Practice 4. Functions of a real variable: integration

1. Integral Calculus

To calculate indefinite integrals and definite integrals, Matlab has the command `int`:

`int(f,x)`	Calculates the indefinite integral $\int f(x)\,dx$, without the additive constant, that is, it calculates an antiderivative of f. If the variable is not specified, MATLAB will choose one by default, always giving preference to the variable x.
`int(f,x,a,b)`	Calculates the definite integral $\int_a^b f(x)\,dx$.

For instance, to calculate the following integrals:

$$\int xe^x\,dx\ ,\quad \int_0^1 xe^x\,dx\ ,\quad \int \cos(xy)\,dx\ ,\quad \int \cos(xy)\,dy\ ,\quad \int_a^b \cos x\,dx\ ,\quad \int_1^{+\infty} \frac{dx}{x^2}\ ,\quad \int_1^{+\infty} \frac{dx}{x}$$

```
>> syms x y
>> int(x*exp(x))
>> int(x*exp(x),0,1)
>> int(cos(x*y))
>> int(cos(x*y),y)
>> syms a b
>> int(cos(x),a,b)
>> int(1/x^2,1,inf)
>> int(1/x,1,inf)
```

obtaining the following results:

$$\int xe^x\,dx = e^x(x-1)+c\ ,\quad \int_0^1 xe^x\,dx = 1\ ,\quad \int \cos(xy)\ dx = \frac{\sin(xy)}{y}+c\ ,\quad \int \cos(xy)\ dy = \frac{\sin(xy)}{x}+c$$
$$\int_a^b \cos x\ dx = \sin b - \sin a\ ,\quad \int_1^{+\infty} \frac{dx}{x^2} = 1\ ,\quad \int_1^{+\infty} \frac{dx}{x} = \infty.$$

1.1. Example

Calculate the area of the region bounded by the curve of equation $y = 2x^3$ and the line $y = 8x$, performing the following steps:

1. Calculate the intersection between the curve and the line with the command `solve`.
2. Plot the curve and the line on an interval which contains the intersection points.
3. Calculate the requested area.

Solution.-

1. Calculate the intersection points between the curve and the line, calling f the expression for the curve and g the expression for the line:

```
>> syms x
>> f=2*x^3
>> g=8*x
>> solve(f-g,x)
ans =
-2
 0
 2
```

2. We graph f and the line on the interval $[-2, 2]$ (this interval contains all the roots):

```
>> ezplot(f,[-2,2])
>> hold on
>> ezplot(g,[-2,2])
>> grid on
>> hold off
```

3. In the above graph we can see that f is above the line in $[-2, 0]$ and below the line in $[0, 2]$. To find the area we need to solve the integral

$$\int_{-2}^{2} |f(x) - g(x)| dx.$$

If we solve the exercise with pencil and paper we need to perform two integrals to calculate the area, whose calculation with MATLAB is:

```
>> a1=int(f-g,x,-2,0)
a1 =
8
>> a2=int(g-f,x,0,2)
a2 =
8
>> area=a1+a2
area =
16
```

Since MATLAB has the absolute value function we can calculate the area with a single integral:

```
>> area=int(abs(f-g),x,-2,2)
area =
16
```

4. We can fill the area with a color (red) writing the following commands

```
>> x1=-2:0.01:2;
>> y1=8*x1;
>> x2=2:-0.01:-2;
>> y2=2*x2.^3;
>> x=[x1 x2];
>> y=[y1 y2];
>> patch(x,y,'r')
```

1.2. Example

Find the area of the region enclosed by the graph of the function $f(x) = \dfrac{x^2-1}{x^2+1}$ and its horizontal asymptote, carrying out the following steps:

1. Find the horizontal asymptote, calculating the limits at infinity.
2. Analyze if there exists intersection between f and the horizontal asymptote.
3. Determine graphically the relative position of f and its asymptote.
4. Calculate the requested area.

Solution.-

1. We calculate the horizontal asymptote at $+\infty$ and at $-\infty$ (they could be different):

```
>> syms x
>> f = (x^2-1)/(x^2+1); pretty(f)
>> limit(f,x,inf)
ans =
1
>> limit(f,x,-inf)
ans =
1
```

   The horizontal asymptote (from the left and from the right) is $y = 1$.

