Physics 514, Problem Set 1: Special Relativity

due: *Friday, January 22 at 3pm*

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Please turn in your problem set using the Assignment link on mycourses. You are encouraged to discuss these problems with your colleagues, but you must write up your own solutions; the solutions you hand in should reflect your own work and understanding. Late problem sets will be penalized 20% if they are less than one week late and will not be accepted if they are more than one week late, unless an extension has been obtained from me or a TA before the due date.

Reading: Chapter 1 of Carroll.

Note: You may wish to wait until the class on Thursday, January 14, before attempting the last two problems.

1. The two dimensionful constants which appear in the theory of general relativity are the speed of light c and Newton's constant G. This allows us to estimate the size of various effects, even without knowing anything about the theory of general relativity.
   (a) What are the dimensions of c and G?
   (b) Use dimensional analysis to estimate the radius of a black hole of mass M.
   (c) Compute your estimate of this radius numerically for a black hole with mass equal to:
      i. the mass of a proton
      ii. the mass of the earth
      iii. the mass of the sun
      iv. 10^{10} times the mass of the sun (which is roughly the mass of the black hole at the center of M87)
   (d) M87 is about 50 million light years away. Estimate the angular size of the black hole at the center of M87, as seen from Earth. This black hole was recently imaged by the Event Horizon telescope.
   (e) Estimate the surface gravity of the black hole at the center of M87.[1] Compare this answer to the surface gravity on Earth ($g \approx 10\ m/s^2$). You may use dimensional analysis (along with whatever other approximations you think are reasonable) in order to obtain an answer.
   (f) Newtonian gravity describes the orbits of the planets in our solar system with very good accuracy, and the effects of general relativity on these orbits are quite small. Explain why general relativity will have a much greater effect on the orbit of Mercury than on the orbit of any other planet in the solar system. You may use dimensional analysis, along with whatever other approximations you think are reasonable, in order to make this argument.

2. In a particular reference frame $\mathcal{O}$, the three events A,B, and C occur in the order ABC. In another reference frame $\mathcal{O}'$ these events occur in the order CBA. Is it possible for a there to be a third reference frame $\mathcal{O}''$ where these events occur in the order CAB? Support your conclusion by drawing a space-time diagram. (*Hint*: It may help to stare at the formula for a Lorentz transformation and think about what happens when one performs a Lorentz boost in different directions.)

3. Consider Minkowski space in the usual Cartesian coordinates $x^\mu = (t, x, y, z)$. The line element is

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = -dt^2 + dx^2 + dy^2 + dz^2$$

[1] Recall that *surface gravity* is the acceleration experienced by an object which is placed on the surface of a planet/star/black hole/etc.

in these coordinates, where dx^μ is an infinitesimal displacement of the coordinate x^μ. Consider a new coordinate system $x^{\mu'}$ which differs from these Cartesian coordinates; here μ' is an index which runs from 0 to 3. The Cartesian coordinates x^μ can be written as a function of these new coordinates $x^\mu = x^\mu(x^{\mu'})$.

(a) Take a point $x^{\mu'}$ in this new coordinate system, and imagine displacing it by an infinitesimal amount to $x^{\mu'} + dx^{\mu'}$. We want to understand how the x^μ coordinates change to first order in this displacement $dx^{\mu'}$. Argue that

$$dx^\mu = \frac{\partial x^\mu}{\partial x^{\mu'}} dx^{\mu'}$$

(*Hint: Taylor expand* $x^\mu(x^{\mu'} + dx^{\mu'})$)

(b) The sixteen quantities $\frac{\partial x^\mu}{\partial x^{\mu'}}$ are referred to as the Jacobian matrix; we will require this matrix to be invertible. Show that the inverse of this matrix is $\frac{\partial x^{\mu'}}{\partial x^\mu}$. (*Hint: Use the chain rule*)

(c) Consider spherical coordinates, $x^{\mu'} = (t, r, \theta, \phi)$ which are related to the Cartesian coordinates by

$$(t, x, y, z) = (t, r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta)$$

Compute the matrix $\frac{\partial x^\mu}{\partial x^{\mu'}}$. Is this matrix invertible everywhere? Compute the displacements dx^μ in this coordinate system (i.e. write them as functions of $x^{\mu'}$ and the infinitesimal displacements $dx^{\mu'}$).

(d) Compute the line element ds^2 in this coordinate system.

4. Consider a particle moving through Minkowski space with worldline $x^\mu(\lambda)$. Here λ is a continuous parameter which labels different points on the worldline and $x^\mu = (t, x, y, z)$ denotes the usual Cartesian coordinates. We will denote $\frac{\partial}{\partial\lambda}$ by a dot. In this problem we will assume that the trajectory of the particle obeys the equation of motion $\ddot{x}^\mu = 0$.

(a) Show that this trajectory describes a particle moving at constant velocity (assuming that $\dot{t} \neq 0$).

(b) Show that this trajectory is an extremum of the action

$$S = \int ds = \int d\lambda \sqrt{\eta_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}$$

which measures the invariant interval of a worldline.

(c) Consider a new coordinate system $x^{\mu'}$ which differs from the original Cartesian coordinate system; as before, the Cartesian coordinates x^μ can be written as a function of these new coordinates $x^\mu = x^\mu(x^{\mu'})$. Show that the equation of motion can be written in these new $x^{\mu'}$ coordinates as

$$\ddot{x}^{\mu'} + \Gamma^{\mu'}_{\nu'\lambda'}\dot{x}^{\nu'}\dot{x}^{\lambda'} = 0 \tag{1}$$

for some $\Gamma^{\mu'}_{\nu'\lambda'}$ which you must compute; $\Gamma^{\mu'}_{\nu'\lambda'}$ is known as the Christoffel symbol. These extra Christoffel terms in the equation of motion can be thought of as "fictitous forces" that arise in an accelerated reference frame.

(d) Consider the case where the $x^{\mu'} = (t', x', y', z')$ describe a coordinate system which is rotating around the z axis with constant angular velocity ω. This is related to the original cartesian coordinates by

$$(t, x, y, z) = (t', x'\cos\omega t' + y'\sin\omega t', -x'\sin\omega t' + y'\cos\omega t', z')$$

Compute the line element ds^2 in this new coordinate system.

(e) Compute $\Gamma^{\mu'}_{\nu'\lambda'}$ in this coordinate system. (*Hint: Rather than using the explicit formula you derived in part (c), you may find it easier to just compute* $\ddot{x}^{\mu'}$ *for a particle moving at constant velocity. The Christoffel symbols can then be determined using formula (1).*)

(f) (*Optional*) Show that these $\Gamma^{\mu'}_{\nu'\lambda'}\dot{x}^{\nu'}\dot{x}^{\lambda'}$ terms describe the usual Centrifugal and Coriolis forces that arise in a rotating coordinate system, which you have already encountered in classical mechanics.