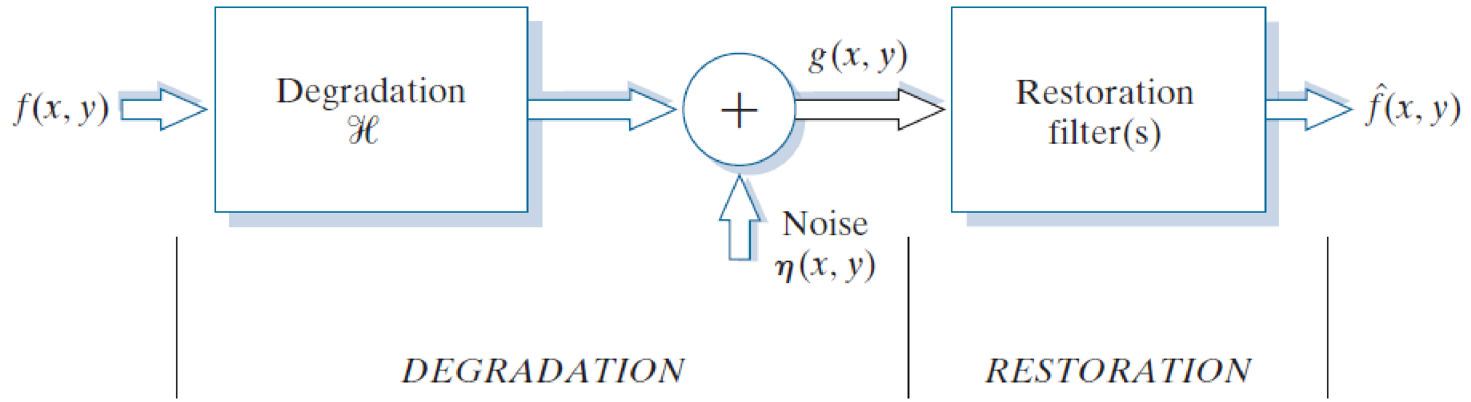
Image Restoration and Reconstruction

ICT4201:DIP

Preview

- Image enhancement is applied to improve an image in some predefined sense.
- Restoration attempts to recover an image that has been degraded by using a priori knowledge of the degradation phenomenon.
- enhancement is largely a subjective process, while image restoration is for the most part an objective process.
- Restoration process is meant for modeling the degradation and applying the inverse process in order to recover the original image.

Noise Model



$$g(x,y) = h(x,y) * f(x,y) + \eta(x,y)$$
$$G(u,v) = H(u,v)F(u,v) + N(u,v)$$

Additive noise

$$\hat{f}(x,y) = R\{g(x,y)\}$$

For Example, in acquiring images with a CCD camera, light levels and sensor temperature are major factors affecting the amount of noise in the resulting image.

Spatial and Frequency Properties of Noise

- When the Fourier spectrum of noise is constant, the noise is called white noise.
  - white light-- which contains all frequencies in the visible spectrum in equal proportions.
- There is no correlation between pixel values and the values of noise components

Some Important Noise functions

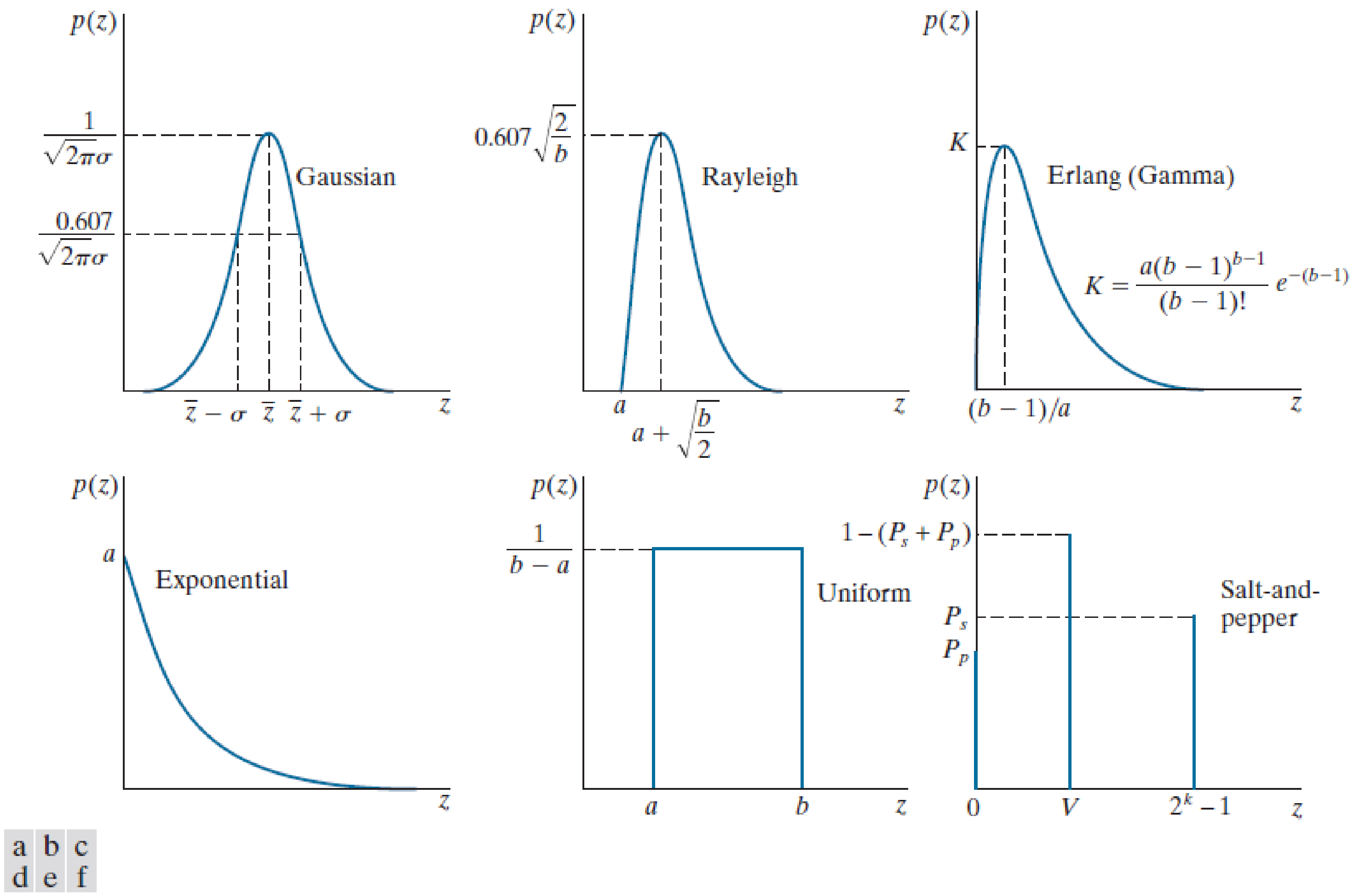


FIGURE 5.2 Some important probability density functions.

