CALCULUS
DEGREE IN SOFTWARE ENGINEERING
CHAPTER 13. THE INDEFINITE INTEGRAL

Till now, we have studied how to find the derivative of known functions and how to apply the derivative to graphing functions or to find their extrema. However, there are relevant scientific problems in which we know the derivative of an unknown function and we want to recover this function from the knowledge of its derivative. For instance, we may know the velocity of a moving body and we need to calculate the distance it travels in a certain time. Or in general, we have been able to write equations involving derivatives of an unknown function (differential equations modeling radioactive decay, growth population, the spread of an infection, etc) and we want to deduce the function itself.
Though this general case, solving differential equations, deserves a complete specific course, the applied techniques are based, in many cases, on finding a function from its derivative.
Therefore, we will devote a few chapters to the solution of the following equation

$$F'(x) = f(x)$$

that is, we know $f(x)$ and try to find $F(x)$, the function whose derivative is $f(x)$. Below, we write this in a more precise way.

DEFINITION OF ANTIDERIVATIVE (PRIMITIVE)

A function F is an antiderivative (or primitive) of f on an interval I if $F'(x) = f(x)$ for all x in I. The process of calculating F from f is called antidifferentiation, or more usually, integration. As we will show in a future chapter, if f is continuous on I, the existence of F is guaranteed.

EXAMPLE

Imagine that $f(x) = x^2$, a very simple continuous function, and we wish to solve $F'(x) = x^2$. It is clear that $F(x) = x^3/3$ will do, but $F(x) = x^3/3 + 5$ or $F(x) = x^3/3 + 9$ will also be primitives of $f(x) = x^2$. It is easy to see that any function $F(x) = x^3/3 + C$ with C an arbitrary constant is a primitive of x^2. And it is also very simple to prove that if $F(x)$ is a primitive of $f(x)$, for any $f(x)$, if we add an arbitrary constant to $F(x)$, we have constructed another primitive. We simply use that the derivative of a constant is zero.
However, we could now ask an interesting question: are all the antiderivatives of $f(x)$ of the form $F(x)+C$, with $F(x)$ a particular primitive and C any constant. Or if we put this question in another way: are there primitives of $f(x)$ that cannot be written as $F(x)+C$. Before we answer this question, I want to draw your attention to its practical importance. If the answer is affirmative, we just need to find a

primitive and we will be able to write all of them just by adding constants. That is, from a particular solution of the problem $F'(x) = f(x)$, we will be able to write the general solution $F(x) + C$.
The following theorem tells us that the answer to the question is yes and its proof rests on the Mean Value Theorem (a very useful theorem indeed)

THEOREM

If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is $F(x) + C$, with C an arbitrary constant.

Proof:

Imagine we have two different antiderivatives of $f(x)$, $G(x)$ and $F(x)$. According to the definition of antiderivative $G'(x) - F'(x) = 0$. And according to the Mean Value Theorem (Corollary 1), see Chapter 10, $G(x) = F(x) + C$, that is any antiderivative can be written as one of them, for instance $F(x)$, plus a constant.
This theorem leads us to the following definition

DEFINITION. THE INDEFINITE INTEGRAL

The collection of all antiderivatives of f is called the indefinite integral of f with respect to x, and is denoted by

$$\int f(x)\, dx$$

The symbol $\int$ is an integral sign. The function f is the integrand of the integral, and x is the variable of integration.
The logic of the symbols used here will be explained later when we present the definite integral. dx is the differential of x and will show its usefulness when we integrate by parts or change variables.

EXAMPLES

$$\int x^6\, dx = x^7/7 + C$$

$$\int e^x\, dx = e^x + C$$

$$\int \sin x\, dx = -\cos x + C$$

In this way, we could show many different examples of indefinite integrals. However, if we want to integrate more complicated functions, we need to use some basic properties of the indefinite integral.

PROPERTY 1. LINEARITY

If $F(x)$ is an antiderivative of $f(x)$, $G(x)$ an antiderivative of $g(x)$ and a and b two constants, then $a\,F(x) + b\,G(x)$ is an antiderivative of $a\,f(x) + b\,g(x)$. The proof is trivial. We take into account that

$$(a\,F(x) + b\,G(x))' = a\,F'(x) + b\,G'(x)) = a\,f(x) + b\,g(x)$$

This implies, given the definition of indefinite integral

$$\int (a\,f(x) + b\,g(x))\,dx = a\int f(x)\,dx + b\int g(x)\,dx$$

This property allows us to calculate expressions such as

$$\int (5x^7 + 10\cos x)\,dx = 5x^8/8 + 10\sin x + C$$

PROPERTY 2. INTEGRATION BY PARTS

Taking into account that

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$$

we can write

$$f(x)g'(x) = (f(x)g(x))' - f'(x)g(x)$$

Calculating the indefinite integral, we obtain the Integration by Parts Formula

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx$$

If we call $u = f(x)$, $v = g(x)$, $du = f'(x)\,dx$, $dv = g'(x)\,dx$, the previous formula becomes

$$\int u\,dv = uv - \int v\,du$$

A compact expression which is very easy to remember
This formula enables us to simplify some indefinite integrals. For instance

