A Categorical Introduction to Dynamical Systems and Chaos

Category Barc : objects are multisets of intervals,

morphisms are partial matchings

- 2.11. Monomorphisms, epimorphisms, opposite category, commutative diagrams and initial/final objects, functors, natural transformations, category of functors, equivalence of categories [Bra, 2.4–3.6,6.7], [Lei, 1.2–1.3]

What is a dynamical system?

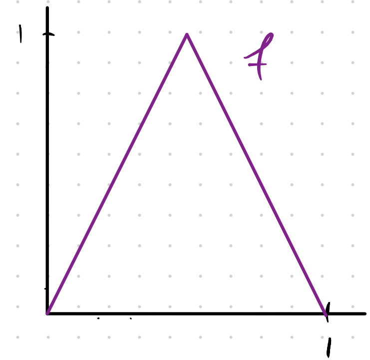
Flows "lorenz63"

$$\dot{x} = \sigma \cdot (y - x)$$
$$\dot{y} = x \cdot (\rho - z) - y$$
$$\dot{z} = x \cdot y - \beta \cdot z$$

$\phi_t\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ "flow map"

https://youtu.be/B4ftJl77PTw?si=LXj4AHLhlt8W7oOb

Iterated maps "Tent map"



$$\phi_t(x) = \underbrace{f \circ \dots \circ f}_{t \text{ times}}(x)$$

here: metric space

"time domain", often $\mathbb{N}_0, \mathbb{Z}, [0,\infty), \mathbb{R}$

Unified language

Let X be a space, $\mathcal{T}$ be an additive semigroup. A dynamical system is a family of maps $\{\phi_t\}_{t \in \mathcal{T}}$ with the properties

- $\phi_0 = id_X$
- $\forall\, t, s \in \mathcal{T}$ the diagram commutes:

$$\begin{tikzcd} X \arrow[r, "\phi_s"] \arrow[dr, "\phi_{t+s}"'] & X \arrow[d, "\phi_t"] \\ & X \end{tikzcd}$$

Functors In the spirit of unifying language...

Definition

A functor F between two categories $\mathcal{U}$ & $\mathcal{B}$ maps objects and morphisms in $\mathcal{U}$ to objects and morphisms in $\mathcal{B}$, in the following structure-preserving way:

- $F(id_A) = id_{F(A)} \quad \forall A \in ob(\mathcal{U})$
- $\forall A_1, A_2, A_3 \in ob(\mathcal{U})$ the diagram commutes:

$$\begin{tikzcd} F(A_1) \arrow[r, "F(g)"] \arrow[dr, "F(f\circ g)"'] & F(A_2) \arrow[d, "F(f)"] \\ & F(A_3) \end{tikzcd}$$

Viewing Υ as a single-object category, we notice a dynamical system is just a functor into **MetTop**.

category of metrizable spaces and continuous maps

$$\Upsilon: \begin{tikzcd} * \arrow[loop above, "t"] \arrow[loop below, "s"] \end{tikzcd} \dashrightarrow \begin{tikzcd} \phi(*) = X \arrow[r, "\phi_t"] \arrow[dr, "\phi_{s+t}"'] & X \arrow[d, "\phi_s"] \\ & X \end{tikzcd}$$

Chaos

$\exists r>0 : \left(\forall x \neq y \in X \ \exists t \in \mathcal{T} : d(\phi_t x, \phi_t y) > r\right)$

A dynamical system is chaotic if it has sensitive dependence on initial conditions, is topologically transitive, & periodic orbits are dense.

$\exists x \in X : \overline{\bigcup_{t \in \mathcal{T}} \{\phi_t x\}} = X$ $\qquad$ $\forall x \in X, \delta > 0 \ \exists y \in X, T \in \mathcal{T} : d(x,y) < \delta \wedge \phi_T y = y$

Left shift map

$X = \Sigma_2, \quad \mathcal{T} = \mathbb{N}_0, \quad \sigma_1(x_0 x_1 x_2 \dots) = (x_1 x_2 x_3 \dots)$

space of binary representations of numbers in $[0,1]$

This system is chaotic.

sensitive dependence: left shift "doubles all differences"

topological transitivity: concatenate all finite sequences

periodic orbits are dense: repeated sequences $1\underbrace{00\dots0}_{n}1\underbrace{00\dots0}_{n}\dots$

That's great and all, but how can we verify chaos in "real" systems?

