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# S3-Euclidean Spaces Part II

Linear Algebra

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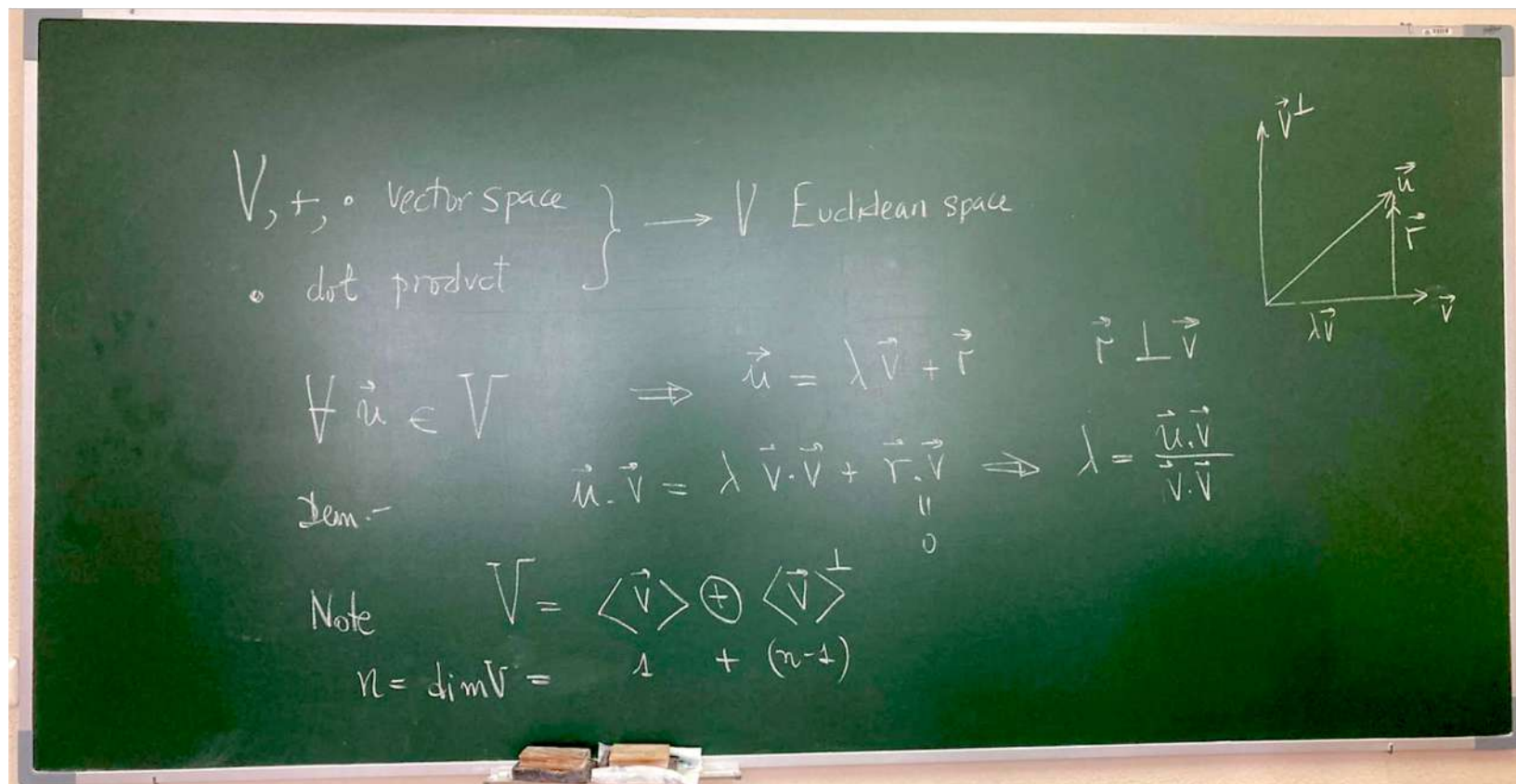
## Orthogonality

- **Orthogonality:** Two vectors  $\mathbf{u}$ ,  $\mathbf{v}$  are orthogonal if  $\mathbf{u} \cdot \mathbf{v} = 0$ .
- **Unit vector:**  $\frac{\mathbf{u}}{\|\mathbf{u}\|}$
- **Orthogonal basis:**
  - $V = \langle \mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n \rangle$
  - $\mathbf{v}_1 \cdot \mathbf{v}_2 = 0$ .
- **Theorem:** Any vector in an Euclidean project any non-null vector  $\mathbf{u}$ , can be decomposed in a vector in the direction of  $\mathbf{v}$  and another which is orthogonal to  $\mathbf{v}$ .

