PRACTICE 5

1 Continuous and Discrete Approximation

(1) Discrete Approximation - Data Fitting

- **Data:** A finite set of points (x_i, y_i), where $i = 1, 2, \dots, m$.
- **Representation:** The data are considered as vectors in $\mathbb{R}^m$.
- **Inner Product:** The canonical inner product in $\mathbb{R}^m$ is used:

$$\langle \mathbf{u}, \mathbf{v} \rangle = \sum_{i=1}^{m} u_i v_i$$

- **Objective:** Find a function $f(x)$ that minimizes the error between the observed values y_i and the predicted values $f(x_i)$, typically using the least squares method.

(2) Continuous Approximation

- **Data:** Instead of a finite set of points, there is a function $g(x)$ defined on an interval $[a, b]$.
- **Representation:** The data are considered as functions belonging to a continuous function space, typically $C([a, b])$.
- **Inner Product:** An inner product defined on functions is used, for example:

$$\langle f, g \rangle = \int_a^b f(x)g(x)\,dx$$

- **Objective:** Find a function $f(x)$ that approximates $g(x)$ over the entire interval, minimizing the error in an appropriate norm, such as the L^2 norm.

1.1 Discrete Approximation - Data fitting

Given the data set $(x_1, y_1),\ (x_2, y_2), \dots,\ (x_m, y_m)$, the objective consists of adjusting the parameters of a model function $y = f(x)$ to best fit a data set, that is if we approximate $y_i \approx f(x_i)$ we obtain the least possible error.

Model Definition:

$$f(x) = \sum_{j=1}^{n} a_j\, \varphi_j(x)$$

where:

- a_j are the parameters (coefficients) to be determined.
- $\varphi_j(x)$ are the basis functions that describe the data, such as polynomials, trigonometric functions (Fourier basis), exponential functions, etc.
- n is the number of basis functions used in the model.

To fit the model to the observed data, we construct a linear system:

$$A\mathbf{x} = \mathbf{b}$$

where:

- $A \in \mathbb{R}^{m\times n}$ is the matrix of basis functions evaluated at the data points:

$$A = \begin{bmatrix} \varphi_1(x_1) & \varphi_2(x_1) & \cdots & \varphi_n(x_1) \\ \varphi_1(x_2) & \varphi_2(x_2) & \cdots & \varphi_n(x_2) \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_1(x_m) & \varphi_2(x_m) & \cdots & \varphi_n(x_m) \end{bmatrix}$$

- $\mathbf{x} = [a_1, a_2, \ldots, a_n]^T$ is the vector of parameters to be determined.
- $\mathbf{b} = [y_1, y_2, \ldots, y_m]^T$ is the vector of observed data.

Solution Approach:

The parameters $\mathbf{x}$ are typically found by solving the system in the least squares sense, minimizing the residual:

$$\min_{\mathbf{x}} \|A\mathbf{x} - \mathbf{b}\|^2$$

that is minimizing the sum of squared residuals $\displaystyle\sum_{i=1}^{m} [y_i - f(x_i)]^2$.

This leads to the normal equations:

$$A^T A\mathbf{x} = A^T\mathbf{b}$$

And the least squares solution is:

$$\mathbf{x}_L S = (A^T A)^{-1} A^T\mathbf{b}$$

If the model that we want to use is a polynomials of degree k, we obtain the *polynomial curve fitting of degree k*:

$$p = a_0 + a_1 x + ... + a_k x^k$$

The basis functions in this case are:

$$\varphi_1(x) = 1, \quad \varphi_2(x) = x, \quad \varphi_3(x) = x^2, \quad \ldots, \quad \varphi_{k+1}(x) = x^k$$

