Phys 514

Problem Set 2

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Winter 2021

Contents

1 A bit of calculus

Consider $\mathbb{R}^3$ in the Cartesian coordinates $x^i = (x, y, z)$. Hmmmm... Wait a minute, something's not right. Ah, there we go. Consider $\mathbb{R}^3$ in the Cartesian coordinates $🥺^i = (🥺, 😖, 😜)$ with spherical coordinates $🥺^{i'} = (🦄, 🦖, 🔥)$ as defined[1] by

$$
\begin{aligned}
🥺 &= 🦄 \sin 🦖 \cos 🔥 \\
😖 &= 🦄 \sin 🦖 \sin 🔥 \\
😜 &= 🦄 \cos 🦖
\end{aligned}
$$

1.1 Particle's path

Consider the world line $🥺^i(🐵) = (\cos 🐵, \sin 🐵, 🐵)$. We want to express this world line in the spherical coordinates as defined above such that

$$
🥺^i = \begin{pmatrix} 🥺 \\ 😖 \\ 😜 \end{pmatrix} = \begin{pmatrix} \cos 🐵 \\ \sin 🐵 \\ 🐵 \end{pmatrix}
$$

If we wish to find a parametrization such that $(🥺, 😖, 😜) \to (🦄, 🦖, 🔥)$ then it follows that

$$
\begin{aligned}
🦄^2 &= 🥺^2 + 😖^2 + 😜^2 \\
🦖 &= \arctan\left(\frac{😖}{🥺}\right) \\
🔥 &= \arctan\left(\frac{\sqrt{🥺^2 + 😖^2}}{😜}\right)
\end{aligned}
$$

So $🥺^2 + 😖^2 = \cos^2 🐵 + \sin^2 🐵 = 1$ implies that

$$
\begin{aligned}
🦄 &= \pm\sqrt{🐵^2 + 1} \\
🦖 &= \arctan\left(\frac{\sin 🐵}{\cos 🐵}\right) = 🐵 \\
🔥 &= \arctan\left(\frac{1}{🐵}\right) = \cot^{-1} 🐵
\end{aligned}
$$

Hence, $🥺^i(🐵) = (\sqrt{🐵^2 + 1}, 🐵, \cot^{-1} 🐵)$.

[1] This document is written in LaTeX, so clearly using emojis is true and validated.

1.2 Tangential Vector

Recall the tangential vector $T^i(\text{🐵}) = \frac{d\text{😟}^i}{d\text{🐵}}$. Thus,

$$T^i(\text{🐵}) = \frac{d}{d\text{🐵}}\begin{pmatrix}\cos\text{🐵}\\ \sin\text{🐵}\\ \text{🐵}\end{pmatrix} = \begin{pmatrix}-\sin\text{🐵}\\ \cos\text{🐵}\\ 1\end{pmatrix}$$

The Chain Rule becomes extremely useful when considering the change of coordinates 😟^i such that

$$\bar{T}^i(\text{🐵}) = \frac{d\bar{\text{😟}}^i}{d\text{🐵}} = \frac{d\bar{\text{😟}}^i}{d\text{😟}^j}\frac{d\text{😟}^j}{d\text{🕰}}$$

where 🕰 represents time. Using the result from 1.a we get

$$\bar{T}^i(\text{🐵}) = \frac{d}{d\text{🐵}}\begin{pmatrix}\sqrt{\text{🐵}^2+1}\\ \text{🐵}\\ \cot^{-1}\text{🐵}\end{pmatrix} = \begin{pmatrix}\frac{\text{🐵}}{\sqrt{\text{🐵}^2+1}}\\ 1\\ -\frac{1}{\text{🐵}^2+1}\end{pmatrix}$$

Spherical coordinates is easy in these cases. But what if we were the type of person that uses Arch Linux? Then we obviously would want to find the Cartesian coordinates:

- $$\begin{aligned}\bar{T}^1 &= \frac{\partial\left(\sqrt{\text{😟}^2+\text{😖}^2+\text{🤪}^2}\right)}{\partial\text{😟}}(-\sin\text{🐵}) + \frac{\partial\left(\sqrt{\text{😟}^2+\text{😖}^2+\text{🤪}^2}\right)}{\partial\text{😖}}(\cos\text{🐵})\\ &\quad + \frac{\partial\left(\sqrt{\text{😟}^2+\text{😖}^2+\text{🤪}^2}\right)}{\partial\text{🤪}}(1)\\ &= \frac{\text{😟}}{\text{🦄}}(-\sin\text{🐵}) + \frac{\text{😖}}{\text{🦄}}(\cos\text{🐵}) + \frac{\text{🤪}}{\text{🦄}}(1)\\ &= -\frac{\cos\text{🐵}\sin\text{🐵}}{\text{🦄}} + \frac{\cos\text{🐵}\sin\text{🐵}}{\text{🦄}} + \frac{\text{🤪}}{\text{🦄}}\\ &= \boxed{\frac{\text{🐵}}{\sqrt{\text{🐵}^2+1}}}\end{aligned}$$
- $$\begin{aligned}\bar{T}^2 &= \frac{\partial\arctan\left(\frac{\text{😖}}{\text{🤪}}\right)}{\partial\text{😟}}(-\sin\text{🐵}) + \frac{\partial\arctan\left(\frac{\text{😖}}{\text{🤪}}\right)}{\partial\text{😖}}\cos\text{🐵} + \frac{\partial\arctan\left(\frac{\text{😖}}{\text{🤪}}\right)}{\partial\text{🤪}}(1)\\ &= \frac{\text{😖}}{\text{😟}^2+\text{😖}^2}\sin\text{🐵} + \frac{\text{😟}}{\text{😟}^2+\text{😖}^2}\cos\text{🐵}\\ &= \boxed{1}\end{aligned}$$

- $$\bar{T}^3 = \frac{\partial \arctan\left(\frac{\sqrt{😟^2 + 😖^2}}{🤪}\right)}{\partial 😟}(-\sin 🐵) + \frac{\partial \arctan\left(\frac{\sqrt{😟^2 + 😖^2}}{🤪}\right)}{\partial 😖}(\cos 🐵) + \frac{\partial \arctan\left(\frac{\sqrt{😟^2 + 😖^2}}{🤪}\right)}{\partial 🤪}(1)$$
$$= -\frac{😟\, 🤪 \sin 🐵}{\sqrt{😟^2 + 😖^2}\, 🦄^2} + \frac{😖\, 🤪 \cos 🐵}{\sqrt{😟^2 + 😖^2}\, 🦄^2} - \frac{\sqrt{😟^2 + 😖^2}}{🦄^2}$$
$$= \frac{🤪}{\sqrt{😟^2 + 😖^2}\, 🦄}\left(-\cos 🐵 \sin 🐵 + \sin 🐵 \cos 🐵\right) - \frac{\sqrt{😟^2 + 😖^2}}{🦄}$$
$$= \boxed{-\frac{1}{🐵^2 + 1}}$$

Therefore:

$$T^i(🐵) = \begin{pmatrix} \cos 🐵 \\ \sin 🐵 \\ 🐵 \end{pmatrix} \qquad \bar{T}^i(🐵) = \begin{pmatrix} \frac{🐵}{\sqrt{🐵^2 + 1}} \\ 1 \\ -\frac{1}{🐵^2 + 1} \end{pmatrix}$$

2 Vector fields

2.1 Infinitesimal vector translations

Alright, no more emojis for this assignment 😟. Let $\dot{x} = \dot{x}^\mu d\mu = \frac{\partial \alpha^\mu}{\partial \lambda}\partial_\mu$ where $\partial_\mu = (\partial_1, \partial_2, \partial_3)$. Thus, it follows that

$$x^\mu \to \alpha^\mu(x^\mu, \lambda)$$

Now consider the transformation

$$T = I + \lambda \dot{x}$$

Then we have

$$\begin{aligned}
\alpha &= Tx \\
&= (I + \lambda\dot{x})x \\
&= \left(I + \lambda \frac{\partial \alpha^\mu}{\partial \lambda \frac{\partial}{\partial x^\mu}}\right) x \\
&= x + \lambda \frac{\partial \alpha^\mu}{\partial \lambda \frac{\partial x}{\partial x^\mu}} \\
&= x + \lambda \frac{\partial \alpha^\mu}{\partial \lambda} \\
&= x + \lambda \dot{x}^\mu
\end{aligned}$$

