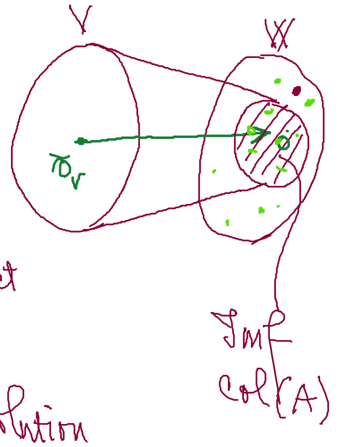
$f: V \xrightarrow{A} W$ $\boxed{AX = b}$ $b \in W$




$\boxed{\exists \text{ solution}}$
$b \in Col(A)$
Imf

Unique: $\ker f = \{\vec{0}\} \Rightarrow f$ injective
$\rightarrow$ If $\exists$ sol for $\forall b \in W \Rightarrow \operatorname{Im} f = W$
f surject

Infinity of sol: Underdetermined system
$\dim \ker f > 0 \rightarrow$ min norm-solution
$(m < n)$, $r = \min(m, n)$

$\boxed{\nexists \text{ solution}}$
$b \notin \operatorname{Im} f$

$r = \min(m, n)$ $(m > n)$ $\rightarrow$ Overdetermined syst
$x_{LS}$ = approx. sol

$r < \min(m, n) \rightarrow$ pinv(A) Rank Deficient Syst.

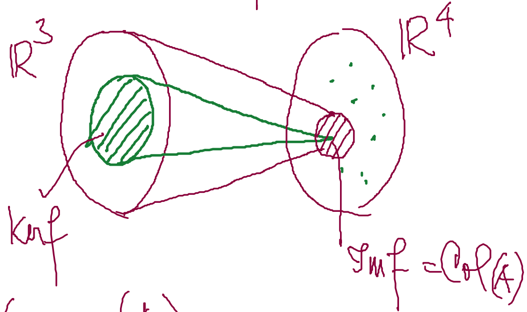
Example: $\boxed{1}$ $f: \mathbb{R}^3 \rightarrow \mathbb{R}^4$, $\ker f = \langle \bar{e}_1, \bar{e}_2 \rangle$, $f(\bar{e}_3) = (1,1,1,1)$

$B = \{\bar{e}_1, \bar{e}_2, \bar{e}_3\}$ st. basis in $\mathbb{R}^3$

$$A = \begin{pmatrix} f(\bar{e}_1) & f(\bar{e}_2) & f(\bar{e}_3) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow r = 1$$

$\bar{e}_1 \in \ker f \Rightarrow f(\bar{e}_1) = \vec{0}_{\mathbb{R}^4}$ $\operatorname{Im} f = \langle f(\bar{e}_3) \rangle \rightarrow \dim \operatorname{Im} f = 1$
$\bar{e}_2 \in \ker f \Rightarrow f(\bar{e}_2) = \vec{0}_{\mathbb{R}^4}$

$\dim \ker f = 2 \rightarrow \dim \operatorname{Im} f = 3 - 2 = 1$




(1) $b \notin \operatorname{Im} f$ . $b \neq \lambda(1,1,1,1) \Rightarrow \nexists$ sol!!

$b = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} \longrightarrow X = b$ $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$ $m = 4$, $n = 3$, $r = 1 \neq \min(m, n)$

$\Rightarrow$ Rank Def. Sys. $\longrightarrow$ pinv(A)·b app. of the sol.

(2) $b \in \operatorname{Im} f = Col(A) \Rightarrow b = \lambda \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \rightarrow b = \begin{pmatrix} 2 \\ 2 \\ 2 \\ 2 \end{pmatrix}$

$\Rightarrow \exists$ sol $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ \ldots \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 2 \\ \ldots \end{pmatrix} \Rightarrow r = 1 = m \Rightarrow$ Underdet.

$$\Rightarrow (\exists \text{ sol}) \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 2 \\ 2 \end{pmatrix} \Rightarrow r = 1 = m \Rightarrow \underline{\text{Underdet.}}$$

$\dim \operatorname{Ker} f = 2$ $\qquad m < n$

$$\boxed{x_{min} = X_0 + \alpha \bar{e}_1 + \beta \bar{e}_2} \qquad z = 0$$

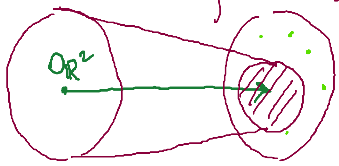
$$\begin{cases} x_0 = 0 \\ y_0 = 0 \\ z_0 = 0 \end{cases} \Rightarrow x_{min} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$\boxed{2}$ $f: \mathbb{R}^2 \to \mathbb{R}^3$, $f(\bar{e}_1) = (1,1,1)$ ; $f(\bar{e}_2) = (1,1,0)$ $\qquad \boxed{AX = b}$

$$A = \begin{pmatrix} f(\bar{e}_1) & f(\bar{e}_2) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 0 \end{pmatrix} \Rightarrow r = \operatorname{rank}(A) = 2$$