Substituting the data points $(x_1, y_1),\ (x_2, y_2), \ldots,\ (x_m, y_m)$ in the model, we obtain:

$$\begin{cases} \alpha_0 + \alpha_1 x_1 + \alpha_2 x_1^2 + \cdots + \alpha_k x_1^k = y_1 \\ \alpha_0 + \alpha_1 x_2 + \alpha_2 x_2^2 + \cdots + \alpha_k x_2^k = y_2 \\ \quad \vdots \\ \alpha_0 + \alpha_1 x_m + \alpha_2 x_m^2 + \cdots + \alpha_k x_m^k = y_m \end{cases}$$

Or in matrix form:

$$\begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^k \\ 1 & x_2 & x_2^2 & \cdots & x_2^k \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_m & x_m^2 & \cdots & x_m^k \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_k \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}$$

Which is the linear system $A\mathbf{x} = \mathbf{b}$, with:

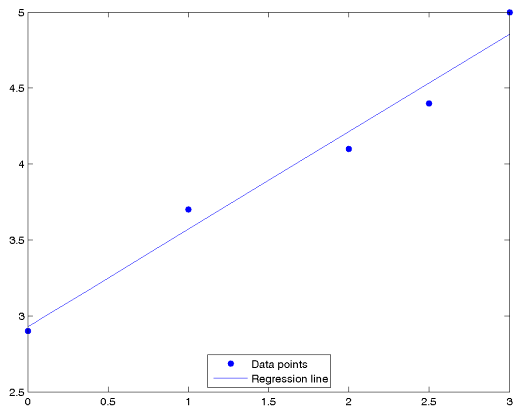
$$A = \begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^k \\ 1 & x_2 & x_2^2 & \cdots & x_2^k \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_m & x_m^2 & \cdots & x_m^k \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_k \end{bmatrix}, \quad \textit{and} \quad \mathbf{b} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}$$

In **MATLAB** we have the command:
`polyfit(x,y,k)`: finds the coefficients $a_0, a_1, ..., a_k$ of a polynomial $p(x)$ of degree k that fits the data, $p(x(i))$ to $y(i)$, in a least squares sense. The output argument p is a row vector of length $k+1$ containing the polynomial coefficients in descending powers.

Example 1.1 *Fit, using the regression line, the data:*

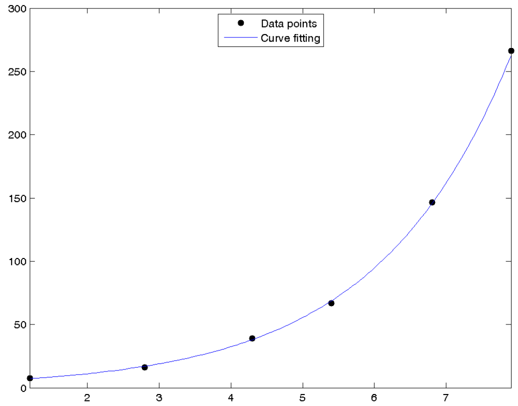
x	0	1	2	2.5	3
y	2.9	3.7	4.1	4.4	5

Compare the results given by polyfit and those obtained by using the first degree polynomial model.



Example 1.2 *Determine the parameters a and b of the function $y = ae^{bx}$ (exponencial model), to fit the data points:*

x	1.2	2.8	4.3	5.4	6.8	7.9
y	7.5	16.1	38.9	67	146.6	266.2



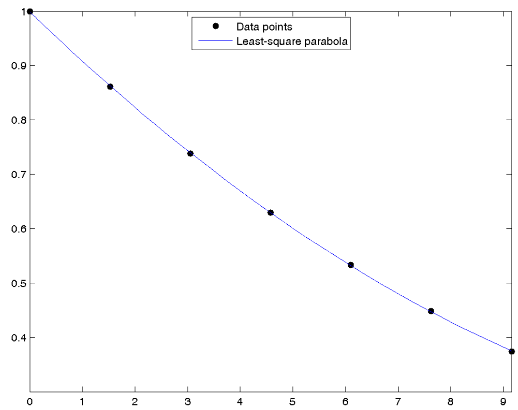
Example 1.3 *Determine the parameters a_0 and a_1 of the function $y = a_0 t^{a_1}$ (potential model), to fit the data points:*

t	1.5	2.7	3.9	5.2	6.6	7.8
y	5.2	10.9	21.4	36.1	60.3	85.7

Example 1.4 *The relative density d of air depends on the height h, the following measurements were obtained:*

h (in km)	0	1.525	3.05	4.575	6.1	7.625	9.15
d (in kg/m^3)	1	0.8617	0.7385	0.6292	0.5328	0.4481	0.3741