# Orthogonal projections

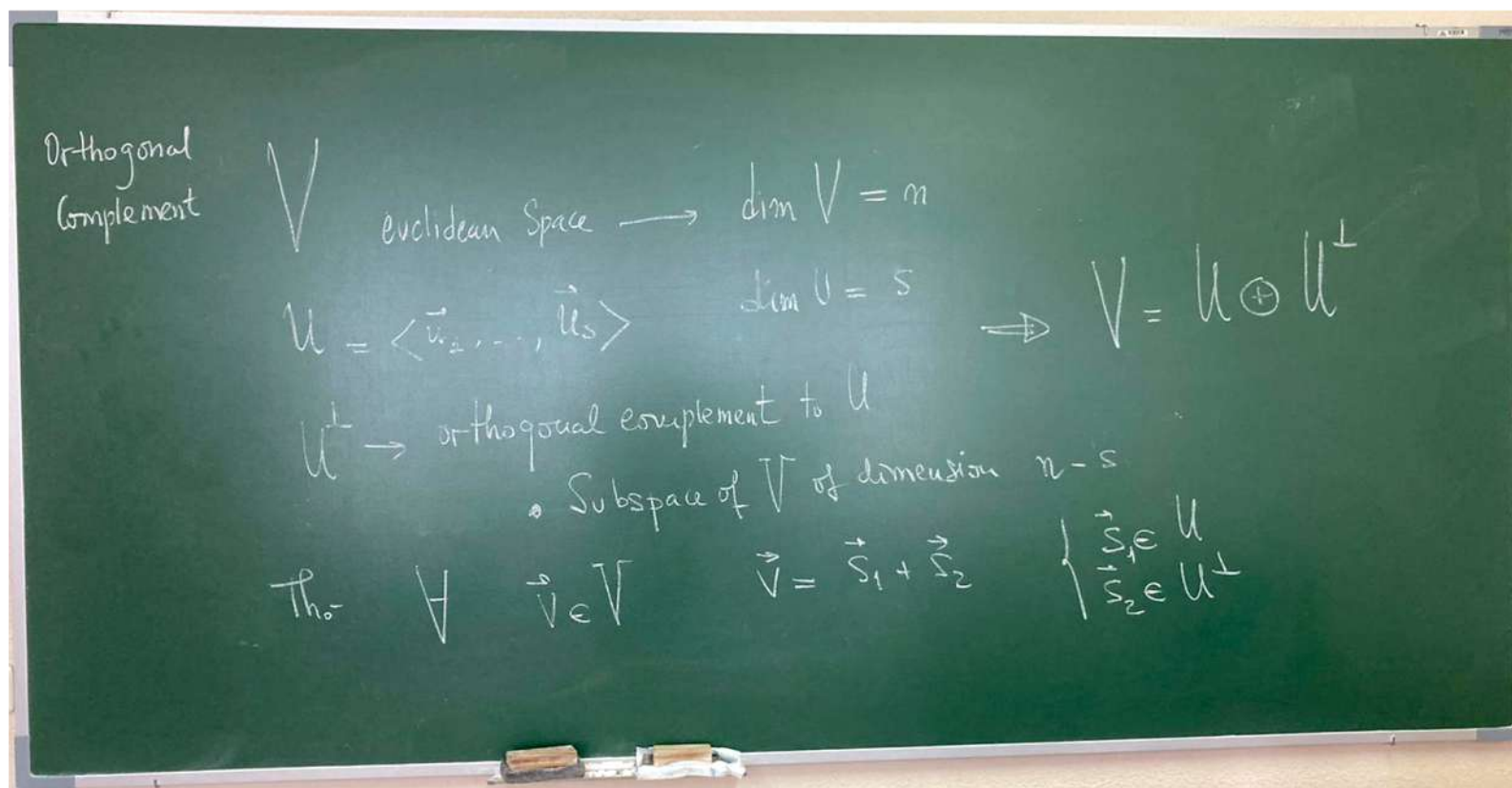


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## Orthogonal projections



# Orthogonal complement



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Orthogonal  
Complement

$$U = \langle \vec{u}_1, \dots, \vec{u}_s \rangle$$

$$U^\perp = \{ \vec{w} \in V : \vec{w} \cdot \vec{u} = 0 \quad \forall \vec{u} \in U \}$$

$$\vec{u} = \sum_{k=1}^s \alpha_k \vec{u}_k \Rightarrow \vec{w} \cdot \vec{u} = \sum_{k=1}^s \alpha_k (\vec{w} \cdot \vec{u}_k) = 0 \quad \forall \vec{w}$$

$$\vec{w} \cdot \vec{u}_k = 0 \iff k=1, s.$$

$\vec{w}$  has to be orthogonal to any basis set of  $U$ .





