CALCULUS
DEGREE IN SOFTWARE ENGINEERING
CHAPTER 2. ELEMENTARY REAL FUNCTIONS OF A SINGLE VARIABLE.

In the following, we will briefly study the basic real functions we are going to use during this course, showing their domains, ranges and graphs. We will begin with polynomials

1. **Polynomials**

   Polynomials are linear combinations of natural powers of a real variable x with real coefficients a_n

   $$P(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0$$

   The number n is the degree of the polynomial, a polynomial of degree two is also called quadratic, if the degree is three, we call it cubic..... x^2 is said x squared, x^3 is x cubed, x^4 is x to the four and so on.

   The domain of polynomials is the whole set of real numbers $\mathbb{R}$. The range depends on the degree of the polynomial. If the degree is odd, it is easy to see that the range is also $\mathbb{R}$, we will prove it by using the Intermediate Value Theorem in a future chapter. However, if the degree is even, the range is either $[m, \infty]$ or $[-\infty, M]$, with m the global minimum of the polynomial and M the global maximum. The first case corresponds to $a_n > 0$ and the second to $a_n < 0$. Some examples of graphs of polynomials are shown in the file Figures.

2. **Rational Functions**

   Rational functions are quotients of polynomials

   $$R(x) = \frac{P(x)}{Q(x)}$$

   The domain is the set $D = \{x \in \mathbb{R}/Q(x) \neq 0\}$. The range depends on the particular function we are studying and will usually be obtained by plotting the function. We will show examples in Figures and in the chapter dedicated to graphs.

3. **Algebraic Functions**

   Though their precise mathematical definition is more complicated - an algebraic function is any root of a polynomial equation with polynomial coefficients- in practice, we will consider functions involving the basic algebraic operations:

addition, subtraction, multiplication, division and raising to a fractional power. For instance

$$(x^2 + 5x + 6)^{1/3}$$

or

$$\frac{\sqrt{1 + x^3}}{(x^2 - 2x + 5)^{1/4}}$$

are algebraic functions. When we consider $P(x)^{1/n}$, with $P(x)$ a polynomial and n a natural number, the domain is $\mathbb{R}$ if n is odd, but it is the set of values of x for which the radicand is greater than or equal to zero if n is even. We will do exercises of this type, determining the domain of algebraic functions, in Exercises 1.

4. **Exponential Functions**

   General exponential functions are those that can be written as a^x, with $a > 0$. Now, the variable is the exponent. All general exponential functions can also be expressed as e^{bx} with e the number e, base of the natural logarithms. We call e^x, the exponential function. $e = 2.718481848....$ is, together with π, the most famous irrational number, as it appears naturally in real problems such as: radioactive decay, population growth, spread of an infection.. it can be proved that it appears in the solution of basic differential equations of the type

   $$\frac{dy}{dx} = ky$$

   it can also be defined by calculating the area under a hyperbola over a certain interval. Therefore, e^{bx} is a ubiquitous function. If $b > 0$ it grows very, very fast, exponentially, and if $b < 0$, it decreases very quickly.

5. **Logarithmic Functions**

   If $y = e^x$, x is called the natural logarithm of y, so we can define the function $y = ln(x)$. In general, if $y = a^x$, x is the logarithm of y to base a. $y = log_a(x)$ is the logarithmic function to base a. There are three bases that are very common because of their use in science and computing: e, 10 and 2. To pass from any base to the natural logarithm, we use the formula

   $$log_a(x) = \frac{ln(x)}{ln(a)}$$

   The domain of the exponential function is $\mathbb{R}$, but its range is the set of positive real numbers. Because of this, the domain of any logarithmic function is the set of positive reals and the range, the whole set of real numbers. A logarithmic function to base a is always increasing if $a > 1$, the typical case, and always decreasing if $a < 1$. For $a = 1$, the function is not defined, as $y = 1^x$ is constant, 1. Though we have not defined inverse functions yet, it is easy to see that the logarithmic function to base a is the inverse function of

$y = a^x$. The general exponential function is one-to-one and that enables us to define its inverse function: the logarithmic function.

In the past, logarithms were very useful to carry out multiplications and divisions of large numbers, applying that $log_a(xy) = log_a(x) + log_a(y)$. Base-ten logarithms were used to construct tables of logarithms, which made the performance of these operations easier. In the 1970s, with the advent of calculators, tables of logarithms became obsolete.

6. **Trigonometric Functions**

Trigon comes from three angles in ancient Greek, triangle, and the trigonometric functions are those constructed from the ratios between the sides of a right triangle. I remind you of the basic definitions: if we take an angle α beween the hypotenuse of length c and the adjacent side, a, and call b the length of the opposite side, we define the trigonometric functions: sine, cosine, tangent, cotangent, secant and cosecant in the following way

$$\sin x = \frac{b}{c} \ , \cos x = \frac{a}{c} \ , \tan x = \frac{b}{a} \ , \cot x = \frac{a}{b} \ , \sec x = \frac{c}{a} \ , \csc x = \frac{c}{b}$$

For these functions, it is very convenient to work with the angle in radians. A radian is the angle subtended at the centre of a circle by an arc equal in length to the radius. A radian is equal to about 57.296 degrees. We show the graphs of the trigonometric functions in the file Figures. Now, we will explore the domain and ranges of these functions. The domain of the sine is $D = \mathbb{R}$ and the range is the closed interval $[-1, 1]$. The graph of the cosine can be obtained by using the property, $\cos x = \sin(x + \pi/2)$, that is, the graph of the cosine is that of the sine shifted $\pi/2$ units to the left on the x-axis. Of course, both sine and cosine have the same domain and range.
The domain of the tangent is $D = \{x \in \mathbb{R}/x \neq (2k+1)\pi/2, k \in \mathbb{Z}\}$ and the range $(-\infty, \infty)$. The domain excludes all the numbers for which the cosine is zero. The domain of the cotangent is $D = \{x \in \mathbb{R}/x \neq k\pi, k \in \mathbb{Z}\}$ and the range is $\mathbb{R}$. Finally, the domain of the secant and the tangent is the same, as are the same those of cosecant and cotangent. The range of the secant and the cosecant is

$$f(D) = (-\infty, -1] \cup [1, \infty)$$

Throughout the course, we will review the basic properties of these functions. For now, I recommend you look at your high school mathematics books and/or the internet when you need to use some basic property of the trigonometric functions. Check the file Figures for the basic graphs. In the next chapter, we will define inverse function and then will focus on the inverse trigonometric functions.