Technische Universität München

Department of Mathematics

Bachelor's Thesis

GAIO.jl: Set-oriented Methods for Approximating Invariant Objects, and their Implementations in julia

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I assure the single handed composition of this bachelor's thesis only supported by declared resources.

Garching,

Zusammenfassung

Bei einer in englischer Sprache verfassten Arbeit muss eine Zusammenfassung in deutscher Sprache vorangestellt werden. Dafür ist hier Platz.

Contents

1 Dynamical Systems

1.1 Motivation

Our goal is to investigate the qualitative, long-term behavior of systems in which a given function describes the trajectory of a point in an ambient space. Such dynamical systems are used in modelling physcial phenomena, economic forecasting, differential equations, etc. We wish to construct topological *closed covers* of sets which describe the infinite dynamics of some portions of the system, as well as statistical *invariant measures* which describe much larger sets in the space, but with less information.

The algorithms described in the present paper have been previously implemented in the statistical programming language `matlab` [1], but is now being fully refactored and reimplemented in the open-source, composable language `julia` [2]. The reason for this change is julia's high-level abstraction capabilities, just-in-time compilation, and in-built set-theoretical functions, which create short, elegant code which is nonetheless more performant.

1.2 Definitions

This section should be treated as an index of definitions, to be referred back to as necessary during reading. In the following, we assume M is a compact or a smooth manifold in $\mathbb{R}^d$, endowed with a metric d, and the map $f : M \to M$ is at least $\mathcal{C}^0$. Our setting is a *discrete, autonomous dynamical system*, that is, a system of the form:

$$x_{k+1} = f(x_k), \quad k = 0,\ 1,\ 2,\ \dots \tag{1}$$

A continuous dynamical system $\dot{x} = F(x)$ can be *discretized* by, for example, considering the *Poincaré time-t map* over some $d-1$ dimensional hyperplane, or by setting one "step" of the system as integrating F for a set time t.

We begin by giving a set of topological definitinos of sets we wish to approximate.

Definition 1.1 ((Forward-, Backward-) Invariant)**.** [3] A set A is called *forward-invariant* if $f(A) \subset A$, *backward-invariant* if $f^{-1}(A) \subset A$, and *invariant* if it is both forward- and backward-invariant.

Definition 1.2 (Attracting Set)**.** [4] An invariant set A is called *attracting* with *fundamental neighborhood U* if for every open set $V \supset A$ there is an $N \in \mathbb{N}$ such that the tail

$$\left\{ f^k(U) \mid k \geq N \right\} \tag{2}$$

lies entirely within A, ie $f^k(U) \subset A \; \forall \, k \geq N$. The attracting set is also called *global* if the *basin of attraction*

$$B(A) = \bigcap_{k \geq 0} f^{-k}(U) \tag{3}$$

is the whole of $\mathbb{R}^n$.

The basin of attraction acts in some sense as the set for which all points eventually arrive in A. Since the map f is smooth, then the closure $\bar{A}$ is invariant too. With continuity it becomes clear that

$$A = \bigcap_{k \geq 0} f^k(U). \tag{4}$$

The global attractor is maximal in the sense that it contains all backward-invariant sets within the system. In particular, it contains all unstable manifolds.

Definition 1.3 (Stable and Unstable Manifolds)**.** [5] Let $\bar{x}$ be a fixed point of the diffeomorphism f, and U a neighborhood of x. Then the *local unstable manifold* is given by

$$W^u(\bar{x}, U) = \left\{ x \in U \mid \lim_{k \to \infty} d(f^{-k}(x),\ \bar{x}) = 0 \textit{ and } f^{-k}(x) \in U \; \forall \, k \geq 0 \right\}. \tag{5}$$

The *global unstable manifold* is given by

$$W^u(\bar{x}) = \bigcup_{k \geq 0} f^k(\, W^u(\bar{x}, U)\,). \tag{6}$$

The dual definition of the *(local) stable manifold* is obtained by reversing the sign of k in the above equations.

