Phys 514

Problem Set 4

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Contents

1 Geodesic equations

Recall the principle of least action yields a geodesic when we have that the action S is extremized such that we have negligent variation over the action $\delta S = 0$. We wish to parametrize the action such that we have λ as an affine parameter where scalars over the manifold take the value $f(\lambda) = 0$, leading to a simpler form of the geodesic equation:

$$S'[x^\mu(\lambda)] = \int g_{\mu\nu} \frac{\partial x^\mu}{\partial \lambda} \frac{\partial x^\nu}{\partial \lambda} d\lambda$$

$$\therefore \delta S' \left[x^\mu(\lambda)\right] = \int \delta \left(g_{\mu\nu} \frac{\partial x^\mu}{\partial \lambda} \frac{\partial x^\nu}{\partial \lambda} \right) d\lambda$$

So the coordinate variation is given as

$$x^\mu \to x^\mu + \delta x^\mu$$

$$g_{\mu\nu} \to g_{\mu\nu} + \left(\partial_\alpha g_{\mu\nu}\right) \partial x^\alpha$$

By our good old friend, the Chain rule, we have

$$\begin{aligned}
\delta S &= \int \delta \left(g_{\mu\nu} \frac{\partial x^\mu}{\partial \lambda} \frac{\partial x^\nu}{\partial \lambda} \right) d\lambda \\
&= \int \left(\frac{\partial x^\mu}{\partial \lambda} \frac{\partial x^\nu}{\partial \lambda} \left(\partial_\alpha g_{\mu\nu}\right) \delta x^\alpha + g_{\mu\nu} \frac{\partial(\delta x^\mu)}{\partial \lambda} \frac{\partial x^\nu}{\partial \lambda} + g_{\mu\nu} \frac{\partial x^\mu}{\partial \lambda} \frac{\partial(\delta x^\nu)}{\partial \lambda} \right) d\lambda \\
&= \int \left(\frac{\partial x^\mu}{\partial \lambda} \frac{\partial x^\nu}{\partial \lambda} \partial_\alpha g_{\mu\nu} \right) \delta x^\alpha d\lambda + \int \left(g_{\mu\nu} \frac{\partial(\delta x^\mu)}{\partial \lambda} \frac{\partial x^\nu}{\partial \lambda} + g_{\mu\nu} \frac{\partial x^\mu}{\partial \lambda} \frac{\partial(\delta x^\nu)}{\partial \lambda} \right) d\lambda
\end{aligned}$$

Hence,

$$\begin{aligned}
\int g_{\mu\nu} \frac{\partial(\delta x^\mu)}{\partial \lambda} \frac{\partial x^\nu}{\partial \lambda} d\lambda &= -\int \left(g_{\mu\nu} \frac{\partial^2 x^\mu}{\partial \lambda^2} + \frac{\partial g_{\mu\nu}}{\partial \lambda} \frac{\partial x^\nu}{\partial t} \right) \delta x^\mu d\lambda \\
&= -\int \left(g_{\mu\nu} \frac{\partial^2 x^\mu}{\partial \lambda^2} + \frac{\partial g_{\mu\nu}}{\partial x^\mu} \frac{\partial x^\mu}{\partial \lambda} \frac{\partial x^\nu}{\partial t} \right) \delta x^\mu d\lambda \\
&= -\int \left(g_{\mu\nu} \frac{\partial^2 x^\mu}{\partial \lambda^2} + \partial_\mu g_{\mu\nu} \frac{\partial x^\mu}{\partial \lambda} \frac{\partial x^\nu}{\partial t} \right) \delta x^\mu d\lambda
\end{aligned}$$

Similarly,

$$\begin{aligned}
\int g_{\mu\nu} \frac{\partial x^\mu}{\partial \lambda} \frac{\partial \delta(x^\nu)}{\partial \lambda} d\lambda &= -\int \left(g_{\mu\nu} \frac{\partial^2 x^\nu}{\partial \lambda^2} + \frac{\partial g_{\mu\nu}}{\partial \lambda} \frac{\partial x^\mu}{\partial t} \right) \delta x^\nu d\lambda \\
&= -\int \left(g_{\mu\nu} \frac{\partial^2 x^\nu}{\partial \lambda^2} + \frac{\partial g_{\mu\nu}}{\partial x^\nu} \frac{\partial x^\nu}{\partial \lambda} \frac{\partial x^\mu}{\partial t} \right) \delta x^\nu d\lambda \\
&= -\int \left(g_{\mu\nu} \frac{\partial^2 x^\nu}{\partial \lambda^2} + \partial_\nu g_{\mu\nu} \frac{\partial x^\nu}{\partial \lambda} \frac{\partial x^\mu}{\partial t} \right) \delta x^\nu d\lambda
\end{aligned}$$