2. We study whether the function intersects the asymptote by solving the equation $f(x) = 1$:

```
>> solve(f-1,x)
ans =

Empty sym: 0-by-1
```

   Note that the answer is that there is no solution of the equation $f(x) - 1 = 0$, then the function f does not intersect the asymptote.

3. We graph f and the asymptote, choosing a suitable interval for the function, for example $[-5, 5]$:

```
>> ezplot(f,[-5,5])
>> hold on
>> ezplot('1',[-5,5])
>> grid on
>> hold off
```

4. We plot the domain in blue

```
>> x1=-5:0.01:5;
>> x2=5:-0.01:-5;
>> y1=(x1.^2-1)./(x1.^2+1);
>> y2=ones(size(x2));
>> xn=[x1 x2];
>> yn=[y1 y2];
>> patch(xn,yn,'b')
```

5. Finally, we calculate the area

```
>> area = int(1-f,x,-inf,inf)
area =
2*pi
```

1.3. Example

Find the area of the region bounded above by the function

$$f(x) = (-x^2 + x + 3)\ln x$$

and bounded below by the x-axis. Represent graphically this region.

Solution.- First, we calculate the intersection points of f with the x-axis

```
>> syms x
>> f=(-x^2+x+3)*log(x);  pretty(f)
>> sol=solve(f,x)
sol =
1
1/2+1/2*13^(1/2)
1/2-1/2*13^(1/2)
>> sol=double(sol)
sol =
1.0000
2.3028
-1.3028
>>sol=sort(sol)
sol =
-1.3028
1.0000
2.3028
```

The root, -1.3028, is not valid since f is not defined for negative values.

Now we represent graphically the function f on the interval $[1, 2.3028]$

```
>> ezplot(f,[sol(2),sol(3)])
>> grid on
```

Since the function is positive on the interval, we calculate the area by performing the integral:

```
>> int(f,x,sol(2),sol(3))
>> double(ans)
ans =
0.8404
```

We can also calculate the volume of the solid generated by rotating this region around the x-axis. The formula we must use is

$$V_x = \pi \int_a^b f^2\,dx$$

where a and b are the endpoints of the region on the x-axis. If the region is rotated around the y-axis, the formula is

$$V_y = 2\pi \int_a^b xf\,dx$$

. Applying both expressions, we obtain

```
>> double(pi*int(f^2,sol(2),sol(3)))

ans =

    2.0408

>> double(2*pi*int(x*f,sol(2),sol(3)))

ans =

    8.7232
```

1.4. Example

The antiderivatives of the function $f(x) = e^{-x^2}$ cannot be expressed by elementary functions. Then, the integral

$$\int_0^1 e^{-x^2}\,dx$$

has to be calculated by approximations. We will approximate its value by using Taylor polynomials.

First, we will calculate the value of the definite integral with the command `int`, and, to see more digits, we choose `format long`:

```
>> format long
>> syms x
>> f=exp(-x^2)
>> int(f,0,1)
ans =
1/2*erf(1)*pi^(1/2)
>> double(ans)
ans =
0.746824132812427
```

We calculate MacLaurin polynomials of f of order 2, 4, 6, 10 y 14.

Remember that the command `taylor(f,x,a,'order',n)` calculates the Taylor polynomial of f of order $n-1$ at the point a.

```
>> p2=taylor(f,x,0,'order',3),p4=taylor(f,x,0,'order',5),p6=taylor(f,x,0,'order',7),
p10=taylor(f,x,0,'order',11),p14=taylor(f,x,0,'order',15)
p2 =
1-x^2
p4 =
1-x^2+1/2*x^4
p6 =
1-x^2+1/2*x^4-1/6*x^6
p10 =
1-x^2+1/2*x^4-1/6*x^6+1/24*x^8-1/120*x^10
p14 =
1-x^2+1/2*x^4-1/6*x^6+1/24*x^8-1/120*x^10+1/720*x^12-1/5040*x^14
```