As expected, we have $\alpha^\mu = x^\mu + \lambda \dot{x}^\mu$. We can now change coordinates such that

$$\alpha^i = x^i + \lambda \frac{\partial \alpha^i}{\partial \lambda} \frac{\partial x^i}{\partial x^\mu}$$

where we have $\partial_i = \frac{\partial}{\partial x^i}$ for $i = 1, 2, 3$. Thus:

$$\alpha^i = x^i + \lambda \frac{\partial \alpha^i}{\partial \lambda}(x^1, x^2, x^3)$$

where $i \neq 1 \vee 2 \vee 3 \implies x^1 \vee x^2 \vee x^3 = 0$, and $i = 1 \vee 2 \vee 3 \implies x^1 \vee x^2 \vee x^3 = x^i$. So for $i = 1$ we have $\frac{\partial x^1}{\partial x^1} = (x^1, 0, 0)$, for $i = 2$ we have $\frac{\partial x^i}{\partial x^i} = (x^1, 0, 0)$, and for $i = 3$ we have $(0, 0, x^3)$. Hence, considering x^1:

$$\begin{aligned}
\alpha^1 &= x^1 + \lambda \frac{\partial \alpha^i}{\partial \lambda}(x^1, 0, 0) \\
&= x^1 \left(1 + \lambda \frac{\partial \alpha^1}{\partial \lambda}\right)
\end{aligned}$$

As $\lambda \to 0$, it is clear that α^1 represents an infinitesimal displacement of x^1 on the x^1 axis. It follows similarly for $i = 2$ and $i = 3$.

2.2 Infinitesimal vector rotations

Given

$$\alpha^{\mu} = \begin{pmatrix} v \\ w \\ z \end{pmatrix} = \begin{pmatrix} x^2\partial_3 - x^3\partial_2 \\ x^3\partial_1 - x^1\partial_3 \\ x^1\partial_2 - x^2\partial_1 \end{pmatrix}$$

we have the following Jacobians for each vector respectively:

$$J_v = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad J_w = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \qquad J_z = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

for $x^{\mu'} = (I + \lambda J)x^{\mu}$. Therefore, the rotations with respect to each vector will be given by

$$T_{1,\mu} = x^{\mu} + \lambda v = \begin{pmatrix} x^1 \\ x^2 - \lambda x^3 \\ x^3 + \lambda x^2 \end{pmatrix}, \quad T_{2,\mu} = x^{\mu} + \lambda w = \begin{pmatrix} x^1 + \lambda x^3 \\ x^2 \\ x^3 - \lambda x^1 \end{pmatrix}, \quad T_{3,\mu} = x^{\mu} + \lambda v = \begin{pmatrix} x^1 - \lambda x^2 \\ x^2 + \lambda x^1 \\ x^3 \end{pmatrix}$$

For the limit of $\lambda \to 0$ as T^{ν}_{μ} forms an infinitesimal rotation about x^{ν}, as desired.

2.3 Radial coordinate r

It is clear that

$$r^2 = (x^1)^2 + (x^2)^2 + (x^3)^2$$

So the dot products are given by:

- $$\begin{aligned} vr &= x^2 \frac{\partial}{\partial x^3}\sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2} - x^3 \frac{\partial}{\partial x^2}\sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2} \\ &= \frac{x^2 x^3}{r} - \frac{x^3 x^2}{r} = 0 \end{aligned}$$

- $$\begin{aligned} wr &= x^3 \frac{\partial}{\partial x^1}\sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2} - x^1 \frac{\partial}{\partial x^3}\sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2} \\ &= \frac{x^3 x^1}{r} - \frac{x^1 x^3}{r} = 0 \end{aligned}$$

- $$\begin{aligned} zr &= x^1 \frac{\partial}{\partial x^2}\sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2} - x^2 \frac{\partial}{\partial x^1}\sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2} \\ &= \frac{x^1 x^2}{r} - \frac{x^2 x^1}{r} = 0 \end{aligned}$$

