Phys 514

Problem Set 8

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Contents

1 Lie derivatives

The equation for the Lie derivative as given in the book for this tensor is clearly

$$\mathcal{L}_v T_{\mu\nu} = v^\sigma \partial_\sigma T_{\mu\nu} + \left(\partial_\mu v^\lambda\right) T_{\lambda\nu} + \left(\partial_\nu v^\lambda\right)$$

So now assuming this holds for covariant derivatives such that

$$\mathcal{L}_v T_{\mu\nu} = v^\sigma \nabla_\sigma T_{\mu\nu} + \left(\nabla_\mu v^\lambda\right) T_{\lambda\nu} + \left(\nabla_\nu v^\lambda\right)$$

Where the covariant derivative is

$$\nabla_\alpha T_{\mu\nu} = \partial_\alpha T_{\mu\nu} - \Gamma^\lambda_{\alpha\mu} T_{\mu\nu} - \Gamma^\lambda_{\alpha\nu} T_{\mu\nu}$$

And the Christoffel symbol is

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\lambda} \left(g_{\lambda\mu;\nu} + g_{\lambda\nu;\mu} - g_{\mu\nu;\lambda}\right)$$

So now we let $\sigma \to \alpha$ to get

$$\begin{aligned}
v^\alpha \nabla_\alpha T_{\mu\nu} &= v^\alpha \partial_\alpha T_{\mu\nu} - v^\alpha \left(\Gamma^\lambda_{\alpha\mu} T_{\lambda\nu} + \Gamma^\lambda_{\alpha\nu} T_{\mu\lambda}\right) \\
\Gamma^\lambda_{\alpha\mu} T_{\lambda\nu} &= \frac{1}{2} g^{\lambda\beta} \left(g_{\beta\alpha;\mu} + g_{\beta\mu;\alpha} - g_{\alpha\mu;\lambda}\right) \\
\Gamma^\lambda_{\alpha\nu} T_{\lambda\mu} &= \frac{1}{2} g^{\lambda\beta} \left(g_{\beta\alpha;\nu} + g_{\beta\nu;\alpha} - g_{\alpha\nu;\lambda}\right)
\end{aligned}$$

By the above we get

$$\begin{aligned}
\Gamma^\lambda_{\alpha\mu} T_{\lambda\nu} + \Gamma^\lambda_{\alpha\nu} T_{\mu\lambda} &= \frac{1}{2} g^{\lambda\beta} \left[g_{\beta\alpha;\mu} T_{\lambda\nu} + g_{\beta\mu;\alpha} T_{\lambda\nu} - g_{\alpha\mu;\lambda} T_{\lambda\nu} + g_{\beta\alpha;\nu} T_{\mu\lambda} + g_{\beta\nu;\alpha} T_{\mu\lambda} - g_{\beta\nu;\lambda} T_{\mu\lambda}\right] \\
&= \frac{1}{2} \left[g_{\beta\alpha;\mu} T^\beta_{\ \nu} + g_{\beta\mu;\alpha} T^\beta_{\ \nu} - g_{\alpha\mu;\lambda} T^\beta_{\ \nu} + g_{\beta\alpha;\nu} T_\mu^{\ \beta} + g_{\beta\nu;\alpha} T_\mu^{\ \beta} - g_{\beta\nu;\lambda} T_\mu^{\ \beta}\right] \\
&= \frac{1}{2} \left[\partial_\mu T_{\alpha\nu} + \partial_\mu T_{\mu\nu} - g_{\alpha\mu;\lambda} T^\beta_{\ \nu} + \partial_\nu T_{\mu\alpha} + \partial_\alpha T_{\mu\nu} - g_{\alpha\nu;\lambda} T_\mu^{\ \beta}\right]
\end{aligned}$$