Let us now calculate the definite integrals of these polynomials on the interval $[0,1]$. Notice how the approximation improves as we consider Taylor polynomials of a higher order:

```
>> double(int(p2,0,1))
ans =
   0.666666666666667
>> double(int(p4,0,1))
ans =
   0.766666666666667
>> double(int(p6,0,1))
ans =
   0.742857142857143
>> double(int(p10,0,1))
ans =
   0.746729196729197
>> double(int(p14,0,1))
ans =
   0.746822806822807
```

2. Exercises

1. Calculate the following antiderivatives:

$$a)\ \int \frac{dx}{1+e^x} \qquad b)\ \int \sec x\,dx \qquad c)\ \int e^{ax}\sin bx\,dx$$

$$d)\ \int x^3 \ln x\,dx \qquad e)\ \int \sin^{-1} x\,dx \qquad f)\ \int x\tan^{-1}\sqrt{x^2-1}\,dx$$

Sol.:

a) $x - \ln(1 + e^x) + c$ *b)* $\ln|secx + tanx| + c$

c) $\dfrac{e^{ax}}{a^2 + b^2}(a \sin bx - b \cos bx) + c$ *d)* $x^4 \left(\dfrac{\ln x}{4} - \dfrac{1}{16}\right) + c$

e) $x \sin^{-1} x + \sqrt{1 - x^2} + c$ *f)* $\dfrac{1}{2}\left(x^2 \tan^{-1}\sqrt{x^2 - 1} - \sqrt{x^2 - 1}\right) + c$

2. Calculate the value of the following improper integrals.

a) $\displaystyle\int_2^{+\infty} \frac{dx}{x^2 - 1}$ *b)* $\displaystyle\int_e^{+\infty} \frac{dx}{x \ln^2 x}$ *c)* $\displaystyle\int_{-\infty}^{0} xe^x \, dx$

d) $\displaystyle\int_{-\infty}^{+\infty} \frac{dx}{x^2 + 1}$ *e)* $\displaystyle\int_3^{5} \frac{xdx}{\sqrt{x^2 - 9}}$ *f)* $\displaystyle\int_{-\infty}^{+\infty} \sin x \, dx$

Sol.: *a)* 0.5493 *b)* 1 *c)* -1 *d)* π *e)* 4 *f)* It does not exist.

3. Represent graphically and find the area of the region bounded by the graph of the function

$$f(x) = \frac{x + 1}{x^2 + x + 1}$$

and the x-axis between 0 and 1.

Sol.: 0.8516

4. Represent graphically the region enclosed by the graph of the function $f(x) = \dfrac{x - 1}{(x + 1)^2}$ and the lines $y = x$, $x = 0$ and $x = 5$. Calculate the area of the given region.

Sol: 12.3749

5. Given the curve $f(x) = \sin x + \cos x$

   a) Calculate the volume of the solid generated by revolving the region between the curve and the x-axis on the interval $[0, \pi/2]$, about the x-axis.

   b) Calculate the volume of the solid generated by revolving the region between the curve and the x-axis on the interval $[0, \pi/2]$, about the y-axis.

Sol.: *a)* $\pi\left(\dfrac{\pi}{2} + 1\right)$ *b)* π^2

6. Let us consider the function

$$f(x) = \frac{x^2}{x^2 - 1}.$$

   a) Graph the function f.

   b) Approximate the function f by a parabola in a neighbourhood of 0.

   c) Find the area of the region bounded by the graph of f and the x-axis between $-1/2$ y $1/2$.

   d) Find the area of the region enclosed by the graph of f and the parabola between $-1/2$ y $1/2$.

Sol.: *b)* $P(x) = -x^2$ *c)* area = 0.0986 *d)* area = 0.0153

7. Let $f(x) = \dfrac{1}{x^2 - 4}$.

a) Graph f. Choose a suitable interval to plot the function.

b) Calculate the area of the enclosure bounded by the graph of f and the x-axis on the interval $[3, +\infty)$.

c) Calculate the volume generated by revolving this region about the x-axis.

d) Calculate the volume generated by revolving the same region about the y-axis.

Sol.: *b*) 0.4024 *c*) 0.0776 *d*) Infinite.