $f$, and $U$ a neighborhood of $x$. Then the *local unstable manifold* is given by

$$W^u(\bar{x}, U) = \left\{ x \in U \mid \lim_{k \to \infty} d(f^{-k}(x),\ \bar{x}) = 0 \text{ and } f^{-k}(x) \in U \ \forall\, k \geq 0 \right\}. \tag{1.4}$$

The *global unstable manifold* is given by

$$W^u(\bar{x}) = \bigcup_{k \geq 0} f^k(\, W^u(\bar{x}, U)\,). \tag{1.5}$$

The dual definition of the *(local) stable manifold* is obtained by reversing the sign of $k$ in the above equations.

**Definition 1.4** (Pseudoperiodic)**.** [9] Let $n \in \mathbb{N}$. A set $\{x_k \mid k \in \{0, \ \ldots, \ n\}\}$ is called *$\epsilon$-pseudoperiodic* if for any $k$, $\quad d(x_{k \bmod n},\ x_{k+1 \bmod n}) < \epsilon$.

As the name suggests, an $\epsilon$-pseudoperiodic orbit is "almost" periodic inthe sense that it represents a "small" perturbation of a theoretically periodic orbit. In practice, such direct orbits may not be known, but it will preresent a naturally useful definition in our approximations.

**Definition 1.5** (Chain Recurrent)**.** [9] The point $\bar{x} \in M$ is called *chain recurrent* if for any $\epsilon > 0$ there exists an $\epsilon$-pseudoperiodic orbit. The *chain recurrent set* $R_M(f)$ is the set of all chain recurrent points in $M$.

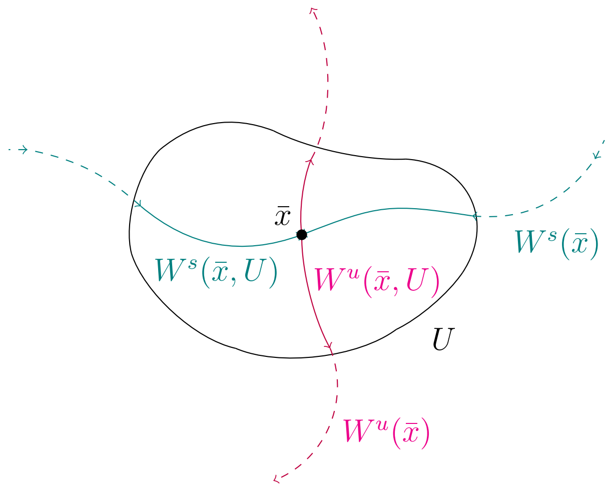


Figure 1.1: [6] Stable and unstable manifolds, local and global

As shown in [5] we have the inclusion $R_M(f) \subset \bigcap_{k\geq 0} f^k(M)$, which also shows that $R_M(f)$ is an invariant set.

We continue with a set of measure-theoretical definitions for types of measures we wish to approximate.

Since our goal is to partition the manifold into a finite set of boxes, we must accept some amount of "uncertainty" in how our sets look, and how *exactly* $f$ maps such a set. We describe this noise using a stochastic transition function.

**Definition 1.6** (Transition Function)**.** [4] Let $\mathfrak{B}$ be the Borel $\sigma$-agebra on $M$. A function $p : M \times \mathfrak{B} \to [0, 1]$ is called *transition function* if

1. $q(\cdot, A) : M \to [0, 1]$ is measurable for all $A \in \mathfrak{B}$,
2. $q(x, \cdot) : \mathfrak{B} \to [0, 1]$ is a probability measure for all $x \in M$.

*Example.*
- [4] We can model the deterministic system using the dirac measure $p(x,\, A) = \delta_{f(x)}(A)$.
- The approximate box version of the system can be modelled as using a uniform probability density: Let $\mathcal{P}$ be a partition of $Q$ into equally sized, disjoint boxes (think of a checkerboard). Then for a point $x$, find the box $B \in \mathcal{P}$ with which contains $x$ and map it forward to $f(B)$. Finally, let $\mathcal{B} \subset \mathcal{P}$ be a cover of $f(B)$.

$$p(x,\, A) = \frac{\mathcal{L}\left(A \,\cap\, \bigcup_{B\in\mathcal{B}} B\right)}{\mathcal{L}\left(\bigcup_{B\in\mathcal{B}} B\right)}, \tag{1.6}$$

