GAIO.jl

Preparing for a 1.0 release using julia 1.9

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Motivation

Attractors

Definition 1. *An invariant set A is attracting if there is a neighborhood U of A such that for every open set $V \supset A$ there is $K \in \mathbb{N}$ such that*

$$f^k(U) \subset V \quad \text{for all } k \geq K.$$

Proposition 1. *If A is a closed attracting set then*

$$A = \bigcap_{k \in \mathbb{N}} f^k(U).$$

Basic idea: Successively refine an approximation of A using subdivision

Computing Attractors

The subdivision algorithm

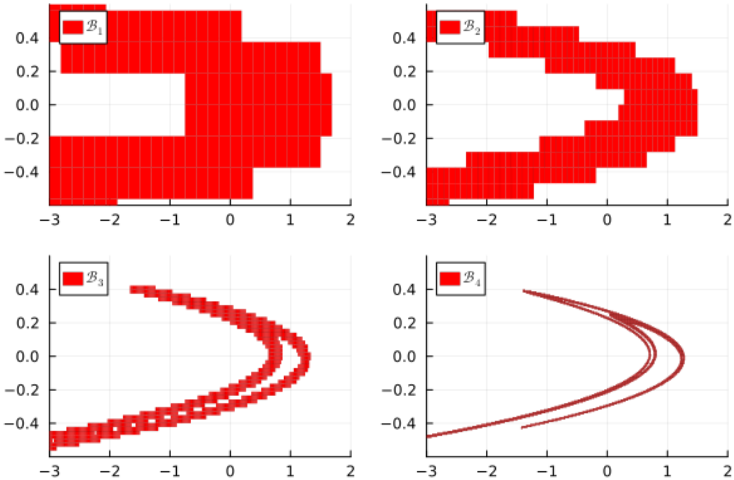
Generate a sequence $\mathcal{B}_0, \mathcal{B}_1, \mathcal{B}_2, \ldots$ of finite families of compact sets as follows:

Let $\mathcal{B}_0 = \{Q\}$, $\theta \in (0,1)$. For $k = 1, 2, \ldots$ do

- construct $\hat{\mathcal{B}}_k$ such that
$$|\hat{\mathcal{B}}_k| = |\mathcal{B}_{k-1}| \quad \text{and} \quad \operatorname{diam} \hat{\mathcal{B}}_k \leq \theta \cdot \operatorname{diam} \mathcal{B}_{k-1}.$$
- set
$$\mathcal{B}_k = f(\hat{\mathcal{B}}_k) \cap \hat{\mathcal{B}}_k, \quad \text{where}$$
$$f(\hat{\mathcal{B}}_k) \cap \hat{\mathcal{B}}_k \stackrel{def}{=} \{B \in \hat{\mathcal{B}}_k \mid \exists B' \in \hat{\mathcal{B}}_k : f(B') \cap B \neq \varnothing\}.$$

Theorem 1. *$|\mathcal{B}_k| \to A_Q$ as $k \to \infty$ in the Hausdorff metric.*

Computing Attractors
The subdivision algorithm



Representation of Cubical Complexes

`BoxPartition`: partition the domain into equally sized grid of hypercubes, or ”Boxes”

```
struct BoxPartition{N,T,I<:Integer}
    domain::Box{N,T}
    dims::SVector{N,I}
end
```

`TreePartition`: binary tree holding successive subdivisions

```
struct Node{I<:Integer}
    left::I
    right::I
end

struct TreePartition{N,T,I,V<:AbstractArray{Node{I}}}
    domain::Box{N,T}
    nodes::V
end
```

Representation of Cubical Complexes

BoxSet: collections of Boxes within a partition

```
struct BoxSet{B,P<:AbstractBoxPartition{B},S<:AbstractSet} <: AbstractSet{B}
    partition::P
    indices::S
end
```

We can use the built-in set data types and setwise operations for BoxSets using multiple-dispatch

```
function Base.⊆(B₁::BoxSet, B₂::BoxSet)
( B₁.partition == B₂.partition ) && ( B₁.indices ⊆ B₂.indices )
end
```