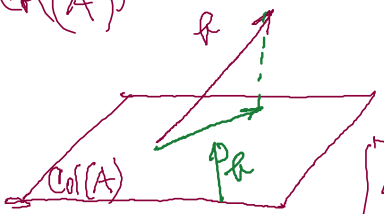
$$\Rightarrow \dim \operatorname{Im} f = 2 \Rightarrow \dim \operatorname{Ker} f = 2 - 2 = 0 \Rightarrow \operatorname{Ker} f = \{\bar{0}_{\mathbb{R}^2}\} \Rightarrow f \text{ inject}$$

(1) $\underline{b \notin \operatorname{Im} f}$ $\qquad \underbrace{\operatorname{Im} f}_{\dim 2} \subset \underbrace{\mathbb{R}^3}_{\dim 3}$




$$b \neq \alpha \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \longrightarrow b = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \notin \underset{\operatorname{Col}(A)}{\operatorname{Im} f} \Rightarrow \nexists \text{ sol.} \Rightarrow LS$$

$\underline{r = 2 = \min(m, n)}$ $\begin{cases} m = 3 \\ n = 2 \end{cases}$ $\qquad AX = b$




Approximation of a sol. by LS: $\qquad \boxed{AX = p_b}$

$$x_{LS} = \underbrace{(A^t \cdot A)}^{-1} \cdot A^t \cdot b = \begin{pmatrix} 3 \\ -3/2 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 2 \\ 2 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 3/2 \end{pmatrix}$$

$\boxed{\text{Ex.1}}$ $f: \mathbb{R}_2[x] \to \mathbb{R}^3$, $f(a + bx + cx^2) = (b + c, 0, a)$ $\qquad \boxed{AX = b}$

$$A = \begin{pmatrix} f(1) & f(x) & f(x^2) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad r = 2 \Rightarrow \operatorname{Im} f = \langle \underbrace{(0,0,1)}_{\bar{e}_3}, \underbrace{(1,0,0)}_{\bar{e}_1} \rangle$$

$$\Rightarrow \dim \operatorname{Ker} f = 3 - 2 = 1$$

$$\underline{\operatorname{Ker} f}: AX = 0 \Rightarrow \begin{cases} x = 0 \\ y = \alpha \\ z = -\alpha \end{cases} \Rightarrow \operatorname{Ker} f = \langle (0, 1, -1) \rangle$$

(1) $b \in \operatorname{Im} f \Rightarrow b = \alpha \bar{e}_1 + \beta \bar{e}_3 = \begin{pmatrix} \alpha \\ 0 \\ \ldots \end{pmatrix}$ $\qquad b = \begin{pmatrix} 1 \\ 0 \\ \ldots \end{pmatrix}$

(1) $\underline{b \in \operatorname{Im} f} \Rightarrow b = \alpha \bar{e}_1 + \beta \bar{e}_2 = \begin{pmatrix} \alpha \\ 0 \\ \beta \end{pmatrix}$ $\qquad b = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$

$AX = b \Rightarrow \exists$ sol $\qquad \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \Rightarrow m = 2, \; n = 3$

<u>Underdet. syst</u> $\qquad r = \min(m, n)$

$x_{min} = X_0 + \lambda(0, 1, -1) \quad ; \quad X_0 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$

$y + z = 1$

$x_{min} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + \lambda(0, 1, -1) = \begin{pmatrix} 2 \\ 1+\lambda \\ -\lambda \end{pmatrix}, \; \lambda \in \mathbb{R}$

(2) $b \notin \operatorname{Im} f$ $(\operatorname{Col}(A))$, $\; b = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \in \operatorname{Im} f \Rightarrow \nexists$ sol $\qquad \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ $\qquad 0 = 2$, $\nexists$ b

$r = 2 < m = n = 3$

$\Rightarrow$ rank Def. Syst $\rightarrow$ pinv(A) ...

$\boxed{\text{EX. 4}}$

$$f : \mathbb{R}^4 \xrightarrow{M} \mathbb{R}_2[x] \qquad B_{S_1} \xrightarrow{P} B_1$$

```tikzcd
B_1 \arrow[d, dashed, "P^{-1}"'] & B_{S_2} \arrow[d, dashed, "I"] \\
B_{S_1} & B_{S_2} \arrow[l, "A"']
```

$A = I^{-1} M P^{-1} = \underline{MP^{-1}} \quad , \quad M = \begin{pmatrix} 1 & 0 & -1 & 2 \\ 1 & 2 & -1 & 3 \\ 0 & 0 & 0 & -1 \end{pmatrix}$

$P = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$

- $\operatorname{Ker} f$
- $\operatorname{Im} f$