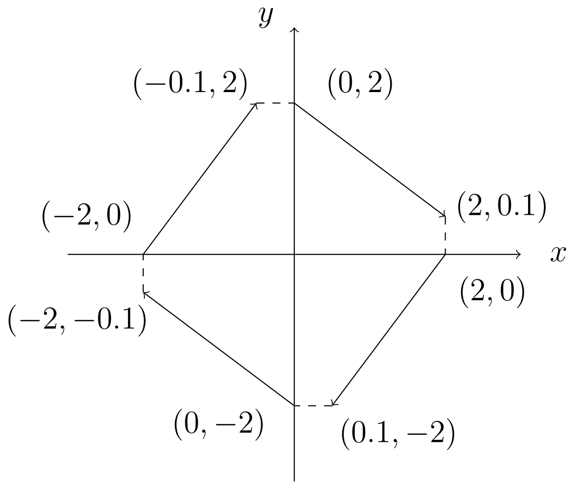


Figure 1.2: [9] A 0.1-pseudoperiodic orbit of the map $f(x, y) = (y,\ 0.05\,(1 - x^2)\,y - x)$

where $\mathcal{L}$ represents the $d$-dimensional Lebesque measure.

**Definition 1.7** (Perron-Frobenius Operator, Invariant Measure)**.** [4] Let $p$ be a stochastic transition function, and $\mu$ a measure on $M$. We define the *Perron-Frobenius operator* as

$$(P\mu)(A) = \int p(x,\ A)\ d\mu(x) \tag{1.7}$$

A measure $\mu$ is called *invariant* if it is a fixed point of $P$.

*Remark.* The Perron-Frobenius operator is often also called *transfer operator*.

*Example.* [4] We calculate

$$(P\mu)(A) = \int \delta_{f(x)}(A)\ d\mu(x) = \int \chi_A(f(x))\ d\mu(x) = \mu \circ f^{-1}(A). \tag{1.8}$$

In this case $P$ simply becomes the *pushforward operator*.

An invariant measure can be used to understand the global behavior of a dynamical system, with more $\mu$-mass assigned to regions which are visited frequently over long trajectories, and less $\mu$-mass to regions visited less frequently.

Our "noisy" approximated system poses the benefit that while deterministic dynamical systems generally support the existence of multiple invariant measures, the stochastic system will (under mild assumptions) have a unique invariant measure, as shown in *WHERE IS THIS SHOWN?*.

A fixed point - or eigenmeasure with eigenvalue 1 - is not the only object of interest when considering the operator $P$. Suppose instead we have a deterministic dynamical system and a finite (complex valued) measure with $P\nu = \lambda\nu$ for a $\lambda = -1$. Then, using finiteness and borel measurability, we can find a partition of M in two disjoint subsets $A_1, A_2$ such that $\nu(A_1) = -\nu(A_2)$. In particular, this implies that $f$ maps $A_1$ to $A_2$, and $A_2$ to $A_1$ (since $P^2\nu = \nu$). This paritition forms a *two-cycle.*

Finally, we set some notation for convenience.

**Definition 1.8** (Image of a Box Set)**.** For a partition $\mathcal{P}$ of $Q$ into boxes, and a subset $\mathcal{B} \subset \mathcal{P}$, we will call the *image of* $\mathcal{B}$ *under* $f$ the set of boxes which intersect with the image $f(B)$, for at least one $B \in \mathcal{B}$. More precisely, it is

$$f(\mathcal{B}) = \left\{ R \in \mathcal{P} \quad | \quad f^{-1}(R) \cap \bigcup_{B \in \mathcal{B}} B \neq \emptyset \right\}. \tag{1.9}$$

**Theorem 1.1** (Image of a Box Set)**.** $f(\mathcal{B})$ is the inclusion-minimal cover of $f(\bigcup_{B\in\mathcal{B}} B)$ with boxes from $\mathcal{P}$.

*Proof.* We have the equivalent characterisation

$$f(\mathcal{B}) = \{R \in \mathcal{P} \quad | \quad \exists\ B \in \mathcal{B} \text{ and } x \in B \ :\ f(x) \in R\}. \tag{1.10}$$

Hence if we remove one box $R$ from $f(\mathcal{B})$, then there exists an $x \in \bigcup_{B\in\mathcal{B}} B$ which maps outside of the created box set.