We can thus change indices on the second order partial derivatives to yield

$$\begin{aligned}\delta S &= \int \left(\frac{\partial x^\mu}{\partial \lambda} \left(\partial_\alpha g_{\mu\nu}\right) - 2 g_{\mu\nu} \frac{\partial^2 x^\mu}{\partial \lambda^2} - \partial_\mu g_{\mu\nu} \frac{\partial x^\mu}{\partial \lambda} \frac{\partial x^\nu}{\partial \lambda} \right) \delta x^\alpha d\lambda = 0 \\ &= - \int \left(2 g_{\mu\nu} \frac{\partial^2 x^\mu}{\partial \lambda^2} + \left(\partial_\mu g_{\nu\alpha} + \partial_\nu g_{\mu\alpha} - \partial_\alpha g_{\mu\nu}\right) \frac{\partial x^\mu}{\partial \lambda} \frac{\partial x^\nu}{\partial \lambda} \right) x^\alpha d\lambda\end{aligned}$$

We multiply the integrand by ${}^{g_{\mu\nu}}/_{2}$ and integrate

$$\begin{aligned}&= \frac{\partial^2 x^\mu}{\partial \lambda^2} \left(\frac{1}{2} g^{\mu\nu} \left(\partial_\mu g_{\nu\alpha} + \partial_\nu g_{\mu\nu} - \partial_\alpha g_{\mu\nu}\right) \right) \frac{\partial x^\mu}{\partial \lambda} = 0 \\ &= \frac{\partial^2 x^\mu}{\partial \lambda^2} + \Gamma^\alpha_{\mu\nu} \frac{\partial x^\mu}{\partial \lambda} \frac{\partial x^\nu}{\partial \lambda} = 0\end{aligned}$$

And Bob's your uncle.

2 Christoffel symbols of two-sphere

The Christoffel symbol for this problem is

$$\Gamma^{\nu}_{\alpha\beta} = \frac{1}{2} g^{\mu\nu} \left(\frac{\partial g_{\lambda\alpha}}{\partial x^{\beta}} + \frac{\partial g_{\lambda\beta}}{\partial x^{\alpha}} - \frac{\partial g_{\alpha\beta}}{\partial x^{\nu}} \right)$$

Moreover, the line element in this metric is given by

$$ds^2 = d\theta^2 + \sin^2\theta d\phi^2 \iff \mathscr{L} = \dot{\theta}^2 + \sin^2\theta\dot{\phi}^2$$

For a geodesic, $\mathscr{L}$ will hold such that we have

$$\frac{\partial \mathscr{L}}{\partial \theta} = \frac{\partial \mathscr{L}}{\partial \lambda}\left(\frac{\partial \mathscr{L}}{\partial \dot{\theta}}\right), \qquad \frac{\partial \mathscr{L}}{\partial \phi} = \frac{\partial}{\partial \lambda}\left(\frac{\partial \mathscr{L}}{\partial \dot{\phi}}\right)$$

Thus,

$$\begin{aligned}
\frac{\partial \mathscr{L}}{\partial \theta} &= 2\sin\theta\cos\theta\dot{\phi}^2, & \frac{\partial}{\partial \lambda}\left(\frac{\partial \mathscr{L}}{\partial \dot{\theta}}\right) &= 2\ddot{\theta} \\
\frac{\partial \mathscr{L}}{\partial \phi} &= 0, & \frac{\partial}{\partial \lambda}\left(\frac{\partial \mathscr{L}}{\partial \dot{\theta}}\right) &= \frac{\partial}{\partial \lambda}\left(2\sin^2\theta\dot{\phi}\right) \\
& & &= 2\sin^2\theta\ddot{\phi} + 4\sin\theta\cos\theta\dot{\theta}\dot{\phi}
\end{aligned}$$

