Phys 514

Problem Set 3

April Sada Solomon - 260708051

Winter 2021

Contents

1 Tensors on Manifolds 1

1.1 Tensor weight . . . 1

1.2 Vectors and tensors . . . 2

1.3 Computation . . . 2

2 General Questions 3

2.1 Dummy (**t h i c c**) indices . . . 3

2.2 Independent components . . . 4

2.3 Sum of tensors . . . 4

2.4 Independent components pt. 2 . . . 4

2.5 Independent components (bonus) . . . 4

3 Higher-dimensional Space 4

3.1 Transformations . . . 4

3.2 Transformations 2 . . . 4

3.3 Metric on Cartesian Coordinates . . . 4

3.4 Equivalent metrics . . . 4

4 Minkowski metric problem 4

1 Tensors on Manifolds

Consider a D-dimensional manifold. In any coordinate system x^μ, the Kronecker delta symbol δ^μ_ν is defined to be

$$\delta^\mu_\nu = \begin{cases} 1 & \text{if} \quad \mu = \nu \\ 0 & \text{if} \quad \mu \neq \nu \end{cases}$$

1.1 Tensor weight

We need to show that δ^μ_ν is a tensor of weight $(1,1)$. We apply the tensor transformation laws such that $\delta^{\mu'}_{\nu'}$ is a Kronecker delta over a new set of coordinates $x^{\mu'}$ given by

$$\delta^{\mu'}_{\nu'} = \delta^\mu_\nu \frac{\partial x^{\mu'}}{\partial x^\mu} \frac{\partial x^\nu}{\partial x^{\nu'}}$$

We notice that we only consider the terms where $\mu = \nu$, as they will be the only non-zero terms. We can thus define $\mu = \nu = \alpha$ such that:

$$\begin{aligned}
\delta^{\mu'}_{\nu'} &= \sum_\alpha \frac{\partial x^{\mu'}}{\partial x^\alpha} \frac{\partial x^\alpha}{\partial x^{\nu'}} \\
&= \sum_\alpha \frac{\partial x^\alpha}{\partial x^{\nu'}} \partial \frac{\partial x^{\mu'}}{\partial x^\alpha} \\
&= \frac{\partial x^{\mu'}}{\partial x^{\nu'}} \\
&= \begin{cases} 1 & \text{if } \mu' = \nu' \\ 0 & \text{if } \mu' \neq \nu' \end{cases} \\
&= \delta^\mu_\nu
\end{aligned}$$

Now, considering $\hat{e_\mu}$ as the basis vectors for coordinates x^μ, and $\hat{e_{\nu'}}$ as the basis vectors for coordinates $x^{\nu'}$, it follows that for any transformation of coordinates $\Lambda^{\nu'}_\mu$ we have

$$\hat{e}_\mu = \Lambda^{\nu'}_\mu \hat{e}_{\nu'}$$

And to isolate the basis $\hat{e}_{\nu'}$, we must apply the inverse of $\Lambda^{\nu'}_\mu$, namely $\Lambda^\rho_{\sigma'}$ such that

$$\Lambda^\mu_{\nu'} \Lambda^{\nu'}_\rho = \delta^\mu_\rho, \qquad \Lambda^{\sigma'}_\lambda \Lambda^\lambda_{\tau'} = \delta^{\sigma'}_{\tau'}$$

Which clearly implies that

$$\hat{e}_{\nu'} = \Lambda^\mu_{\nu'} \hat{e}_\mu$$

So δ^μ_ν is a tensor of weight $(1,1)$.

1.2 Vectors and tensors

We need to show that $\delta^\mu_\nu v^\nu = v^\mu$ and $\delta^\mu_\nu w_\mu = w_\nu$ where v^ν and w_μ are a vector and one form respectively. As seen in the second part of my answer to 1.1, δ^μ_ν represents a transformation multiplied by its own inverse. As such, we can clearly see that

$$\delta^\mu_\nu v^\nu = \Lambda^\mu_{\nu'}\Lambda^{\nu'}_\nu v^\nu = \Lambda^\mu_{\nu'}(\Lambda^{\nu'}_\nu v^\nu) = \Lambda^\mu_{\nu'} v^{\nu'} = v^\mu$$

$$\delta^\mu_\nu w_\mu = \Lambda^\mu_{\nu'}\Lambda^{\nu'}_\nu w_\mu = \Lambda^{\nu'}_\nu(\Lambda^\mu_{\nu'} w_\mu) = \Lambda^{\nu'}_\nu w_{\nu'} = w_\nu$$

As required.

1.3 Computation

First we have δ^μ_μ, which seems very sneaky. A quick guess would assume that

$$\delta^\mu_\mu = \begin{cases} 1 & \text{if} \quad \mu = \mu \\ 0 & \text{if} \quad \mu \neq \mu \end{cases}$$

So written in transformation matrices:

$$\delta^\mu_\mu = \Lambda^\mu_{\nu'}\Lambda^{\nu'}_\mu = I$$

This can be denoted as an identity matrix, as it will return the same vector or one-form under the same coordinates when multiplying either.