- Gaussian : electronic circuit noise, sensor noise
- Rayleigh : range imaging
- Exponential and Gamma : laser imaging
- Uniform : quantization
- Impulse : defective pixels, faulty switching

Gaussian Noise

- Because of its mathematical simplicity, the Gaussian noise model is often used in practice and even in situations where they are marginally applicable at best. Here, m is the mean and σ^2 is the variance.
- Gaussian noise arises in an image due to factors such as electronic circuit noise and sensor noise due to poor illumination or high temperature.
- The PDF of a Gaussian random variable, *z*, is given by

$$p(z) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(z-\bar{z})^2}{2\sigma^2}} \qquad -\infty < z < \infty$$

  - Where, *z* represents intensity, z is the mean (average) value of $\bar{z}$, and σ is its standard deviation.
  - 70% of its values are in the range $-\sigma \leq \bar{z} \leq \sigma$
  - Central limit theorem : The sum of a sufficiently large number of identically distributed independent random variables each with finite mean and variance will be approximately normally (Gaussian) distributed.

Rayleigh Noise

- Rayleigh noise is usually used to characterize noise phenomena in range imaging.
- Figure 5.2(b) shows a plot of the Rayleigh density. Note the displacement from the origin, and the fact that the basic shape of the density is skewed to the right.
- The Rayleigh density can be quite useful for modeling the shape of skewed histograms.

The PDF of *Rayleigh* noise is given by

$$p(z) = \begin{cases} \frac{2}{b}(z - a)e^{-(z-a)^2/b} & z \geq a \\ 0 & z < a \end{cases} \tag{5-4}$$

The mean and variance of z when this random variable is characterized by a Rayleigh PDF are

$$\bar{z} = a + \sqrt{\pi b/4} \tag{5-5}$$

and

$$\sigma^2 = \frac{b(4 - \pi)}{4} \tag{5-6}$$

Erlang (Gamma) Noise

- Gamma noise density finds application in laser imaging.

The PDF of Erlang noise is

$$p(z) = \begin{cases} \dfrac{a^b z^{b-1}}{(b-1)!} e^{-az} & z \geq 0 \\ 0 & z < 0 \end{cases} \tag{5-7}$$

where the parameters are such that $a > b$, b is a positive integer, and "!" indicates factorial. The mean and variance of z are

$$\bar{z} = \frac{b}{a} \tag{5-8}$$

and

$$\sigma^2 = \frac{b}{a^2} \tag{5-9}$$

Figure 5.2(c) shows a plot of this density. Although Eq. (5-9) often is referred to as the *gamma* density, strictly speaking this is correct only when the denominator is the gamma function, $\Gamma(b)$. When the denominator is as shown, the density is more appropriately called the *Erlang* density.

Exponential Noise

The PDF of *exponential* noise is given by

$$p(z) = \begin{cases} ae^{-az} & z \geq 0 \\ 0 & z < 0 \end{cases} \tag{5-10}$$

where $a > 0$. The mean and variance of z are

$$\bar{z} = \frac{1}{a} \tag{5-11}$$

and

$$\sigma^2 = \frac{1}{a^2} \tag{5-12}$$

Note that this PDF is a special case of the Erlang PDF with $b = 1$. Figure 5.2(d) shows a plot of the exponential density function.

Exponential noise is also commonly present in cases of laser imaging.

Uniform Noise

Uniform noise is not practically present but is often used in numerical simulations to analyze systems.

The PDF of *uniform* noise is

$$p(z) = \begin{cases} \dfrac{1}{b - a} & a \leq z \leq b \\ 0 & \text{otherwise} \end{cases} \tag{5-13}$$

The mean and variance of z are

$$\bar{z} = \frac{a + b}{2} \tag{5-14}$$

and

$$\sigma^2 = \frac{(b - a)^2}{12} \tag{5-15}$$

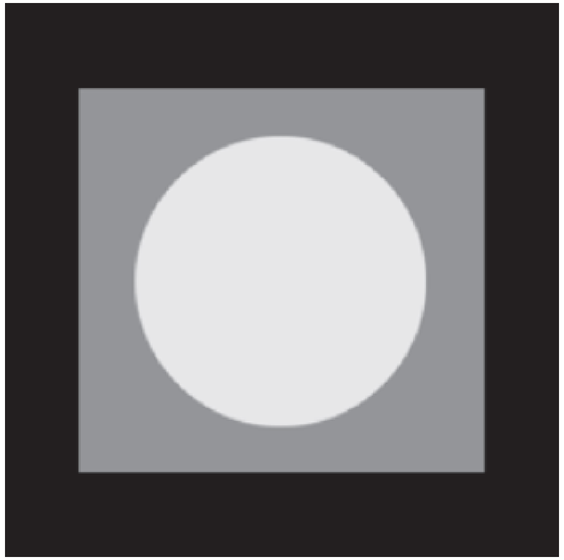
Figure 5.2(e) shows a plot of the uniform density.

