Phys 514

Relativity

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Course Description

Notes taken directly from the lectures given by Dr. Maloney (not proof read by him). Special Relativity, Manifolds, Spacetime Curvature, Gravitation, Schwarzchild Solution, Black Holes, Perturbations, Radiation, Introduction to Cosmology, Introduction to QFT in Spacetime.

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1 A Brief Review of Special Relativity

1.1 Introduction

General Relativity is the modern theory of classical gravity. What do we mean by "classical"? Simply, it is a nomenclature to distinguish from Quantum Mechanics, so those effects are ignored. The basic idea from Newtonian Mechanics is that gravity is a force, and as we know, most forces are described by fields.

The simplest example of a field that describes a force is the Newtonian gravitational potential $\Phi(\vec{r})$:

$$\frac{d}{dt}\frac{\partial \mathscr{L}}{\partial \dot{x}} = \frac{\partial \mathscr{L}}{\partial x} \implies \vec{F}(\vec{v}) = -\frac{\partial}{\partial \vec{r}}\Phi(\vec{r}) \tag{1.1}$$

If we are studying electromagnetism, we would study the electromagnetic fields $\vec{E}$ and $\vec{B}$. Let us list the Maxwell Equations, that we will review again later on:

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \tag{1.2}$$

$$\nabla \cdot \vec{B} = 0 \tag{1.3}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{1.4}$$

$$\nabla \times \vec{B} = \mu_0 \left(\vec{J} + \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \right) \tag{1.5}$$

A classical field theory like the theory of Newtonian gravity or the theory of Electromagnetism has two parts that describe it:

1. **Field equation**

   A field equation is an equation of motion that describes exactly how a certain field is determined by some set of sources. In other words, it is usually a second order differential equation that has to be solved to determine the field from some collection of sources.

   Example 1.1. Newtonian gravity field equation

   $$\nabla^2 \Phi = 4\pi G \rho$$

   where ρ is the mass density. We solve this Laplace's equation to determine the field. If ρ is a $\nabla(\vec{r})$ function such that it describes a point mass, which implies that $\Phi \propto {}^{1}\!/\!_{\vec{r}}$.

   Example 1.2. Electromagnetic field equations. See the Maxwell's Equation's above. In equation (1.2), we have that ρ represents the charge density to determine the field in terms of the forces from some collection of charge sources.

2. **Force law**

The force law is an equation that determines how an object moves in the presence of a field. So we start by describing a collection of sources and how they describe some force field, and then we use the force law to determine how bodies in motion are affected by the force field using the force law.

Example 1.3. Newtonian force law

$$\vec{F}(t, \vec{x}, \dot{\vec{x}}) = \frac{m}{2} \frac{d}{dt} \frac{\partial}{\partial \vec{v}} \vec{v}^2 = m\dot{\vec{v}} = -m\vec{\nabla}\Phi(t, \vec{x})$$

where $\dot{\vec{v}} = {}^{d^2\vec{x}}/_{dt^2} = \ddot{\vec{x}}$. This force law describes how some gravitational potential affects the motion of bodies in the presence of the field described by the potential.

Example 1.4. Lorentzian force law

$$\vec{F}\left(t, \vec{x}, \dot{\vec{x}}\right) = q(\vec{E} + \vec{v} \times \vec{B})$$

where $\vec{v} = {}^{d\vec{x}}/_{dt} = \dot{\vec{x}}$.

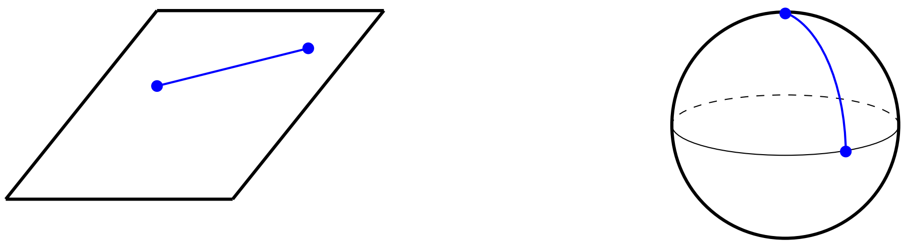
The important thing here is that fields are just functions of points in spacetime. For example, the gravitational potential $\Phi(t, \vec{x})$ yields a number that depends on where you are in space time, and the electric field $\vec{E}(t, \vec{x})$ is also a vector that depends on where you are in space and time. We should think of force laws as equations that describe how the motion of objects deviate from being "straight lines".

