CALCULUS
DEGREE IN COMPUTER SOFTWARE ENGINEERING
CHAPTER 19. VOLUMES

In this chapter we will study how to find the volume of a solid by applying the definite integral. Firstly, we will use the method of cross-sections; secondly, we will use the disk and washer methods for solids of revolution and thirdly, we will present the shell method. We will make continuous references to "Presentation 7. The definite integral. Volumes" a power point presentation you can find in the Virtual Campus, especially to the figures in the presentation.

CROSS-SECTIONS

Have a look at Figure 6.1 (Presentation 7), you can see a solid S- a body or geometrical figure having three dimensions- referenced to a system of axes. The x-coordinate runs from a to b in this case. If we draw a plane perpendicular to the x-axis, P_x, through x in $[a,b]$, we intersect the solid, creating a plane figure which is called a cross-section $S(x)$ of S. Now, our goal is to calculate the volume of S. The basic idea is to approximate the solid by general cylinders. A general cylinder can be seen in Figure 6.2, it is a solid generated by lifting its base a certain height h. All the cross-sections of the cylinder are equal and the volume of this general cylinder is just the product of the area of its base times its height. That will remind you of the volume of a circular cylinder, a cylinder whose base is a disk. We consider a slab of the solid (Figure 6.3) with x running from x_{k-1} to x_k. We can define a partition P of $[a,b]$ as in the definition of the definite integral $P = \{x_0, x_1, .., x_k, ...x_n\}$. By using this partition, we can split the solid into thin slabs and it is clear that the volume of the solid is the sum of the volumes of the slices.
In the next figure (Figure 6.4), we approximate a thin slab, enlarged in the figure, by a cylindrical solid with base $S(x_k)$, the cross-section corresponding to $x = x_k$. The volume of this cylindrical slice will be $V(x_k) = A(x_k)\Delta x_k$, that is, the area of the cross-section $S(x_k)$ times the height of the slice $\Delta x_k = x_k - x_{k-1}$, what makes the total volume of the union of cylindrical slices

$$\sum_{k=1}^{n} V(x_k) = \sum_{k=1}^{n} A(x_k)\Delta x_k$$

If we consider finer and finer partitions, the approximation becomes better and better and we can define the volume V of the solid as

$$V = \lim_{||P||\to 0} \sum_{k=1}^{n} V(x_k) = \lim_{||P||\to 0} \sum_{k=1}^{n} A(x_k)\Delta x_k$$

You will probably have realized that the sums are Riemann sums and the limit is the definite integral, assuming that $A(x)$, the area of the cross-sections, is a continuous function of x- what will be true in the usual cases. Therefore,

$$V = \int_a^b A(x)\,dx$$

So, we have a recipe for finding the volume of a solid by applying the cross-section method.

1. Sketch the solid and a typical cross-section.
2. Calculate the area $A(x)$ of a general cross-section.
3. Find the limits of integration a and b.
4. Solve the definite integral using the Fundamental Theorem.

An example is the formula for the volume of a sphere (see Figure 6.9). The equation of a sphere (surface) of radius a centered at the origin of coordinates is

$$x^2 + y^2 + z^2 = a^2$$

If we consider the cross-sections $S(x)$, they are disks bounded by circles of equation $y^2 + z^2 = a^2 - x^2$ and the area of the cross-sections is then

$$A(x) = \pi(a^2 - x^2)$$

The limits of integration are $-a$ and a. The volume can be written

$$V = \pi \int_{-a}^{a} (a^2 - x^2)\,dx = \pi(a^2x - x^3/3)\Big|_{-a}^{a} = 4/3\pi a^3$$

A primitive of the integrand at the upper limit minus the same primitive at the lower limit. Finally, we obtain the famous formula for the volume of a sphere.

SOLIDS OF REVOLUTION. THE DISK METHOD. THE WASHER METHOD

Imagine a plane region bounded by the lines $x = a$, $x = b$, the graph of a continuous non-negative function $y = f(x)$ and the x-axis. We know how to calculate the area of this region. Now, imagine you rotate this region around the x-axis-an example can be seen in Figure 6.8- the generated solid has circular cross-sections, that is, its cross-sections are disks of radius $f(x)$ for each value of x between a and b. The generated solid is called a solid of revolution and the method for the calculation of the volume is the disk method. According to the previous section, the formula for the volume is

