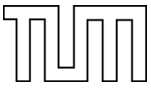

Technische Universität München

Department of Mathematics

Bachelor's Thesis

GAIO.jl: Set-oriented Methods for Approximating Invariant Objects, and their Implementations in julia

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Submission Date: ...

I assure the single handed composition of this bachelor's thesis only supported by declared resources.

Garching,

Zusammenfassung

Bei einer in englischer Sprache verfassten Arbeit muss eine Zusammenfassung in deutscher Sprache vorangestellt werden. Dafür ist hier Platz.

Contents

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1 Dynamical Systems

1.1 Motivation

Our goal is to investigate the qualitative, long-term behavior of systems in which a given function describes the trajectory of a point in an ambient space. Such dynamical systems are used in modelling physcial phenomena, economic forecasting, differential equations, etc. We wish to construct topological *closed covers* of sets which describe the infinite dynamics of some portions of the system, as well as statistical *invariant measures* which describe much larger sets in the space, but with less information.

The basic technique of all the topological algorithms is to split a compact set Q into a partition $\mathcal{P}$ of *boxes* - that is, generalized rectangles, each with center vector c and componentwise radii r. The algorithms will begin with a set of boxes $\mathcal{B}$, and then repeatedly subdivide each box in $\mathcal{B}$ into two (or more) smaller boxes, examine the dynamics of the subdivided boxes, and refine the box set to include only the boxes we are interested in.

The algorithms described in the present paper have been previously implemented in the statistical programming language `matlab` [15], but is now being fully refactored and reimplemented in the open-source, composable language `julia` [3]. The reason for this change is julia's high-level abstraction capabilities, just-in-time compilation, and in-built set-theoretical functions, which create short, elegant code which is nonetheless more performant. Source code for `GAIO` in `matlab` and `julia` and be found in [9] and [22], respectively.

1.2 Definitions

This section should be treated as an index of definitions, to be referred back to as necessary during reading. In the following, we assume M is a compact or a smooth manifold in $\mathbb{R}^d$, endowed with a metric d, and the map $f : M \to M$ is at least $\mathcal{C}^0$. Our setting is a *discrete, autonomous dynamical system*, that is, a system of the form:

$$x_{k+1} = f(x_k), \quad k = 0,\ 1,\ 2,\ \ldots \tag{1.1}$$

A continuous dynamical system $\dot{x} = F(x)$ can be *discretized* by, for example, considering the *Poincaré time-t map* over some $d-1$ dimensional hyperplane, or by setting one "step" of the system as integrating F for a set time t.

We begin by giving a set of topological definitions of sets we wish to approximate.

Definition 1.1 ((Forward-, Backward-) Invariant)**.** [7] A set A is called *forward-invariant* if $f(A) \subset A$, *backward-invariant* if $f^{-1}(A) \subset A$, and *invariant* if it is both forward- and backward-invariant.

Definition 1.2 (Attracting Set)**.** [5] An invariant set A is called *attracting* with *fundamental neighborhood* U if for every open set $V \supset A$ there is an $N \in \mathbb{N}$ such that the tail

$\bigcup_{k\geq N} f^k(U)$ lies entirely within A. The attracting set is also called *global* if the *basin of attraction*

$$B(A) = \bigcap_{k\geq 0} f^{-k}(U) \tag{1.2}$$

is the whole of $\mathbb{R}^n$.

The basin of attraction acts in some sense as the set for which all points eventually arrive in A. Since the map f is smooth, then the closure $\bar{A}$ is invariant too. With continuity it becomes clear that

$$A = \bigcap_{k\geq 0} f^k(U). \tag{1.3}$$

The global attractor is maximal in the sense that it contains all backward-invariant sets within the system. In particular, it contains local unstable manifolds.

