An introduction to pseudospectra and application to validated computational dynamics

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Pseudospectra

Definition 1. *[Trefethen, Landau, Varah, Godunov, Hinrichsen & Pritchard] The pseudospectrum of a closed linear operator $M : \mathcal{X} \to \mathcal{X}$ over a Banach space $\mathcal{X}$ is the set*

$$\sigma_\epsilon(M) = \bigcup_{\|E\|_{op} \leq \epsilon} \sigma(M + E) \tag{1}$$

228 L.N. Trefethen

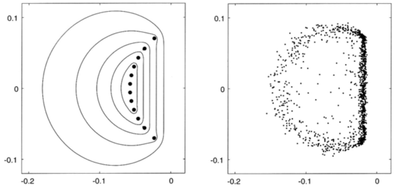

Fig. 2.2. Pseudospectra of a 12×12 Legendre spectral differentiation matrix. The left side shows the eigenvalues (solid dots) and the boundaries of the 2-norm ϵ-pseudospectra for $\epsilon = 10^{-3}, 10^{-4}, \ldots, 10^{-7}$ (from outer to inner). The right side shows 1200 of the 10^{-3}-pseudo-eigenvalues of A—specifically, a superposition of the eigenvalues of 100 randomly perturbed matrices $A + E$, where each E is a matrix with independent normally distributed complex entries of mean 0 scaled so that $\|E\| = 10^{-3}$. If all possible perturbations with $\|E\| = 10^{-3}$ were considered instead of just 100 of them, the dots on the right would exactly fill the outermost curve on the left.

Pseudospectra

Theorem 1. *[Trefethen] The following formulations are equivalent:*

(i)

$$\sigma_\epsilon(M) = \bigcup_{\|E\|_{op} \leq \epsilon} \sigma(M + E)\,, \tag{2}$$

(ii)

$$\sigma_\epsilon(M) = \left\{ z \in \mathbb{C} \mid \|(M - z)^{-1}\|_{op} \geq 1/\epsilon \right\}, \tag{3}$$

(iii)

$$\sigma_\epsilon(M) = \left\{ z \in \mathbb{C} \mid \inf_{\|v\|=1} \|(M - z)v\| \leq \epsilon \right\}, \tag{4}$$

(iv) *If* $\dim(\mathcal{X}) < \infty$ *is an inner product space,*

$$\sigma_\epsilon(M) = \{ z \in \mathbb{C} \mid s_{min} \leq \epsilon \}\,. \tag{5}$$

Theoretical results

Geometry of pseudospectra

Theorem 2. *[Trefethen]*

(i) *If M is normal, $\sigma_\epsilon(M) = \sigma(M) + B_\epsilon$ are ϵ-balls around the spectrum.*

(ii) $\bigcap_{\epsilon>0} \sigma_\epsilon(M) = \sigma(M)$ *and conversely* $\sigma_{\epsilon+\delta}(M) \supset \sigma_\epsilon + B_\delta$

Theoretical results

Almost invariant sets

Theorem 3. *[Junge]*

From now on we assume that $\lambda \neq 1$ is an eigenvalue of P_ε with corresponding real valued eigenmeasure $\nu \in \mathcal{M}_{\mathbb{C}}$, that is,

$$P_\varepsilon \nu = \lambda \nu.$$

PROPOSITION 5.7 *Suppose that ν is scaled so that $|\nu| \in \mathcal{M}$, and let $A \subset X$ be a set with $\nu(A) = \frac{1}{2}$. Then*

$$\delta + \sigma = \lambda + 1, \tag{5.3}$$

if A is δ-almost invariant and $X - A$ is σ-almost invariant with respect to $|\nu|$.

Theorem 4. *Suppose ν is an ϵ-pseudoeigenmeasure for the pseudoeigenvalue $\lambda \neq 1$ of a Perron-Frobenius operator P, where $\epsilon = o(1-\lambda)$. Suppose further that ν is scaled so $|\nu| \in \mathcal{M}$ and A is a set with $\nu(A) = 1/2$. Then*

$$\delta + \sigma = 1 + \lambda + o(1-\lambda) \tag{6}$$

if A is δ-almost invariant and $X - A$ is σ-almost invariant with respect to $|\nu|$.

A note on backward-stability

Theorem 5. *Let X be a metric space with Borel measure dx. Let $S, \hat{S} : X \to X$ be two continuous functions with $d^{\infty}_{ess}(S, \hat{S}) > 0$* [1]*. Then the induced Perron-Frobenius (pushforward) operators $P_S, P_{\hat{S}} : \mathcal{M} \to \mathcal{M}$ satisfy*

$$\|P_S - P_{\hat{S}}\|_{op} \geq 2. \tag{7}$$

This remains true (under adjustment of the const 2) if $P_S, P_{\hat{S}}$ are induced by (sufficiently) small random perturbations in the sense of [Kifer].

> Note that this does not contradict the continuous dependence of eigenvalues of P_S.

