

Fast, Set-Oriented Numerical Analysis using GAIO.jl

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Dynamical systems

Consider a continuous map

$$f : \mathbb{R}^d \rightarrow \mathbb{R}^d.$$

The map defines a **discrete dynamical system** by iteration:

$$x_{k+1} = f(x_k), \quad k = 0, 1, 2, \dots$$

Basic question: *What is the fate of some x_0 as $k \rightarrow \infty$?*

Motivation

Attractors

Definition 1. An invariant set A is *attracting* if there is a neighborhood U of A such that for every open set $V \supset A$ there is $K \in \mathbb{N}$ such that

$$f^k(U) \subset V \quad \text{for all } k \geq K.$$

Proposition 1. If A is a closed attracting set then

$$A = \bigcap_{k \in \mathbb{N}} f^k(U).$$

Basic idea: Successively refine an approximation of A using *subdivision*

Computing Attractors

The subdivision algorithm

Generate a sequence $\mathcal{B}_0, \mathcal{B}_1, \mathcal{B}_2, \dots$ of finite families of compact sets as follows:

Let $\mathcal{B}_0 = \{Q\}$, $\theta \in (0, 1)$. For $k = 1, 2, \dots$ do

- construct $\hat{\mathcal{B}}_k$ such that

$$|\hat{\mathcal{B}}_k| = |\mathcal{B}_{k-1}| \quad \text{and} \quad \text{diam } \hat{\mathcal{B}}_k \leq \theta \cdot \text{diam } \mathcal{B}_{k-1}.$$

- set

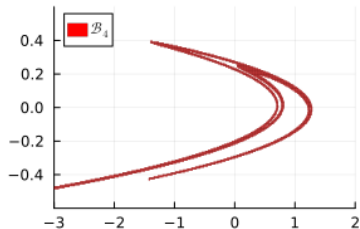
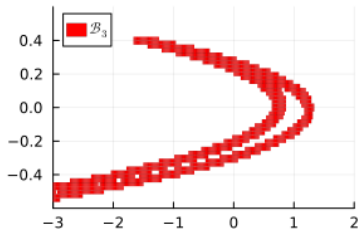
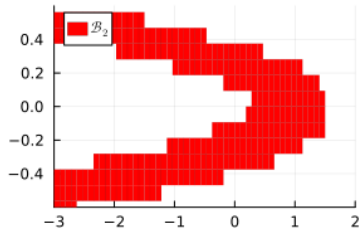
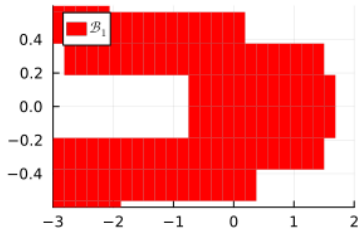
$$\mathcal{B}_k = f(\hat{\mathcal{B}}_k) \cap \hat{\mathcal{B}}_k, \quad \text{where}$$

$$f(\hat{\mathcal{B}}_k) \cap \hat{\mathcal{B}}_k \stackrel{\text{def}}{=} \{B \in \hat{\mathcal{B}}_k \mid \exists B' \in \hat{\mathcal{B}}_k : f(B') \cap B \neq \emptyset\}.$$

Theorem 1. $|\mathcal{B}_k| \rightarrow A$ as $k \rightarrow \infty$ in the Hausdorff metric.

Computing Attractors

The subdivision algorithm



Representation of Cubical Sets

Simple idea: partition the domain into equally sized grid of hypercubes

```
struct Box{N, T}  
  center::SVector{N, T}  
  radius::SVector{N, T}  
end  
  
struct BoxPartition{B <: Box}  
  domain::B  
  size::CartesianIndex{N}  
end  
  
struct BoxSet{P <: BoxPartition, S <: AbstractSet{<:CartesianIndex}}  
  partition::P  
  cartesian_indices::S  
end
```

Representation of Cubical Sets

We can use the built-in set data types and setwise operations for BoxSets using **multiple-dispatch**

```
function Base.⊆( B1::BoxSet, B2::BoxSet )
    B1.partition == B2.partition &&
        B1.cartesian_indices ⊆ B2.cartesian_indices
end

function Base.rand( B::Box{N,T} ) where {N,T}
    c = B.center; r = B.radius
    c .+ r .* rand(-1:eps(T):1, N)
end
```

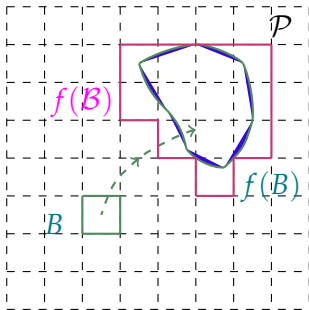
Combined with BoxSet, the BoxPartition serves as a σ -algebra over the domain. We can generate sets with cover:

```
cover(partition, points), cover(partition, other_boxset)
```

Cell Mapping

So how do we compute

$$f(\mathcal{B}) = \{B \in \mathcal{P} \mid \exists B' \in \mathcal{B} : f(B') \cap B \neq \emptyset\} ?$$



Cell Mapping

test point sampling

```
function map_boxes(f, source::BoxSet; n_samples)

    B() = empty(source)                # Initialize empty BoxSet
    P = source.partition

    @floop for box in source
        for k = 1:n_samples
            p = rand(box)                # Generate sample point
            image_p = f(p)
            hit = cover(P, image_p)      # Box in P covering the point fp
            @reduce(image = B() ∪ hit)    # Each thread collects hits,
        end                             # after loop completion the
    end                                 # result is reduced by ∪

    return image
end
```

Cell Mapping

test point sampling

- predefine test points and (affine) transform them to $B \in \mathcal{B}$ as needed
- memory-efficient "lazy" test point sampling with Generators
- ensure type-stability with FunctionWrappers to shorten generated llvm
- spread load across multiple compute threads using @floop macro
- collect hits and reduce per-thread result into single result using @reduce macro
- "embarrassingly parallel" - can harness the GPU using CUDA.jl

Usage: Constructing an Attractor algorithm

Compare the pseudocode algorithm

Require: f , \mathcal{B}_0 , n_{steps}

- 1: **for** $k = \{1, \dots, n_{\text{steps}}\}$ **do**
- 2: $\mathcal{B}_k \leftarrow \text{SUBDIVIDE}(\mathcal{B}_{k-1})$
- 3: $\mathcal{B}_k \leftarrow \mathcal{B}_k \cap f(\mathcal{B}_k)$
- 4: **return** \mathcal{B}_n

to the julia implementation

```
function relative_attractor(F::BoxMap, B::BoxSet, steps)
    for k = 1:steps
        B = subdivide(B)
        B = B ∩ F(B)
    end
    return B
end
```



`github.com/gaioguys/GAIO.jl`

- [1] A. GAIO.jl: Set-oriented Methods for Approximating Invariant Objects, and their Implementation in Julia. Thesis. 2022. URL: <https://github.com/April-Hannah-Lena/schoolwork/blob/2eada059678d91bad8a813c3e6b657a1ac72e86f/Thesis/main.pdf>.
- [2] Michael Dellnitz and Andreas Hohmann. “A subdivision algorithm for the computation of unstable manifolds and global attractors”. English. In: *Numer. Math.* 75.3 (1997), pp. 293–317. ISSN: 0029-599X. DOI: 10.1007/s002110050240.
- [3] Michael Dellnitz and Oliver Junge. “On the Approximation of Complicated Dynamical Behavior”. In: *SIAM Journal on Numerical Analysis* 36.2 (1999), pp. 491–515. DOI: 10.1137/S0036142996313002.