## Orthogonal projections into a subspace

Orthogonal Complement

$$U = \langle \vec{u}_1, \dots, \vec{u}_s \rangle$$

$$\forall \vec{v} \in V \quad \vec{v} = \sum_{k=1}^s \alpha_k \vec{u}_k + \vec{r} \quad \vec{r} \in U^\perp$$

$$\vec{v} \cdot \vec{u}_j = \sum_{k=1}^s \alpha_k (\vec{u}_k \cdot \vec{u}_j) + \vec{r} \cdot \vec{u}_j$$

$$\begin{pmatrix} \vec{u}_1 \cdot \vec{u}_1 & \vec{u}_1 \cdot \vec{u}_2 & \dots & \vec{u}_1 \cdot \vec{u}_s \\ \vdots & \vdots & \ddots & \vdots \\ \vec{u}_s \cdot \vec{u}_1 & \dots & \dots & \vec{u}_s \cdot \vec{u}_s \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_s \end{pmatrix} = \begin{pmatrix} \vec{v} \cdot \vec{u}_1 \\ \vec{v} \cdot \vec{u}_2 \\ \vdots \\ \vec{v} \cdot \vec{u}_s \end{pmatrix}$$

orthogonal & orthonormal basis !!

$j = 1, \dots, s$

if  $\vec{u}_i \cdot \vec{u}_j = 0$  for  $i \neq j$

$\Rightarrow$  This system is solved very easily



## Bit of Practice

Considering in  $\mathbb{R}^2$  the Gram matrix:

$$G = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

1. Finding the norm of  $\mathbf{u} = (1,1)$ .
2. The angle that forms  $\mathbf{u} = (1,1)$  and  $\mathbf{v} = (1, -1)$ . Are they orthogonal?
3. Find one vector orthogonal to  $\mathbf{u}$ .

Considering in  $\mathbb{R}^3$  the scalar product that in the canonic basis has the following Gram matrix:

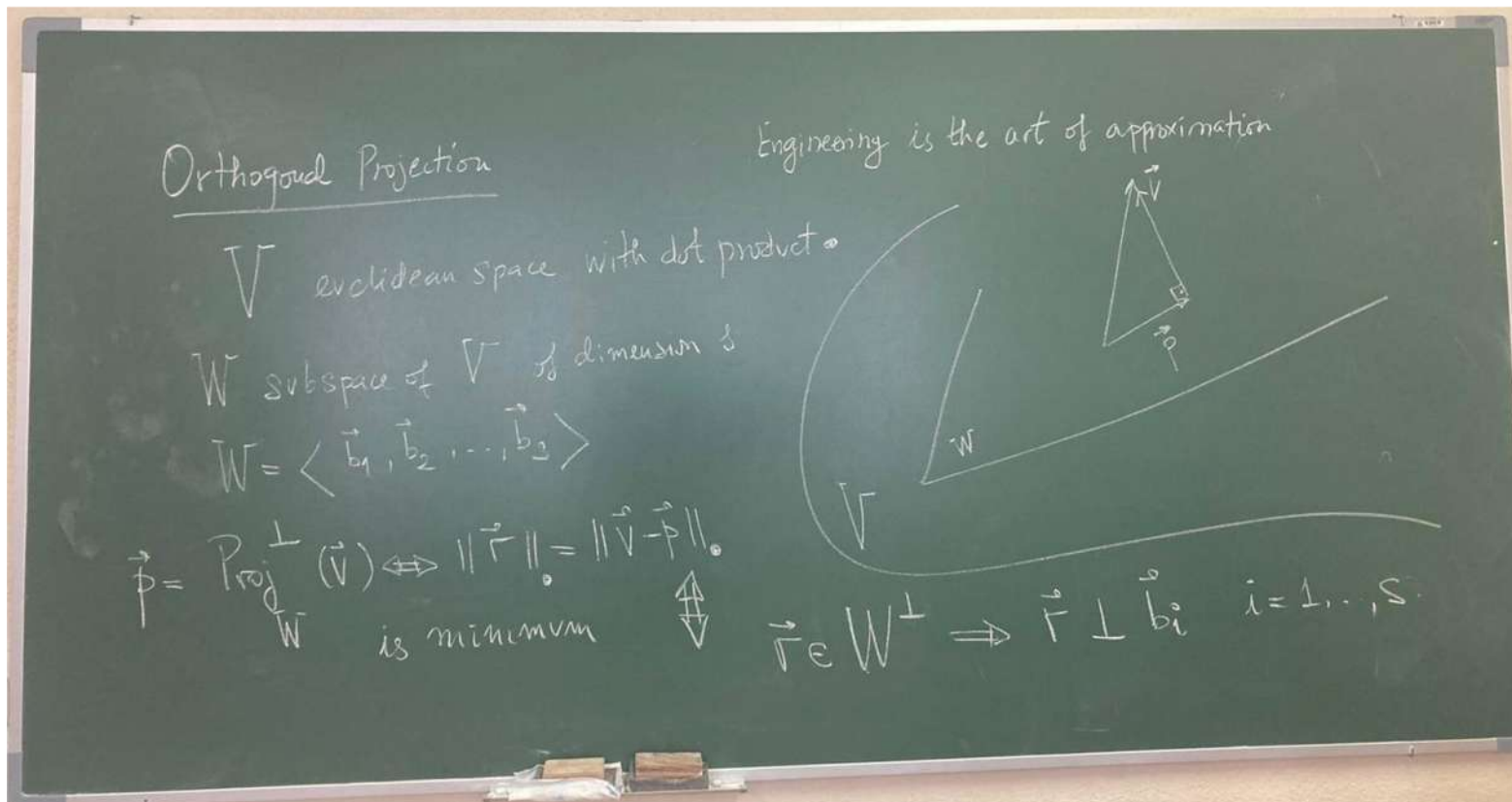
$$G = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

1. Finding the angle that forms  $\mathbf{u} = (1,0,0)$  and  $\mathbf{v} = (0,1,0)$ . Are they orthogonal?
2. Confirm that  $\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| \cos \langle \mathbf{u}, \mathbf{v} \rangle$

# Orthogonal projection



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**V can be vector space of infinite dimension**

**W is always a subspace of finite dimension**



# Orthogonal projection- The normal equations



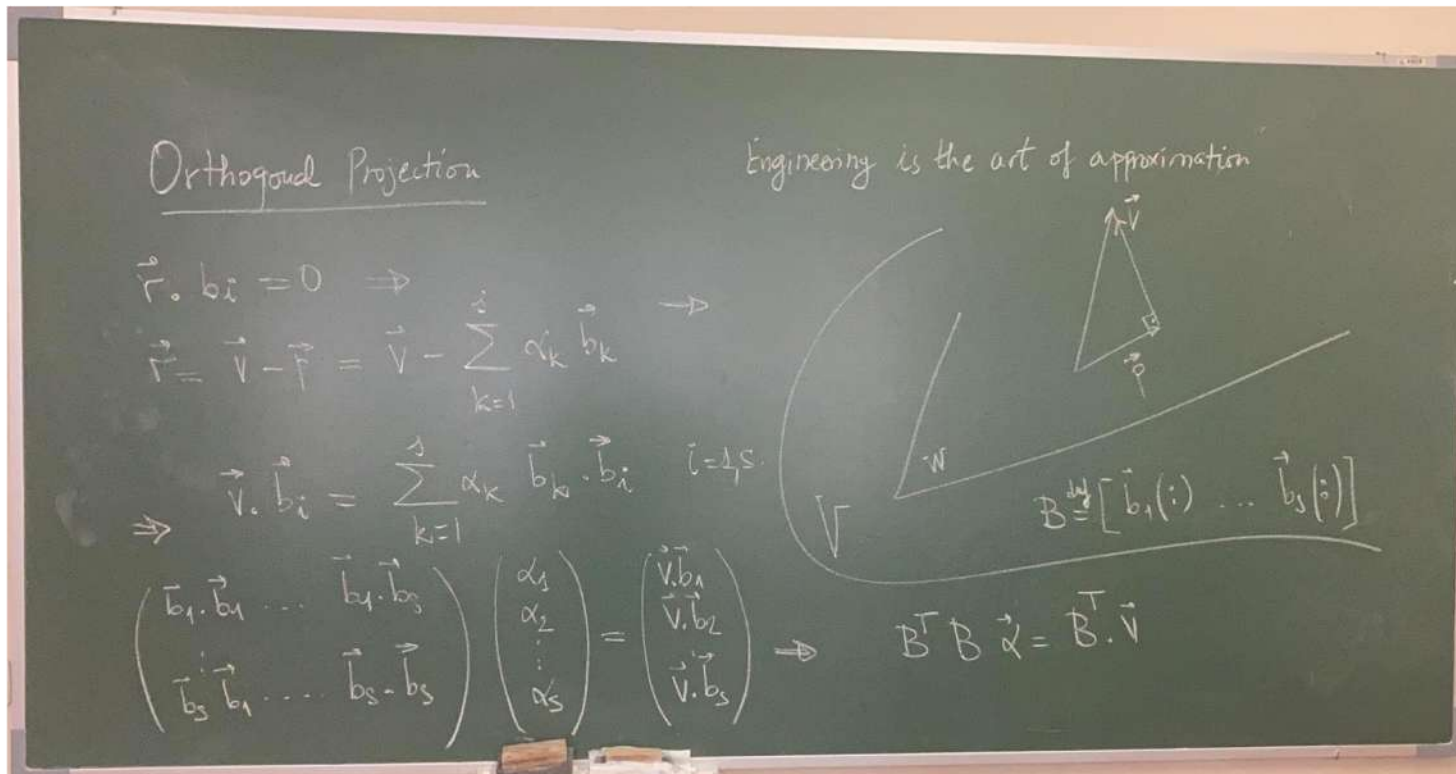
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What happens if the basis set of  $\mathbf{W}$  is orthogonal?