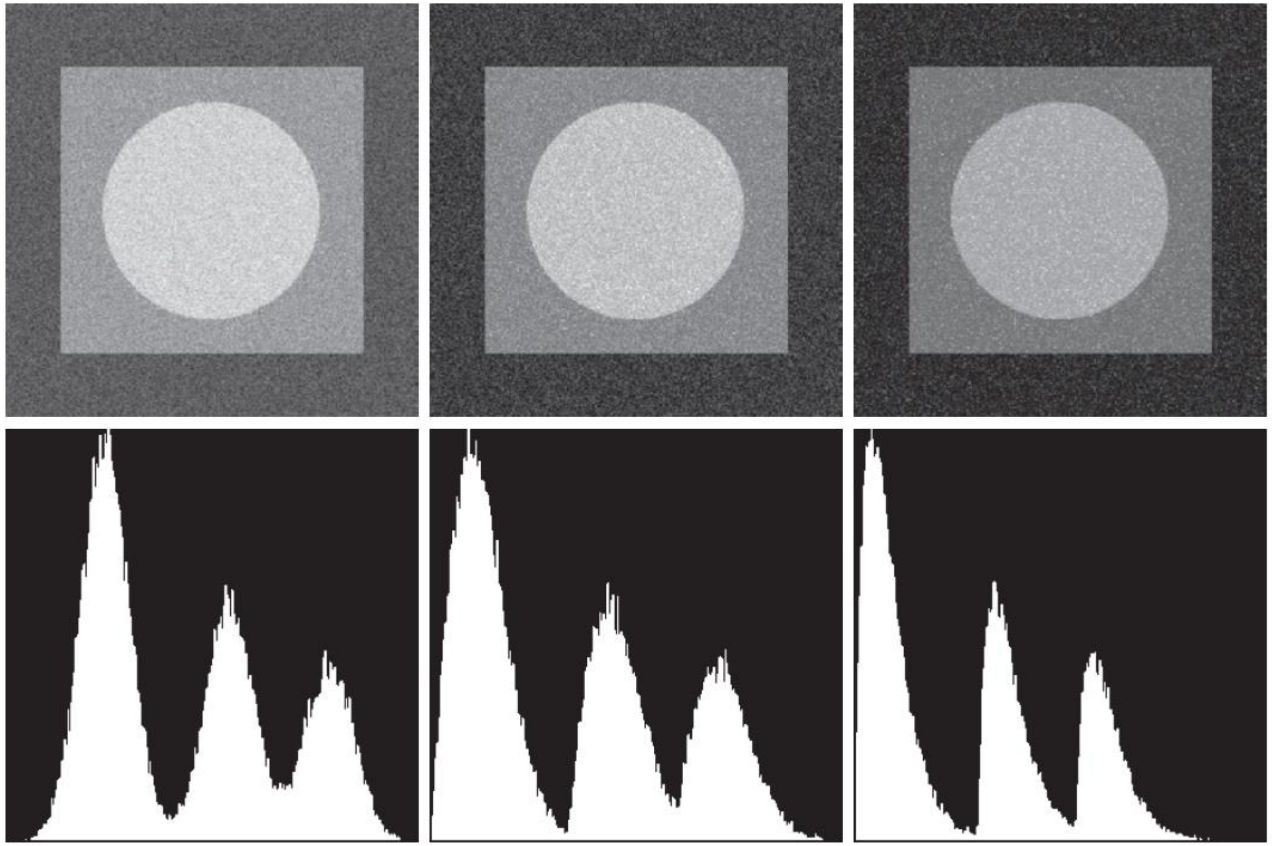
Salt-and-Pepper Noise

- Also known as Impulse noise.
- If p > s, intensity p will appear as a light dot in the image. Conversely, level s will appear like a black dot in the image. Hence, this presence of white and black dots in the image resembles to salt-and-pepper granules, hence also called salt-and-pepper noise. When either P_s or P_p is zero, it is called unipolar noise. The origin of impulse noise is quick transients such as faulty switching in cameras or other such cases.
- where V is any integer value in the range $0 < V < 2^k - 1$.

$$p(z) = \begin{cases} P_s & \text{for } z = 2^k - 1 \\ P_p & \text{for } z = 0 \\ 1 - (P_s + P_p) & \text{for } z = V \end{cases}$$

Example



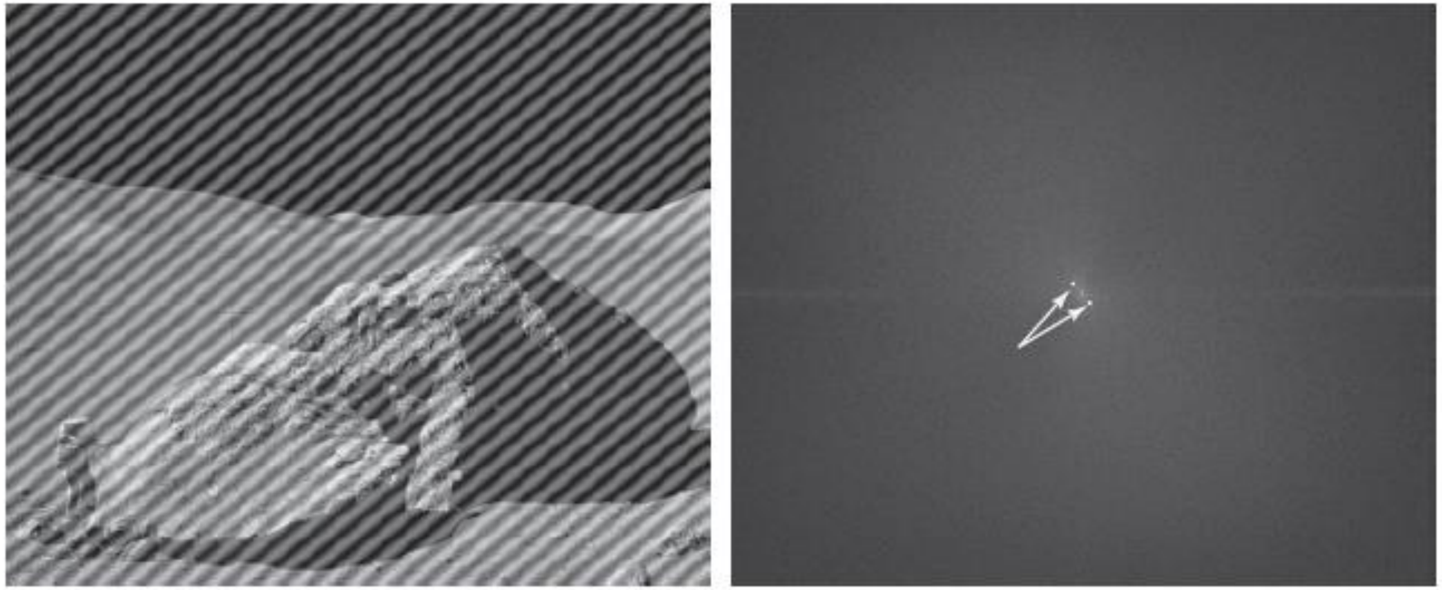


a b c
d e f

FIGURE 5.4 Images and histograms resulting from adding Gaussian, Rayleigh, and Erlanga noise to the image in Fig. 5.3.

PERIODIC NOISE

- Periodic noise in images typically arises from electrical or electromechanical interference during image acquisition.
- This is the only type of spatially dependent noise we will consider in this chapter.
- periodic noise can be reduced significantly via frequency domain filtering.



a b

FIGURE 5.5 (a) Image corrupted by additive sinusoidal noise. (b) Spectrum showing two conjugate impulses caused by the sine wave. (Original image courtesy of NASA.)

