Determinants of Trace Class Operators - Lidskii's Theorem

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Definition 0.1 (Notation)**.** We consider a separable Hilbert space X with inner product $(\cdot, \cdot)$. $A \in K(X)$ is a trace-class operator.

Let $\lambda_1, \lambda_2, \ldots$ (resp. $s_1, s_2, \ldots$) be the eigenvalues (resp. singular values) of A in descending order:

$$A\, v_j = \lambda_j v_j, \quad |\lambda_1| \geq |\lambda_2| \geq \ldots, \quad s_1 \geq s_2 \geq \ldots . \tag{1}$$

A can be decomposed as $A = U\,|A|$ where U is unitary and $|A|$ is positive and self-adjoint. $\{z_j\}_j$ is an orthonormal basis of $\overline{\mathrm{Range}\,|A|}$. Writing $w_j = U z_j$, we have

$$A = \sum_{j=1}^{\infty} s_j(\cdot, z_j) w_j\,. \tag{2}$$

0.1 Lidskii's Theorem

Theorem 0.2 (Lidksii)**.** Let X be a Hilbert space and $A \in L(X)$ be a trace-class operator with eigenvalues $\lambda_1, \lambda_2, \ldots$. Then

$$\mathrm{Tr}\, A = \sum_{n=1}^{\infty} \lambda_n\ . \tag{3}$$

0.2 Tools: Generalized Eigenspaces and a Singular Value Inequality

Lemma 0.3 (Generalized eigenspaces)**.** Let $0 \neq \lambda \in \sigma(A)$. Then there exist subspaces $N = N(\lambda)$, $R = R(\lambda)$ of X with

$$\text{a) (finiteness)} \quad \dim N < \infty, \tag{4}$$

$$\text{b) (splitting)} \quad X = N \oplus R, \tag{5}$$

$$\text{c) (invariance)} \quad A(N) \subset N, \ \ A(R) \subset R, \tag{6}$$

$$\text{d) (isolation)} \quad \lambda \in \sigma(A\,|\,N), \ \ \lambda \in \sigma(A\,|\,R). \tag{7}$$

Write $P_\lambda : X \twoheadrightarrow N$ for the orthogonal projection of X into N.

Lemma 0.4 (Lalesco-Schur-Weyl)**.** For any $N \in \mathbb{N}$, $\prod_{j=1}^{N} |\lambda_j| \leq \prod_{j=1}^{N} s_j$. In particular we can conclude the eigenvalue - singular value inequality

$$\sum_{j=1}^{\infty} |\lambda_j| \leq \sum_{j=1}^{\infty} s_j. \tag{8}$$

Lemma 0.5 (Hadamard factorization)**.** Let $f : \mathbb{C} \circlearrowleft$ be an entire function with $f(0) = 1$. Let $\{z_j\}_j$ be the zeros of f and assume $\sum_{j=1}^{\infty} |z_j|^{-1} < \infty$. Finally suppose the boundedness condition

$$\forall\, \epsilon > 0 \ \ \exists\, C > 0 : \ \ |f(z)| \leq C \cdot e^{\epsilon |z|} \ \ \forall\, z \in \mathbb{C}. \tag{9}$$

Then

$$f(z) = \prod_{j=1}^{\infty} (1 - \frac{z}{z_j}). \tag{10}$$

0.3 (Antisymmetric) Tensor Products in Hilbert Spaces

Definition 0.6. For $x_1, x_2, \ldots, x_n \in X$, let $x_1 \otimes x_2 \otimes \ldots \otimes x_n$ be the multilinear map

$$(x_1 \otimes x_2 \otimes \ldots \otimes x_n) : X \times X \times \ldots \times X \to \mathbb{C}, \quad (y_1, y_2, \ldots, y_n) \mapsto \prod_{j=1}^{n} (x_j, y_j)\,. \tag{11}$$

The metric completion of the span of such $x_1 \otimes \ldots \otimes x_n$ (w.r.t. the natural inner product) forms a Hilbert space, denoted $X \otimes X \otimes \ldots \otimes X$.

If $\{e_j\}_j$ is an orthonormal basis of X, then $\{e_{\iota_1} \otimes e_{\iota_2} \otimes \ldots \otimes e_{\iota_n}\}_{\iota_1 < \iota_2 < \ldots < \iota_n}$ is an orthonormal basis of $X \otimes X \otimes \ldots \otimes X$.

Operators $A_1, A_2, \ldots, A_n \in L(X)$ induce a map $X \otimes \ldots \otimes X \circlearrowleft$:

$$(A_1 \otimes A_2 \otimes \ldots \otimes A_n)(\ell) : \ (y_1, y_2, \ldots, y_n) \mapsto \ell(A_1^* y_1, A_2^* y_2, \ldots, A_n^* y_n) \tag{12}$$

Denote the antisymmetrization of $x_1 \otimes \ldots \otimes x_n$ as

$$x_1 \wedge x_2 \wedge \ldots \wedge x_n = \frac{1}{\sqrt{n}} \sum_{\pi \in S_n} (-1)^\pi \, x_{\pi(1)} \otimes x_{\pi(2)} \otimes \ldots \otimes x_{\pi(n)} \tag{13}$$

where S_n is the symmetric group with n elements, $(-1)^\pi$ is the sign of a permutation.

Denote by $\Lambda^n(X)$ the (Hilbert) span of such $x_1 \wedge x_2 \wedge \ldots \wedge x_n$. $\Lambda^0(X) = \mathbb{C}$, $\Lambda^1(X) = X$. The same statement about orthonormal bases can be made for $\Lambda^n(X)$.

Proposition 0.7 (Properties of the tensor products)**.**

a) $(A_1 \otimes A_2 \otimes \ldots \otimes A_n)(x_1 \otimes x_2 \otimes \ldots \otimes x_n) = (A_1 x_1) \otimes (A_2 x_2) \otimes \ldots \otimes (A_n x_n),$ (14)

b) $|\Lambda^n(A)| = \Lambda^n(|A|),$ (15)

c) $\Lambda^n(A)$ has singular values $s_{\iota_1} \cdot s_{\iota_2} \cdot \ldots \cdot s_{\iota_n} \ \forall \iota_1 < \iota_2 < \ldots < \iota_n,$ (16)

d) If $\dim X = d < \infty$, then $\operatorname{Tr} \lambda^d(A) = \det A$, $\operatorname{Tr} \Lambda^{d+m}(A) = 0 \ \forall m > 0.$ (17)

0.4 Operator Determinants

Definition 0.8 (Fredholm's determinant)**.**

$$\det(I + zA) = \sum_{k=0}^{\infty} z^k \cdot \operatorname{Tr} \Lambda^k(A). \tag{18}$$

The technique to prove Lidskii's Theorem 0.2 is to show that Lemma 0.5 applies. This shows that the preceeding definition is equivalent to the following:

Definition 0.9 (Groh'berg-Krein's determinant)**.**

$$\det(I + zA) = \prod_{j=1}^{\infty} (1 + \lambda_j z). \tag{19}$$