Next we have $\delta^\mu_\nu\delta^\mu_\rho$. We again write this down in transformations:

$$\delta^\mu_\nu\delta^\mu_\rho = \Lambda^\mu_{\nu'}\Lambda^{\nu'}_\nu\Lambda^\mu_{\sigma'}\Lambda^{\sigma'}_\rho$$

Which means that

$$\delta^\mu_\nu\delta^\mu_\rho = \begin{cases} 1 & \text{if} \quad \mu = \nu = \rho \\ 0 & \quad \text{otherwise} \end{cases}$$

2 General Questions

2.1 Dummy (t h i c c) indices

We must show by manipulating dummy indices that

$$Z_{\mu\nu}v^{\mu}v^{\nu} = \frac{1}{2}\left(Z_{\mu\nu} + Z_{\nu\mu}\right)v^{\mu}v^{\nu}$$

where we assume $Z_{\mu\nu}$, $Z_{\nu\mu}$ are both rank (0,2) tensors that are also symmetric matrices. Hence, we know that $Z_{\mu\nu} = Z^T_{\nu\mu}$ (see equation 1.69 in the book for an example, where the tensor is antisymmetric instead of symmetric), so clearly:

$$\begin{aligned}
\frac{1}{2}(Z_{\mu\nu} + Z_{\nu\mu}) &= \frac{1}{2}\left(\begin{pmatrix} z_{1,1} & \cdots & z_{1,\nu} \\ \vdots & \ddots & \vdots \\ z_{\mu,1} & \cdots & z_{\mu,\nu} \end{pmatrix} + \begin{pmatrix} z_{1,1} & \cdots & z_{1,\mu} \\ \vdots & \ddots & \vdots \\ z_{\nu,1} & \cdots & z_{\nu,\mu} \end{pmatrix}\right) \\
&= \frac{1}{2}\begin{pmatrix} 2\cdot z_{1,1} & \cdots & z_{1,\nu} + z_{1,\mu} \\ \vdots & \ddots & \vdots \\ z_{\mu,1} + z_{\nu,1} & \cdots & \underbrace{2z_{\mu,\nu}}_{=2z_{\nu,\mu}} \end{pmatrix} \\
&= \begin{pmatrix} z_{1,1} & \cdots & \frac{z_{1,\nu}+z_{1,\mu}}{2} \\ \vdots & \ddots & \vdots \\ \frac{z_{\mu,1}+z_{\nu,1}}{2} & \cdots & z_{\mu,\nu} \end{pmatrix} = \begin{pmatrix} z_{1,1} & \cdots & \frac{z_{1,\mu}+z_{1,\nu}}{2} \\ \vdots & \ddots & \vdots \\ \frac{z_{\nu,1}+z_{\mu,1}}{2} & \cdots & z_{\mu,\nu} \end{pmatrix}
\end{aligned}$$

Recall that both tensors are symmetric

$$\begin{aligned}
&= \begin{pmatrix} z_{1,1} & \cdots & z_{1,\nu} \\ \vdots & \ddots & \vdots \\ z_{\mu,1} & \cdots & z_{\mu,\nu} \end{pmatrix} \\
&= Z_{\mu\nu}
\end{aligned}$$

Therefore,

$$Z_{\mu\nu}v^{\mu}v^{\nu} = \frac{1}{2}\left(Z_{\mu\nu} + Z_{\nu\mu}\right)v^{\mu}v^{\nu}$$

and the dummy index is thicc indeed.

2.2 Independent components

For any (0,2) tensor, we have 2 covariant indices and no contravariant indices. Thus, we can get the number of independent components for any such tensor in D dimensions simply by squaring the dimension. For a 3D (0,2) tensor, we have 3^2 =9 independent components, for a 4D we have 4^2 =16, etc.

Now if we consider symmetric tensors, then we know that $T_{ij} = T_{ji}$, so we may only consider $\sum_{k=1}^{D} k$ independent elements. In 3D, this yields 1+2+3 = 6, in 4D we have 1+2+3+4 = 10, etc.

For an antisymmetric tensor where $T_{ij} = -T_{ji}$ for $i \neq j$ and $T_{ij} = 0$ for $i = j$, we consider the same amount of independent components as the symmetric case but without the diagonal, so we have $\sum_{k=1}^{D-1} k$ independent components. In 3D we have 1+2 =3 independent component, in 4D we have 1+2+3 = 6, and so on.

2.3 Sum of tensors

2.4 Independent components pt. 2

2.5 Independent components (bonus)

3 Higher-dimensional Space

3.1 Transformations

3.2 Transformations 2

3.3 Metric on Cartesian Coordinates

3.4 Equivalent metrics

4 Minkowski metric problem