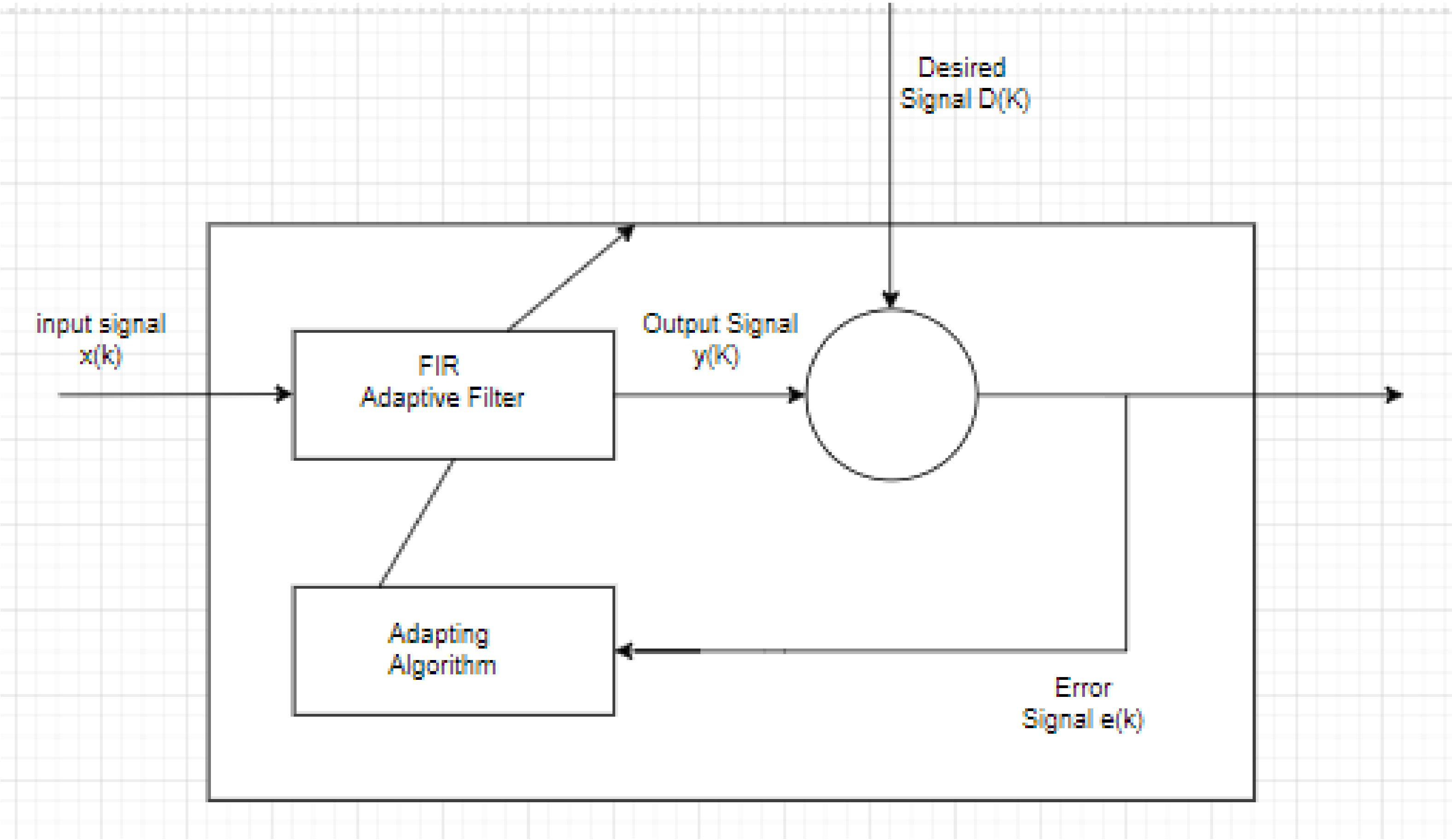
Restoration in the Presence of Noise Only - Spatial Filtering

- Mean filter
  - Arithmetic mean
  - Geometric mean
  - Harmonic mean
  - Contraharmonic mean •
- Order-statistics filter
  - Median
  - Max and min
  - Midpoint (min+max)/2
  - Alpha-trimmed mean : delete the d/2 lowest and the d/2 highest and average remaining pixels

Adaptive Filters

- Adaptive filters are also Digital filters that change their coefficients with the intention of bringing the filter closer to its optimal state.
- A cost function, typically the mean square of the error signal between the adaptive filter’s output and the desired signal, serves as the optimization criterion.
- The mean square error (MSE) converges to its minimal value as the filter adjusts its coefficients. The coefficients have converged to a solution and the filter has been modified at this point.
- The desired signal, d(k), is said to be very closely matched by the filter output, y(k). The filter adjusts to the new environment by generating a new set of coefficients for the new data when the characteristics of the input data are altered, a process that is referred to as the “filter environment.”