Now set $\alpha \to \beta$ such that

$$\begin{aligned}
&= \partial_\mu T_{\alpha\nu} + \partial_\nu T_{\mu\alpha} - \frac{1}{2} \left[g_{\alpha\mu;\lambda} T^\alpha_{\ \nu} + g_{\alpha\nu;\lambda} T_\mu^{\ \alpha}\right] \\
&= \partial_\mu T_{\alpha\nu} + \partial_\nu T_{\mu\alpha} - g_{\alpha\mu;\lambda} g^{\alpha\mu} T_{\mu\nu}
\end{aligned}$$

Finally we set $\lambda \to \alpha$ to get

$$v^\alpha \nabla_\alpha T_{\mu\nu} = 2 v^\alpha \partial_\alpha T_{\mu\nu} - v^\alpha \partial_\mu T_{\alpha\nu} - v^\alpha d_\nu T_{\mu\alpha}$$

Now

$$\begin{aligned}
\nabla_\mu v^\alpha T_{\alpha\nu} &= \left(\partial_\mu v^\alpha + \Gamma^\alpha_{\mu\lambda} v^\lambda\right) T_{\alpha\nu} \\
&= \partial_\mu v^\alpha T_{\alpha\nu} + \frac{1}{2}\left(g_{\beta\mu;\lambda} T^\beta{}_\nu + g_{\beta\lambda;\mu} T^\beta{}_\nu - g_{\mu\lambda;\beta} T^\beta{}_\nu\right) v^\lambda \\
&= \partial_\mu v^\alpha T_{\alpha\nu} + \frac{1}{2}\left(v^\lambda \partial_\lambda T_{\mu\nu} + v^\lambda \partial_\mu T_{\lambda\nu} - \partial_\beta v_\mu \left(g^{\beta\mu} T_{\mu\nu}\right)\right) \\
&= \partial_\mu v^\alpha T_{\alpha\nu} + \frac{1}{2}\left[v^\lambda \partial_\lambda T_{\mu\nu} - \left(\partial_\beta v^\beta\right) T_{\mu\nu} + v^\lambda \partial_\mu T_{\lambda\nu}\right]
\end{aligned}$$

Similarly,

$$\nabla_\nu v^\alpha T_{\mu\alpha} = \partial_\nu v^\alpha T_{\mu\alpha} + \frac{1}{2}\left[v^\lambda \partial_\nu T_{\mu\lambda} + v^\lambda d_\lambda \Gamma_{\mu\nu} - \partial_\alpha v^\alpha T_{\mu\nu}\right]$$

Now putting everything together yields

$$\begin{aligned}
v^\alpha \nabla_\alpha T_{\mu\nu} + \nabla_\nu v^\alpha T_{\alpha\nu} + \nabla_\mu v^\alpha T_{\mu\alpha} &= 2v^\alpha \partial_\alpha T_{\alpha\nu} - v^\alpha \partial_\nu T_{\mu\alpha} + \partial_\mu v^\alpha T_{\alpha\nu} \\
&\quad + \frac{1}{2}\left[v^\lambda \partial_\lambda T_{\mu\nu} + v^\lambda \partial_\mu T_{\lambda\nu} - T_{\mu\nu}\left(\partial_\beta v^\beta\right)\right] \\
&\quad + \frac{1}{2}\left[v^\lambda \partial_\lambda T_{\mu\nu} + v^\lambda \partial_\nu T_{\mu\lambda} - T_{\mu\nu}\left(\partial_\beta v^\beta\right)\right] \\
&= \left(v^\alpha \partial_\alpha T_{\mu\nu} + \partial_\mu v^\alpha T_{\alpha\nu} + \partial_\nu v^\alpha T_{\mu\alpha}\right) \\
&\quad + v^\alpha \partial_\alpha T_{\mu\nu} - v^\alpha \partial_\mu T_{\alpha\nu} - v^\alpha \partial_\nu T_{\mu\alpha} + v^\lambda \partial_\lambda T_{\mu\nu}
\end{aligned}$$