EXAMPLE

$$\int \ln x\,dx$$

We call $u = \ln x$, $dv = dx$. Then, $du = dx/x$ and $v = x$. Using integration by parts

$$\int \ln x\,dx = x\ln x - \int dx = x\ln x - x + C$$

Whereas knowing by heart an antiderivative of $\ln x$ is tricky, integration by parts allows us to write it in terms of known antiderivatives. We will see several more examples in Exercises 6.

PROPERTY 3. THE SUBSTITUTION RULE

If $u = g(x)$ is a differentiable function whose range is an interval I, and f is continuous on I, then

$$\int f(g(x))\, g'(x)\, dx = \int f(u)\, du$$

Proof: If F is an antiderivative of f, $(F \circ g)(x)$ is an antiderivative of $(f \circ g)(x)\, g'(x)$, since

$$(F \circ g)'(x) = F'(g(x))\, g'(x) = f(g(x))\, g'(x)$$

according to the Chain Rule. Then

$$\int f(g(x))\, g'(x)\, dx = F(g(x)) + C = F(u) + C = \int f(u)\, du$$

We have used the Chain Rule, running it backwards. The gist of this method is that we can change an indefinite integral into a simpler one, $\int f(u)\, du = F(u) + C$, by substituting u for $g(x)$. However, note that you have to change back to the original variable at the end of the process. That is, your solution is not $F(u) + C$, but $F(g(x)) + C$. Let us see it with an example

EXAMPLE 1. SUBSTITUTION RULE

Let us calculate

$$\int x^2 \, \sin x^3 \, dx$$

Without much practice, it seems impossible to work out this indefinite integral directly. But we can notice that $u = x^3$ is a suitable substitution. Carrying out this change of variable and taking into account that $du = 3x^2 dx$

$$\int x^2 \, \sin x^3 \, dx = \int \frac{1}{3} \sin u \, du$$

Now, the integration is trivial

$$\int \frac{1}{3} \sin u \, du = -\frac{1}{3} \, \cos u + C$$

However, we are not done yet. We have to transform back to the original variable x. Doing this,

$$\int \frac{1}{3} \sin u \, du = -\frac{1}{3} \, \cos u + C = -\frac{1}{3} \, \cos x^3 + C$$

You can check that everything is right by differentiating this last expression.

We can also write the Substitution Rule in a slightly different way

$$\int f(x)\,dx = \int f(g(t))\,g'(t)\,dt$$

Note that this formula is like the Substitution Rule, but we use t instead of x and x instead of u. Besides, we change sides: we write the previous left-hand side as the right-hand side and viceversa. In this way, we use the change $x = g(t)$ to simplify the indefinite integral. Then, we calculate the corresponding antiderivative

$$\int f(g(t))\,g'(t)\,dt = F(g(t)) + C$$

Our antiderivative is a function of t, but we have to return to the original variable x by writing $t = g^{-1}(x)$, using the inverse function of $x = g(t)$.
This will be more clear with an example

EXAMPLE 2. SUBSTITUTION RULE

$$\int \frac{dx}{(1+x^2)^{(3/2)}}$$

In principle, we have no idea of how to solve this integral. I will give you a hint that will be explained later. Use the change

$$x = \tan t$$

Then, since

$$dx = (1+\tan^2 t)\,dt$$

$$\int \frac{dx}{(1+x^2)^{(3/2)}} = \int \frac{dt}{(1+\tan^2 t)^{1/2}}$$

With a little trigonometry, this becomes

$$\int \cos t\,dt = \sin t + C$$

We have to revert to the original variable. Since $x = \tan t$,

$$\sin t = \frac{x}{\sqrt{1+x^2}}$$

and the final indefinite integral is

$$\int \frac{dx}{(1+x^2)^{(3/2)}} = \frac{x}{\sqrt{1+x^2}} + C$$

Of course, it was not easy to guess this final result.
In the next chapter, we will write a basic table of indefinite integrals, by using these methods for solving them.