As we saw in the rest of Problem 2, $x^\mu \times \partial_\mu$ represents a rotation about an axis in the coordinate system. As rotations change the direction of a vector but not its magnitude, we conclude that:

$$(x^\mu \times \partial_\mu) r = \vec{0}$$

2.4 Spherical z vector components

We have $z = x^1\partial_2 - x^2\partial_1 = (-x^2, x^1, 0)$ where $\partial_\mu = \frac{\partial}{\partial x^\mu}$. We want to find spherical coordinates $x^{\mu'}$. By Chain Rule, we know that

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = \frac{\partial x^{\mu'}}{\partial x^\mu}\frac{\partial}{\partial x^{\mu'}} = J\partial_{\mu'}$$

$$\therefore \partial_{\mu'} = J^{-1}\partial_\mu = \frac{\partial x^\mu}{\partial x^{\mu'}}\partial_\mu$$

Thus, it follows that:

$$z(x^\mu) = x^1\partial_2 - x^2\partial_1$$

$$\therefore z(x^{\mu'}) = x^1(x^{\mu'})\frac{\partial x^{\mu'}}{\partial x^2}\frac{\partial}{\partial x^{\mu'}} - x^2(x^{\mu'})\frac{\partial x^{\mu'}}{\partial x^1}\frac{\partial}{\partial x^{\mu'}}$$

So the inverse Jacobian $J^{-1} = \frac{\partial x^{\mu'}}{\partial x^\mu}$ is

$$\frac{\partial x^{\mu'}}{\partial x^\mu} = \begin{pmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} & \frac{\partial r}{\partial z} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} & \frac{\partial \theta}{\partial z} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \end{pmatrix} = \begin{pmatrix} 2x & 2y & 2z \\ \frac{xz}{r\sin\theta} & \frac{yz}{r\sin\theta} & -r\sin\theta \\ -\frac{y}{(r\sin\theta)^2} & \frac{x}{(r\sin\theta)^2} & 0 \end{pmatrix}$$

for $\frac{\partial}{\partial x^{\mu'}} = \left(\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}, \frac{\partial}{\partial \phi}\right)$, $x^1(x^{\mu'}) = r\sin\theta\cos\phi$, and $x^2(x^{\mu'}) = r\sin\theta\sin\phi$. Hence,

$$\frac{\partial x^{\mu'}}{\partial x^1}\frac{\partial}{\partial x^{\mu'}} = \begin{pmatrix} 2x\frac{\partial}{\partial r} \\ \frac{xz}{r\sin\theta}\frac{\partial}{\partial\theta} \\ \frac{-y}{(r\sin\theta)^2}\frac{\partial}{\partial\phi} \end{pmatrix} \qquad \frac{\partial x^{\mu'}}{\partial x^2}\frac{\partial}{\partial x^{\mu'}} = \begin{pmatrix} 2y\frac{\partial}{\partial r} \\ \frac{yz}{r\sin\theta}\frac{\partial}{\partial\theta} \\ \frac{x}{(r\sin\theta)^2}\frac{\partial}{\partial\phi} \end{pmatrix}$$

$$\therefore z(x^{\mu'}) = x^1(x^{\mu'})\frac{\partial x^{\mu'}}{\partial x^2}\frac{\partial}{\partial x^{\mu'}} - x^2(x^{\mu'})\frac{\partial x^{\mu'}}{\partial x^1}\frac{\partial}{\partial x^{\mu'}} = r\sin\theta\cos\phi\begin{pmatrix} 2y\frac{\partial}{\partial r} \\ \frac{yz}{r\sin\theta}\frac{\partial}{\partial\theta} \\ \frac{x}{(r\sin\theta)^2}\frac{\partial}{\partial\phi} \end{pmatrix} - r\sin\theta\sin\phi\begin{pmatrix} 2y\frac{\partial}{\partial r} \\ \frac{yz}{r\sin\theta}\frac{\partial}{\partial\theta} \\ \frac{x}{(r\sin\theta)^2}\frac{\partial}{\partial\phi} \end{pmatrix}$$

Now, notice that $(x^1)^2 + (x^2)^2 = r^2\sin\theta$ and

$$\begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} r\sin\theta\cos\phi \\ r\sin\theta\sin\phi \\ r\cos\theta \end{pmatrix}$$

