CALCULUS
DEGREE IN SOFTWARE ENGINEERING
CHAPTER 11. CONCAVITY AND CURVE SKETCHING.

In this chapter we will see that the second derivative (the derivative of the derivative) gives us information about how the graph of a differentiable function bends or turns. With this knowledge, coupled with our previous use of the first derivative to determine where a function is increasing or decreasing and its extreme values, we can now draw an accurate graph of a function. By organizing all this knowledge and the information about asymptotic behaviour and symmetries, we will present a method for sketching graphs and revealing visually the key features of functions.

CONCAVITY

The graph of a differentiable function is **concave up** on an open interval I if f' is increasing on I.
The graph of a differentiable function is **concave down** on an open interval I if f' is increasing on I.

If the function has a second derivative, we can apply Corollary 2 of the Mean Value Theorem to the first derivative and conclude

THE SECOND DERIVATIVE TEST FOR CONCAVITY

Let $y = f(x)$ be twice-differentiable on an open interval I

1) If $f'' > 0$ on I, the graph of f over I is concave up.

2) If $f'' < 0$ on I, the graph of f over I is concave down.

EXAMPLE

The function $y = x^3$ has $y' = 3x^2$ as first derivative. We see that it is positive on $(-\infty, 0)$ and $(0, \infty)$. So, the function is incresing on $(-\infty, \infty)$. However, its second derivative $y'' = 6x$ is negative on $(-\infty, 0)$ (concave down) and positive on $(0, \infty)$ (concave up). The basic idea is that the function is always increasing, but its increase slows down (the increase is slower as x increases) for $x < 0$ and speeds up (it is faster as x increases) for $x > 0$. Graphically, the curve rises above its tangents if $x > 0$ (concave up), but if $x < 0$ its behaviour is different: the curve falls below its tangents (concave down).
What happens if the function is decreasing, for instance $y = -x^3$. In this case, if $x < 0$, the graph is concave up, $y'' = -6x$, and the function decreases slower as we move from left to right. If $x < 0$, the graph is concave down and the function

decreases faster. Graphically, the curve rises above its tangents if $x < 0$ (concave up), but if $x > 0$ the curve falls below its tangents (concave down).
We can see both graphs below. In both cases, there is a change of concavity at $x = 0$. It is a critical point but it is not a maximum point or a minimum point. The corresponding point of the graph is called a **point of inflection** and we define it precisely hereunder.

POINT OF INFLECTION. DEFINITION

A point where the graph of a function has a tangent line and where the concavity changes is a **point of inflection**.
At a **point of inflection** $(c, f(c))$, either $f''(c) = 0$ or $f''(c)$ fails to exist.

It is interesting to point out that the function can have a vertical tangent at a point of inflection (infinite derivative). For instance, $y = x^{1/3}$ has a vertical tangent at $x = 0$, $y'(0) = \infty$, the second derivative does not exist at that point, but $y''(x) = -2/9\,x^{-5/3}$ $y'' > 0$ for $x < 0$ (concave up) and $y'' < 0$ for $x > 0$, (concave down) so that there is a change of concavity at $(0, 0)$ and, according to the definition, this point of the graph is a point of inflection.
If f'' exists and is continuous over an interval containing c, a simple application of the Intermediate Value Theorem allows us to prove that f'' must be zero at c.
Our last result before setting out the basic rules for graphing functions concerns the use of second derivatives to classify critical points and identify local maxima and minima:

THE SECOND DERIVATIVE TEST FOR LOCAL EXTREMA

Suppose f'' is continuous on an open interval that contains $x = c$.

1) If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at $x = c$.
2) If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at $x = c$.
3) If $f'(c) = 0$ and $f''(c) = 0$, then the test fails. The function may have a local maximum, a local minimum or neither .

The proof rests on the fact that if, for instance, $f''(c) < 0$, f' will be decreasing on an open interval around $x = c$ and since $f'(c) = 0$, it will be positive (f increasing) for $x < c$ and negative for $x > c$ (f decreasing) . Then, the function will have a local maximum (First Derivative Test for Local Extrema). The case $f''(c) > 0$ can be proved in a similar way. Try to do it .
We will see a few examples of the application of this theorem when we graph functions in the next section.
Note that if $f'(c) = 0$ and $f''(c) = 0$ the graph of the function need not have a point of inflection at $x = c$. It could also have a maximum value, for instance $y = -x^4$ at $x = 0$, or a minimum value, $y = x^4$ at $x = 0$. The fact that $f''(c) = 0$ is a necessary condition, if the second derivative exists at the point, for a point of inflection, but it is not a sufficient condition.

