ODEs - General

Picard Lindelöf

f locally L-continuous wrt x $\exists$ unique soln that extends to boundary of Ω

Gronwall

$\phi, \psi : [a,b] \to [0,\infty)$ continuous, $\rho \geq 0$, $\psi(t) \leq \rho + \int_a^b \phi(s)\psi(s)ds$ $\psi(s) \leq \rho \exp\left(\int_a^b \phi(s)ds\right)$

consistency error

$\epsilon(t,\tau,x) = \Phi^{t+\tau,t}x - \Psi^{t+\tau,t}x$

consisency order

$\epsilon(t,\tau,x) = O(\tau^{p+1})$

convergence order

$x_\Delta : \Delta \to \mathbb{R},\ x : I \to \mathbb{R},\ \Delta_n with \tau_n \to 0,\ \|\epsilon_{\Delta_n}\| = \max_{t\in\Delta_n} \|x(t) - x_{\Delta_n}(t)\| = O(\tau^p)$

convergence theorem

ψ scheme with consisency order p, ψ locally L-continuous wrt x ψ has convergence order p

$\dot{x} = Ax$ stability

$Re\lambda \leq 0,\ Re\lambda = 0 \Rightarrow i(\lambda) = 1\ \forall\lambda \in \sigma(A)$

$x_{k+1} = Mx_k$ stability

$\rho(M) \leq 1,\ |\lambda| = 1 \Rightarrow i(\lambda) = 1\ \forall\lambda \in \sigma(M)$

Leibnitz rule

$\frac{d}{dx}\int_{a(x)}^{b(x)} f(t,x)dt =$

$b'(x)f(b(x),x) - a'(x)f(a(x),x) + \int_{a(x)}^{b(x)} \frac{d}{dx}f(t,x)dt$

Runge Kutta Schemes

Form of RK schemes

$k_i = f\left(t + c_i\tau,\ x + \sum_{j=1}^s a_{ij}k_j\right),$

$\Psi^{t+\tau,t}x = x + \tau\sum_{i=1}^s b_ik_i$

consistent

$\sum_{i=1}^s b_i = 1$

invariant under autonomization

$\sum_{j=1}^s a_{ij} = c_i$

max order

explicit $\Rightarrow \leq s$, implicit $\Rightarrow \leq 2s$

Rooted trees

$f : \mathbb{R}^d \to \mathbb{R}^d$ autonomous

$f^\beta = f^n \cdot (f^{\beta_1}, \ldots, f^{\beta_n}),$

$f^{[]} = f(x),$

$\#\beta = \#\beta_1 + \cdots + \#\beta_n + 1,$

$\beta! = \#\beta \cdot \beta_1! \ldots \beta_n!,$

$[]! = 1$

Taylor expansions (!!autonomous)

$\Phi^\tau x = x + \sum_{\#\beta\leq p} \frac{\tau^{\#\beta}}{\beta!}\alpha f^\beta(x) + O(\tau^{p+1})$

$\alpha_\beta = \frac{\delta_\beta}{n!}\alpha_{\beta_1}\ldots\alpha_{\beta_n},\ \delta_\beta =$ nr of orderings of β

$\Psi^\tau x = x + \sum_{\#\beta\leq p} \tau^{\#\beta}\alpha_\beta b^T A^\beta f^\beta(x) + O(\tau^{p+1})$

$\left(A^\beta\right)_i = \left(A \cdot A^{\beta_1}\right)_i \ldots \left(A \cdot A^{\beta_n}\right)_i, \quad A^{[]} = \mathbf{1} \in \mathbb{R}^s$

Butcher

RK has order p

$\forall f \in \mathcal{C}^p \quad \Leftrightarrow \quad b^T A^\beta = 1/\beta!\ \forall\#\beta \leq p$

order 1 $\Leftrightarrow$ consistent

order 2 $\Leftrightarrow \sum_{i=1}^s c_ib_i = 1/2$

order 3 $\Leftrightarrow \sum_{i=1}^s b_ic_i^2 = 1/3\ \wedge\ Ac = 1/6$

Adaptive Schemes

Use higher order scheme $\hat{\Psi}$, use max τ with $[\epsilon] = \Psi^\tau x - \hat{\Psi}^\tau x$ has norm $\leq$ TOL

Embedded Schemes

Given $\hat{\Psi} = (\hat{b}, \hat{A})$, construct $\Psi = (b, A)$ with order $p < \hat{p}$ to reduce number of f evals