By manipulation of indices we have that

$$\begin{aligned}
&v^\alpha \partial_\alpha T_{\mu\nu} - v^\alpha \partial_\mu T_{\alpha\nu} = 0 && \text{As } \alpha \to \mu \\
&v^\lambda \partial_\lambda T_{\mu\nu} - v^\alpha \partial_\mu T_{\alpha\nu} = v^\nu \partial_\nu T_{\mu\nu} - v^\nu \partial_\nu T_{\mu\nu} && \text{As } \lambda \to \alpha \to \nu
\end{aligned}$$

$$\mathcal{L}_r T_{\mu\nu} = v^\alpha \nabla_\alpha T_{\mu\nu} + \nabla_\mu v^\alpha T_{\alpha\nu} + \nabla_\nu v^\alpha T_{\mu\alpha}$$

For the metric, we have

$$\mathcal{L}_\nu g_{\mu\nu} = v^\alpha \nabla_\alpha g_{\mu\nu} + \nabla_\mu v^\alpha g_{\alpha\nu} + \nabla_\nu v^\alpha g_{\mu\alpha}$$

Hence, as $\nabla_\alpha g_{\mu\nu} = 0$, we have

$$\mathcal{L}_v = g_{\mu\nu} = \nabla_\mu v_\nu + \nabla_\nu v_\mu$$

As expected.

2 Killing vectors

2.1 Continuity

Assuming that we are given a stress tensor $T_{\mu\nu}$, we define $j_\mu = T_{\mu\nu}k^\nu$ such that

$$\begin{aligned}\mathcal{L}_k g_{\mu\nu} &= \nabla_\mu k_\nu + \nabla_\nu k_\mu \\ &= \nabla_\mu k_\nu = -\nabla_\nu k_\mu\end{aligned}$$

Considering the metric $g_{\alpha\nu}$ we have

$$\begin{aligned}\nabla_\mu \left(g_{\alpha\nu}k^\alpha\right) &= -\nabla_\nu \left(g_{\alpha\mu}k^\alpha\right) \\ \therefore \nabla_\mu k^\alpha &= \delta^\mu_\nu \left(-\nabla_\nu k^\alpha\right) = -\nabla_\mu k^\alpha\end{aligned}$$

Which implies that $\nabla_\mu k^\alpha = 0$ as desired. For more detail, look at problem 1.

2.2 Noether charge

We must show that $I(\lambda)$ is conserved such that it does not change with respect to λ. Define a tangent vector $v^\mu = \frac{\partial x^\mu}{\partial \lambda} = \partial_\lambda x^\mu$ where $x^\mu = (x^0(\lambda), x^1(\lambda), \dots)$. So by conservation we have

$$\begin{aligned}v^\nu \nabla_\nu I(\lambda) &= 0 \\ &= v^\nu \nabla_\nu \left(k_\mu v^\mu\right) \\ &= v^\nu \left(\nabla_\nu k_\mu\right) v^\mu + v^\nu k_\mu \left(\nabla_\nu v^\mu\right)\end{aligned}$$

Which implies that $v^\nu \left(\nabla_\nu v^\mu\right) = 0$ for a tangent vector along the worldline defined. Furthermore, $\nabla_\nu k_\mu = -\nabla_\nu k_\mu = 0$, so $v^\nu \nabla_\nu I(\lambda) = 0 \implies \frac{dI(\lambda)}{d\lambda} = 0$. To show that indeed $\frac{dI(\lambda)}{d\lambda} = 0$, we have

$$\begin{aligned}\frac{dI(\lambda)}{d\lambda} &= \frac{d}{d\lambda} k_\mu v^\mu \\ &= \frac{dk_\mu}{d\lambda}\frac{dx^\mu}{d\lambda} + k_\mu \frac{d^2 x^\mu}{d\lambda^2} \\ &= \frac{\partial k^\mu}{\partial x^\nu}\frac{dx^\nu}{d\lambda}\frac{dx^\mu}{d\lambda} + k_\mu \frac{d^2 x^\mu}{d\lambda^2} \\ &= \left(\nabla_\nu k_\alpha + \Gamma^\alpha_{\mu\nu} k_\alpha\right)\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} + k_\alpha \frac{d^2 x^\alpha}{d\lambda^2}\end{aligned}$$