So we get

$$\begin{aligned}
2\sin\theta\cos\theta\dot{\phi}^2 &= 2\ddot{\theta} \\
0 &= 2\sin^2\theta\ddot{\phi} + 4\sin\theta\cos\theta\dot{\theta}\dot{\phi}
\end{aligned}$$

$$\begin{aligned}
\ddot{\theta} - \sin\theta\cos\theta\dot{\phi}^2 &= 0 \\
\ddot{\phi} + 2\cot\theta\dot{\theta}\dot{\phi} &= 0
\end{aligned}$$

So the geodesic equation is clearly

$$\frac{d^2x^{\mu}}{d\lambda^2} + \Gamma^{\mu}_{\alpha\beta}\frac{dx^{\alpha}}{d\lambda}\frac{dx^{\beta}}{d\lambda}$$

Considering our Euler-Lagrangian second forms, we have

$$\ddot{\phi} + \Gamma^{\phi}_{\phi\theta}\dot{\phi}\dot{\theta} + \Gamma^{\phi}_{\theta\phi}\dot{\theta}\dot{\phi} = 2\Gamma^{\phi}_{\theta\phi}\dot{\theta}\dot{\phi} = 0$$

Therefore,

$$\Gamma^{\theta}_{\phi\phi} = -\sin\theta\cos\theta$$
$$\Gamma^{\phi}_{\theta\phi} = \cot\theta = \Gamma^{\phi}_{\phi\theta}$$

And otherwise, the Christoffel symbols are zero. Now, assume that $\dot{\theta} = 0$ such that the system becomes

$$\sin\theta\cos\theta\dot{\phi}^2 = 0$$
$$\ddot{\phi} = 0$$

Using the first equation of the system, it is easy to see that $\phi = 0$ implies this equation, so we instead focus on $\sin\theta\cos\theta$, which for the equation to hold, has a domain $\{0, \frac{\pi}{2}, \pi\}$. However, the Sonic the Hedgehog (gotta go phast) Theorem states that the coordinate system will degenerate at the poles as we have that $A \to B \vee \theta = 0 \vee \theta = \pi$.

For the second equation, we need ϕ to be linear with respect to λ as we have that the only possible value for θ is $\theta = \frac{\pi}{2}$, considering invariable latitude.

Assuming invariant longitude, the Sonic the Hedgehog theorem says that

$$\ddot{\phi} + 2\cot\theta\dot{\phi}\dot{\theta} = 0$$

$$\therefore 0 = 0$$

$$\therefore \ddot{\theta} - 3\cos\theta\sin\theta\dot{\phi}^2 = 0 \implies th\ddot{e}ta = 0$$

So this holds linearly for $\theta(\lambda)$.

3 Christoffel symbols of new metric

Take the line element for coordinates $x^\mu = (t(\lambda), r(\lambda), \theta(\lambda), \phi(\lambda))$ to be

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

We consider the Lagrangian is given by $\mathscr{L} = g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = g_{\mu\nu}\frac{\partial x^\mu}{\partial \lambda}\frac{\partial x^\nu}{\partial \lambda}$, hence

$$\mathscr{L} = -e^{2\alpha}\dot{t}^2 + e^{2\beta}\dot{r}^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2$$

Let us consider the second Euler-Lagrangian equation

$$\frac{\partial \mathscr{L}}{\partial x^\mu} = \frac{\partial \mathscr{L}}{\partial \lambda}\left(\frac{\partial \mathscr{L}}{\partial \dot{x}^\mu}\right)$$

On the LHS, for each coordinate we get

$$\frac{\partial \mathscr{L}}{\partial t} = 0 \qquad \frac{\partial \mathscr{L}}{\partial r} = -2\dot{\alpha}e^{2\alpha}\dot{t}^2 + 2\dot{\beta}e^{2\beta}\dot{r}^2 + 2r\dot{\theta}^2 + 2r\sin^2\theta\dot{\phi}^2$$
$$\frac{\partial \mathscr{L}}{\partial \theta} = 2r^2\sin\theta\cos\theta\dot{\phi}^2 \qquad \frac{\partial \mathscr{L}}{\partial \phi} = 0$$

