Phys 514

Problem Set 2

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Contents

1 A bit of calculus

Consider $\mathbb{R}^3$ in the Cartesian coordinates $x^i = (x, y, z)$. Hmmmm... Wait a minute, something's not right. Ah, there we go. Consider $\mathbb{R}^3$ in the Cartesian coordinates $😟^i = (😟, 😖, 😜)$ with spherical coordinates $😟^{i'} = (🦄, 🦖, 🔥)$ as defined by

$$
\begin{aligned}
😟 &= 🦄 \sin 🦖 \cos 🔥 \\
😖 &= 🦄 \sin 🦖 \sin 🔥 \\
😜 &= 🦄 \cos 🦖
\end{aligned}
$$

1.1 Particle's path

Consider the world line $😟^i(🐵) = (\cos 🐵, \sin 🐵, 🐵)$. We want to express this world line in the spherical coordinates as defined above such that

$$
😟^i = \begin{pmatrix} 😟 \\ 😖 \\ 😜 \end{pmatrix} = \begin{pmatrix} \cos 🐵 \\ \sin 🐵 \\ 🐵 \end{pmatrix}
$$

If we wish to find a parametrization such that $(😟, 😖, 😜) \to (🦄, 🦖, 🔥)$ then it follows that

$$
\begin{aligned}
🦄^2 &= 😟^2 + 😖^2 + 😜^2 \\
🦖 &= \arctan\left(\frac{😖}{😟}\right) \\
🔥 &= \arctan\left(\frac{\sqrt{😟^2 + 😖^2}}{😜}\right)
\end{aligned}
$$

So $😟^2 + 😖^2 = \cos^2 🐵 + \sin^2 🐵 = 1$ implies that

$$
\begin{aligned}
🦄 &= \pm\sqrt{🐵^2 + 1} \\
🦖 &= \arctan\left(\frac{\sin 🐵}{\cos 🐵}\right) = 🐵 \\
🔥 &= \arctan\left(\frac{1}{🐵}\right) = \cot^{-1} 🐵
\end{aligned}
$$

Hence, $😟^i(🐵) = (\sqrt{🐵^2 + 1}, 🐵, \cot^{-1} 🐵)$.

1.2 Tangential Vector

Recall the tangential vector $T^i(🐵) = \frac{d😟^i}{d🐵}$. Thus,

$$T^i(🐵) = \frac{d}{d🐵}\begin{pmatrix} \cos 🐵 \\ \sin 🐵 \\ 🐵 \end{pmatrix} = \begin{pmatrix} -\sin 🐵 \\ \cos 🐵 \\ 1 \end{pmatrix}$$

The Chain Rule becomes extremely useful when considering the change of coordinates $😟^i$ such that

$$\bar{T}^i(🐵) = \frac{d\bar{😟}^i}{d🐵} = \frac{d\bar{😟}^i}{d😟^j}\frac{d😟^j}{d🕰}$$

where 🕰 represents time. Using the result from 1.a we get

$$\bar{T}^i(🐵) = \frac{d}{d🐵}\begin{pmatrix} \sqrt{🐵^2+1} \\ 🐵 \\ \cot^{-1} 🐵 \end{pmatrix} = \begin{pmatrix} \frac{🐵}{\sqrt{🐵^2+1}} \\ 1 \\ -\frac{1}{🐵^2+1} \end{pmatrix}$$

Spherical coordinates is easy in these cases. But what if we were the type of person that uses Arch Linux? Then we obviously would want to find the Cartesian coordinates:

- $$\begin{aligned} \bar{T}^1 &= \frac{\partial\left(\sqrt{😟^2+😖^2+😜^2}\right)}{\partial 😟}\left(-\sin 🐵\right) + \frac{\partial\left(\sqrt{😟^2+😖^2+😜^2}\right)}{\partial 😖}\left(\cos 🐵\right) \\ &\quad + \frac{\partial\left(\sqrt{😟^2+😖^2+😜^2}\right)}{\partial 😜}(1) \\ &= \frac{😟}{🦄}\left(-\sin 🐵\right) + \frac{😖}{🦄}\left(\cos 🐵\right) + \frac{😜}{🦄}(1) \\ &= -\frac{\cos 🐵 \sin 🐵}{🦄} + \frac{\cos 🐵 \sin 🐵}{🦄} + \frac{😜}{🦄} \\ &= \boxed{\frac{🐵}{\sqrt{🐵^2+1}}} \end{aligned}$$
- $$\begin{aligned} \bar{T}^2 &= \frac{\partial \arctan\left(\frac{😖}{😜}\right)}{\partial 😟}(-\sin 🐵) + \frac{\partial \arctan\left(\frac{😖}{😜}\right)}{\partial 😖}\cos 🐵 + \frac{\partial \arctan\left(\frac{😖}{😜}\right)}{\partial 😜}(1) \\ &= \frac{😖}{😟^2+😖^2}\sin 🐵 + \frac{😟}{😟^2+😖^2}\cos 🐵 \\ &= \boxed{1} \end{aligned}$$

- $\bar{T}^3 = \dfrac{\partial \arctan\left(\frac{\sqrt{😟^2+😖^2}}{😜}\right)}{\partial 😟}(-\sin 🐵) + \dfrac{\partial \arctan\left(\frac{\sqrt{😟^2+😖^2}}{😜}\right)}{\partial 😖}(\cos 🐵)$
$+ \dfrac{\partial \arctan\left(\frac{\sqrt{😟^2+😖^2}}{😜}\right)}{\partial 😜}(1)$
$= -\dfrac{😟\, 😜 \sin 🐵}{\sqrt{😟^2+😖^2}\,🦄^2} + \dfrac{😖\, 😜 \cos 🐵}{\sqrt{😟^2+😖^2}\,🦄^2} - \dfrac{\sqrt{😟^2+😖^2}}{🦄^2}$
$= \dfrac{😜}{\sqrt{😟^2+😖^2}\,🦄}\left(-\cos 🐵 \sin 🐵 + \sin 🐵 \cos 🐵\right) - \dfrac{\sqrt{😟^2+😖^2}}{🦄}$
$= \boxed{-\dfrac{1}{🐵^2+1}}$

Therefore:

$$T^i(🐵) = \begin{pmatrix} \cos 🐵 \\ \sin 🐵 \\ 🐵 \end{pmatrix} \qquad \bar{T}^i(🐵) = \begin{pmatrix} \frac{🐵}{\sqrt{🐵^2+1}} \\ 1 \\ -\frac{1}{🐵^2+1} \end{pmatrix}$$

2 Vector fields

2.1 Infinitesimal vector translations

Alright, no more emojis for this assignment 😟. Let $\dot{x} = \dot{x}^\mu d\mu = \frac{\partial \alpha^\mu}{\partial \lambda} \partial_\mu$ where $\partial_\mu = (\partial_1, \partial_2, \partial_3)$. Thus, it follows that

$$x^\mu \to \alpha^\mu(x^\mu, \lambda)$$

Now consider the transformation

$$T = I + \lambda \dot{x}$$

Then we have

$$\begin{aligned}
\alpha &= Tx \\
&= (I + \lambda \dot{x})x \\
&= \left(I + \lambda \frac{\partial \alpha^\mu}{\partial \lambda \frac{\partial}{\partial x^\mu}} \right) x \\
&= x + \lambda \frac{\partial \alpha^\mu}{\partial \lambda \frac{\partial x}{\partial x^\mu}} \\
&= x + \lambda \frac{\partial \alpha^\mu}{\partial \lambda} \\
&= x + \lambda \dot{x}^\mu
\end{aligned}$$