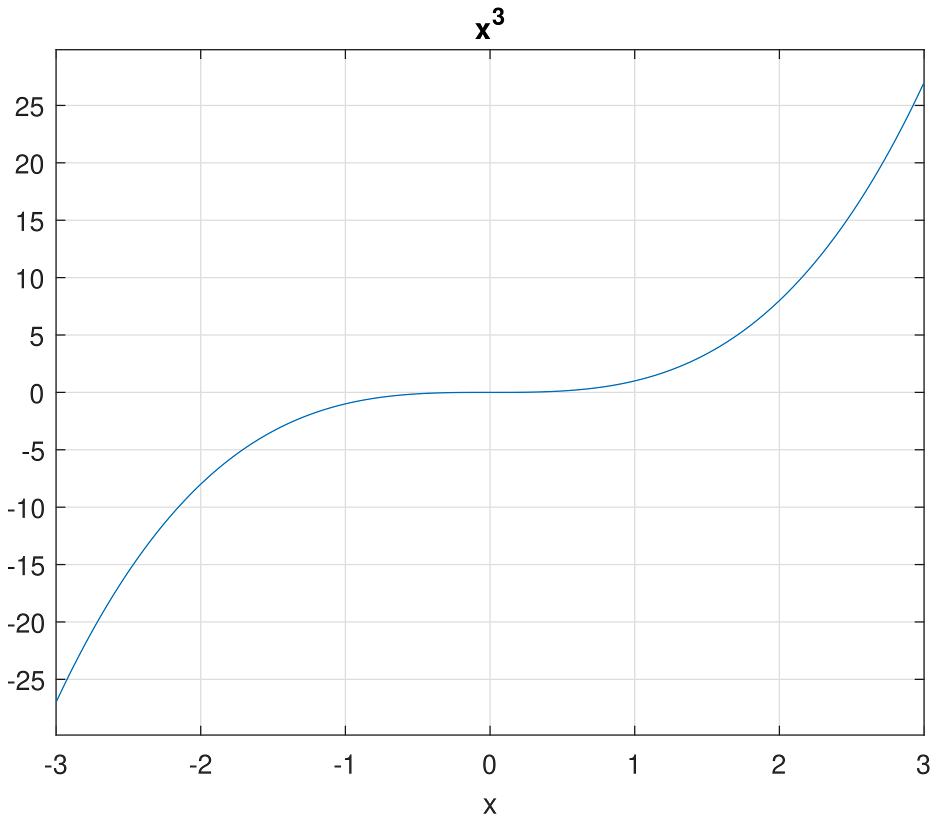

Figure 1: $y = x^3$

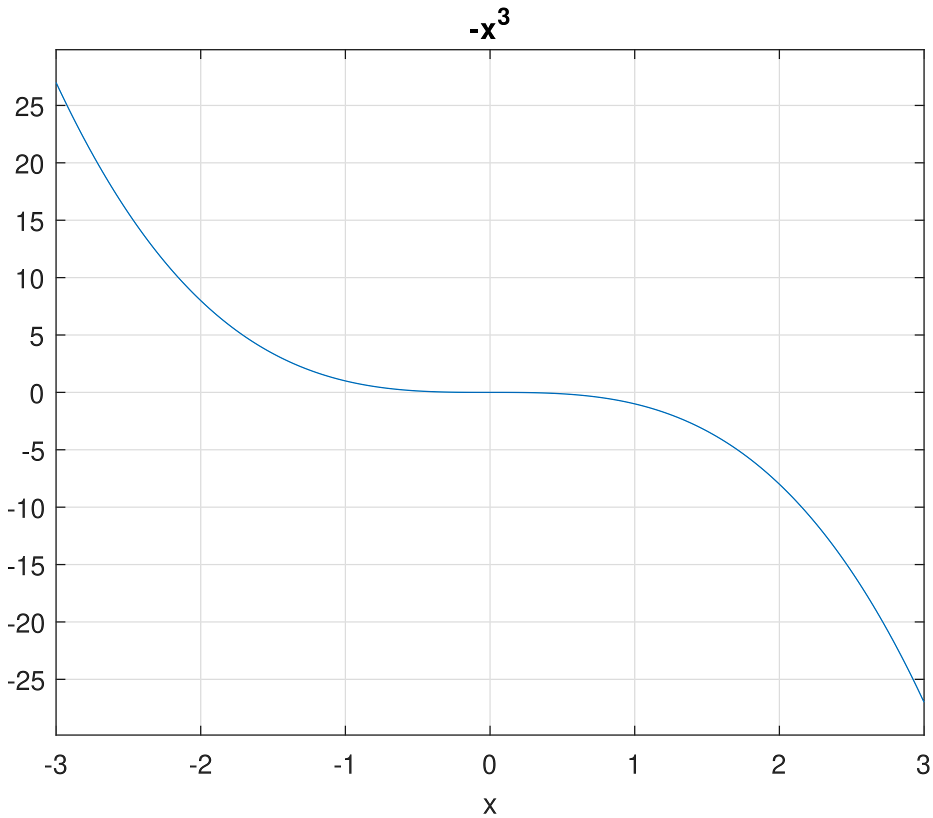

Figure 2: $y = -x^3$

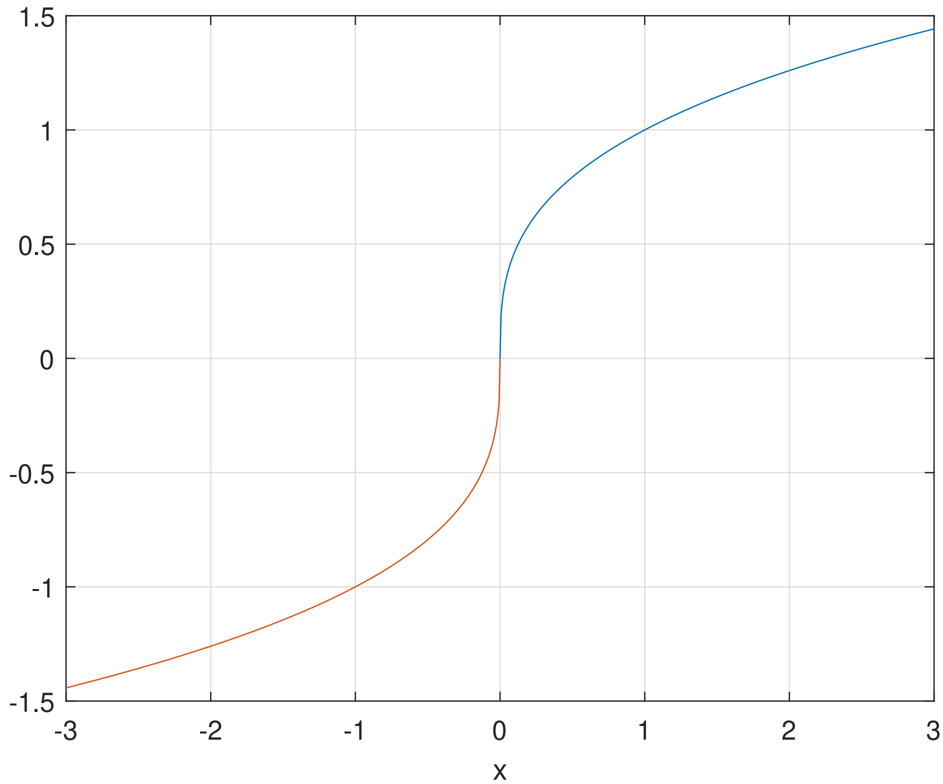

Figure 3: $y = x^{1/3}$

The function $y = x^2$ is the prototypical example of a local minimum (at $x = 0$), $x = 0$ is a critical point and $f''(0) = 2 > 0$. The function is concave up and in this case, since $f''(x) > 0$ on the whole domain, the local minimum is global (absolute). Can you see why?, $y = -x^2$ would also be a typical example of a local maximum. Is it absolute? Check that the graph is always concave down.
Finally, we will put together the results presented in this chapter and the previous ones in order to create a procedure for graphing the key features of a function.

PROCEDURE FOR GRAPHING A FUNCTION Y=F(X)

1) Identify the domain of f and any symmetries the curve may have.
2) Find the derivatives y' and y''.
3) Find the critical points of f, if any, and identify the function's behaviour at each one.
4) Find where the curve is increasing and where it is decreasing.
5) Find the points of inflection, if any occur, and determine the concavity of the curve.
6) Identify any asymptotes that may exist.
7) Plot key points, such as the intercepts and the points found in Steps 3-5, and sketch the curve together with any asymptotes that may exist.

EXAMPLE. PLOT THE GRAPH OF

$$y = \frac{x^2 + 4}{2x}$$

1) The function is defined at all points except $x = 0$. It is clear that $f(x)$ is odd, since $f(-x) = -f(x)$ and the graph will then be symmetric with respect to the origin of coordinates.

2) The derivatives are $y' = \dfrac{x^2 - 4}{2x^2}$ and $y'' = \dfrac{4}{x^3}$.