Adaptive Filters



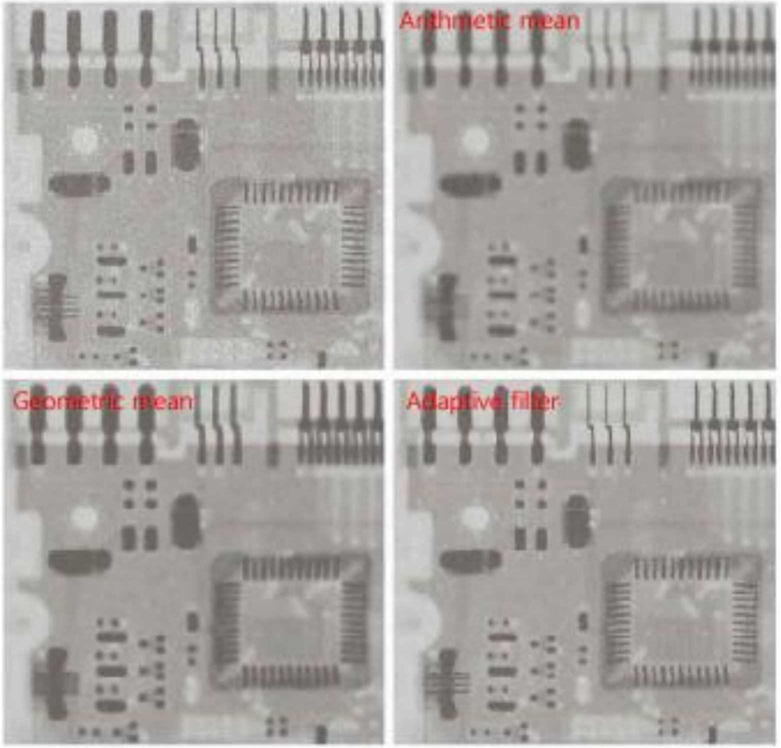
Application of Adaptive Filter

1. **System Identification: Identifying Unknown Systems Using an Adaptive Filter-** Identifying an unknown system, such as the response of an unknown communications channel or the frequency response of an auditorium, to select fairly distinct applications is one common application of adaptive filters. Channel identification and echo cancellation are two additional applications.
2. **Using an Adaptive Filter to Remove Noise from an Unknown System: Noise or Interference Cancellation-** In commotion crossing out, versatile channels let you eliminate clamor from a sign continuously. Noise and desired information are combined here into the desired signal, the one to be cleaned. Feed the adaptive filter a signal n'(k) that is correlated to the noise to be removed from the desired signal to get rid of the noise.
3. **Identification of an Inverse Response to an Unknown System via Inverse System-** adaptive filter becomes the inverse of the unknown system as e(k) decreases when the unknown system is placed in series with it. In order to keep the data at the summation synchronized, the procedure necessitates the addition of a delay to the desired signal d(k) path, as depicted in the figure. The system remains causal when the delay is added.
4. **A Periodic Signal's Future Values Can Be Predicted Through Prediction-** In order to predict signals, you must make important assumptions. Assume that the signal is periodic and either steady or slowly changing over time.

Adaptive Filter

- Image characteristics vary from one point to another.
- Adaptive filter : Its behavior changes based on statistical characteristics inside the filter region.
- Improved performance ↔ Filter complexity
- Simple adaptive filter
  - If $\sigma_n^2=0$ → there is no noise → $\hat{f}(x,y)=g(x,y)$
  - If $\sigma_L^2 \gg \sigma_n^2$ → high contrast (edge) → $\hat{f}(x,y)\cong g(x,y)$
  - If $\sigma_L^2 \approx \sigma_n^2$ → noise in a smooth area → $\hat{f}(x,y)\cong m_L$

$$\Rightarrow \hat{f}(x,y)=g(x,y)-\frac{\sigma_n^2}{\sigma_L^2}[g(x,y)-m_L]$$



Adaptive Median Filter

- Main purposes
  - To remove salt-and-pepper noise
  - To provide smoothing of other noise (not impulsive)
  - To reduce distortion such as excessive thinning or thickness of object boundaries

- Stage A: (S_{max}= maximum allowed size of window)

  If $z_{min} < z_{med} < z_{max}$, go to stage B

  Else increase the window size

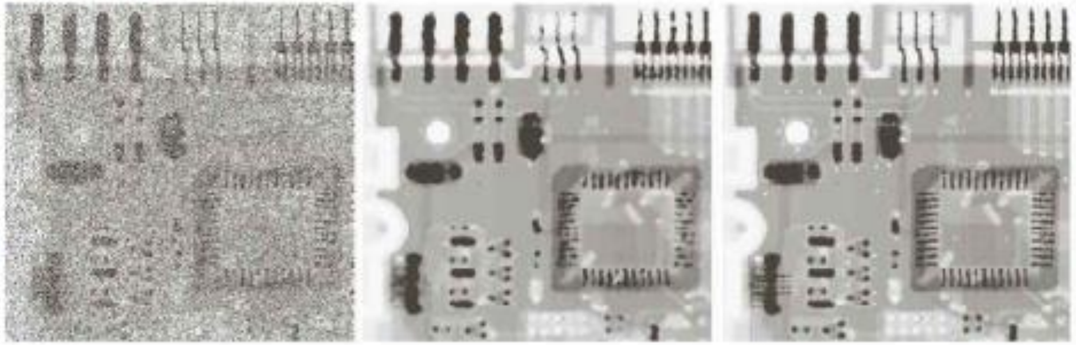
  If window size $\leq S_{max}$, repeat stage A

  Else output z_{med}

- Stage B:

  If $z_{min} < z_{xy} < z_{max}$, output z_{xy}

  Else output z_{med}



a b c

FIGURE 5.14 (a) Image corrupted by salt-and-pepper noise with probabilities $P_a = P_b = 0.25$. (b) Result of filtering with a 7×7 median filter. (c) Result of adaptive median filtering with $S_{max} = 7$.

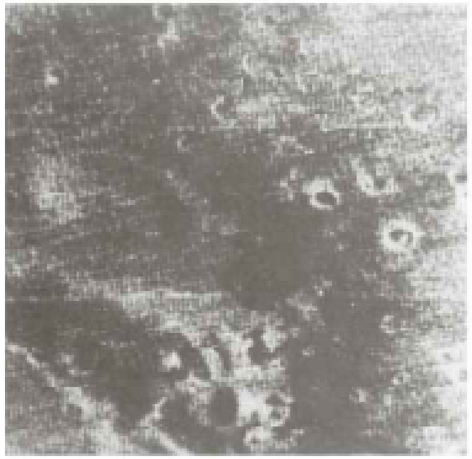
Periodic Noise Reduction
- Optimum Notch Filtering

- Optimum : Minimize local variances of the restored image
  1) Extract the principal frequency components $n(x,y)$ of the interference pattern
  2) Subtract a variable, weighted portion of the pattern from the corrupted image $g(x,y)$

1) $N(u,v) = H_{NP}(u,v)G(u,v)$

$n(x,y) = \Im^{-1}\{N(u,v)\}$

2) $\hat{f}(x,y) = g(x,y) - w(x,y)n(x,y)$



- $w(x,y)$: weighting (modulation) function

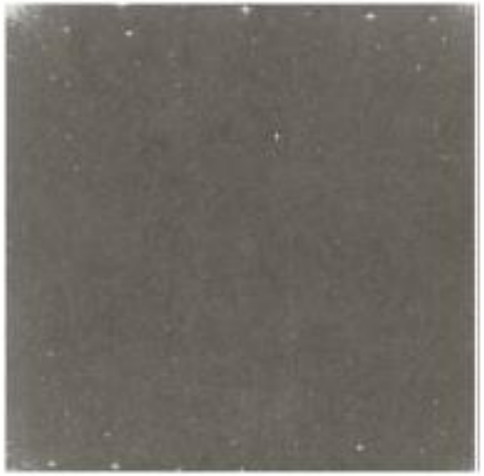
  → minimize the local variance of $\hat{f}(x,y)$