As expected, we have $\alpha^\mu = x^\mu + \lambda \dot{x}^\mu$. We can now change coordinates such that

$$\alpha^i = x^i + \lambda \frac{\partial \alpha^i}{\partial \lambda} \frac{\partial x^i}{\partial x^\mu}$$

where we have $\partial_i = \frac{\partial}{\partial x^i}$ for $i = 1, 2, 3$. Thus:

$$\alpha^i = x^i + \lambda \frac{\partial \alpha^i}{\partial \lambda} (x^1, x^2, x^3)$$

where $i \neq 1 \vee 2 \vee 3 \implies x^1 \vee x^2 \vee x^3 = 0$, and $i = 1 \vee 2 \vee 3 \implies x^1 \vee x^2 \vee x^3 = x^i$. So for $i = 1$ we have $\frac{\partial x^1}{\partial x^1} = (x^1, 0, 0)$, for $i = 2$ we have $\frac{\partial x^i}{\partial x^i} = (x^1, 0, 0)$, and for $i = 3$ we have $(0, 0, x^3)$. Hence, considering x^1:

$$\begin{aligned}
\alpha^1 &= x^1 + \lambda \frac{\partial \alpha^i}{\partial \lambda} (x^1, 0, 0) \\
&= x^1 \left(1 + \lambda \frac{\partial \alpha^1}{\partial \lambda} \right)
\end{aligned}$$

As $\lambda \to 0$, it is clear that α^1 represents an infinitesimal displacement of x^1 on the x^1 axis. It follows similarly for $i = 2$ and $i = 3$.

2.2 Infinitesimal vector rotations

Given

$$\alpha^{\mu} = \begin{pmatrix} v \\ w \\ z \end{pmatrix} = \begin{pmatrix} 0 & x^2\partial_3 & -x^3\partial_2 \\ x^3\partial_1 & 0 & -x^1\partial_3 \\ x^1\partial_2 & -x^2\partial_1 & 0 \end{pmatrix}$$

we have the following rotation matrices

$$R_{x^1}(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \approx \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -\theta \\ 0 & \theta & 1 \end{pmatrix}$$

$$R_{x^2}(\theta) = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \approx \begin{pmatrix} 1 & 0 & \theta \\ 0 & 1 & 0 \\ -\theta & 0 & 1 \end{pmatrix}$$

$$R_{x^1}(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \approx \begin{pmatrix} 1 & -\theta & 0 \\ \theta & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

for $\theta \to 0$ angle of rotation. The image of $\Omega = \left(\begin{smallmatrix} v \\ w \\ z \end{smallmatrix}\right)$ is thus given by

$$\text{Image}(\Omega) = \begin{pmatrix} 0 & -r^3 & r^2 \\ r^3 & 0 & -r^1 \\ -r^2 & r^1 & 0 \end{pmatrix}$$

Therefore,

$$\begin{aligned} x^{\mu'} &= \text{Image}(\Omega)x^{\mu} \\ &= \begin{pmatrix} 0 & -r^3 & r^2 \\ r^3 & 0 & -r^1 \\ -r^2 & r^1 & 0 \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -r^3x^2 & r^2x^3 \\ r^3x^1 & 0 & -r^1x^3 \\ -r^2x^1 & r^1x^2 & 0 \end{pmatrix} \end{aligned}$$

which can be rearranged to α^{μ} by replacing in $-r^i = \partial_i$, as expected.

2.3 Radial coordinate r

2.3.1 Dot products

It is clear that

$$r^2 = (x^1)^2 + (x^2)^2 + (x^3)^2$$

So the dot products are given by:

- $$\begin{aligned} vr &= x^2 \frac{\partial}{\partial x^3} \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2} - x^3 \frac{\partial}{\partial x^2} \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2} \\ &= \frac{x^2 x^3}{r} - \frac{x^3 x^2}{r} = 0 \end{aligned}$$

- $$\begin{aligned} wr &= x^3 \frac{\partial}{\partial x^1} \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2} - x^1 \frac{\partial}{\partial x^3} \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2} \\ &= \frac{x^3 x^1}{r} - \frac{x^1 x^3}{r} = 0 \end{aligned}$$

- $$\begin{aligned} zr &= x^1 \frac{\partial}{\partial x^2} \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2} - x^2 \frac{\partial}{\partial x^1} \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2} \\ &= \frac{x^1 x^2}{r} - \frac{x^2 x^1}{r} = 0 \end{aligned}$$

As we saw in the rest of Problem 2, $x^\mu \times \partial_\mu$ represents a rotation about an axis in the coordinate system. As rotations change the direction of a vector but not its magnitude, we conclude that:

$$(x^\mu \times \partial_\mu) r = \vec{0}$$