Definition 1.3 (Stable and Unstable Manifolds)**.** [19] Let $\bar{x}$ be a fixed point of the diffeomorphism f, and U a neighborhood of x. Then the *local unstable manifold* is given by

$$W^u(\bar{x}, U) = \left\{ x \in U \mid \lim_{k\to\infty} d(f^{-k}(x),\ \bar{x}) = 0 \text{ and } f^{-k}(x) \in U\ \forall\, k \geq 0 \right\}. \tag{1.4}$$

The *global unstable manifold* is given by

$$W^u(\bar{x}) = \bigcup_{k\geq 0} f^k(\, W^u(\bar{x}, U)\,). \tag{1.5}$$

The dual definition of the *(local) stable manifold* is obtained by reversing the sign of k in the above equations.

Definition 1.4 (Pseudoperiodic)**.** [19] Let $n \in \mathbb{N}$. A set $\{x_k \mid k \in \{0,\ \ldots,\ n\}\}$ is called *ϵ-pseudoperiodic* if for any k, $\quad d(x_{k \bmod n},\ x_{k+1 \bmod n}) < \epsilon$.

As the name suggests, an ϵ-pseudoperiodic orbit is "almost" periodic inthe sense that it represents a "small" perturbation of a theoretically periodic orbit. In practice, such direct orbits may not be known, but it will preresent a naturally useful definition in our approximations.

Definition 1.5 (Chain Recurrent)**.** [19] The point $\bar{x} \in M$ is called *chain recurrent* if for any $\epsilon > 0$ there exists an ϵ-pseudoperiodic orbit. The *chain recurrent set* $R_M(f)$ is the set of all chain recurrent points in M.

As shown in [7] we have the inclusion $R_M(f) \subset \bigcap_{k\geq 0} f^k(M)$, which also shows that $R_M(f)$ is an invariant set.

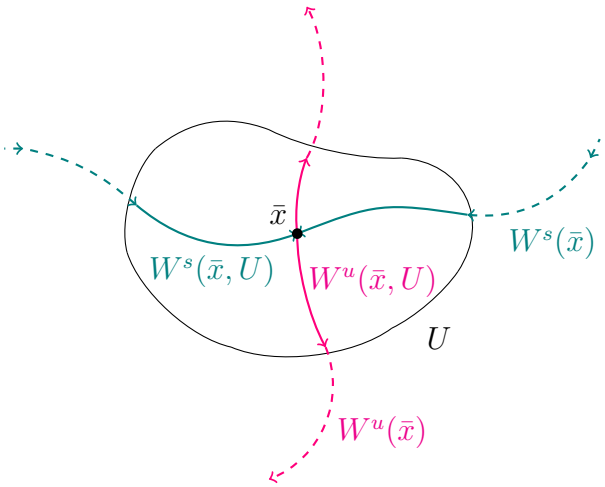

Figure 1.1: [8] Stable and unstable manifolds, local and global

We continue with a set of measure-theoretical definitions for types of measures we wish to approximate.

Since our goal is to partition the manifold into a finite set of boxes, we must accept some amount of "uncertainty" in how our sets look, and how *exactly* f maps such a set. We describe this noise using a stochastic transition function.

Definition 1.6 (Transition Function)**.** [6] Let $\mathcal{A}$ be a σ-agebra on M. A function $p : M \times \mathcal{A} \to [0, 1]$ is called *transition function* if

1. $p(\cdot, A) : M \to [0, 1]$ is measurable for all $A \in \mathcal{A}$,
2. $p(x, \cdot) : \mathcal{A} \to [0, 1]$ is a probability measure for all $x \in M$.

Example 1.7.

- [6] We can model the deterministic system using the dirac measure $p(x, A) = \delta_{f(x)}(A)$.
- The approximate box version of the system can be modelled as using a uniform probability density: Let $\mathcal{P}$ be a partition of Q into equally sized, (up to a lebesque null set) disjoint closed boxes (think of a checkerboard). Then for a point x, find the box $B \in \mathcal{P}$ with which contains x and map it forward to $f(B)$. Finally, let $\mathcal{B} \subset \mathcal{P}$ be a cover of $f(B) \cap Q$.

$$p(x, A) = \frac{\mathcal{L}\left(A \cap \bigcup_{B \in \mathcal{B}} B\right)}{\mathcal{L}\left(\bigcup_{B \in \mathcal{B}} B\right)}, \tag{1.6}$$

where $\mathcal{L}$ represents the d-dimensional Lebesque measure.