3) The critical points are $x = 2$ and $x = -2$. Since $f''(-2) = -1/2 < 0$ and $f''(2) = 1/2 > 0$, we see that a local maximum occurs at $x = -2$ with $f(-2) = -2$ and a local minimum occurs at $x = 2$ with $f(2) = 2$. Note that the local minimum value is greater than the local maximum value. Is this strange ?

4) On the interval $(-\infty, -2)$, the derivative is positive, because $x^2 - 4 > 0$ and f is incresing. On $(-2, 0)$, the derivative is negative $x^2 - 4 < 0$, and f is decreasing. Similarly, the function is decreasing on $(0, 2)$, negative derivative, and increasing on $(2, \infty)$, positive derivative. From this, we could have guessed that f has a local maximum at $x = -2$ and a local minimum at $x = 2$, without making use of the second derivative.

5) There are no points of inflection. The second derivative is negative on $(-\infty, 0)$, then the graph is concave down and it is positive on $(0, \infty)$, the graph is concave up.

6) The limits at $x = 0$ are

$$\lim_{x \to 0^+} f(x) = \infty$$

$$\lim_{x \to 0^-} f(x) = -\infty$$

Then, there is a vertical asymptote at $x = 0$. At ∞ and $-\infty$, the limits are infinite. Thus, there are no horizontal asymptotes. However, we can explore the possible existence of oblique asymptotes. These occur when the curve approaches a straight line (tilted) at ∞. We calculate

$$\lim_{x \to \infty} f(x)/x = 1/2$$

and

$$\lim_{x \to -\infty} f(x)/x = 1/2$$

If we now subtract $f(x) - x/2$ and calculate

$$\lim_{x \to \infty} f(x) - x/2 = 0$$

and

$$\lim_{x \to -\infty} f(x) - x/2 = 0$$

Therefore, $y = x/2$ is an oblique asymptote of $y = f(x)$

Putting Steps 1-6 together, we sketch the graph of the function and its oblique asymptote. For a better visualization, we have plotted the function with Matlab.

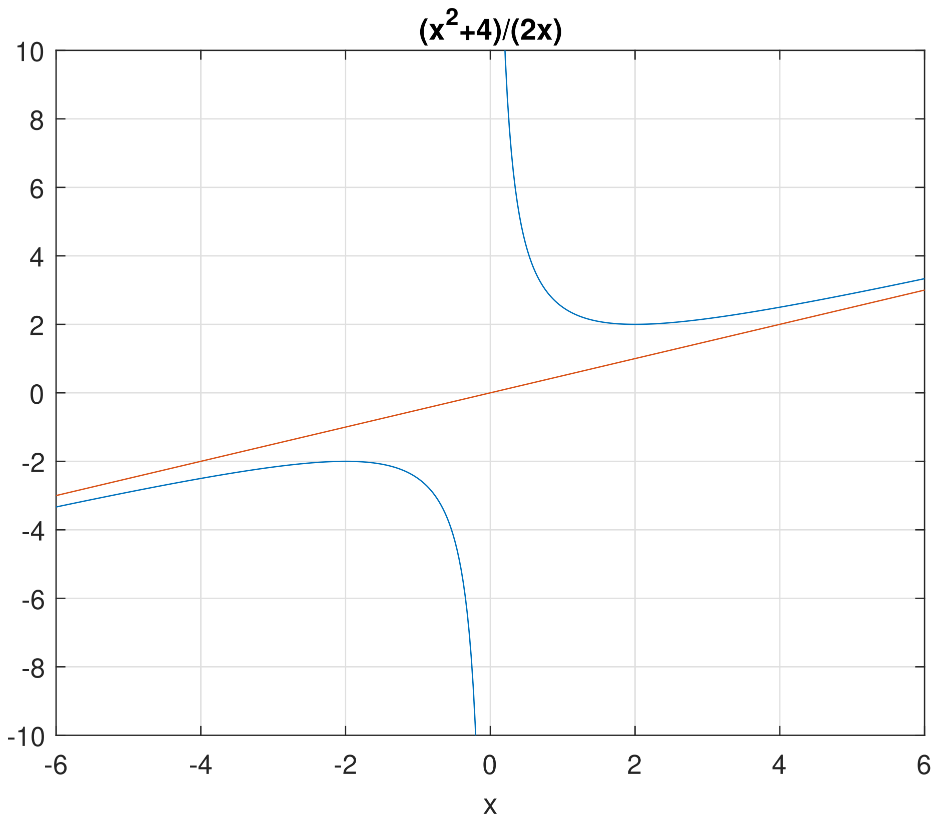

Figure 4: $y = \dfrac{x^2+4}{2x}$