On the RHS,

$$\frac{\partial}{\partial \lambda}\left(\frac{\partial \mathscr{L}}{\partial \dot{t}}\right) = -4\dot{\alpha}e^{2\alpha}\dot{r}\dot{t} - 2e^{2\alpha}\ddot{t} \qquad \frac{\partial}{\partial \lambda}\left(\frac{\partial \mathscr{L}}{\partial \dot{r}}\right) = 4\dot{\beta}e^{2\beta}\dot{r}^2 + 2e^{2\beta}\ddot{r}$$
$$\frac{\partial}{\partial \lambda}\left(\frac{\partial \mathscr{L}}{\partial \dot{\theta}}\right) = 4r\dot{r}\dot{\theta} + 2r^2\ddot{\theta} \qquad \frac{\partial}{\partial \lambda}\left(\frac{\partial \mathscr{L}}{\partial \dot{\phi}}\right) = 4r\sin^2\theta\dot{r}\dot{\phi} + 4r^2\sin\theta\cos\theta\dot{\theta}\dot{\phi} + 2r^2\sin^2\theta\ddot{\theta}$$

We can then equate each RHS to their corresponding LHS such that we get a 4 equation system:

$$\begin{aligned} -4\dot{\alpha}e^{2\alpha}\dot{r}\dot{t} - 2e^{2\alpha}\ddot{t} &= 0 \\ 4\dot{\beta}e^{2\beta}\dot{r}^2 + 2e^{2\beta}\ddot{r} + 2\dot{\alpha}e^{2\alpha}\dot{t}^2 - 2\dot{\beta}e^{2\beta}\dot{r}^2 - 2r\dot{\theta}^2 - 2r\sin^2\theta\dot{\phi}^2 &= 0 \\ 4r\dot{r}\dot{\theta} + 2r^2\ddot{\theta} - 2r^2\sin\theta\cos\theta\dot{\phi}^2 &= 0 \\ 4r\sin^2\theta\dot{r}\dot{\phi} + 4r^2\sin\theta\cos\theta\dot{\theta}\dot{\phi} + 2r^2\sin^2\theta\ddot{\theta} &= 0 \end{aligned}$$

Hence,

$$
\begin{aligned}
\ddot{t} + 2\dot{\alpha}\dot{r}\dot{t} &= 0 \\
\ddot{r} + \dot{\beta}\dot{r}^2 + \dot{\alpha}e^{2\alpha-2\beta}\dot{t}^2 - re^{-2\beta}\dot{\theta}^2 - r\sin^2\theta e^{-2\beta}\dot{\phi}^2 &= 0 \\
\ddot{\theta} + 2\frac{\dot{r}}{r}\dot{\theta} - \sin\theta\cos\theta\dot{\phi}^2 &= 0 \\
\ddot{\phi} + \frac{2}{r}\dot{r}\dot{\phi} + 2\cot\theta\dot{\theta}\dot{\phi} &= 0
\end{aligned}
$$

From Problem 1, recall the geodesic equation for an affine parameter λ is

$$
\ddot{x}^\mu + \Gamma^\mu_{\nu\lambda}\dot{x}^\nu\dot{x}^\lambda = 0
$$

Therefore, the Christoffel symbol for each coordinate is

$$
x^0: \qquad \Gamma^t_{rt} = \Gamma^t_{tr} = \dot{\alpha} \implies \Gamma^t_{\mu\nu} = \begin{pmatrix} 0 & \alpha & 0 & 0 \\ \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
$$

$$
x^1: \qquad \Gamma^r_{tt} = \dot{\alpha}e^{2(\alpha-\beta}, \quad \Gamma^r_{rr} = \dot{\beta}, \quad \Gamma^r_{\theta\theta} = -re^{-2\beta}\dot{\theta}^2, \quad \Gamma^r_{\phi\phi} = -r\sin^2\theta e^{-2\beta}
$$

$$
\therefore \Gamma^\theta_{\mu\nu} = \begin{pmatrix} \dot{\alpha}e^{2(\alpha-\beta)} & 0 & 0 & 0 \\ 0 & \dot{\beta} & 0 & 0 \\ 0 & & -re^{-2\beta}\dot{\theta}^2 & 0 \\ 0 & 0 & 0 & -r\sin^2\theta e^{-2\beta} \end{pmatrix}
$$