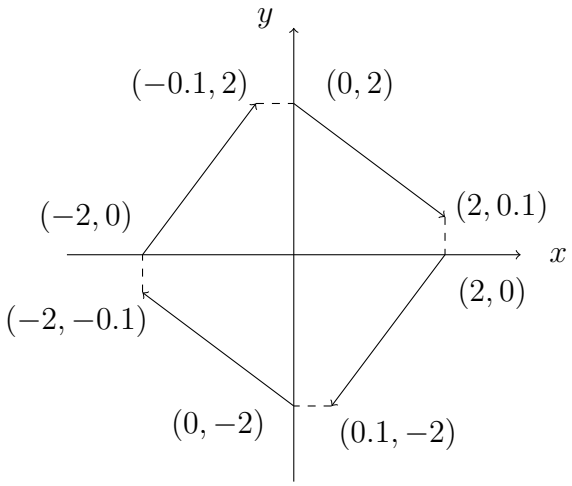

Figure 1.2: [19] A 0.1-pseudoperiodic orbit of the map $f(x, y) = (y,\ 0.05\,(1 - x^2)\,y - x)$

Definition 1.8 (Perron-Frobenius Operator, Invariant Measure)**.** [6] Let p be a stochastic transition function, and μ a measure on M. We define the *Perron-Frobenius operator* as

$$(P\mu)(A) = \int p(x,\ A)\ d\mu(x) \tag{1.7}$$

A measure μ is called *invariant* if it is a fixed point of P.

Remark. The Perron-Frobenius operator is often also called *transfer operator*.

Example 1.9. [6] We calculate

$$(P\mu)(A) = \int \delta_{f(x)}(A)\ d\mu(x) = \int \chi_A(f(x))\ d\mu(x) = \mu \circ f^{-1}(A). \tag{1.8}$$

In this case P simply becomes the *pushforward operator*.

An invariant measure can be used to understand the global behavior of a dynamical system, with more μ-mass assigned to regions which are visited frequently over long trajectories, and less μ-mass to regions visited less frequently.

Our "noisy" approximated system poses the benefit that while deterministic dynamical systems generally support the existence of multiple invariant measures, the stochastic system will (if the transition function has a strictly positive desnity) have a unique invariant measure, as shown in [12].

A fixed point - or eigenmeasure with eigenvalue 1 - is not the only object of interest when considering the operator P. Suppose instead we have a deterministic dynamical system and a finite (complex valued) measure with $P\nu = \lambda\nu$ for a $\lambda = -1$. Then, using finiteness and borel measurability, we can find a partition of M in two disjoint subsets A_1, A_2 such that $\nu(A_1) = -\nu(A_2)$. In particular, this implies that f maps A_1 to A_2, and A_2 to A_1

(since $P^2\nu = \nu$). This paritition forms a *two-cycle*.

Finally, we set some notation for convenience.

Definition 1.10 (Image of a Box Set)**.** For a partition $\mathcal{P}$ of Q into boxes, and a subset $\mathcal{B} \subset \mathcal{P}$, we will call the *image of* $\mathcal{B}$ *under* f the set of boxes which intersect with the image $f(B)$, for at least one $B \in \mathcal{B}$. More precisely, it is

$$f(\mathcal{B}) = \left\{ R \in \mathcal{P} \quad | \quad f^{-1}(R) \cap \bigcup_{B \in \mathcal{B}} B \neq \emptyset \right\}. \tag{1.9}$$

Theorem 1.11 (Image of a Box Set)**.** $f(\mathcal{B})$ is the inclusion-minimal cover of $f(\bigcup_{B\in\mathcal{B}} B)$ with boxes from $\mathcal{P}$.

Proof. We have the equivalent characterisation

$$f(\mathcal{B}) = \{ R \in \mathcal{P} \quad | \quad \exists\, B \in \mathcal{B} \text{ and } x \in B \ : \ f(x) \in R \}. \tag{1.10}$$

Hence if we remove one box R from $f(\mathcal{B})$, then there exists an $x \in \bigcup_{B\in\mathcal{B}} B$ which maps outside of the created box set.

