CALCULUS
DEGREE IN SOFTWARE ENGINEERING
CHAPTER 7. CONTINUITY THEOREMS.

First, we state Bolzano's theorem, a fundamental result that guarantees the existence of a zero of a function, that is, a point c at which $f(c) = 0$, under very simple conditions.

BOLZANO'S THEOREM

If $f(x)$ is continuous on $[a, b]$ and $f(a)f(b) < 0$, then there exists at least one number c in (a, b) such that $f(c) = 0$.

That is, if f is continuous on a closed, bounded interval and there is a change of sign, the graph of the function must cross the x-axis. This looks very intuitive; since the function is continuous, there cannot be jumps and when we draw the graph, we must intersect the x-axis, $f(c) = 0$.
However, the theorem is not true for a function defined on the rationals. For instance, x^2-2 on the interval $[1, 2]$ has a zero, $x = \sqrt{2}$, but it is not rational. In fact, the proof makes use of the existence of a lowest upper bound or supremum, a fundamental property of real numbers and of the sign preserving property of continuous functions. You can see (if you are interested) a proof of Bolzano's theorem on the internet. The idea is to use the method of bisection: you divide the interval $[a, b]$ into two subintervals again and again, and always choose the subinterval where there is a change of sign. In this way, you obtain smaller and smaller subintervals whose endpoints converge to a point inside the interval $[a, b]$. At this point the function cannot be positive or negative, taking into account the continuity of the sign.
Bolzano's theorem bears the name of Bernard Bolzano (1781-1848), an important Czech mathematician. Have a look at his biography on the internet, for instance in the English Wikipedia.

EXAMPLE

$f(x) = \cos x + x$ changes sign on the interval $[-\pi/2, 0]$, given that $f(-\pi/2) = -\pi/2$ and $f(0) = 1$. Taking also into account that the function is continuous on that interval, we can apply Bolzano's theorem and prove that there is at least a zero of the function on $(-\pi/2, 0)$. A simple procedure for approaching the zero would be choosing the midpoint of the interval and calculating the sign of the function there, $\cos(\pi/4) - \pi/4 = -0.078..$, which is negative. Therefore, there must be a zero in $[-\pi/4, 0]$. By this method, called bisection, we could continue to divide the interval and corner the root of the equation. The method can be useful and easy, but there are faster methods that use the derivative of the function.

A simple consequence of Bolzano's theorem is the Intermediate Value Theorem.

THE INTERMEDIATE VALUE THEOREM

If f is continuous on $[a, b]$ and if y_0 is any value between $f(a)$ and $f(b)$, then $y_0 = f(c)$ for some c in (a, b).

The proof is very simple: Imagine $f(a) < y_0 < f(b)$, we define $g(x) = f(x) - y_0$, then $g(a) < 0$ and $g(b) > 0$. According to Bolzano's theorem, there must be a number c in (a, b) such that $g(c) = 0$. Therefore $f(c) = y_0$. If $f(a) > y_0 > f(b)$, there is a slight change in the demonstration. Do the proof in this case.
The meaning of the theorem is crystal-clear: A continuous function on a closed and bounded interval takes on all the values between two given values of the function. In mathematical terms $[f(a), f(b)] \subseteq f([a, b])$

EXAMPLE

The function $f(x) = x^3 - x$ is equal to 1 at a certain point in $(1, 2)$. We check that $f(1) = 0$ and $f(2) = 6$. Therefore, since the function is continuous, it takes on all the values between 0 and 6 in that interval, 1 in particular. In this way, we have proved that $x^3 - x - 1 = 0$ has a root in $(1, 2)$.
Finally, we will state the following theorems:

BOUNDEDNESS THEOREM

If f is continuous on $[a, b]$, then it is bounded on $[a, b]$

Though the proof is not easy, the idea is quite simple: in order to be unbounded, the function must have an infinite discontinuity, what is impossible if the function is continuous. However, it is essential that the function be continuous on $[a, b]$. If the endpoints are not included, the function could go to ∞ or $-\infty$ at one of them. If the function has a finite number of jump discontinuities (finite), the theorem is also true.

THE EXTREME VALUE THEOREM

If f is continuous on a closed interval $[a, b]$, then f attains both an absolute maximum value M and an absolute minimum value m in $[a, b]$. That is, there are numbers x_1 and x_2 in $[a, b]$ with $f(x_1) = m$ and $f(x_2) = M$ and $m \leq f(x) \leq M$ for every other x in $[a, b]$.
We will see examples of the application of this theorem in the chapter dedicated to the study of maxima and minima.