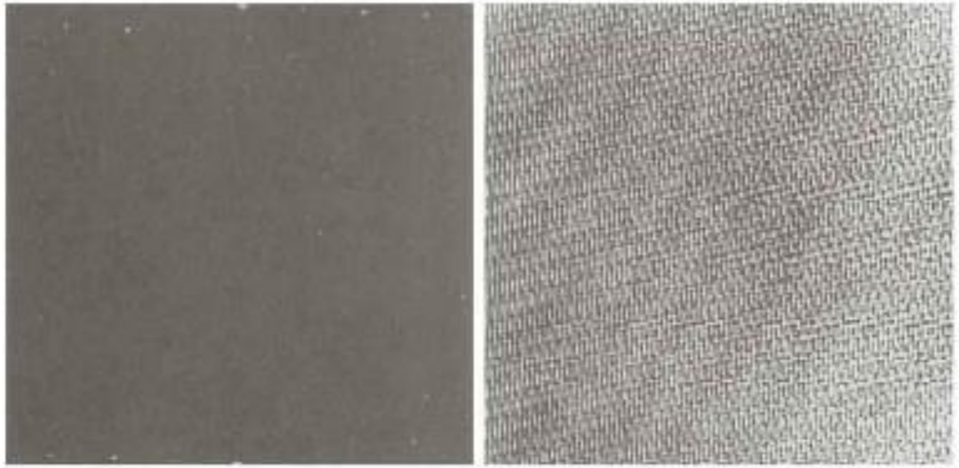
$$\sigma^2(x,y) = \frac{1}{(2a+1)(2b+1)} \sum_{s=-a}^{a} \sum_{t=-b}^{b} [\hat{f}(x+s,y+t) - \bar{\hat{f}}(x,y)]^2$$

To minimize $\sigma^2(x,y)$, $\dfrac{\partial \sigma^2}{\partial w} = 0$ $\quad (w(x,y) \cong w(x,y))$

$$w(x,y) = \frac{\overline{g(x,y)n(x,y)} - \bar{g}(x,y)\bar{n}(x,y)}{\overline{n^2}(x,y) - \bar{n}^2(x,y)}$$

Result





a b

FIGURE 5.22 (a) Fourier spectrum of $N(u, v)$, and (b) corresponding noise interference pattern $\eta(x, y)$. (Courtesy of NASA.)

a=b=15

Linear, Position-Invariant Degradations

$$g(x,y) = h(x,y) * f(x,y) + n(x,y)$$

$$G(u,v) = H(u,v)F(u,v) + N(u,v)$$

- $h(x, y)$ is called point spread function (PSF)
- $H(u, v)$ is the frequency response of a linear space-invariant system
- The estimate of $f(x, y)$ is obtained by means of deconvolution methods (deconvolution filters applying the process in reverse)
- First, we need to estimate the degradation functions
  - estimation by image observation
  - estimation by experimentation
  - estimation by mathematical modeling
- $\hat{F}(u,v) = G(u,v) / H(u,v)$

- The process of restoring an image by using a degradation function sometimes is called ***blind deconvolution***, due to the fact that the true degradation function is seldom known completely.

Estimating the Degradation Function (1) Estimation by Image Observation

- Suppose that we are given a degraded image without any knowledge about the degradation function $H(u, v)$.
- Estimation ← Information from the image itself
  - Example: If the image is blurred, we can look at a small section of the image containing simple structures, like part of an object and the background
  1) Look for an area in which the signal content is strong (e.g., an area of high contrast)
  2) Process the subimage to arrive at a result is as unblurred as possible

- $H(u,v) \cong H_s(u,v) = G_s(u,v) / \hat{F}_s(u,v)$

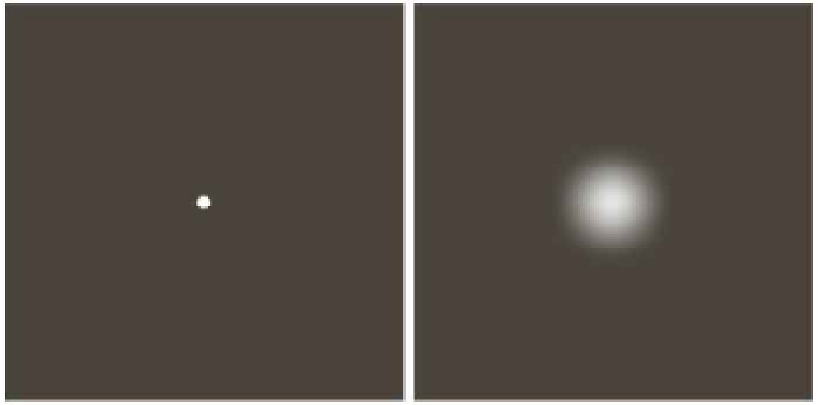
- Laborious process → Not often applicable in practice
- Application example : Restoring an old photograph of historical value

(2) Estimation by Experimentation

- If equipment similar to the equipment used to acquire the degraded image is available, it is possible in principle to obtain an accurate estimate of the degradation.
- If the acquisition device is available, we can obtain an accurate estimate of the PSF.
- Acquire the image of a point (a small dot of light) under the same working conditions

a b

FIGURE 5.24 Degradation estimation by impulse characterization. (a) An impulse of light (shown magnified). (b) Imaged (degraded) impulse.



- $H(u, v) = G(u, v)/A$

(3) Estimation by Modeling

- Build a physical model of the degradation process
- Ex) Atmospheric turbulence $H(u, v)=\exp\{-k(u^2+v^2)^{5/6}\}$

a b
c d

FIGURE 5.25 Illustration of the atmospheric turbulence model. (a) Negligible turbulence. (b) Severe turbulence, $k = 0.0025$. (c) Mild turbulence, $k = 0.001$. (d) Low turbulence, $k = 0.00025$. (Original image courtesy of NASA.)

Mathematical Model

- Ex) Blur by uniform linear motion

$$g(x,y)=\int_0^T f[x-x_0(t),y-y_0]dt$$

$$G(u,v)=\Im\{g(x,y)\}=F(u,v)\int_0^T e^{-j2\pi[ux_0(t)+vy_0(t)]}dt$$

$$H(u,v)=\frac{G(u,v)}{F(u,v)}=\int_0^T e^{-j2\pi[ux_0(t)+vy_0(t)]}dt$$

$$=\frac{T}{\pi(ua+vb)}\sin[\pi(ua+vb)]e^{-j\pi(ua+vb)} \qquad x_0(t)=\frac{at}{T},y_0(t)=\frac{bt}{T}$$

a b

FIGURE 5.26
(a) Original image.
(b) Result of blurring using the function in Eq. (5.6-11) with $a = b = 0.1$ and $T = 1$.