□

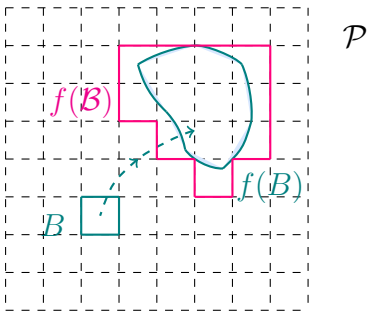

Figure 1.3: Image of the simple box set $\mathcal{B} = \{B\}$

2 Algorithms

2.1 Relative Attractor

The construction of a fundamental neighborhood U for a global attractor A is relatively difficult, but the description Eq. 1.2 lends to a natural *ansatz* for its approximation using a compact subdomain $Q \subset M$.

Definition 2.1 (Relative Global Attractor)**.** Let Q be compact. Then we define the *attractor relative to* A as

$$A_Q = \bigcap_{k \geq 0} f^k(Q) \tag{2.1}$$

Remark. It follows from the definition that the relative global attractor is a subset of the global attractor.

The idea to approximate the relative gloabl attractor is in two steps: first, we subdivide each of the boxes and second, discard all those boxes which do not intersect with the previous box set. The algorithm requires a map f, a box set $\mathcal{B}$, and a predefined number of steps n.

Algorithm 1 Relative Attractor

```
B_0 ← B
for i = {1, ..., n} do
    B_i ← SUBDIVIDE( B_{i-1} )
    B_i ← B_i ∩ f( B_i )
return B_n
```

Remark.

- Optionally, the set $\{\mathcal{B}_0,\ \mathcal{B}_1,\ \ldots,\ \mathcal{B}_n\}$ can be returned instead. This will be true for all algorithms of this type.
- The precise technique for subdivision can be tuned depending on the situation. In `GAIO.jl`, boxes are bisected evenly along one dimension $k \in \{1,\ \ldots,\ d\}$. The dimension k along which to bisect is cycled through during the steps.

Proposition 2.2. [7, 5] Set $Q_i = \bigcup_{B \in \mathcal{B}_i} B, \quad Q_\infty = \bigcap_{n \geq 1} Q_n$. For all i we have

1. $Q_{i+1} \subset Q_i$
2. $A_Q \subset Q_i$
3. $A_Q = Q_\infty$

In particular, this shows that Q_∞ is backward-invariant.

2.2 Unstable Manifold

From the definition of the local unstable manifold $W^u(\bar{x}, U)$ we see that the relative gloabl attractor $R_Q(f)$ contains the local unstable manifold, and, provided the set Q is sufficiently small, $W^u(\bar{x}, U)$ coincides with $R_Q(f)$. For further details, see [4, 20]. Using this knowledge, we can approximate the global unstable manifold $W^u(\bar{x})$:

1. first, we perform an *initialiation step*: replace the calculation of the local unstable manifold with the calculation of the relative attractor for a small set Q' surrounding the fixed point $\bar{x}$, using Algorithm 1.

2. Second, we repeat the following *continuation step*: map the current box set forward one iteration, and note any new boxes which are hit. These new boxes get added to the box set. Repeat until there are no new boxes added to the set.

Algorithm 2 Continuation Step

```
1: B_0 ← B
2: B_1 ← B
3: while B_1 ≠ ∅ do
4:     B_1 ← f( B_1 )
5:     B_1 ← B_1 \ B_0
6:     B_0 ← B_1 ∪ B_0
7: return B_0
```

Proposition 2.3. [4] The algorithm in general cannot guarantee covering of the entire unstable manifold, nor can it guarantee covering of the entirety of $W^u(\bar{x}) \cap Q$. This is because $W^u(\bar{x})$ could in theory exit Q, but return at another point. The algorithm can however guarantee covering of the connected component of $W^u(\bar{x}) \cap Q$ which contains $\bar{x}$.

