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S3-Euclidean Spaces PART I

Linear Algebra

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Plan

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- Euclidean Spaces. Definition
- Dot products in $\mathbb{R}^2$, $\mathbb{R}^3$ and $\mathbb{R}^n$.
- Matrix form of scalar products: the Gram matrix
- Length and angle. Orthogonality.
- Euclidean spaces of digital signals, analogic signals and images.
- Orthogonal Projections.
- Gram-Schmidt method.
- To practice

Khan Academy

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https://www.khanacademy.org/math/linear-algebra/vectors-and-spaces/dot-cross-products/v/vector-dot-product-and-vector-length

Euclidean Spaces

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If **V** is a vector space, a dot product (or scalar product) is any application · :

$$\cdot: \quad \begin{array}{ccc} V \times V & \to & \mathbb{R} \\ (\vec{v}, \vec{u}) & \to & \vec{v} \cdot \vec{u} \end{array}$$

fulfilling the following conditions:

1. **Linearity**

$$\forall \vec{v}_1, \vec{v}_2, \vec{u} \in V \ \forall \alpha, \beta \in \mathbb{R}$$

$$\begin{aligned} (\alpha \vec{v}_1 + \beta \vec{v}_2) \cdot \vec{u} &= \alpha \vec{v}_1 \cdot \vec{u} + \beta \vec{v}_2 \cdot \vec{u} \\ \vec{u} \cdot (\alpha \vec{v}_1 + \beta \vec{v}_2) &= \alpha \vec{u} \cdot \vec{v}_1 + \beta \vec{u} \cdot \vec{v}_2 \end{aligned}$$

2. **Symmetry**

$$\forall \vec{v}, \vec{u} \in V \quad \vec{v} \cdot \vec{u} = \vec{u} \cdot \vec{v}$$

3. **Positivity** $\forall \vec{v} \in V \quad \vec{v} \cdot \vec{v} \geq 0$ $\quad \vec{v} \cdot \vec{v} = 0 \to \vec{v} = \vec{0}$

Euclidean Spaces

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- The Vector space **V** provided with the scalar product $\cdot$, $(\mathbf{V}, \cdot)$ is called **Euclidean Space**
- The scalar products serve to **compare vectors**, for instance, to compare two digital or analogic signals and see if they are similar or not.

Norm (or length)of a vector: $\|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}} = \sqrt{\vec{v}^{\,2}}$.

The norm of a vector is its length according to the the scalar product $\cdot$

Note: $\dfrac{\mathbf{v}}{\|\mathbf{v}\|}$ is the unit vector in the direction of $\mathbf{v}$.

Scalar product in $\mathbb{R}^n$

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- **Euclidean scalar product**

$$\mathbf{u} \cdot \mathbf{v} = \sum_{k=1}^{n} u_k \, . v_k = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

satisfies these properties of scalar product.

Prove the 3 conditions that has to fulfill any scalar product.

- **The norm:**

$$\|\boldsymbol{u}\| = \sqrt{\mathbf{u} \cdot \ \mathbf{u}} = \sqrt{\ u_1^2 + \cdots + u_n^2} > 0.$$

Scalar product in Matrix form

- $B = \{\mathbf{v_1}, \mathbf{v_2}, \dots, \mathbf{v_n}\}$ basis set of V

- $\mathbf{x} = x_1\,\mathbf{v_1} + x_2\,\mathbf{v_2} + \cdots + x_n\,\mathbf{v_n}$
- $\mathbf{y} = y_1\,\mathbf{v_1} + y_2\,\mathbf{v_2} + \cdots + y_n\,\mathbf{v_n}$