□

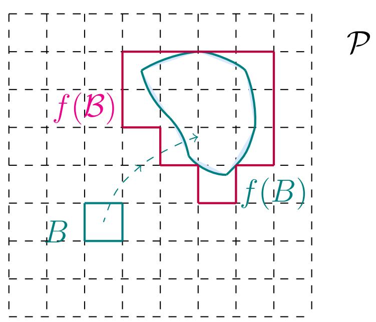


Figure 1.3: Image of the simple box set $\mathcal{B} = \{B\}$

# 2 Algorithms

## 2.1 Relative Attractor

The construction of a fundamental neighborhood $U$ for a global attractor $A$ is relatively difficult, but the description $A = \bigcap_{k\geq 0} f^k(U)$ lends to a natural *ansatz* for its approximation using a compact subdomain $Q \subset M$.

**Definition 2.1** (Realtive Global Attractor)**.** Let $Q$ be compact. Then we define the *attractor relative to* $A$ as

$$A_Q = \bigcap_{k\geq 0} f^k(Q) \tag{2.1}$$

*Remark.* It follows from the definition that the relative global attractor is a subset of the global attractor.

The idea to approximate the relative gloabl attractor is in two steps: first, we subdivide each of the boxes and second, discard all those boxes which do not intersect with the previous box set. The algorithm requires a map $f$, a box set $\mathcal{B}$, and a predefined number of steps $n$.

**Algorithm 1** Relative Attractor

```
B_0 ← B
for i = {1, ..., n} do
    B_i ← SUBDIVIDE( B_{i-1} )
    B_i ← B_i ∩ f( B_i )
return B_n
```

*Remark.*

- Optionally, the set $\{\mathcal{B}_0,\ \mathcal{B}_1,\ \ldots,\ \mathcal{B}_n\}$ can be returned instead. This will be true for all algorithms of this type.
- The precise technique for subdivision can be tuned depending on the situation. In `GAIO.jl`, boxes are bisected evenly along one dimension $k \in \{1,\ \ldots,\ d\}$. The dimension $k$ along which to bisect is cycled through during the steps.

**Proposition 2.1.** [5, 3] Set $Q_i = \bigcup_{B\in\mathcal{B}_i} B$. For all $i$ we have

1. $Q_{i+1} \subset Q_i$
2. $A_Q \subset Q_i$

Further, if we set $Q_\infty = \bigcap_{n \geq 1} Q_n$, then $A_Q = Q_\infty$. In particular, this shows that $\mathcal{B}_\infty$ is backward-invariant.

## 2.2 Unstable Manifold

From the definition of the local unstable manifold $W^u(\bar{x}, U)$ we see that the relative gloabl attractor $R_Q(f)$ contains the local unstable manifold, and, provided the set $Q$ is sufficiently small, $W^u(\bar{x}, U)$ coincides with $R_Q(f)$. For further details, see [2, 10].

Using this knowledge, we can approximate the global unstable manifold $W^u(\bar{x})$: first, we replace the calculation of the local unstable manifold with the calculation of the relative attractor for a small set $Q'$ surrounding the fixed point $\bar{x}$, using algorithm 1. We then repeat the following *continuation step*: we map the current box set forward one iteration, and note any new boxes which are hit. These new boxes get added to the box set. The algorithm requires a map $f$, a box set $\mathcal{B}$, and a number of steps $n$.

**Algorithm 2** Continuation Step

1: $\mathcal{B}_0 \leftarrow \mathcal{B}$
2: **for** $i = \{1, \ldots, n\}$ **do**
3: $\quad \mathcal{B}' \leftarrow f(\mathcal{B}')$
4: $\quad \mathcal{B}' \leftarrow \mathcal{B}' \setminus \mathcal{B}_{i-1}$
5: $\quad \mathcal{B}_i \leftarrow \mathcal{B}' \cup \mathcal{B}_{i-1}$
6: **return** $\mathcal{B}_n$

*Remark.*

- To avoid having to repeat mapping the same boxes over and over again, the algorithm maps only the "new" boxes found in the last iteration. This is what line 4 achieves. A simpler version of the algorithm might simply run $\mathcal{B}_i \leftarrow \mathcal{B}_{i-1} \cup f(\mathcal{B}_{i-1})$ in each iteration.
- [2] The algorithm in general cannot guarantee covering of the entire unstable manifold, nor can it guarantee coverng of the entirety of $W^u(\bar{x}) \cap Q$. This is because $W^u(\bar{x})$ could in theory exit $Q$, but return at another point. The algorithm can however guarantee convergence to the connected component of $W^u(\bar{x}) \cap Q$ which contains $\bar{x}$.

### 2.3 Chain Recurrent Set

## 3 Julia

### 3.1 Introduction to the Julia Language

```
print("Here's some placeholder code")
```

## 4 Parallelization using the CPU

### 4.1 CPU Architechture

## 5 Parallelization using the GPU

### 5.1 GPU Architechture

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