2.3 Chain Recurrent Set

The algorithm to compute the chain recurrent set is first due to [18]. The idea is to construct a directed graph whose vertices are the box set $\mathcal{B}$, and for which edges are drawn from B_1 to B_2 if f maps any part of B_1 into B_2. Call this graph $\textsc{graph}(\mathcal{B})$, and call $\bar{d}$ the maximum *diameter* of a box in our partition, ie $\bar{d} = \max_{\substack{B \in \mathcal{P}, \\ x, y \in B}} d(x, y)$.

We can now ask for a subset of the vertices, for which each vertex is part of a directed cycle. We can equivalently characterize this set as follows:

Definition 2.4 (Strongly Connected)**.** For a directed graph $G = (V, E)$, a subset $H \subset V$ of vertices is called *strongly connected* if for all $u, v \in H$ there exist paths in both directions between v and u. Denote by $\textsc{scc}(G, v)$ the strongly connected subgraph which includes v, and by $\textsc{sccs}(G) = \bigcup_{v \in V} \textsc{scc}(G, v)$ the subgraph induced by union of strongly connected components.

Algorithm 3 Chain Recurrrent Set

```
B_0 ← B
for i = {1, ..., n} do
    B_i ← SUBDIVIDE( B_{i-1} )
    G ← GRAPH(B_{i-1})
    B_i ← SCCS(G)
return B_n
```

Proposition 2.5. A cycle $\{B_0,\ B_1,\ \ldots,\ B_{n-1}\}$ exists in G of and only if there exists an ϵ-pseudoperiodic orbit $\{x_0,\ x_1,\ \ldots,\ x_{n-1}\}$ with $x_i \in B_i$. In particular, the vertices of G form a covering of the chain recurrent set with Hausdorff distance at most $\bar{d}$.

Proof. Consider now a cycle in G. Then by construction, for each edge (B_i, B_{i+1}) in the cycle we can find a point $x_i \in B_i$ which gets mapped to B_{i+1}. Doing this for all edges we get a set $\{x_0,\ x_1,\ \ldots,\ x_{n-1}\}$ which satisfies $d(x_{i \bmod n},\ x_{i+1 \bmod n}) \leq \bar{d}$, ie a $\bar{d}$-pseudoperiodic orbit.

Conversely, consider an ϵ-pseudoperiodic orbit $\{x_0,\ x_1,\ \ldots,\ x_{n-1}\}$, $\epsilon \leq \bar{d}$. Then each x_i is in a box B_i, and since $d(x_{i \bmod n},\ x_{i+1 \bmod n}) \leq \bar{d}$, there is an edge (B_i, B_{i+1}) in G. Hence $\{B_0,\ B_1,\ \ldots,\ B_{n-1}\}$ is a cycle in G.

□

Corollary 2.6. Proposition 2.2 holds for Algorithm 3. For further details, see [18].

2.4 Invariant Measure

We shift focus to approximating invariant measures for the Perron-Frobenius operator P. For simplicity we will work only with measures absolutely continuous with respect to the Lebesque measure on M, ie measures for which there exists a *kernel* k with

$$p(x, A) = \int_A k(x, y)\, dy. \tag{2.2}$$

In this case we can define $P : L^1 \to L^1$ as

$$(P\phi)(y) = \int \phi(x)\, k(x, y)\, dx. \tag{2.3}$$

Note that ϕ is the density of an invariant measure $\mu_\phi(A) = \int_A \phi(x)\, dx$.