Definition 1.4 (Pseudoperiodic)**.** [5] Let $n \in \mathbb{N}$. A set $\{x_k \mid k \in \{1,\ \ldots,\ n\}\}$ is called *ϵ-pseudoperiodic* if for any k

$$d(x_k,\ x_{k+1 \bmod n}) < \epsilon. \tag{7}$$

As the name suggests, an ϵ-pseudoperiodic orbit is ”almost” periodic inthe sense that it represents a ”small” perturbation of a theoretically periodic orbit. In practice, such direct orbits may not be known, but it will preresent a naturally useful definition in our approximations.

Definition 1.5 (Chain Recurrent)**.** [5] The point $\bar{x} \in M$ is called *chain recurrent* if for any $\epsilon > 0$ there exists an ϵ-pseudoperiodic orbit. The *chain recurrent set* $R_M(f)$ is the set of all chain recurrent points in M.

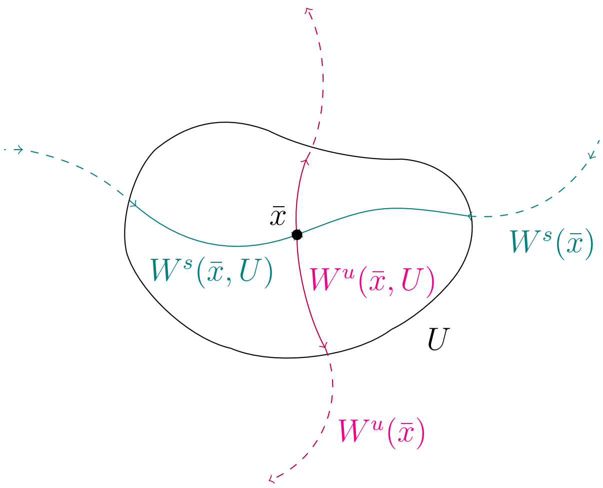

Figure 1.1: [6] Stable and unstable manifolds, local and global

As shown in [3] we have the inclusion $R_M(f) \subset \bigcap_{k \geq 0} f^k(M)$, which also shows that $R_M(f)$ is an invariant set.

We finish with a measure-theoretical definition for types of measures we wish to approximate.

Definition 1.6 (Perron-Frobeniuns Operator, Invariant Measure)**.** [7] Let $\mathcal{M}$ be the set of probability measures on M. We define the *Perron-Frobenius operator* $P : \mathcal{M} \to \mathcal{M}$ pointwise on the level of sets as

$$(P\mu)(A) = \mu \circ f^{-1}(A). \tag{8}$$

A measure μ is called *invariant* if it is an eigenmeasure of P with eigenvalue 1, ie $P\mu = \mu$.

Remark. The Perron-Frobenius operator is often also called *transfer operator* or *pushforward operator*.

An invariant measure can be used to understand the global behavior of a dynamical system, with more μ-mass assigned to regions which are visited frequently over long trajectories, and less μ-mass to regions visited less frequently.

An eigenvalue of 1 is not the only object of interest when considering the operator P. Suppose instead we have a finite (complex valued) measure with $P\nu = \lambda\nu$ for a $\lambda = -1$. Then, using finiteness and borel measurability, we can find a partition of M in two disjoint subsets A_1, A_2 such that $\nu(A_1) = -\nu(A_2)$. In particular, this implies that f maps A_1 to A_2, and A_2 to A_1 (since $P^2\nu = \nu$). This paritition forms a *two-cycle*.

