DMD Error Representation

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Given

- system $S : \Omega \to \Omega$
- quadrature scheme $\{(w_i, x_i)\}_i$
- samples $\{y_i\}_i$, $y_i = S(x_i)$
- family $\{\psi_j\}_j$ spanning a Hilbert space $\mathscr{H}$

Construct the matrices

$$\Psi_X = \begin{matrix} \psi_1(x_1) & \dots & \psi_N(x_1) \\ \vdots & & \vdots \\ \psi_1(x_M) & \dots & \psi_N(x_M) \end{matrix}, \quad \Psi_Y = \begin{matrix} \psi_1(y_1) & \dots & \psi_N(y_1) \\ \vdots & & \vdots \\ \psi_1(y_M) & \dots & \psi_N(y_M) \end{matrix}.$$

EDMD constructs the Galerkin matrices

$$G = \Psi_X^* W \Psi_X, \quad A = \Psi_X^* W \Psi_Y, \quad L = \Psi_Y^* W \Psi_Y.$$

We have

- $\lim_{M\to\infty} G_{jk} = \langle \psi_j, \psi_k \rangle =: \mathbb{G}_{jk}$,
- $\lim_{M\to\infty} A_{jk} = \langle \psi_j, \mathscr{K}\psi_k \rangle =: \mathbb{A}_{jk}$,
- $\lim_{M\to\infty} L_{jk} = \langle \mathscr{K}\psi_j, \mathscr{K}\psi_k \rangle =: \mathbb{L}_{jk}$.

Consider a candidate eigenpair (λ, g), $g = \sum_j^\infty a_j \psi_j$ for $\mathscr{K}$.

$$\begin{aligned} \|(\mathscr{K} - \lambda I)g\|_{\mathscr{H}}^2 &= \langle (\boldsymbol{\mathscr{K}} - \lambda I)g, (\boldsymbol{\mathscr{K}} - \lambda I)g \rangle \\ &= \langle \boldsymbol{\mathscr{K}} g, \boldsymbol{\mathscr{K}} g \rangle - \lambda \langle \boldsymbol{\mathscr{K}} g, g \rangle - \bar{\lambda} \langle g, \boldsymbol{\mathscr{K}} g \rangle + |\lambda|^2 \langle g, g \rangle \\ &= a^* \left(\mathbb{L} - \lambda \mathbb{A}^* - \bar{\lambda}\mathbb{A} + |\lambda|^2 \mathbb{G} \right) a. \end{aligned}$$

Now if we consider the "standard" EDMD residual

$$\begin{aligned} \left\|(\mathbb{G}^{-1}\mathbb{A} - \lambda)g\right\|_{\mathscr{H}}^2 &= \left\langle (\mathbb{G}^{-1}\mathbb{A} - \lambda)g, (\mathbb{G}^{-1}\mathbb{A} - \lambda)g \right\rangle \\ &= \left\langle \mathbb{G}^{-1}\mathbb{A}g, \mathbb{G}^{-1}\mathbb{A}g \right\rangle - \lambda \left\langle \mathbb{G}^{-1}\mathbb{A}g, g \right\rangle - \bar{\lambda} \left\langle g, \mathbb{G}^{-1}\mathbb{A}g \right\rangle + |\lambda|^2 \langle g, g \rangle \\ &= a^* (\mathbb{A}^* (\mathbb{G}\mathbb{G}^*)^{-1} \mathbb{A} - \lambda (\mathbb{G}^{-1}\mathbb{A})^* - \bar{\lambda}\mathbb{G}^{-1}\mathbb{A} + |\lambda|^2 \mathbb{G}) a. \end{aligned}$$

Let us assume that $\{\psi_i\}_i$ is an orthonormal basis, so that $\mathbb{G} = I$.

$$\left\|(\mathbb{G}^{-1}\mathbb{A} - \lambda)g\right\|_{\mathscr{H}}^2 - \|(\mathscr{K} - \lambda I)g\|_{\mathscr{H}}^2 = a^* (\mathbb{A}^*\mathbb{A} - \mathbb{L}) a.$$

Now, viewing $\mathbb{A}$ as an "infinite matrix" on $l^2(\mathbb{N}_0)$

$$
\begin{aligned}
(\mathbb{A}^*\mathbb{A})_{ij} &= \sum_l \mathbb{A}^*_{il}\mathbb{A}_{lj} \\
&= \sum_l \langle \boldsymbol{\mathscr{K}}\psi_i, \psi_l\rangle \ \langle \psi_l, \boldsymbol{\mathscr{K}}\psi_j\rangle \\
&= \left\langle \sum_l \langle \boldsymbol{\mathscr{K}}\psi_i, \psi_l\rangle \, \psi_l, \boldsymbol{\mathscr{K}}\psi_j \right\rangle \\
&= \langle \boldsymbol{\mathscr{K}}\psi_i, \boldsymbol{\mathscr{K}}\psi_j\rangle \\
&= \mathbb{L}_{ij}
\end{aligned}
$$

This is good. If we however only have access to the first N basis elements, that is, we have a finite matrix $\tilde{\mathbb{A}} = (\mathbb{A}_{ij})_{i,j=1}^N$, then

$$
\begin{aligned}
\left(\tilde{\mathbb{A}}^*\tilde{\mathbb{A}}\right)_{ij} &= \sum_{l=1}^N \tilde{\mathbb{A}}^*_{il}\tilde{\mathbb{A}}_{lj} \\
&= \sum_{l=1}^N \langle \boldsymbol{\mathscr{K}}\psi_i, \psi_l\rangle \ \langle \psi_l, \boldsymbol{\mathscr{K}}\psi_j\rangle \\
&= \langle \boldsymbol{\mathscr{K}}\psi_i, \boldsymbol{\mathscr{K}}\psi_j\rangle - \left\langle \sum_{l=N+1}^{\infty} \langle \boldsymbol{\mathscr{K}}\psi_i, \psi_l\rangle \, \psi_l, \boldsymbol{\mathscr{K}}\psi_j \right\rangle \\
&= \mathbb{L}_{ij} - \left\langle P^\perp \boldsymbol{\mathscr{K}}\psi_i, \boldsymbol{\mathscr{K}}, \psi_j \right\rangle
\end{aligned}
$$

where P is the orthogonal projection onto $\operatorname{span}\{\psi_1, \dots, \psi_N\}$, $P^\perp = I - P$.