Isomorphic systems

Conjugacy

We can study the dynamics of one system using a simpler "conjugate" system, that is, a system $(Y, \mathcal{T}, \psi)$ such that the diagram commutes for some homeomorphism h.

$$\begin{tikzcd} X \arrow[r, "\phi_t"] \arrow[d, "h"'] & X \\ Y \arrow[r, "\psi_t"'] & Y \arrow[u, "h^{-1}"'] \end{tikzcd}$$

The tent map is chaotic:

"Itinerary map" technique: Let $x \in [0,1]$, define

$$h(x) = y_0 y_1 y_2 \cdots \quad \text{where} \quad y_i = \begin{cases} 1, & \phi_i x \geq \frac{1}{2}, \\ 0, & \text{else.} \end{cases}$$

With some work, h is a homeomorphism. And by definition,

$$\begin{tikzcd} x \arrow[r, mapsto] \arrow[d, mapsto, "h"'] & \phi_1 x \arrow[d, mapsto, "h"] \\ y_0 y_1 \cdots \arrow[r, mapsto] & y_1 y_2 \cdots \end{tikzcd}$$

Natural transformations

What did we just do?

Definition
Given categories $\mathcal{A}$ and $\mathcal{B}$, a natural transformation is a morphism in the category of functors from $\mathcal{A}$ to $\mathcal{B}$. ← denoted $[\mathcal{A},\mathcal{B}]$

We write $\eta : F \Rightarrow G$. If the functors $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{A}$ satisfy

∃ invertible natural transformation

$$G \circ F \cong \mathrm{id}_{\mathcal{A}}, \quad F \circ G \cong \mathrm{id}_{\mathcal{B}}$$

then we say $\mathcal{A}$ is equivalent to $\mathcal{B}$, write $\mathcal{A} \simeq \mathcal{B}$.

Specifically, a natural transformation is a collection of morphisms

$$\eta = \{\eta_A\}_{A \in ob(\mathcal{A})}$$

such that $\forall f : \underbrace{A_1 \longrightarrow A_2}_{\in ob(\mathcal{A})?}$, the diagram commutes.

$$\begin{tikzcd}
F(A_1) \arrow[r, "F(f)"] \arrow[d, "\eta_{A_1}"'] & F(A_2) \arrow[d, "\eta_{A_2}"] \\
G(A_1) \arrow[r, "G(f)"'] & G(A_2)
\end{tikzcd}$$

Historical Note

RESEARCH ANNOUNCEMENT

APPEARED IN BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 32, Number 1, January 1995, Pages 66-72

CHAOS IN THE LORENZ EQUATIONS: A COMPUTER-ASSISTED PROOF

KONSTANTIN MISCHAIKOW AND MARIAN MROZEK

ABSTRACT. A new technique for obtaining rigorous results concerning the global dynamics of nonlinear systems is described. The technique combines abstract existence results based on the Conley index theory with computer-assisted computations. As an application of these methods it is proven that for an explicit parameter value the Lorenz equations exhibit chaotic dynamics.

The proof of Theorem 1 has five distinct components:

1. algebraic invariants based on the Conley index theory which guarantee the structure of the global dynamics, in this case the semi-conjugacy to the full two shift;
2. an extension of these invariants to multivalued maps;
3. a theory of finite representable multivalued maps which, when combined with the above-mentioned algebraic invariants, serves to bridge the gap between the continuous dynamics (in this case the Lorenz equations) and the finite dynamics of the computer;
4. the numerical computations of the finite multivalued map of interest;
5. the combinatorial computations of the Conley index for the multivalued map.

The rough idea of the general scheme of the proof is as follows. Choose a potential isolating neighborhood N for the Poincare map g. Select a finite representable multivalued map $\mathcal{G}$ (definitions follow) such that $\mathcal{G}$ is an *extension* of g, i.e. $g(x) \in \mathcal{G}(x)$ for all $x \in \operatorname{dom} \mathcal{G}(x)$. Perform a computer calculation both to determine $\mathcal{G}$ and to check whether N is an isolating neighborhood for $\mathcal{G}$. Theorem

Postscript Equivalent equivalence

Previously mentioned

If the functors $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{A}$ satisfy

$$G \circ F \cong \mathrm{id}_{\mathcal{A}}, \quad F \circ G \cong \mathrm{id}_{\mathcal{B}}$$

then we say $\mathcal{A}$ is equivalent to $\mathcal{B}$, write $\mathcal{A} \simeq \mathcal{B}$.

Theorem

F induces an equivalence $\mathcal{A} \simeq \mathcal{B}$ iff F is

- fully faithful: $f \mapsto F(f)$ is a bijection
$$\mathrm{Hom}_{\mathcal{A}}(A_1, A_2) \longrightarrow \mathrm{Hom}_{\mathcal{B}}(F(A_1), F(A_2))$$
for any $A_1, A_2 \in \mathrm{ob}(\mathcal{A})$,
- essentially surjective: every $B \in \mathcal{B}$ is isomorphic to some $F(A)$, $A \in \mathcal{A}$.