Inverse Filtering

- H : degradation function, G : degraded image

$$\hat{F}(u,v) = \frac{G(u,v)}{H(u,v)}$$

- Take noise into account

$$\hat{F}(u,v) = F(u,v) + \frac{N(u,v)}{H(u,v)}$$

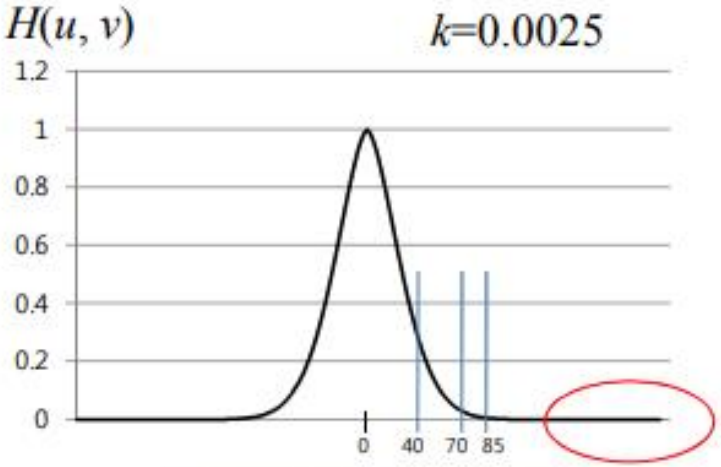
- If $H(u,v)$ has zero or very small values, $N(u,v) / H(u,v)$ could easily dominate the estimate $\hat{F}(u,v)$

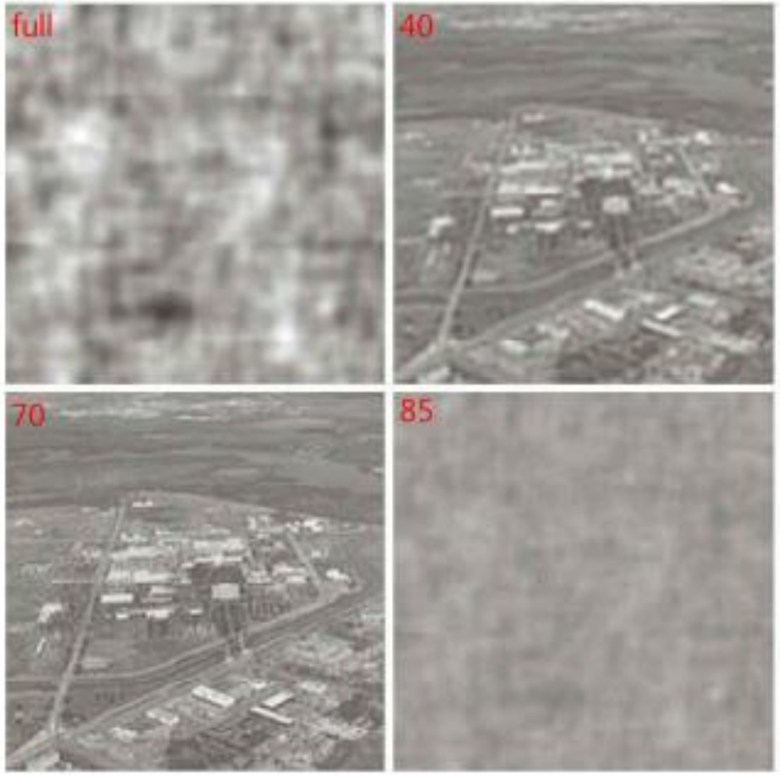
  → *This is frequently the case!*

  → Limit the filter frequencies to values near the origin

Example

- Turbulence degradation $H(u, v)=\exp\{-k(u^2+v^2)^{5/6}\}$





a b
c d

FIGURE 5.27 Restoring Fig. 5.25(b) with Eq. (5.7-1). (a) Result of using the full filter. (b) Result with H cut off outside a radius of 40; (c) outside a radius of 70; and (d) outside a radius of 85.

Minimum Mean Square Error (Wiener) Filtering

- Incorporate both the degradation function and statistical characteristics of noise into the restoration process.
- Assume
  - Consider images and noise as uncorrelated stationary random variables.
  - One or the other has zero mean.
  - The intensity levels in the estimate are a linear function of the levels in the degraded image.
- Find an estimate $\hat{f}$ of the uncorrupted image f such that the **MSE** between them is minimized.

  MSE $e^2 = E\{(f - \hat{f})^2\}$

- Wiener filter, optimal filter, least square error (LSE) filter

$$\hat{F}(u,v) = \left[\frac{1}{H(u,v)} \frac{|H(u,v)|^2}{|H(u,v)|^2 + S_n(u,v)/S_f(u,v)} \right] G(u,v)$$

H: degradation function
$S_n = E\{|N(u,v)|^2\}$: power spectrum of the noise
$S_f = E\{|F(u,v)|^2\}$: power spectrum of the undegraded image

Derivation (Spatial Domain)

$\hat{f}(m) = w(m) * g(m)$ w: Wiener filter (linear filter)

error $e(m) = f(m) - \hat{f}(m)$

MSE $E\{e^2\} = E\{(f - \hat{f})^2\}$

$$= E\{f^2\} - 2E\{f \cdot \hat{f}\} + E\{\hat{f}^2\}$$

$$= E\{f^2\} - 2E\{f \cdot (w * g)\} + E\{(w * g)^2\}$$

$$= E\{f^2\} - 2E\{f \cdot \sum_{i=1}^{L} w(i) g(m-i)\} + E\{\left[\sum (w(i) g(m-i)\right]^2\}$$

$$\frac{\partial}{\partial w(j)} E\{e^2\} = 0 - 2E\{g(m-i) f\} + 2E\{\left[\sum w(i) g(m-i)\right] g(m-j)\}$$

$$= -2E\{\sum g(m-i) f\} + 2\sum w(i) E\{g(m-i) g(m-j)\}$$

cross-correlation $R_{gg}(m) = E\{g(n) g(n+m)\}$

autocorrelation $R_{fg}(m) = E\{f(n) g(n+m)\}$

$$\frac{\partial}{\partial w(j)} E\{e^2\} = -2R_{gf}(j) + 2\sum w(i) R_{gg}(i-j)$$

$$= 0$$

$$\sum w(i) R_{gg}(i-j) = R_{gf}(j)$$

$$w * R_{gg} = R_{gf}$$

$$S_g = \Im\{R_{gg}\},\ S_{gf} = \Im\{R_{gf}\}$$

$$W \cdot S_g = S_{gf}$$

$$g = f + n$$

$$R_{gf} = R_{(f+n)f} = R_{ff},\quad R_{gg} = R_{ff} + R_{nn}$$

$$\therefore\ W = \frac{S_{gf}}{S_g} = \frac{S_f}{S_f + S_n} = \frac{1}{1 + S_n / S_f}$$