[1] $d^{\infty}_{ess}(S, \hat{S}) = \operatorname{ess\,sup}_x \; d(\, S(x), \, \hat{S}(x)\,)$ refers to the essential supremum distance

A note on backward-stability

Proposition 1. [2] *Let X be a metric space with Borel measure dx. Let $S, \hat{S} : X \to X$ be two continuous functions. Then*

$$d^{\infty}_{ess}(S, \hat{S}) < \epsilon \quad \Leftrightarrow \quad W_1(P_S\varphi, P_{\hat{S}}\varphi) < \epsilon \quad \forall \varphi \geq 0, \|\varphi\|_{L^1} = 1. \tag{8}$$

Conjecture 1. *This remains true (under potential adjustments of constants) if $P_S, P_{\hat{S}}$ are induced by (sufficiently) small random perturbations in the sense of [Kifer].*

This seems to suggest considering a different notion of pseudoeigenvalues

$$W_1(P_S\varphi, \lambda\varphi) \leq \epsilon \quad \forall \varphi \geq 0, \|\varphi\|_{L^1} = 1 \tag{9}$$

though this would likely be much more difficult.

[2] W_1 is the Wasserstein-1 metric.

How to compute the pseudospectrum

General inner approximation

Lemma 1. *Let* $(V_d)_d$ *be a collection of projections which converge pointwise to the identity* [3]. *Let*

$$(\lambda, x) \text{ be an } \epsilon\text{-pseudoeigenpair for } V_d M V_d. \tag{10}$$

Then $\lambda \in \sigma_{\epsilon+\delta}(M)$ *for all* δ *and sufficiently large* $d = d(\delta, x)$.

Note that this does not necessarily imply that $\sigma_\epsilon(V_d M V_d) \nearrow \sigma_\epsilon(M)$ as $d \to \infty$.

3 $V_d x \xrightarrow{d\to\infty} x \quad \forall x$

How to compute the pseudospectrum

ResDMD [Colbrook, Townsend]

- Given:
  - quadrature scheme $(\,(w^i, x^i)\,)_{i=1}^m$
  - dictionary $(\,\psi_j\,)_{j=1}^N$ of L^2 observables
- Data matrices:

$$\Psi_X = \Psi.(\mathbf{x}) = (\psi_1 \ldots \psi_N)\ .\begin{pmatrix} x^1 \\ \vdots \\ x^m \end{pmatrix} = \begin{pmatrix} \psi_1(x^1) & \cdots & \psi_N(x^1) \\ \vdots & & \vdots \\ \psi_1(x^m) & \cdots & \psi_N(x^m) \end{pmatrix} \tag{11}$$

$$\Psi_Y = (\Psi\ .\circ\ S).(\mathbf{x}) \tag{12}$$

How to compute the pseudospectrum

ResDMD [Colbrook, Townsend]

$$\begin{array}{ccccc} \text{Graham matrix} & & \text{EDMD matrix} & & \text{ResDMD matrix} \\ G = \Psi_X' W \Psi_X & & A = \Psi_X' W \Psi_Y & & L = \Psi_Y' W \Psi_Y \\ \downarrow & {\scriptstyle m\to\infty} & \downarrow & {\scriptstyle m\to\infty} & \downarrow \\ \langle \psi_i,\ \psi_j\rangle_{i,j} & & \langle \psi_i,\ K\psi_j\rangle_{i,j} & & \langle K\psi_i,\ K\psi_j\rangle_{i,j} \end{array} \tag{13}$$

Now

$$\inf_{\|v\|_{L^2}=1} \|(k-\lambda)v\|^2 = \inf_{\|v\|=1} \langle v,\ Kv\rangle - \bar{\lambda}\,\langle v,\ Kv\rangle - \lambda\langle Kv,\ v\rangle + |\lambda|^2\langle v,\ v\rangle \tag{14}$$

$$\approx \inf_{\mathbf{v}'G\mathbf{v}=1} v'(\,L - \bar{\lambda}\,A - \lambda\,A' + |\lambda|^2\,G\,)v \tag{15}$$

How to compute the pseudospectrum
Residual Ulam

Theorem 6. *Let S be an a.e. diffeomorphism onto its image, $P = P_S$ the induced transfer operator. Consider a box partition $\mathcal{P} = \{A_1, \ldots, A_N\}$ of the phase space X. Then*

$$\inf_{\|v\|_{L^2}=1} \|(P-\lambda)v\|^2 \approx \inf_{\mathbf{v}'G\mathbf{v}=1} v'(\, L - \bar{\lambda}\, A - \lambda\, A' + |\lambda|^2\, G\,)v \tag{16}$$

where

$$L_{i,j} = \int_{A_i \cap A_j} \frac{dx}{|\det DS(x)|}\,, \quad A_{i,j} = \underbrace{m(A_i \cap S^{-1}(A_j))}_{\text{(scaled) Ulam matrix}}\,, \quad G_{i,j} = m(A_i \cap A_j). \tag{17}$$