Simplicial Sets

1. Geometric Simplicial Complexes

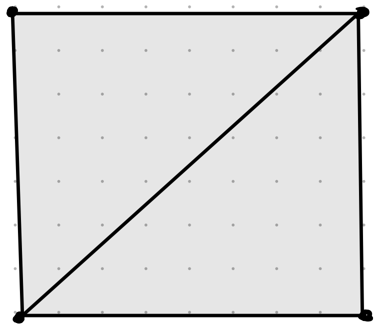
Definition

A geometric simplicial complex is a collection X of simplices such that

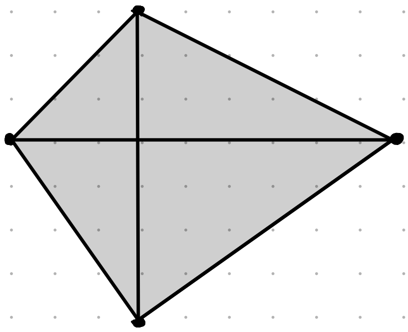
1. every face of a simplex is in X,
2. the intersection of any two simplices (if nonempty) is a mutual face.

Examples

standard n-simplex $|\Delta^n|$



$|\Delta^3| =$



Remark

We can describe a geom. simpl. compl. using the skeleta X^k, $k = 0, 1, 2, \ldots$ ie we write

"vertices" comprise X^0

$$[v_{i_0} \cdots v_{i_{k-1}}] \in X^k \leftarrow$$

X^k is a collection of subsets of X^0 with cardinality k.

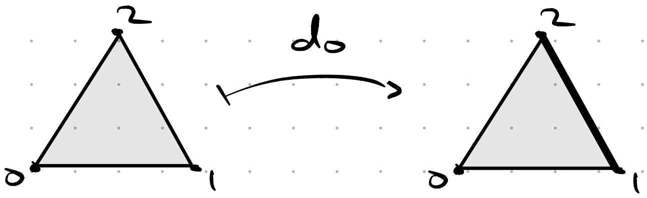
We can strictly order the vertices such that

$$[v_{i_0} \cdots v_{i_{k-1}}] \in X^k \Rightarrow i_0 < \cdots < i_{k-1}.$$

Definition

Face map:

$$d_i[i_0 \dots i_n] = [i_0 \dots \hat{i}_i \dots i_n]$$



Remark

a) $d_i d_j = d_{j-1} d_i \quad \forall\, i<j$

b) $d_i[i_0 \dots i_n] \neq d_j[i_0 \dots i_n] \quad \forall\, i \neq j$

$\Rightarrow$ Geometric simplicial complexes are determined uniquely by their face map relations

Removing b) and taking a) as a definition yeilds "Delta sets".

2. Simplicial Maps

Definition

A map $f: X \longrightarrow Y$ of simplicial complexes is determined by its action on vertices $X^0 \ni [i] \longmapsto [f([i])] \in Y^0$ such that

$\{i_0 \dots i_{k-1}\}$ are vertices of an X-simplex

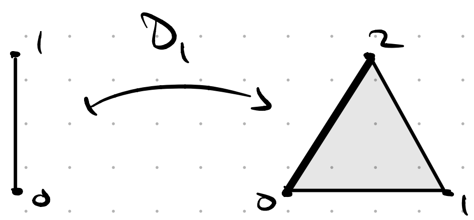
$\Rightarrow \{f[i_0] \dots f[i_k]\}$ are vertices of a Y-simplex.

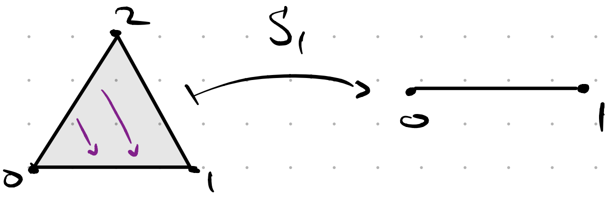
we do not require that they be unique!