And as we saw that $\nabla_\nu k_\alpha = 0$, it follows that

$$k_\alpha \left[\frac{d^2 x^\mu}{d\lambda^2} + \Gamma^\alpha_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}\right] = 0$$

Which clearly implies that $\frac{dI(\lambda)}{d\lambda} = 0$ as the geodesic equation vanishes along a worldline parametrized by λ, a parametrization of the geodesic itself.

2.3 Moving particle

We have that $g_{\mu\nu} = \eta_{\mu\nu}$ and $x^\mu = (t, x, y, z)$. Thus, the momentum 4-vector is

$$p^\mu = \left(\frac{E}{c^2}, p_x, p_y, p_z\right) = (E, \gamma m\vec{v})$$

As we let $c = 1$. Now recalling the previous parts of question 2, we note that the 4 velocity to the worldline is the tangent vector $v^\mu = \frac{dx^\mu}{d\lambda}$ for a time-like geodesic. The momentum vector is then

$$p^\mu = \gamma m v^\mu = m u^\mu$$

For $k = \partial_t$ such that $k_\mu = (1, 0, 0, 0)$, we can write

$$I(\lambda) = k_\mu v^\mu = \frac{dt}{d\lambda}$$

Since $p^0 = E = mu^0 = \gamma m \frac{dt}{d\lambda}$, then $\frac{dt}{d\lambda} = \frac{E}{\gamma m}$ So with $K = \partial_t$, then $I(\lambda) = \frac{dt}{d\lambda} \propto E$ as expected. On the other hand, $k = \partial_i$ implies $I(\lambda) = \partial_i \frac{dx^\mu}{d\lambda}$ for spatial coordinates i. This redefines the momentum vector as

$$p^\mu = \gamma m v^\mu$$

Where we have

$$p^i = \gamma m v^i = \gamma m \frac{dx^i}{d\lambda} = \gamma\left(m v_i\right)$$

Hence

$$\frac{dx^i}{d\lambda} = \frac{p^i}{\gamma m} \qquad \text{AND} \qquad I(\lambda) = \frac{dx^i}{d\lambda} \propto p^i$$

For momentum in x^i direction.

2.4 Null geodesics

$$\begin{aligned}\frac{d\log I}{d\lambda} &= \frac{1}{I}\frac{dI(\lambda)}{d\lambda} \\ &= \frac{1}{I}\left[\frac{\partial k_\nu}{\partial x^\mu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} + k_\alpha\frac{d^2x^\alpha}{d\lambda^2}\right] \\ &= \frac{K_\alpha}{I}\left[\frac{d^2x^\alpha}{d\lambda^2} + \Gamma^\alpha_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}\right]\end{aligned}$$

Now,

$$\begin{aligned}u^\mu\nabla_\mu v^\nu &= v^\mu\left(\partial_\mu v^\nu + \Gamma^\nu_{\mu\alpha}v^\alpha\right) \\ &= v^\mu\partial_\mu v^\nu + \Gamma^\nu_{\mu\alpha}v^\mu v^\alpha \\ &= \frac{d}{d\lambda}\frac{dx^\nu}{d\lambda} + \Gamma^\nu_{\mu\alpha}\frac{dx^\mu}{d\lambda}\frac{dx^\mu}{d\lambda} \\ &= \frac{d^2x^\mu}{d\lambda^2} + \Gamma^\nu_{\mu\alpha}\frac{dx^\mu}{d\lambda}\frac{dx^\alpha}{d\lambda} = f(\lambda)v^\nu\end{aligned}$$

Therefore

$$\begin{aligned}\frac{d\log I(\lambda)}{d\lambda} &= \frac{K_\alpha}{I}\left[\frac{d^2x^\alpha}{d\lambda^2} + \Gamma^\alpha_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}\right] \\ &= \frac{K_\alpha}{I}f(\lambda)v^\alpha \\ &= f(\lambda)\end{aligned}$$