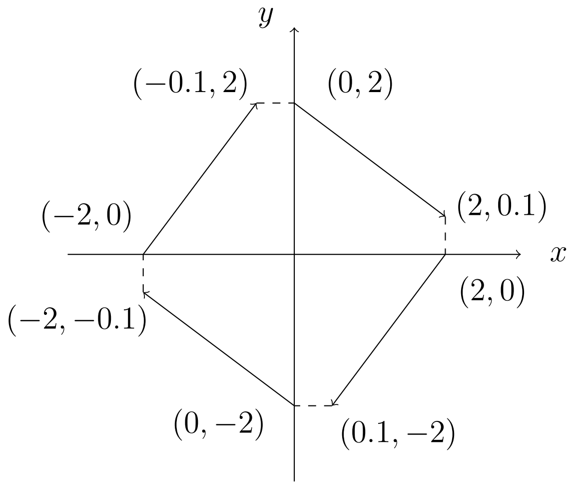

Figure 1.2: [5] A 0.1-pseudoperiodic orbit of the map $f(x, y) = (y,\ 0.05\,(1 - x^2)\,y - x)$

2 Algorithms

The basic technique of all the topological algorithms is to split the compact set Q into a partition $\mathcal{P}$ of *boxes* - that is, generalized rectangles, each with center vector r and componentwise radii r. The algorithms will begin with a set of boxes $\mathcal{B}$, and then repeatedly subdivide each box in $\mathcal{B}$ into two (or more) smaller boxes, examine the dynamics of the subdivided boxes, and refine the box set to include only the boxes we are interested in.

Definition 2.1 (Image of a Box Set)**.** For a partition $\mathcal{P}$ of Q into boxes, and a subset $\mathcal{B} \subset \mathcal{P}$, we will call the *image of* $\mathcal{B}$ *under* f as the set of boxes which intersect with the image $f(B)$, for at least one $B \in \mathcal{B}$. More precisely, it is

$$f(\mathcal{B}) = \Big\{ B \in \mathcal{P} \quad | \quad \exists\ \tilde{B} \in \mathcal{B},\ x \in \tilde{B}\ :\ f(x) \in B \Big\} \tag{9}$$

2.1 Relative Attractor

The construction of a fundamental neighborhood U for a global attractor A is relatively difficult, but the above description lends to a natural *ansatz* for its approximation using a compact subdomain $Q \subset M$.

Definition 2.2 (Realtive Global Attractor)**.** Let Q be compact. Then we define the *attractor relative to* A as

$$A_Q = \bigcap_{k \geq 0} f^k(Q) \tag{10}$$

Remark. It follows from the definition that the relative global attractor is a subset of the global attractor.

The idea to approximate the relative gloabl attractor is in two steps: first, we subdivide each of the boxes and second, discard all those boxes which do not intersect with the previous box set. The algorithm accepts the map f, the box set $\mathcal{B}$, and a predefined number of steps n.

Algorithm 1 Relative Attractor

$\mathcal{B}_0 \leftarrow \mathcal{B}$
for $i = \{1, \ldots, n\}$ **do**
  $\mathcal{B}_i \leftarrow$ SUBDIVIDE($\mathcal{B}_{i-1}$)
  $\mathcal{B}_i \leftarrow \mathcal{B}_i \cap f(\mathcal{B}_i)$
return $\mathcal{B}_n$
▷ *Optionally, the set* $\{\mathcal{B}_i \mid i \in \{0, \ldots, n\}\}$ *can be returned instead* ◁

Remark. The precise technique for subdivision can be tuned depending on the situation. In `GAIO.jl` [8], boxes are bisected evenly along one dimension $k \in \{1, \ldots, d\}$. The dimension k along which to bisect is cycled through during the steps.

Proposition 2.1. [3, 4] Set $\mathcal{B}_\infty = \bigcap_{n \geq 1} \bigcup_{B \in \mathcal{B}_i} B$.

1. $A_Q \subset \bigcup_{B \in \mathcal{B}_i} B$ for all $i \in \{0, \ldots, n\}$
2. $A_Q = \mathcal{B}_\infty$

In particular, this shows that $\mathcal{B}_\infty$ is backward-invariant.

3 Julia

3.1 Introduction to the Julia Language

```julia
print("Here's some placeholder code")
```

4 Parallelization using the CPU

4.1 CPU Architechture

5 Parallelization using the GPU

5.1 GPU Architechture