Examples

- inclusion:

$$D_i[i_0 \dots i_n] = [i_0 \dots \hat{i}_i \dots i_{n+1}]$$



· collapse: 

In particular, b) does not necessarily hold after applying an s_i.

is a valid simplicial map but cannot be represented in a strictly order preserving way.

=> the need for degeneracy. We remove the requirement for a strict ordering of the vertices.

Equivalent: vertices in a simplex no longer unique.

$\Rightarrow \; s_i[i_0 \dots i_n] = [i_0 \dots i_i \, i_i \dots i_{n-1}]$

Proposition

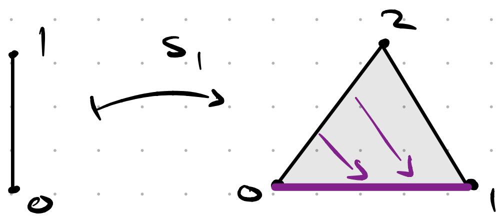
All order preserving maps are generated by D, S.

In particular, all simplicial maps are generated by inclusions and collapses on each of the simplices.

Definition

degeneracy map:

$$s_i[i_0 \dots i_n] = [i_0 \dots i_i \, i_i \dots i_n]$$



Proposition

$$d_i d_j = d_{j-1} d_i \quad \forall i < j$$

$$s_i s_j = s_{j+1} s_i \quad \forall i \leq j$$

$$d_i s_j = \begin{cases} s_{j-1} d_i, & i < j \\ s_j d_{i-1}, & i > j+1 \\ id, & \text{else} \end{cases}$$

*

3. Simplicial Sets

Definition

A simplicial set consists of sets X^k, $k = 0, 1, 2, \ldots$ and face / degeneracy maps satisfying $*$.

Example

Singular chain complexes

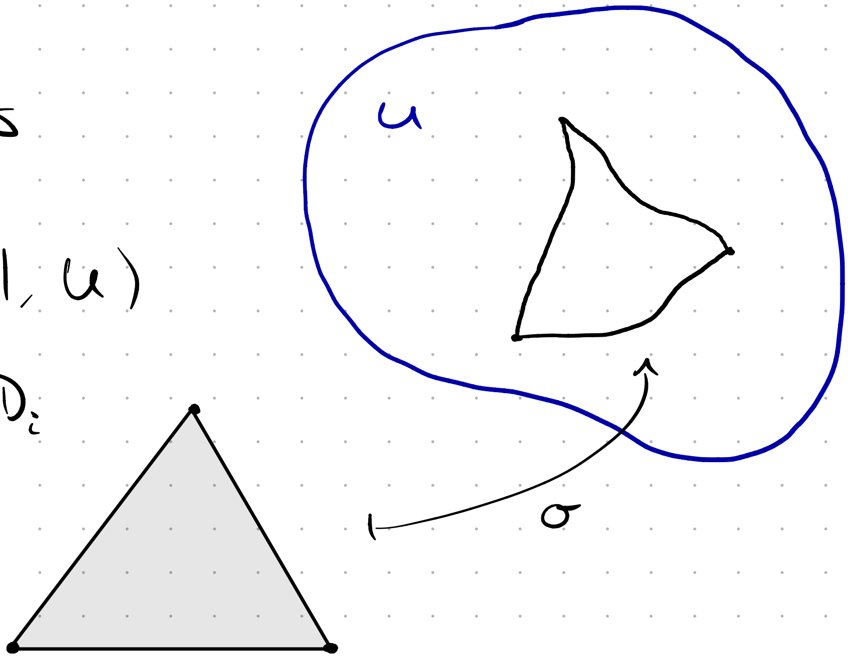
$$\underset{\in}{\overset{\mathrm{Top}}{}}$$

$$\mathrm{sing}(U)^k = \mathrm{Hom}_{\mathrm{Top}}(|\Delta^k|, U)$$

$$d_i \sigma := \sigma|_{d_i[0..k]} = \sigma \circ D_i$$

$$\sigma \in \mathrm{sing}(U)^k$$

$$s_i \sigma := \sigma \circ S_i$$



Definition

The category Δ consists of objects $\{0 \ldots n\}$ and morphisms the order-preserving functions $\{0 \ldots n\} \to \{0 \ldots m\}$.