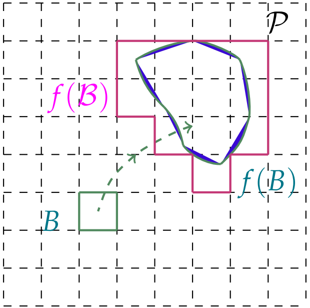
Combined with BoxSet, the BoxPartition (resp. TreePartition) serves as a σ-algebra over the domain. We can generate sets with cover:

```
cover(partition, points),   cover(partition, other_boxset)
```

Cell Mapping

So how do we compute

$$f(\mathcal{B}) = \{B \in \mathcal{P} \mid \exists B' \in \mathcal{B} : f(B') \cap B \neq \varnothing\} \ ?$$



Cell Mapping

test point sampling

```
function map_boxes(g::SampledBoxMap, source::BoxSet)

  B() = empty(source)                  # Function to initialize empty BoxSet
  P = source.partition

  @floop for box in source
    for p in g.domain_points(box)      # Generate sample points
      fp = typesafe_map(g, p)          # Wrap user-function output
      hit = cover(P, fp)               # Box in P covering the point fp
      @reduce(image = B() ∪ hit)       # Each thread collects hits,
    end                                # after loop completion the
  end                                  # result is reduced

  return image
end
```

Cell Mapping

test point sampling

- choose test points within $\mathcal{B}$ and record their images under f
- memory-efficient "lazy" test point sampling with `Generators`
- ensure type-stability to shorten generated code
- spread load across multiple compute threads using `@floop` macro
- collect hits and reduce per-thread result into single result using `@reduce` macro
- can harness the GPU using CUDA.jl

Cell Mapping

All `BoxMap` types work under a common API: they must define

```
map_boxes(g::MyBoxMap, source::BoxSet)
construct_transfers(g::MyBoxMap, domain::BoxSet)
construct_transfers(g::MyBoxMap, domain::BoxSet, codomain::BoxSet)
```

You now have all the knowledge to understand `TransferOperator`, `BoxGraph` as well!

Usage: Constructing an Attractor algorithm

Compare the pseudocode algorithm

Require: f, $\mathcal{B}_0$, n_{steps}

1: **for** $k = \{1, \ldots, n_{\mathsf{steps}}\}$ **do**

2: $\quad \mathcal{B}_k \leftarrow \text{SUBDIVIDE}(\mathcal{B}_{k-1})$

3: $\quad \mathcal{B}_k \leftarrow \mathcal{B}_k \cap f(\mathcal{B}_k)$

4: **return** $\mathcal{B}_n$

to the julia implementation

```julia
function relative_attractor(f::BoxMap, 𝓑::BoxSet, steps)
  for k = 1:steps
    𝓑 = subdivide(𝓑)
    𝓑 = 𝓑 ∩ f(𝓑)
  end
  return 𝓑
end
```

Optimization: Solving ”Time-To-First-Attractor”

Investigating the first execution shows an interesting problem: a GPU kernel gets compiled... even when no GPU is present?

Solve by converting the CUDA dependency to an extension

```
name = "GAIO"
uuid = "33d280d1-ac47-4b0f-9c2e-fa6a385d0226"
authors = ["The GAIO.jl Team"]
version = "1.0.0"

[deps]
...

[weakdeps]
CUDA = "052768ef-5323-5732-b1bb-66c8b64840ba"
...

[extensions]
CUDAExt = "CUDA"
```

Optimization: Solving "Time-To-First-Attractor"

With dynamic dispatch reduced, we can now save precompiled code in a package image. To force precompilation of common workloads, use `PrecompileTools.jl`

```
using PrecomileTools

@setup_workload begin
  # set local variables for common workload, e.g. attractor of Henon map

  @compile_workload begin
    # track which code gets generated and force full compilation
  end

end
```