Stability function

$\dot{x} = Ax \ \Rightarrow\ \Psi^\tau x = R(\tau A)x$

$R(z) = 1 + zb^T(I - zA)^{-1}\mathbf{1}$

Stability domain

$S_R = \{z \in \mathbb{C}\ :\ |R(z)| \leq 1\}$

$A(\alpha)$ stable

$\{z \in \mathbb{C}\ :\ |arg(z)| \leq \alpha\} \subset S_R$

L stable

$\lim_{|z|\to\infty} |R(z)| = 0$

RK method stability

$\tau\lambda \in S_R\ \forall\lambda \in \sigma(A),\ |\tau\lambda| = 1 \ \Rightarrow\ i(\tau\lambda) = 1\ \forall\lambda \in \sigma(A)$

Collocation

Given $0 \leq c_1 \leq \cdots \leq c_s \leq 1$,

$a_{ij} = \int_0^{c_i} L_j(\theta)d\theta, \quad b_j = \int_0^1 L_j(\theta)d\theta$

$\Psi^\tau x_0 = u(\tau),\ \dot{u}(c_i\tau) = f(u(c_i\tau))$ with u as interpolation polynomial

$L_j(x) = \prod_{k\neq j} \frac{x - x_k}{x_j - x_k}$

Gauß schemes $\Rightarrow$ Gauß nodes, $p = 2s$, A-stable

Radau schemes $\Rightarrow$ interpolation nodes hits the grid, $p = 2s - 1$, L-stable

Multistep Schemes

MSS

$\rho(z) = \alpha_0 + \ldots \alpha_k z^k,\ \sigma(z) = \beta_0 + \ldots \beta_k z^k,\ (Ex)(t) = x(t+\tau),\ Ex_j = x_{j+1}$

$\rho(E)x_\tau(t) = \tau\sigma(E)f(t, x_\tau(t))$

MSS Consistency order

$L(x,t,\tau) = \rho(E)x(t) - \tau\sigma(E)\dot{x}(t) = O(\tau^{p+1})$

$\Leftrightarrow \quad L(Q,0,\tau) = 0$

$\Leftrightarrow \quad L(\exp,0,\tau) = O(\tau^{p+1})$

$\Leftrightarrow \quad \sum_{j=0}^k \alpha_j j^l = l\sum_{j=0}^k \beta_j j^{l-1} \quad \forall l = 0,\ldots,p$

$\rho(1) = 0,\ \rho'(1) = \sigma(1) \quad \Rightarrow \quad$ consistent

MSS on $\dot{x} = 0$ stability

$\rho(E)x_\tau = 0$ is stable $\Leftrightarrow |\lambda| \leq 1,\ |\lambda| = 1 \ \Rightarrow\ \lambda$ is a simple root of ρ

MSS Convergence order

$\forall$ IVPs with unique soln:

$\|x_\tau(t) - x(t)\| = O(\tau^p)\ \forall t \in \Delta_\tau \cap [t_0, T]$

$\|x_j - x_0\| = O(\tau^p)\ \forall j = 0,\ldots,k-1$

MSS Convergence theorem

stability & consisteny order p $\Leftrightarrow$ convergence order p

Stability domain

$S = \{z \in \mathbb{C}\ :\ (\rho(E) - x\sigma(E))\,x_\tau = 0$ is stable$\}$

Root locus curve

$S \subset \text{int}\left(\left\{\frac{\rho(e^{i\phi})}{\sigma(e^{i\phi})}\ :\ \phi \in [0, 2\pi]\right\}\right)$

Adams Bashford Schemes

$x_{j+1} = x_{j+k-1} + \tau\sum_{i=0}^{k-1} f_{j+i}\beta_i$ (or if implicit)

$\beta_i = \int_0^1 L_{k-1-i}(\theta)d\theta$

$u(t_{j+i}) = f_\tau(t_{j+i}, x_{j+i})$, order k+1

BDF Schemes

$\rho(z) = \sum_{i=0}^k L'_{k-i}(0)z^i,\ \sigma(z) = z^k$

$u(t_{j+i}) = x_{j+i}\ \forall i = 0,\ldots,k,\ \dot{u}(t_{j+k}) = f_\tau(t_{j+k}, x_{j+k})$

BDF scheme is stable $\Leftrightarrow k \leq 6$

Max rder of MSS

explicit $\Rightarrow \leq 2k - 1$, implicit $\Rightarrow \leq 2k$,

stable $\leq \begin{cases} k,\ \beta_k/\alpha_k = 0 \\ k+1,\ \text{k odd} \\ k+2,\ \text{k even} \end{cases}$