Theorem

A simplicial set is a contravariant functor $\Delta \to \mathrm{Set}$.

Intuition:

including a face into a simpl. $\overset{\text{"opposite"}}{\longleftrightarrow}$ given a simpl., prescribing a face

collapsing into a degen. face $\overset{\text{"opposite"}}{\longleftrightarrow}$ blowing up a face into a degen. simpl.

5. Realization

Definition

Endow each X^n with the discrete topology.

$$|X| = \left(\coprod_n X^n \times |\Delta^n| \right) / \sim$$

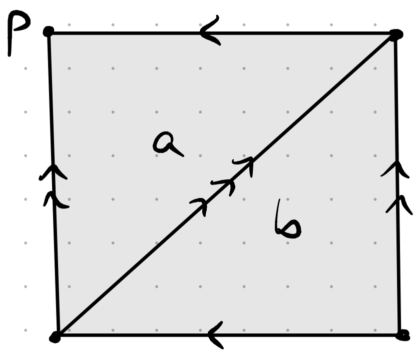
$$\underbrace{(x, D_i p) \sim (d_i x, p)}_{\text{identify common faces}} \quad \text{and} \quad \underbrace{(x, S_i p) \sim (s_i x, p)}_{\text{kill degeneracies}}$$

Examples

- 0-simplex: only nondegenerate simplex is $[0]$

$$X^k = [0 \dots 0] = s_0^k [0] \sim [0]$$

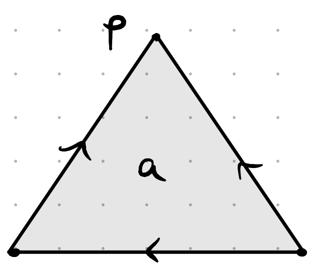
$\Rightarrow$ realization is single point.

- 

$X^0 = \{p\}$

$X^1 = \{ \leftarrow, \twoheadleftarrow, \lll \}$

$X^2 = \{a, b\}$

- 

$X^0 = \{p\}$

$X^1 = \{ \leftarrow \}$

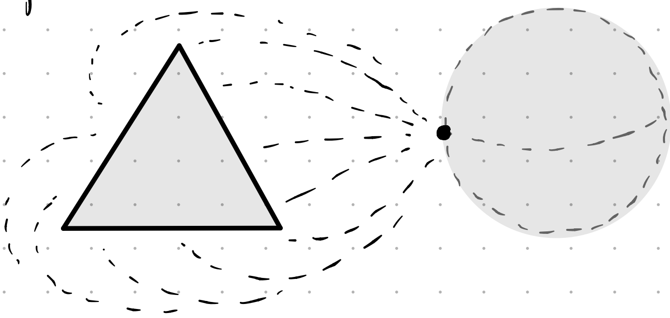
$X^2 = \{a\}$

· Let X have the two nondegenerate simplices $[0]$ and $[0 \dots n-1]$

=> all faces of $[0 \dots n-1]$ are degeneracies of $[0]$

=> all faces collapsed to a point

=> realization is S^{n-1}.



Theorem

$$|\cdot| \dashv \operatorname{sing}(\cdot)$$

construction:

$$\operatorname{Hom}_{Top}(|X|, Y) \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} \operatorname{Hom}_{sSet}(X, \operatorname{sing}(Y))$$

$$g : |X| \ni (x, z) \longmapsto y \in Y$$

$x \in X^n$ (nondegen.), $z \in |\Delta^n|$

$$\Big\downarrow \Phi$$

$$x \longmapsto [g(x, \cdot) : |\Delta^n| \longrightarrow Y] \in \operatorname{sing}(Y)^n$$

$$|X| \ni (x, z) \longmapsto \sigma_x(z) \in Y$$

$x \in X^n$, $z \in |\Delta^n|$

$$\Big\uparrow \Psi$$

$$\sigma : X^n \ni x \longmapsto (\sigma_x : |\Delta^n| \longrightarrow Y) \in \operatorname{sing}(Y)^n$$