To emphasize, let us assume that an object in inertial motion on the $+\hat{x}$ axis, neglecting field interactions with the object. Evidently so, it's acceleration is zero, yielding the differential equation $\ddot{x} = 0$. The solution to this second order differential equation yields motion in a straight line. So the force law considering a field with an object experiencing this motion will be able to distort the object's motion such that it is not a straight line anymore. For instance, a gravitational field will bend the straight line into a curve as it generates acceleration in the $-\hat{y}$ axis, leading to parabolic motion on the $\hat{x}\hat{y}-$plane.

Definition 1.1. General Relativity is a complete reinterpretation of gravitation such that it is not a field using a potential $\Phi(t, \vec{x})$, but instead it is a feature of spacetime itself. In particular, we replace the gravitational potential with a *metric tensor* that describes this feature, particularly, the geometrical curvature of spacetime.

We will be learning pseudo-Riemann topological spaces by employing Einstein's field equation, which determines how spacetime curvature is determined in the presence of matter or energy. In Newtonian gravitation, the source term was a mass (or energy density in Special Relativity where $E = mc^2$). Recall however that mass has no independent meaning in terms of relativity as energy and momentum both depend on the reference frame where they are measured. The "source" of the curvature in general relativity is a term in the Einstein field equations that describes a generalized mass-energy distribution in spacetime of the sources present.

The force law is also replaced by the geodesic equation, which tells us how objects move through some curved spacetime. Particularly, the geometric interpretation of the geodesic is quite simple; it is the statement that *object will move on **geodesics**, which are the shortest paths between two points on a surface.* In a flat surface, this is a straight line, but in curved surfaces, this trajectory will not be a straight line, like two points on the surface of a sphere.



(a) Geodesic over a flat surface

(b) Geodesic over a spherical surface

Figure 1: Examples of geodesic on different surfaces.

So in other words, it is the curve that minimizes the distance between two points over some surface. The force law is therefore just the statement that bodies will move in geodesics over some geometry, even when said geometry is curved. So when we think of gravity as the curvature of spacetime, then geodesics will describe motion over spacetime as it is curved.

1.2 Special Relativity

The discussion of special relativity will serve to introduce spacetime properly. We will develop it further later on, but for now we will proceed with a heuristic manner.

Definition 1.2. Spacetime is a smoothly connected manifold where the points defined are called **events**.

The events in spacetime can be smoothly parametrized using coordinates, such as the most common system of coordinates used, the Cartesian coordinates.

Example 1.5. Cartesian coordinates

$$(t, \vec{x}) = (-c\hat{t}, \hat{x}, \hat{y}, \hat{z})$$

In relativistic mechanics, we usually consider all directions of motion in a single vector $\vec{x} = (\hat{x}, \hat{y}, \hat{z})$ while time is multiplied by the negative speed of light $t = (-c\hat{t})$. We will explain this later on.

Thus, spacetime is a smooth manifold with events as points parametrized by some coordinates, and in relativity, we know that physics should be independent of its coordinate system. This is known as the **principle of general covariance**.

To begin, let us assume we are working with Newtonian mechanics where we have two points in spacetime labelled $(t_1, \vec{x}_1) = (t_1, x_1, y_1, z_1)$ and $(t_2, \vec{x}_2) = (t_2, x_2, y_2, z_2)$ respectively, both of which describe a single event in spacetime given as

$$(\Delta t, \Delta \vec{x})$$

In this event, Δt denotes the **temporal separation** while $\Delta \vec{x}$ denotes the **spatial separation**. Explicitly:

$$\begin{aligned}
&\Delta t = t_2 - t_1 \\
&\Delta \vec{x} = \sqrt{(\vec{x}_2 - \vec{x}_1) \cdot (\vec{x}_2 - \vec{x}_1)} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}
\end{aligned}$$

This makes sense as Newtonian mechanics can always be written in terms of both Δt and $\Delta \vec{x}$, so the numerical value of t_1 will depend on the origin in the frame of reference we are working on. In other words, if we choose our reference frame to be Montreal at 6:00pm and remain at rest until 6:01pm, then we can denote that $\Delta t = 60$ s. So if we now jog in Montreal at constant velocity of 1 m/s in this reference frame, the velocity will depend on the time separation Δt and not on the absolute values of t_1 and t_2. So, if we suddenly change the time frame to be 4:20pm to 4:21pm, the velocity should not vary due to the absolute value of the new points in time, as the time separation Δt would be the equal to the case at 6:00pm. The same would go for the choice of spatial coordinates, so if we move from Montreal to Toronto and the speed remains constant, the space separation $\Delta \vec{x}$ should also not vary. To summarize, the laws of physics would remain invariant even if we change the choice of reference frame where the choice of time is independent of the choice of space, and so they are independent of the entire coordinate system chosen. Albeit elementary to classical mechanics, it is vital to understand this properly to proceed with the philosophy of relativity.