Now if $I_1 = k^1_\mu v^\mu$ and $I_2 = k^2_\mu v^\mu$, then

$$\frac{d\log\left(\frac{I_1}{I_2}\right)}{d\lambda} = \frac{d\log I_1}{d\lambda} - \frac{d\log I_2}{d\lambda}$$

$$\therefore \log\left(\frac{I_1}{I_2}\right) = C$$

Where C is a constant along the geodesic.

2.5 Commuter

We have $[v,u]^\mu = v^\nu \partial_\nu u^\mu - u^\nu \partial_\nu v^\mu$ so

$$\begin{aligned}
[v,u]^\mu &= v^\nu \nabla_\nu u^\mu - u^\nu \nabla_\nu v^\mu \\
&= v^\nu \left[\partial_\nu u^\mu + \Gamma^\mu_{\nu\lambda} u^\lambda\right] - u^\nu \left[\partial_\nu v^\mu + \Gamma^\mu_{\nu\lambda} v^\lambda\right] \\
&= v^\nu \partial_\nu u^\mu - u^\nu \partial_\nu v^\mu + \Gamma^\mu_{\nu\lambda} u^\lambda v^\nu - \Gamma^\mu_{\nu\lambda} v^\lambda u^\nu
\end{aligned}$$

Now we change indices $\lambda \leftrightarrow \nu$ such that we have the Christoffel symbol

$$\Gamma^\mu_{\lambda\nu} u^\nu v^\lambda = \Gamma^\mu_{\nu\lambda} u^\nu v^\lambda$$

So

$$[v,u]^\mu = v^\nu \nabla_\nu u^\mu - u^\nu \nabla_\nu v^\mu = v^\nu \partial_\nu u^\mu - u^\nu d_\nu v^\mu$$

Now, assuming that $[v,u]^\mu$ and v^μ, u^μ are killing vectors, we have

$$\nabla_\nu v^\mu + \nabla_\mu v^\nu = 0 \implies \nabla_\nu v^\mu = -\nabla_\mu v^\nu$$

Hence

$$[v,u]^\mu = v^\nu \nabla_\nu u^\mu - u^\nu \nabla_\nu v^\mu = \mathcal{L}_\nu u^\mu$$

Let $k = [v,u]$ and $\mathcal{L}_k g_{\mu\nu} = 0 = \nabla_\mu k_\nu + \nabla_\nu k_\mu$, so

$$\begin{aligned}
\mathcal{L}_k g_{\mu\nu} &= g_{\mu\nu} \left[\nabla_\mu [v,u]^\mu + \nabla_\nu [v,u]^\nu\right] \\
&= g_{\mu\nu} [\nabla_\mu v^\nu \nabla_\nu u^\mu + v^\nu \nabla_\mu \nabla_\nu u^\mu - \nabla_\mu u^\nu \nabla_\nu v^\mu - u^\nu \nabla_\mu \nabla_\nu v^\mu \\
&\qquad + \nabla_\nu v^\mu \nabla_\mu u^\nu + v^\mu \nabla_\mu \nabla_\nu u^\nu - \nabla_\nu u^\mu \nabla_\mu v^\nu - u^\mu \nabla_\mu \nabla_\nu v^\nu]
\end{aligned}$$

As $\nabla_\nu u^\mu = -\nabla_\mu u^n u$, then

$$= g_{\mu\nu} [v^\nu \nabla_\mu \nabla_\nu u^\mu - u^\nu \nabla_\mu \nabla_\nu v^\mu - u^\mu \nabla_\mu \nabla_\nu v^\nu]$$

Recalling that u^μ, v^μ are killing vectors we get

$$= 0$$

Thus, $[v,u]^\mu$ is a killing vector if v^μ and u^μ are killing vectors.