So we can now finally compute each component of the z vector in spherical coordinates:

$$\begin{aligned}
x^{1'} &= & 2xy\partial r - 2yx\partial r & &= 0 \\
x^{2'} &= & \frac{xyz}{\sqrt{x^2+y^2}}\partial\theta - \frac{yxz}{\sqrt{x^2+y^2}}\partial\theta & &= 0 \\
x^{3'} &= & \frac{x^2}{x^2+y^2}\partial\phi + \frac{y^2}{x^2+y^2}\partial\phi & &= \partial\phi
\end{aligned}$$

$$\therefore \boxed{z = \begin{pmatrix} 0 \\ 0 \\ \partial\phi \end{pmatrix}}$$

2.5 Lorentz boost

From the Carroll book, we have the following definition of a Lorentz boost over Minkowski spacetime:

$$\Lambda^{\mu'}_{\nu} = \begin{pmatrix} \cosh\phi & -\sinh\phi & 0 & 0 \\ -\sinh\phi & \cosh\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

such that $x' = \Lambda x$. Therefore, we define $\mathcal{M}^2$ spacetime (t, x) such that

$$\Lambda = \begin{pmatrix} \cosh\lambda & -\sinh\lambda \\ -\sinh\lambda & \cosh\lambda \end{pmatrix}$$

where λ is the parametrization of the boost over a worldline in the x direction. Hence:

$$\begin{aligned}
\begin{pmatrix} t' \\ x' \end{pmatrix} &= \begin{pmatrix} \cosh\lambda & -\sinh\lambda \\ -\sinh\lambda & \cosh\lambda \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix} \\
&= \begin{pmatrix} t \\ x \end{pmatrix} \cosh\lambda - \begin{pmatrix} x \\ t \end{pmatrix} \sinh\lambda
\end{aligned}$$

Now assume that λ is infinitesimally small such that $\cosh\lambda \to 1$ and $\sinh\lambda \to 2$. Then, it follows that

$$\begin{pmatrix} t' \\ x' \end{pmatrix} = \begin{pmatrix} t \\ x \end{pmatrix} - \lambda \begin{pmatrix} x \\ t \end{pmatrix}$$

$$\therefore x' = x - \lambda v$$

The velocity v is given by

$$v = \begin{pmatrix} x \\ t \end{pmatrix} = x\partial_t + t\partial_x$$

where for infinitesimally small λ we have $\partial_t \to (1,0)$ and $\partial_x \to (0,1)$. Therefore:

$$x\partial_t \to (x,0)$$
$$t\partial_x \to (0,t)$$

So

$$\boxed{v = x\partial_t + t\partial_x = \begin{pmatrix} x \\ t \end{pmatrix}}$$

is the vector yielding a boost on the x direction, with a Lorentzian matrix given by:

$$\Lambda = \begin{pmatrix} 1 & -\lambda \\ -\lambda & 1 \end{pmatrix}$$

3 Lie algebra

3.1 Vector transform

We have the commutator:

$$[\text{😟}, \text{😖}]^{\text{🦄}} = \text{😟}^{\text{😜}} \partial_{\text{😜}} \text{😖}^{\text{🦄}} - \text{😖}^{\text{😜}} \partial_{\text{😜}} \text{😟}^{\text{🦄}}$$

SIKE, just kidding. Imagine doing GR with emojis tho. Einstein would be so proud of us.

$$[v, w]^{\nu} = v^{\mu}\partial_{\mu} w^{\nu} - w^{\mu}\partial_{\mu} v^{\nu}$$

Now we let λ be a smooth function on the manifold on which the vectors are defined such that

$$\begin{aligned}
[v, w]^{\nu}(\lambda) &= [v^{\mu}\partial_{\mu}, w^{\nu}\partial_{\nu}]\,(\lambda) \\
&= v^{\mu}\frac{\partial}{\partial x^{\mu}}\left(w^{\nu}\frac{\partial \lambda}{\partial x^{\nu}}\right) - w^{\mu}\frac{\partial}{\partial x^{\mu}}\left(v^{\nu}\frac{\partial \lambda}{\partial x^{\nu}}\right)
\end{aligned}$$