If we assume white noise (constant power spectral density), the Wiener filter is a low-pass filter.
It introduces significant blurring.

$$g(m) = h(m) * f(m) + n(m)$$

$$R_{gf} = R_{(h*f+n)f} = h(-m) * R_{ff},\quad R_{gg} = h(-m) * h(m) * R_{ff} + R_{nn}$$

$$S_{gf} = H^* \cdot S_f,\quad S_g = S_f H^* H = S_f |H|^2$$

$$\therefore\ W = \frac{S_{gf}}{S_g} = \frac{H^* \cdot S_f}{S_f |H|^2 + S_n} = \frac{1}{H} \frac{|H|^2 \cdot S_f}{S_f |H|^2 + S_n} = \frac{1}{H} \frac{|H|^2}{|H|^2 + S_n / S_f}$$

If noise is zero (S_n=0), the Wiener filter W is the inverse filter.

Derivation (Frequency Domain)

$g = h * f + n$ $\quad$ h:degradation function

$\hat{f} = w * g$ $\quad$ w:Wiener filter (linear filter)

$\hat{F} = W \cdot G = W(HF + N)$

error $\; e = F - \hat{F}$

MSE $\; E\{e^2\} = E\{|F - \hat{F}|^2\}$

$$
\begin{aligned}
&= E\{|F - W(HF+N)|^2\} = E\{|(1-WH)F - WN|^2\} \\
&= E\{[(1-WH)F - WN][(1-WH)^* F^* - W^* N^*]\} \\
&= (1-WH)(1-WH)^* E\{FF^*\} + (1-WH)W^* E\{FN^*\} + W(1-WH)^* E\{F^* N\} + WW^* E\{NN^*\} \\
&= (1-WH)(1-WH)^* E\{|F|^2\} + 0 + 0 + WW^* E\{|N|^2\} \\
&= (1-WH)(1-WH)^* S_f + WW^* S_n
\end{aligned}
$$

$$\frac{\partial}{\partial W} E\{e^2\} = -H(1 - W^* H^*) S_f + W^* S_n = 0$$

$$W^*(HH^* S_f + S_n) = HS_f$$

$$W = \frac{H^* S_f}{|H|^2 S_f + S_n} = \frac{1}{H} \frac{|H|^2 S_f}{|H|^2 S_f + S_n}$$

Note that if noise is zero (S_n=0), the Wiener filter W is the inverse filter $1/H$.

$g = f + n$ $\quad$ (only noise degradation)

$H = 1$

$$W = \frac{S_f}{S_f + S_n} = \frac{S_f}{S_f + c}$$

If we assume white noise (constant power spectral density), the Wiener filter is a low-pass filter. It introduces significant blurring.

Constrained Least Squares Filtering

- Wiener filtering requires knowledge of
  - Degradation function, $H(u,v)$
  - Power spectrum of the undegraded image, $S_f(u,v)$
  - Noise power spectrum, $S_n(u,v)$
- A constant estimate of the ratio $S_n(u,v) / S_f(u,v)$ of the power spectra is not always a suitable solution.

$$W(u,v) = \frac{1}{H(u,v)} \frac{|H(u,v)|^2}{|H(u,v)|^2 + S_n(u,v)/S_f(u,v)} \approx \frac{1}{H(u,v)} \frac{|H(u,v)|^2}{|H(u,v)|^2 + K}$$

- Constrained Least Square Filtering requires
  - $H(u,v)$
  - the mean and the variance of the noise ← degraded image

Constrained Least Squares Filtering

$g=h*f+n$ $$h(x,y)*f(x,y)=\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}f(m,n)h(x-m,y-n)$$

Vector matrix form $\mathbf{g}=\mathbf{Hf}+\mathbf{n}$

- If f, g, and n are of size M×N, vectors $\mathbf{g}$, $\mathbf{f}$, and $\mathbf{n}$ have dimensions $(MN)\times 1$.
- The matrix $\mathbf{H}$ has dimensions $(MN) \times (MN)$.
- The matrix $\mathbf{H}$ is highly sensitive to noise.
- → The problem cannot be solved by a simple matrix manipulation.

Constrained Least Squares Filtering

- One way to alleviate the noise sensitivity problem is to base optimality of restoration on a measure of smoothness, such as the second derivative (the Laplacian) of an image.
- To be meaningful, the restoration must be constrained by the parameters of the problems at hand. → Find the minimum of a criterion function C, defined as

$$C = \sum_{x=0}^{M-1}\sum_{y=0}^{N-1}[\nabla^2 f(x,y)]^2$$

subject to the constraint $\left\|\mathbf{g} - \mathbf{H}\hat{\mathbf{f}}\right\|^2 = \|\mathbf{n}\|^2$

Euclidean vector norm

$$\|\mathbf{w}\|^2 = \mathbf{w}^T\mathbf{w} = \sum_{k=1}^{N} w_k^2$$

- The frequency domain solution to this optimization problem

$$\hat{F}(u,v) = \left[\frac{H^*(u,v)}{|H(u,v)|^2 + \gamma\,|P(u,v)|^2}\right]G(u,v) \qquad p(x,y) = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 0 \end{bmatrix}$$

(if γ=0, it reduces to inverse filtering)

Geometric Mean Filter

- Generalization of the Wiener filter

$$\hat{F}(u,v) = \left[\frac{H^*(u,v)}{|H(u,v)|^2}\right]^{\alpha} \left[\frac{H^*(u,v)}{|H(u,v)|^2 + \beta \dfrac{S_n(u,v)}{S_f(u,v)}}\right]^{1-\alpha} G(u,v)$$

- α=1 → inverse filter
- α=0 → parametric Wiener filter (β=1 → standard Wiener filter)
- β=1
  - As α decreases below ½, the filter behaves more like the Wiener filter.
  - As α increases above ½, the filter behaves more like the inverse filter.

- Useful when implementing restoration filters

  Because it represents a family of filters combined into a single expression.