$$\alpha_k = \frac{\mathbf{v} \cdot \mathbf{b}_k}{\|\mathbf{b}_k\|^2}$$

And if the basis set is orthonormal?

$$\alpha_k = \mathbf{v} \cdot \mathbf{b}_k$$





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## Bit of Practice

1. Finding the orthogonal projection of  $f(x) = e^x, x \in [0,1]$  onto

$$\mathbf{W} = \langle 1, x, x^2 \rangle.$$

2. Finding the orthogonal projection of  $\mathbf{v} = (1, 1, -2)$  onto

$$\mathbf{W} = \{(x, y, z) : x + z = 0\}.$$

3. Finding the orthogonal projection of the matrix  $A = \begin{pmatrix} 1 & 2 \\ 1 & -2 \end{pmatrix}$  into the subspace of symmetric matrices.

Do we obtain the same result using the column-wise and the row-wise scalar product?

# Gram-Schmidt method



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Gram-Schmidt method

$V, +, \cdot$  Euclidean space

$W = \langle \vec{b}_1, \dots, \vec{b}_s \rangle$  subspace  $\Rightarrow$

$\downarrow$  GS algorithm

orthogonal basis

$\{\vec{o}_1, \vec{o}_2, \dots, \vec{o}_s\}$

$W = \langle \vec{b}_1, \dots, \vec{b}_s \rangle = \langle \vec{o}_1, \dots, \vec{o}_s \rangle$

$\vec{o}_1 = \vec{b}_1$

$\vec{o}_2 = \vec{b}_2 - \vec{p}_{\text{proj}}^\perp(\vec{b}_2)_{\langle \vec{o}_1 \rangle}$

$\vec{o}_3 = \vec{b}_3 - \vec{p}_{\text{proj}}^\perp(\vec{b}_3)_{\langle \vec{o}_1, \vec{o}_2 \rangle}$

$\vec{o}_s = \vec{b}_s - \vec{p}_{\text{proj}}^\perp(\vec{b}_s)_{\langle \vec{o}_1, \vec{o}_2, \dots, \vec{o}_{s-1} \rangle}$  last projection



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## Bit of Practice

1. Finding an orthogonal basis  $\mathbf{W} = \langle 1, x, x^2 \rangle, x \in [0,1]$  using the common scalar product defined in  $C_{[0,1]}^0$
2. Finding an orthogonal bases of  
$$\mathbf{W} = \{(x, y, z) : x + z = 0\},$$
using the scalar product in  $\mathbb{R}^3$ .
3. Finding an orthogonal basis of the subspace of symmetric matrices in  $M_{3 \times 3}(\mathbb{R})$  using the columnwise scalar product.