3 Christoffel symbols and Riemann tensors

Recall the Riemann tensor symmetries when considering lowering indices

$$R_{\rho\sigma\mu\nu} = -R_{\rho\sigma\nu\mu}$$
$$R_{\rho\sigma\mu\nu} = -R_{\sigma\rho\nu\mu}$$
$$R_{\rho\sigma\mu\nu} = R_{\mu\nu\rho\sigma}$$

Where we defined a Riemann tensor to be

$$R^{\rho}_{\sigma\mu\nu} = \partial_\mu \Gamma^{\rho}_{\nu\sigma} - \partial_\nu \Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma}$$

So given an interval

$$ds^2 = -e^{2\alpha(r)}dt^2 + e^{2\beta(r)}dr^2 + r^2 d\Omega^2$$

and considering the similarly tedious process in Problem set 4 question 3, we have the Christoffel symbols:

$$\Gamma^{t}_{rt} = \Gamma^{t}_{tr} = \partial_r \alpha \qquad \Gamma^{r}_{\mu\nu} = \begin{pmatrix} \partial_r \alpha e^{2\alpha - 2\beta} & 0 & 0 \\ 0 & \partial_r \beta & 0 \\ 0 & 0 & -re^{-2\beta} \end{pmatrix}$$

$$\Gamma^{\Omega}_{r\Omega} = \Gamma^{\Omega}_{\Omega r} = \frac{1}{r}$$

With the usual definition $\Gamma^{\rho}_{\mu\nu} = \frac{1}{2}g^{\rho\sigma}\left(g_{\sigma\mu;\nu} + g_{\sigma\nu;\mu} - g_{\mu\nu;\sigma}\right)$. We may now compute the Riemann tensors and then the Ricci tensors, which are $R^{\rho}_{\mu\rho\nu} = R_{\mu\nu}$. So the Riemann tensors are

$$R^{t}_{rrt} = -R^{t}_{rtr} = \left(\partial_r^2 \alpha\right) + \left(\partial_r \alpha\right)^2 - \left(\partial_r \alpha\right)\left(\partial_r \beta\right) \qquad R^{r}_{ttr} = -R^{r}_{trt} = e^{2\alpha-2\beta}R^{t}_{rrt} = -e^{2\alpha-2\beta}R^{t}_{rtr}$$

$$R^{t}_{\Omega\Omega t} = -R^{t}_{\Omega t\Omega} = \left(\partial_r \alpha\right)\left(re^{-2\beta}\right) \qquad R^{\Omega}_{tt\Omega} = -R^{\Omega}_{t\Omega t} = -\partial_r \alpha \frac{e^{2\alpha-2\beta}}{r}$$

$$R^{r}_{\Omega r\Omega} = -R^{r}_{\Omega\Omega r} = \left(\partial_r \beta\right)\left(e^{-2\beta}r\right) \qquad R^{\Omega}_{r\Omega r} = -R^{\Omega}_{rr\Omega} = \frac{\partial_r \beta}{r}$$

And the Ricci tensors are

$$R_{rr} = R^{t}_{rtr} + R^{r}_{rrr}R^{\Omega}_{r\Omega r} = \partial_r \alpha \partial_r \beta + \frac{\partial_r \beta}{r} - \left(\partial_r^2 \alpha\right) - \left(\partial_r \alpha\right)^2$$
$$R_{tt} = R^{t}_{ttt} + R^{r}_{trt} + R^{\Omega}_{t\Omega t} = e^{2\alpha-2\beta}\left[-\partial_r \alpha \partial_r \beta + \frac{\partial_r \alpha}{r} + \left(\partial_r^2 \alpha\right) + \left(\partial_r \alpha\right)^2\right]$$
$$R_{\Omega\Omega} = R^{t}_{\Omega t\Omega} + R^{r}_{\Omega r\Omega} + R^{\Omega}_{\Omega\Omega\Omega} = re^{-2\beta}\left(\partial_r \beta - \partial_r \alpha\right)$$