Fit these data points to the least-square parabola and approximate the density of the air to the height of 5 kilometers.



Note: The interpolating polynomial of a set of n data points coincides with the best fitting polynomial of degree $n-1$ (because this polynomial passes through all the data points and then, the square residuals are all zero) so, we can use the `polyfit` command to find the interpolating polynomial.

```
>> x=0:5; y=[1.1 1.5 2.4 2 3 1];
>> L=lagrange(x);
>> p=y*L; % interpolation polynomial
```

We obtain the same with:

```
>> q=polyfit(x,y,5)
```

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2 Continuous Approximation

In the continuous approximation problem, we aim to approximate a function $f(x)$ defined on an interval $[a, b]$ using a combination of basis functions. The approximation $\hat{f}(x)$ is given by:

$$\hat{f}(x) = \sum_{j=1}^{n} \alpha_j \, \varphi_j(x)$$

where:

- α_j are the coefficients to be determined.
- $\varphi_j(x)$ are the basis functions, such as orthogonal polynomials (e.g., Legendre polynomials) or trigonometric functions (e.g., Fourier series).

Objective: Minimize the approximation error in the L^2 norm:

$$\min_{\alpha_1,\ldots,\alpha_n} \left\| f(x) - \hat{f}(x) \right\|_{L^2}^2 = \min_{\alpha_1,\ldots,\alpha_n} \int_a^b \left(f(x) - \sum_{j=1}^{n} \alpha_j \varphi_j(x) \right)^2 dx$$

If the basis functions $\{\varphi_j(x)\}$ are orthogonal with respect to the inner product:

$$\langle f, g \rangle = \int_a^b f(x) g(x) \, dx, \tag{1}$$

Example 2.1 *Approximate the function $f(x) = \sin(\pi x)$ on the interval $[-1, 1]$ using a polynomial of degree 3.*

Find the approximation $\hat{f}(x)$ which in this case is the 3^{rd} degree polynomial of $f(x) = \sin(\pi x)$:

$$\hat{f}(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3.$$

If we use orthogonal polynomials or directly solve the normal equations from the least squares method, we can compute the coefficients explicitly.

MATLAB Implementation:

```
f = @(x) sin(pi * x);

% Generate data points for least squares fitting
n_points = 100;
x_data = linspace(-1, 1, n_points)';
y_data = f(x_data);

% Construct the A matrix (Ax=b)
A = [ones(n_points, 1), x_data, x_data.^2, x_data.^3];
b = y_data;

% Solve the least squares problem
coeffs = A \ b;

% Evaluate the polynomial approximation and the function
t=linspace(-1,1,1000);
f_approx = coeffs(1) + coeffs(2) * t + coeffs(3) * t.^2 + coeffs(4) * t.^3;
f_vals=f(t);

% Plot the original function and approximation
plot(x_vals, f_vals, 'b-', 'LineWidth', 2), hold on
plot(x_vals, f_approx, 'r--', 'LineWidth', 2)
```

2.0.1 Approximation with Fourier Series

If $f(x)$ is a periodic function with period T, the approximation using a basis of trigonometric functions is highly effective. We use the previously defined dot product:

$$\langle f(x), g(x)\rangle = \int_{\lambda}^{\lambda+T} f(x)g(x)\,dx \tag{2}$$

where the integration interval covers a full period.