- $\mathbf{x} \cdot \mathbf{y} = (x_1\,\mathbf{v_1} + x_2\,\mathbf{v_2} + \cdots + x_n\,\mathbf{v_n}) \cdot (y_1\,\mathbf{v_1} + y_2\,\mathbf{v_2} + \cdots + y_n\,\mathbf{v_n})$

$$\mathbf{x} \cdot \boldsymbol{y} = (x_1, x_2, \dots, x_n) \cdot \mathbf{G} . \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \text{ with } \boldsymbol{G} = \begin{pmatrix} \mathbf{v_1} . \mathbf{v_1} & \cdots & \mathbf{v_1} . \mathbf{v_n} \\ \vdots & \ddots & \vdots \\ \mathbf{v_n} . \mathbf{v_1} & \cdots & \mathbf{v_n} . \mathbf{v_n} \end{pmatrix}$$

$$\mathbf{x} \cdot \boldsymbol{y} = \mathbf{x}^{\boldsymbol{T}} \cdot \boldsymbol{G} \cdot \boldsymbol{y}, \quad \text{with} \quad \boldsymbol{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \text{ and } \mathbf{x}^T = (x_1, x_2, \dots, x_n).$$

G is called the Gram matrix of the scalar product in the basis $B = \{\mathbf{v_1}, \mathbf{v_2}, \dots, \mathbf{v_n}\}$.

Note : $\mathbf{x}^{\boldsymbol{T}}$ is always row vector !!

Gram matrix

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The Gram matrix has some important properties that come from the properties of the scalar properties

- **Symmetry:** $\boldsymbol{G}(i,j) = \mathbf{v}_{\boldsymbol{i}}.\mathbf{v}_{\boldsymbol{j}} = \boldsymbol{G}(j,\boldsymbol{i}) = \mathbf{v}_{\boldsymbol{j}}.\mathbf{v}_{\boldsymbol{i}} \leftrightarrow \boldsymbol{G} = \boldsymbol{G}^{\boldsymbol{T}}$
- **Definite positive** $\mathbf{x}^{\boldsymbol{T}} \cdot \boldsymbol{G} \cdot \mathbf{x} > 0 \quad \forall\, \mathbf{x} \neq \mathbf{0}$

Note: any matrix $\boldsymbol{G}$ fulfilling these two conditions correspond to scalar product in V

$$G = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \qquad \mathbf{x}^{\boldsymbol{T}} \cdot \boldsymbol{G} \cdot \mathbf{x} = (x_1, x_2)\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 2x_1^2 + 2x_1x_2 + x_2^2 = x_1^2 + (x_1 + x_2)^2 > 0.$$

What happens wit the Gram matrix when we change the basis?

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- $B = \{\mathbf{v_1}, \mathbf{v_2}, \dots, \mathbf{v_n}\}$ basis set of V
- $B' = \{\mathbf{b_1}, \mathbf{b_2}, \dots, \mathbf{b_n}\}$ basis set of V
- $\mathbf{x} \cdot \boldsymbol{y} = (x_1, x_2, \dots, x_n)_B \cdot \mathbf{G}. \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}_B = (c_1, c_2, \dots, c_n)_{B'} \cdot \boldsymbol{G}^* \begin{pmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{pmatrix}_{B'}$
- $\mathbf{x}=\mathbf{P.c}$; $\mathbf{y}=\mathbf{P.s} \rightarrow \mathbf{x} \cdot \boldsymbol{y} = (\mathbf{Pc})^T \cdot \mathbf{G}.\,(\mathbf{Ps})$
- $\boldsymbol{G}^* = \mathbf{P^T G P}$ (Congruent matrices)

Bit of Practice

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- In $\mathbb{R}^2$ we define the following dot product

$$(x_1, x_2) \cdot (y_1, y_2) = 2x_1y_1 + x_2y_1 + x_1y_2 + x_2y_2$$

- ✓ **Finding the Gram matrix in the canonic basis.**
- ✓ **Finding the Gram matrix in the** $\boldsymbol{basis}\ B = \{(1,1),(-1,1)\}$
- ✓ **Finding the norm of** $(1,1)$.

