

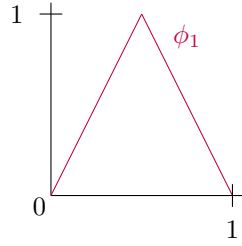
# A Categorical Introduction to Dynamical Systems

Functors, Natural Transformations, Category of Functors, Equivalence of Categories

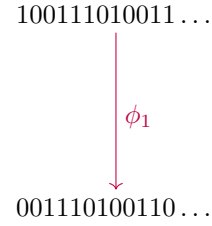
April Herwig

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## 1 Two Dynamical Systems



(a) Tent map



(b) Left shift map on two symbols

## 2 Functors

**Definition 2.1.** A *functor*  $F : \mathcal{A} \rightarrow \mathcal{B}$  between two categories  $\mathcal{A}$  and  $\mathcal{B}$  maps  $\text{ob}(\mathcal{A}) \rightarrow \text{ob}(\mathcal{B})$ ,  $\text{hom}_{\mathcal{A}} \rightarrow \text{hom}_{\mathcal{B}}$  in the following structure-preserving way:

- for all  $A \in \text{ob}(\mathcal{A})$ ,  $F(\text{id}_A) = \text{id}_{F(A)}$ ;
- for all  $A_1, A_2, A_3 \in \text{ob}(\mathcal{A})$ , the diagram commutes

$$\begin{array}{ccc} F(A_1) & \xrightarrow{F(g)} & F(A_2) \\ & \searrow F(g \circ f) & \downarrow F(f) \\ & & F(A_3) \end{array}$$

*Example 2.2.* A *dynamical system* is a functor  $\phi$  from an additive semigroup  $\mathcal{T}$  (viewed as a single-object category) to the category  $\mathbf{MetTop}$  of metrizable spaces with continuous functions.

## 3 Natural Transformations

**Definition 3.1.** A *natural transformation* is a morphism in the category  $[\mathcal{A}, \mathcal{B}]$  of functors from  $\mathcal{A}$  to  $\mathcal{B}$ . Specifically, a natural transformation (*natural morphism*, *natural isomorphism*, *etc.*) is a collection  $\{\eta_A\}_{A \in \text{ob}(\mathcal{A})}$  of morphisms such that

- for all  $A_1, A_2 \in \text{ob}(\mathcal{A})$  and  $f : A_1 \rightarrow A_2$ , the diagram commutes

$$\begin{array}{ccc} F(A_1) & \xrightarrow{F(f)} & F(A_2) \\ \eta_{A_1} \downarrow & & \downarrow \eta_{A_2} \\ G(A_1) & \xrightarrow{G(f)} & G(A_2) \end{array}$$

We write  $\eta : F \Rightarrow G$ , and write  $\mathcal{A} \cong \mathcal{B}$  if there exists a natural isomorphism  $\eta$ .

Moreover, if two functors  $F : \mathcal{A} \rightarrow \mathcal{B}$ ,  $G : \mathcal{B} \rightarrow \mathcal{A}$  satisfy  $G \circ F \cong \text{id}_{\mathcal{A}}$ ,  $F \circ G \cong \text{id}_{\mathcal{B}}$ , then we say  $\mathcal{A}$  and  $\mathcal{B}$  are *equivalent*, write  $\mathcal{A} \simeq \mathcal{B}$ .

*Example 3.2.* Given two dynamical systems, that is two functors  $\phi, \psi : \mathcal{T} \rightarrow \mathbf{MetTop}$ , a homeomorphism  $h$  which satisfies  $\psi_1 = h^{-1} \circ \phi_1 \circ h$  is known in dynamical systems literature as a *topological conjugacy*. Such a homeomorphism is precisely a natural isomorphism when viewed in the categorical context.

*Example 3.3.* The tent map is topologically conjugate to the left shift on two symbols. The conjugacy is constructed using a so-called "itinerary map" technique. Systems which are conjugate share many properties, including *chaos*. This is to say, despite how simple the map looks - a piecewise linear map - after just a few iterations the dynamics become incalculably complicated!

## 4 Equivalence

**Theorem 4.1.** A functor  $F : \mathcal{A} \rightarrow \mathcal{B}$  induces an equivalence of categories  $\mathcal{A} \simeq \mathcal{B}$  if and only if  $F$  is

- *fully faithful*: for all  $A_1, A_2 \in \text{ob}(\mathcal{A})$ , the map  $f \mapsto F(f)$  is a bijection

$$\text{hom}_{\mathcal{A}}(A_1, A_2) \rightarrow \text{hom}_{F(\mathcal{A})}(F(A_1), F(A_2));$$

- *essentially surjective*: every  $b \in \text{ob}(\mathcal{B})$  is isomorphic to some  $F(A)$ ,  $A \in \text{ob}(\mathcal{A})$ .

## References

- [1] Martin Brandenburg. *Einführung in die Kategorientheorie*. Springer Spektrum Berlin, Heidelberg, 2016. DOI: <https://doi.org/10.1007/978-3-662-47068-8>.
- [2] Tom Leinster. *Basic Category Theory*. 2016. arXiv: 1612.09375 [math.CT].