We will use a Galerkin approximation for P which maintains the eigenvalues and cyclic behavior of P. Let $\mathcal{P}$ be a partition of the compact set Q into equally sized, (up to Lebesque null sets) disjoint closed sets. Then we project to a subspace $\chi_\mathcal{P}$ generated by the basis $\{\chi_B \mid B \in \mathcal{P}\}$ of indicator functions on the boxes of our partition. For $\mathcal{P} = \{B_1,\ B_2,\ \ldots,\ B_n\}$ we define the matrix

$$P_{ij} = \frac{\mathcal{L}\left(B_j \cap f(B_i)\right)}{\mathcal{L}(B_j)}, \quad i,\, j = 1, \ldots, n, \tag{2.4}$$

as well as a linear operator $Q_n P : \chi_{\mathcal{P}} \to \chi_{\mathcal{P}}$ as

$$(Q_n P)\, \chi_{B_i} = \sum_{j=1}^{n} P_{ij}\, \chi_{B_j}. \tag{2.5}$$

To realize this approximation, we need to calculate P_{ij}. For this there are two techniques discussed in [7]. Currently the only technique built into `GAIO.jl` is a Monte Carlo approach. Namely, we choose a fixed number r of test points in B_i, and set P_{ij} as the fraction of test points which land in B_j:

Algorithm 4 Invariant Measure

1: **for** $i, j \in \{1,\ \ldots,\ n\}$ **do**
2: $\quad p \ \leftarrow$ Choose $\{p_1,\ \ldots,\ p_r\}$ randomly from a uniform distribution on B_i
3: $\quad P_{ij} \leftarrow \mid B_j \cap f(p) \mid \,/\, \mid p \mid$
4: $v \leftarrow$ Find a fixed point of P
5: $\phi_n \leftarrow \sum_{i=1}^{n}\ v_i\, \chi_{B_i}$
6: **return** ϕ_n

Remark.

- In practice the test points are only generated once from the unit cube $[-1, 1]^d$, and then scaled to fit inside the box B_i.
- Line 4 is acheived using the Fortran library ARPACK [13], which has been wrapped in julia by the creators of Arpack.jl [17]. Recently, the underlying algorithm (known as implicitly restarted Arnoldi method) has been implemented in pure julia under the package name ArnoldiMethod.jl [21].

Proposition 2.7. [12] Let $p(x, A)$ be a transition function with globally lipschitz continuous kernel k. Then the operator P is compact. Further, if k is strictly positive, P has a unique fixed point $\phi \in L^1$.

Proposition 2.8. [6, 14, 7, 11, 10] Suppose the transition function $p = p_\epsilon$ converges weakly to the dirac measure, ie

$$p_\epsilon(x, \cdot) \rightharpoonup \delta_{f(x)} \quad \text{as} \quad \epsilon \to 0. \tag{2.6}$$

(An example of this would be Ex. 1.7 for a sequence of partitions $\mathcal{P}_\epsilon$ with diameters $\bar{d}_\epsilon = \max_{\substack{B \in \mathcal{P}_\epsilon, \\ x,\, y \in B}} d(x, y) \to 0$). Suppose further that the diffeomorphism f has a hyperbolic attracting set A with and an open set $U \supset A$ such that for the kernels we have

$$k_\epsilon(f(x), y) = 0 \quad \text{if} \quad x \in \overline{U},\ y \notin U. \tag{2.7}$$

Let P_ϵ be the Perron-Frobenius operator for the transition function p_ϵ. Then there exist unique fixed points ϕ_ϵ^n of $Q_n P_\epsilon$, and the sequence of fixed points converge to the fixed point ϕ of the true operator P, ie

$$\phi_\epsilon^n \rightharpoonup \phi \quad \text{as} \quad n \to \infty,\ \epsilon \to 0. \tag{2.8}$$

3 Julia

3.1 A (Very) Brief Introduction to the Julia Language

The julia language GitHub page decribes julia as "a high-level, high-performance dynamic language for technical computing" [1]. Its creators come from Matlab, Python, Lisp among others, and they desired a langauge as syntactically simple as Python, as powerful for linear algebra as Matlab, and as fast as C [3]. To achieve this, julia is just-in-time compiled using LLVM to generate native machine code. Further, "julia uses multiple dispatch as a paradigm, which makes it easy to express many object-oriented and functional programming patterns" [2]. As a brief example of multiple dispatch, consider a simple user defined type `2dBox` (similar to the implementation in `GAIO.jl`):

```
struct 2dBox
    center::Tuple{Float64,Float64}
    radius::Tuple{Float64,Float64}
end
```