$$
x^2: \qquad \Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = \frac{1}{r}, \quad \Gamma^\theta_{\phi\phi} = \sin\theta\cos\theta
$$

$$
\therefore \Gamma^\theta_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1/r & 0 \\ 0 & 1/r & 0 & 0 \\ 0 & 0 & 0 & -\sin\theta\cos\theta \end{pmatrix}
$$

$$
x^3: \qquad \Gamma^\phi_{r\phi} = \Gamma^\phi_{\phi r} = \frac{1}{r}, \quad \Gamma^\theta_{\theta\phi} = \Gamma^\theta_{\phi\theta} = \cot\theta
$$

$$
\therefore \Gamma^\theta_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{r} \\ 0 & 0 & 0 & \cot\theta \\ 0 & \frac{1}{r} & \cot\theta & -\sin\theta\cos\theta \end{pmatrix}
$$

4 Contractions and covariant derivatives

We have the scalar gradient $\nabla_\mu f = \partial_\mu f$ and the Kronecker delta is given by $\delta^\rho_\sigma = \frac{\partial x^\rho}{x^\sigma}$. Hence, we have that

$$\nabla_\mu = \frac{\partial}{\partial x^\mu}$$

Such that

$$\begin{aligned}
\nabla_\mu \delta^\rho_\sigma &= \frac{\delta}{\delta x^\mu} \frac{\delta x^\rho}{\delta x^\sigma} \\
&= \sum_n \frac{\partial}{\partial x^n} \frac{\partial x^n}{\partial x^n} \\
&= \sum_n \frac{\partial}{\partial x^n} \\
&= 0
\end{aligned}$$

As expected from the given Kronecker delta δ_{σ^ρ}. With this, we evidently see that the Kronecker delta will never vary over time, meaning it is a constant. Hence, it follows that $\nabla_\mu \delta^\rho_\sigma = 0$. Let us employ Christoffel symbols such that we have

$$\nabla_\mu \delta^\rho_\sigma = \partial_\mu \delta^\rho_\sigma - \Gamma^\rho_{\mu\lambda} \delta^\lambda_\sigma = -\Gamma^\lambda_{\mu\sigma} \delta^\rho_\lambda$$

Thus:

- $\delta^\lambda_\sigma \implies \lambda = \sigma$

  We have $\Gamma^\rho_{\mu\lambda} \delta^\lambda_\sigma = -\Gamma^\rho_{\mu\sigma}$

- $\delta^\rho_\lambda \implies \lambda = \rho$

  We have $\Gamma^\lambda_{\mu\sigma} \delta^\rho_\lambda = \Gamma^\rho_{\mu\sigma}$

- $\delta_\mu \delta^\rho_\sigma = 0$

From the above statements, it is clear that

$$\nabla_\mu \delta^\rho_\sigma = 0 + \Gamma^\rho_{\mu\sigma} - \Gamma^\rho_{\mu\sigma} = 0$$

Now, recall that $\delta^\rho_\sigma = \delta^\rho_\lambda \delta^\lambda_\sigma$. Let us define a contraction c^β_α where we have

$$\delta^\rho_\sigma = \delta^\rho_\lambda \delta^\lambda_\sigma = c^\beta_\alpha \delta^\rho_\alpha \delta^\beta_\sigma$$

Clearly,

$$\nabla_\mu \delta^\rho_\sigma = \nabla_\mu \left(c^\beta_\alpha (delta^\rho_\alpha \delta^\beta_\sigma) \right)$$

$$= c^\beta_\alpha \left(\nabla_\mu \delta^\rho_\alpha \delta^\beta_\sigma\right)$$
$$= c^\beta_\alpha \left(\delta^\rho_\alpha \nabla_\mu \delta^\beta_\sigma + \delta^\beta_\sigma \nabla_\mu \delta^\rho_\alpha\right)$$
$$= \nabla_\mu \delta^\rho_\sigma + \nabla_\mu \delta^\rho_\sigma$$
$$= 2\nabla_\mu \delta^\rho_\sigma$$

Hence, $\nabla_\mu \delta^\rho_\sigma = 2\nabla_\mu \delta^\rho_\sigma \implies \nabla_\mu \delta^\rho_\sigma = 0$ So we have shown that it commutes with contraction as well, as desired.