Angle between 2 vectots

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Angle that form 2 vectors

$$\mathbf{u} \cdot \mathbf{v} = \|\boldsymbol{u}\|\|\mathbf{v}\| cos\langle \mathbf{u},\ \mathbf{v}\rangle$$

$$-1 < cos\langle \mathbf{u},\ \mathbf{v}\rangle = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\boldsymbol{u}\|\|\mathbf{v}\|} < 1$$ **(Cauchy-Schwartz inequality)**

$$-\|\mathbf{u}\|\|\mathbf{v}\| < \mathbf{u} \cdot \mathbf{v} < \|\boldsymbol{u}\|\|\mathbf{v}\|$$

$$(\mathbf{u}+\mathbf{v})^{\mathbf{2}} = (\mathbf{u}+\mathbf{v}).\,(\mathbf{u}+\mathbf{v}) = \|\mathbf{u}+\mathbf{v}\|^{\mathbf{2}} = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 + 2\mathbf{u} \cdot \mathbf{v} \leq (\|\mathbf{u}\| + \|\mathbf{v}\|)^{\mathbf{2}}$$

$$\|\mathbf{u}+\mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$$ **(Triangle Inequality)**

Scalar product in $C^0_{[a,b]}$

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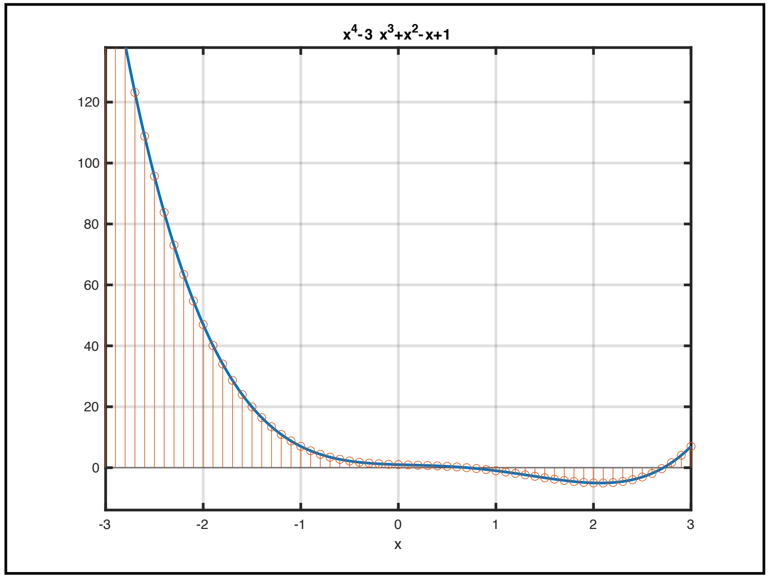

Analogic signal $\xrightarrow{\text{Sampling}}$ Digital signal

$f(x)$ $\xrightarrow{\text{Sampling}}$ $\mathbf{f} \in \mathbb{R}^N$

$g(x)$ $\rightarrow$ $\mathbf{g} \in \mathbb{R}^N$

$\rightarrow$ $\mathbf{f}.\mathbf{g}$

When the sampling interval is approaching to 0 $\rightarrow$ $\langle f, g\rangle = \int_a^b f(x)g(x)dx$

$$cos\langle \boldsymbol{f},\ \boldsymbol{g}\rangle = \frac{\boldsymbol{f}\cdot \boldsymbol{g}}{\|\boldsymbol{f}\|\|\boldsymbol{g}\|} = \frac{\int_a^b f(x)g(x)dx}{\sqrt{\int_a^b f^2(x)dx}\sqrt{\int_a^b g^2(x)dx}}$$

Bit of Practice

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Considering in $\boldsymbol{C^0_{[0,1]}}$ 3 functions:

$$f(x) = e^x, \qquad g(x) = x, \qquad h(x) = x^2,$$

1. Finding the norms of these functions
2. Deciding which pairs of functions are more similar: $(f, g), (f, h), (h, g)$.

Entrywise Scalar Product in $M_{m\times n}(\mathbb{R})$

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- We saw that gray color images are matrices of integers.
- The question is how to compare images?
- $A, B \ \in M_{m\times n}(\mathbb{R})$
- The simplest way is: $\boldsymbol{A}.\boldsymbol{B} = \boldsymbol{A}(:)^{\boldsymbol{T}}\boldsymbol{G}\boldsymbol{B}(:)$ **(elementwise scalar product)**