By the product rule, we get:

$$\begin{aligned}
&= v^{\mu}\frac{\partial w^{\nu}}{\partial x^{\mu}}\frac{\partial \lambda}{\partial x^{\nu}} + v^{\nu}w^{\nu}\frac{\partial^2 \lambda}{\partial x^{\mu}\partial x^{\nu}} - w^{\mu}\frac{\partial v^{\nu}}{\partial x^{\mu}}\frac{\partial \lambda}{\partial x^{\nu}} - w^{\mu}v^{\mu}\frac{\partial^2 \lambda}{\partial x^{\nu}\partial x^{\mu}} \\
&= v^{\mu}\frac{\partial w^{\nu}}{\partial x^{\mu}}\frac{\partial \lambda}{\partial x^{\nu}} - w^{\mu}\frac{\partial v^{\nu}}{\partial x^{\mu}}\frac{\partial \lambda}{\partial x^{\nu}} \\
&= \left(v^{\mu}\frac{\partial w^{\nu}}{\partial x^{\mu}}\frac{\partial}{\partial x^{\nu}} - w^{\mu}\frac{\partial v^{\nu}}{\partial x^{\mu}}\frac{\partial}{\partial x^{\nu}}\right)(\lambda) \\
&= \left(v^{\mu}\partial_{\mu} w^{\nu} - w^{\mu}\partial_{\mu} v^{\nu}\right)(\lambda)
\end{aligned}$$

So therefore,

$$[v, w]^{\nu} = [v^{\mu}\partial_{\mu}, w^{\nu}\partial_{\nu}] = v^{\mu}\frac{\partial w^{\nu}}{\partial x^{\mu}}\frac{\partial}{\partial x^{\nu}} - w^{\mu}\frac{\partial v^{\nu}}{\partial x^{\mu}}\frac{\partial}{\partial x^{\nu}}$$

As needed.

3.2 Commutator computation

We have the commutator:

$$[v,w]^\mu = v^\mu \partial_\mu w^\nu - \omega^\mu \partial_\mu v^\nu = \sum_i \sum_j \left(v^j \partial_j w^i - w^j \partial_j v^i\right) \partial_i$$

Therefore:

$$v = \left[0, -x^3, x^2\right]^T, \qquad w = \left[x^3, 0, -x^1\right]^T, \qquad z = \left[-x^2, x^1, 0\right]^T$$

which leads us to the following Jacobians:

$$J_v = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \qquad J_w = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \qquad J_z = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Now recall the definition of a commutator via Jacobian:

$$[X,Y] := J_y X - J_x Y$$

Therefore:

$$\begin{aligned}
[v,w] &= J_w v - J_v w \\
&= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ -x^3 \\ x^2 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x^3 \\ 0 \\ -x^1 \end{bmatrix} \\
&= \begin{bmatrix} x^2 \\ -x^1 \\ 0 \end{bmatrix} \\
&= x^2 \partial_1 - x^1 \partial_2 \\
&= -z
\end{aligned}$$

From the property $[X,Y] = -[Y,X]$, it follows that $[w,v] = z$. Similarly,

$$\begin{aligned}
[v,z] &= J_z v - J_v z \\
&= \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ -x^3 \\ x^2 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} -x^2 \\ x^1 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} x^3 \\ 0 \\ -x^1 \end{bmatrix} \\
&= x^3 \partial_1 - x^1 \partial_3 \\
&= w
\end{aligned}$$

Again, $[z, v] = -w$. Finally,

$$\begin{aligned}
[w, z] &= J_z w - J_w z \\
&= \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x^3 \\ 0 \\ -x^1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x^3 \\ 0 \\ -x^1 \end{bmatrix} \\
&= \begin{bmatrix} -x^2 \\ x^1 \\ 0 \end{bmatrix} \\
&= x^3 \partial_2 - x^2 \partial_3 \\
&= -v
\end{aligned}$$

So in conclusion

$$\boxed{\begin{aligned} [v, w] = -z = -[w, v] \\ [v, z] = w = -[z, v] \\ [w, z] = -v = -[z, w] \end{aligned}}$$

😟 UwU A tensor transforms like a tensor UwU[2], 😟

[2] To use this awesome Emoji package to make physics easier and clearer, as I have clearly done, simply define a command via \newfontfamily\SegoeEmoji{Segoe UI Emoji}. Proceed to use '\DeclareMathOperator{\<keyword>}{SegoeEmoji <emoji>}' to declare your variable of choice! Use in mathmode and encapsulate in brackets for sub/superscripts to avoid errors.