The basis functions are the set of trigonometric functions:

$$\left\{1, \cos\left(\frac{2\pi x}{T}\right), \sin\left(\frac{2\pi x}{T}\right), \cos\left(\frac{4\pi x}{T}\right), \sin\left(\frac{4\pi x}{T}\right), \ldots, \cos\left(\frac{2n\pi x}{T}\right), \sin\left(\frac{2n\pi x}{T}\right)\right\} \tag{3}$$

that forms an **orthogonal basis** under the dot product defined in equation (2).

Using this basis functions, we solve the linear problem $Ax = b$, with A diagonal, containing

$$... \int_{\lambda}^{\lambda+T} \sin^2\left(k\frac{2\pi x}{T}\right)\,dx \quad , \quad \int_{\lambda}^{\lambda+T} \cos^2\left(k\frac{2\pi x}{T}\right)\,dx, ...$$

on the main diagonal.

$$b = \begin{pmatrix} \int_{\lambda}^{\lambda+T} f(x)\,dx \\ \int_{\lambda}^{\lambda+T} f(x)\cos\left(\frac{2\pi x}{T}\right)\,dx \\ \int_{\lambda}^{\lambda+T} f(x)\sin\left(\frac{2\pi x}{T}\right)\,dx \\ \vdots \\ \int_{\lambda}^{\lambda+T} f(x)\cos\left(\frac{n2\pi x}{T}\right)\,dx \\ \int_{\lambda}^{\lambda+T} f(x)\sin\left(\frac{n2\pi x}{T}\right)\,dx \end{pmatrix}$$

Example 2.2 (Discrete Case) *In this example, we model the temperature variations in Alaska using a trigonometric interpolation based on Fourier series. The temperature data (in Fahrenheit) is measured every 28 days, covering an entire year. Our goal is to fit the data using a Fourier series expansion with up to 6 harmonics.*

Temperature Data *The recorded temperature values are given as:*

$$T = [-14, -8, 2, 15, 35, 52, 62, 63, 58, 50, 34, 12, -5]$$

where each value corresponds to a measurement taken every 28 days.

We use the following set of basis functions for the trigonometric interpolation:

$$\phi_1(x) = 1, \quad \phi_2(x) = \sin(x), \quad \phi_3(x) = \cos(x), \ldots, \phi_{12}(x) = \sin(6x), \quad \phi_{13}(x) = \cos(6x)$$

Using these **basis functions**, we seek to represent the temperature function $T(x)$ as:

$$T(x) \approx c_1 + c_2\sin(x) + c_3\cos(x) + c_4\sin(2x) + c_5\cos(2x) + \cdots + c_{13}\cos(6x)$$

where the coefficients c_i are determined by fitting the model to the given data points.

This Fourier-based approach provides a smooth, continuous approximation of temperature variations, effectively capturing periodic seasonal trends. The method can be extended with additional harmonics for improved accuracy.

Example 2.3 *Approximate by the fifth-order Fourier series for the function*

$$f(x) = x, \quad \textit{for } x \in [0, 3]$$

considering the period $T = 3$.

Solution:

$$f(x) \approx a_0 + \sum_{k=1}^{5}\left(a_k\cos\left(\frac{2\pi kx}{T}\right) + b_k\sin\left(\frac{2\pi kx}{T}\right)\right)$$

where the Fourier coefficients are computed as:

$$a_0 = \frac{1}{T}\int_0^T f(x)\,dx, \quad , \ldots, a_n = \frac{2}{T}\int_0^T f(x)\cos\left(\frac{2\pi nx}{T}\right)dx, \quad b_n = \frac{2}{T}\int_0^T f(x)\sin\left(\frac{2\pi nx}{T}\right)dx.$$

Example 2.4 *Do the same for the function $f(x) = (x-\pi)^2$, in $[0, 2\pi]$, with the period $T = 2\pi$, using the sixth order Fourier series.*