Then we can create a `2dBox` by calling `b = 2dBox((0.0, 0.0), (1.0, 1.0))`. A natural question to ask is whether the vector `x = [0.5, 0.6]` lies within `b`. To do this, we can *overload* the function `in` from julia base:

```
function Base.in(x::Vector, b::2dBox)
    b.center .- b.radius .≤ x .≤ b.center .+ b.radius
end
```

This function uses julia's *dot-syntax* to vectorize operations, eg. `+` by writing `.+`. We can now call the base function `in` or its unicode alias `∈`:

```
x = [0.5, 0.6]
b = 2dBox((0.0, 0.0), (1.0, 1.0))

x ∈ b           # returns true
x .- 2 ∈ b      # returns false
x .- 2 ∉ b      # returns true
```

The above example demonstrates the syntactical simplicity of julia. It should be noted that despite this simplicity, julia can generate highly performant machine code. For further illustration, consider the algorithm for the global unstable manifold (see Algorithm 2) implemented in matlab and in julia. The julia code is one third as long, mirrors the pseudocode much more closely, and still runs faster than the matlab code.

Listing 1: Unstable manifold algorithm in matlab

```
function gum(t, f, X, depth)

dim = t.dim;
none = 0; ins = 2; expd = 4;              % defining flags
nb0 = 0;  nb1 = t.count(depth);           % bookkeeping the no. of boxes
tic; j = 1;
t.set_flags('all', ins, depth);
while nb1 > nb0                           % while new boxes nonempty
t.change_flags('all', ins, expd);         % mark inserted boxes
b = t.boxes(depth); M = size(b,2);        % get the geometry of the boxes
flags = b(2*dim+1, :);
I = find(bitand(flags,expd));             % find boxes to expand
b = b(:,I); N = size(b,2);
S = whos('X'); l = floor(5e7/S.bytes);
for k = 0:floor(N/l),                     % split in chunks of size l
    K = k*l+1:min((k+1)*l,N);
    c = b(1:dim,K);                       % center ...
    r = b(dim+1:2*dim,1);                 % ... and radii of the boxes
    n = size(c,2); E = ones(n,1);
    P = kron(E,X)*diag(r) + ...           % sample points in all boxes
        kron(c',ones(size(X,1),1));
    t.insert(f(P)', depth, ins, none);    % map sample points, insert boxes
end
t.unset_flags('all', expd);               % unflag recently expanded boxes
nb0 = nb1; nb1 = t.count(depth);
j = j+1;
end
```

Listing 2: Unstable manifold algorithm in julia

```
function unstable_set(F::BoxMap, B::BoxSet)
    B₀ = B
    B₁ = B
    while B₁ ≠ ∅
        B₁ = F(B₁)
        B₁ = B₁ \ B₀
        B₀ = B₁ ∪ B₀
    end
    return B₀
end
```

4 Parallelization using the CPU

4.1 CPU Architechture

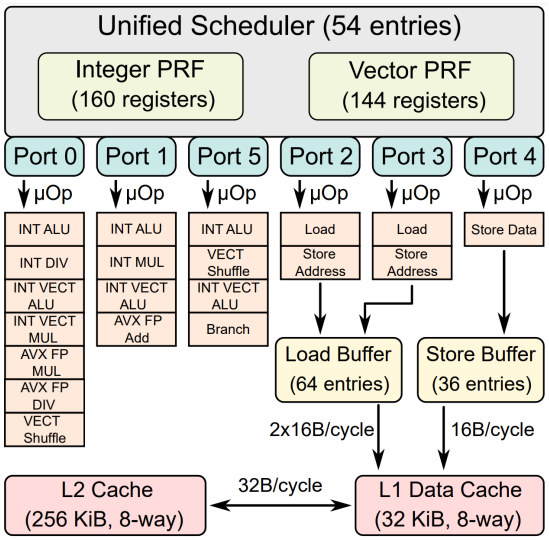

Figure 4.1: [16] CPU microarchitecture

5 Parallelization using the GPU

5.1 GPU Architechture

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