In special relativity, there is no separate notion between space and time, so we denote both as spacetime. This means that the both the separation of time and the separation of space are dependent on each other, which leads to effects like time dilation. Time dilation implies that *the temporal difference Δt of two event in spacetime will depend on the relative position in spacetime where an observer is viewing the event from.* Similarly, the distance or length of an object will depend on where the observer is in spacetime. In other words, we cannot think of Δt and $\Delta \vec{x}$ as independent separations in spacetime, as one is directly related to the other depending on the frame of reference. So, observing the same event in a different reference frame will change the values of Δt and $\Delta \vec{x}$.

However, we can denote an invariant notion to both the spatial and temporal separations called the **interval** between two events as

$$\Delta s^2 = -c^2 \Delta t^2 + \Delta \vec{x}^2 \qquad \text{(in Cartesian coordinates)}$$

Usually, we simplify by allowing $c = 1$ in units in terms of the speed of light [3×10^8 m/s] such that

$$\Delta s^2 = -\Delta t^2 + \Delta \vec{x}^2 \tag{1.6}$$

So when we measure distance with respect to the speed of light, we do so in terms of light-time, such as light-seconds and light-years. It is usually helpful to draw pictures in a spacetime diagram.

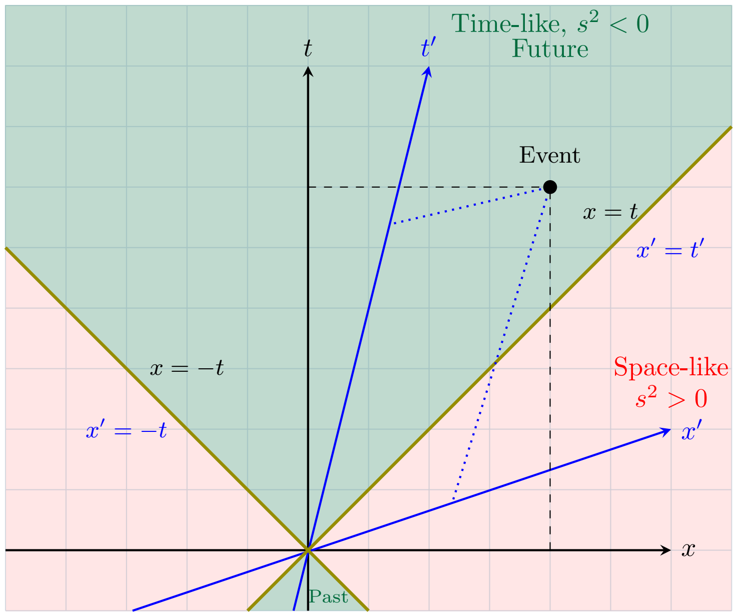


Figure 2: A spacetime diagram of an time-like event with the light cone in two different reference frames $K = (t, x)$ and $K' = (t', x')$. Notice how even if the spatial ($\Delta x \neq \Delta x'$) and temporal ($\Delta t \neq \Delta t'$) separations vary from one reference frame to the other, the interval in spacetime remains invariant $\Delta s^2 = \Delta(s')^2$. We usually denote $(t_1, x_1) = (t_1', x_1') = (0, 0)$ for convenience in either reference frame.

Notice in the spacetime diagram that when $\Delta s^2 = 0$, we have that $\Delta x = \pm\Delta t$, defining a **light cone**. Events defined on the light cone are denoted **light-separated events** or **null-separated events**. As otherwise noted, events where $\Delta s^2 < 0$ are inside the light cone and denoted as **time-like** or **time-separated**, while events where $\Delta s^2 > 0$ lie outside the light cone and are denoted as **space-like** or **space-separated**. Notice that for time-like events, it is clear that when $t_2 > 0$, the event will propagate to the future, while if $t_2 < 0$ it propagates to the past. Also note that space-like events are deemed as impossible, as there is no geodesic inside a light cone that can connect the origin to the event. In other words, an object would have to travel faster than light to arrive at this location in spacetime, which is indeed impossible.

We can thus assume that if we fix Δs^2 in a spacetime phase for the system

$$\begin{pmatrix} \partial(\Delta\vec{x}^2)/\partial(\Delta s^2) \\ \partial(\Delta t^2)/\partial(\Delta s^2) \end{pmatrix} = \begin{pmatrix} 0 & \pm 1 \\ \pm 1 & 0 \end{pmatrix} \begin{pmatrix} \Delta\vec{x}^2 \\ \Delta t^2 \end{pmatrix}$$

We would get hyperbolic equilibrium to a 2D spacetime system with respect to the interval Δs^2. This means a time-like event will always be either time-like or get close to null-like, while a space-like event will also either remain space-like or get close to null-like.