$$V = \int_a^b A(x)\,dx = \int_a^b \pi\, f^2(x)\,dx$$

The integral of the area of the circular cross-sections. An example is the one that appears in Figure 6.8 where $f(x) = \sqrt{x}$, $a = 0$ and $b = 4$. The calculation is straightforward

$$V = \int_0^4 \pi\, x\, dx = \pi x^2/2\Big|_0^4 = 8\pi$$

and the solid is a parabolid of revolution produced by the rotation of a parabola. We will solve more exercises of this type in Exercises 9.
If the plane region is bounded by the lines $y = c$, $y = d$, $x = f(y)$ and the y-axis and we rotate it around the y-axis (see Figure 6.11), the formula for the volume of the generated solid is

$$V = \int_c^d \pi\, f^2(y)\, dy$$

In general, we can use these formulas when the rotation is carried out around an axis perpendicular to the line that rotates. The calculation for Figure 6.11 is

$$V = \int_1^4 \pi\, 4/y^2\, dy = -4\pi/y\Big|_1^4 = 3\pi$$

We can complicate the situation a little more. Now, imagine a region bounded by $x = a$, $x = b$, the x-axis and two graphs $y = f(x)$ and $y = g(x)$ with $f(x) \geq g(x)$. We rotate this region around the x-axis. See Figure 6.13 for an example. It is clear that the cross-sections are shaped like a ring or a washer, thus the method is called the washer method and the formula for the volume is

$$V = \int_a^b \pi\, (f^2(x) - g^2(x))\, dx$$

Be careful, a common mistake is to write $(f(x) - g(x))^2$ instead of $(f^2(x) - g^2(x))$. If $y = f(x)$ and $y = g(x)$ intersect so that $f(x)$ is sometimes above and sometimes below $g(x)$, the formula would be

$$V = \int_a^b \pi\, \Big| f^2(x) - g^2(x) \Big|\, dx$$

We will divide the integral into the sum of several integrals, applying the right sign in each one.
The calculation for Figure 6.14 is carried out taking into account that $a = -2$, $b = 1$, $f(x) = -x + 3$ and $g(x) = x^2 + 1$. Then, the volume is

$$V = \int_{-2}^1 \pi\, ((-x+3)^2 - (x^2+1)^2)\, dx = \int_{-2}^1 \pi\, \Big(-x^4 - x^2 - 6x + 8 \Big)\, dx =$$

$$V = \pi \Big(-x^5/5 - x^3/3 - 3x^2 + 8x \Big)\Big|_{-2}^1 = 117\pi/5$$

SOLIDS OF REVOLUTION. THE SHELL METHOD

In the last section we will present the shell method. We consider a plane region bounded by $x = a$, $x = b$, the x-axis and the graph of a continuous non-negative function $y = f(x)$. Now we revolve this figure around the y-axis. Remember that in the previous cases we rotated the region about the x-axis, but now we rotate it about an axis parallel (not perpendicular) to the line joining $y = 0$ and $y = f(x)$ for each x, that is, the line that rotates is parallel to the axis of rotation (see Figures 6.19 and 6.20 for an example). According to these figures, we can decompose the solid into cylindrical shells that approximate it. We define a partition $P = \{x_0, x_1, .., x_k, ...x_n\}$ and the volume of the shell between x_{k-1} and x_k is

$$V_k = \pi(x_k^2 - x_{k-1}^2)\, f(c_k)$$

where we can choose c_k as the midpoint between x_{k-1} and x_k , then $c_k = (x_k+x_{k-1})/2$ and

$$V_k = 2\pi(x_k - x_{k-1})(x_k + x_{k-1})/2\ f(c_k)$$

or

$$V_k = 2\pi\, c_k f(c_k)\,\Delta x_k$$

This is a Riemann sum and finally,

$$V = \lim_{||P||\to 0} V_k = 2\pi \lim_{||P||\to 0} c_k f(c_k)\,\Delta x_k = 2\pi \int_a^b x f(x)\, dx$$

where we have considered finer and finer partitions making the approximation better and better.

This formula changes if the region is bounded by $y = f(x)$ and $y = g(x)$ with both graphs intersecting at the endpoints or even at intermediate points. It is now

$$V = 2\pi \int_a^b x|f(x) - g(x)|\, dx$$

the axis could be a general line $x = L$ parallel to the y-axis. Then

$$V = 2\pi \int_a^b d|f(x) - g(x)|\, dx$$

with d the radius of the shell, $d = |x - L|$.

The calculation of the volume in Figure 6.20 is very simple

$$V = 2\pi \int_0^4 x\sqrt{x}\, dx = 2\pi\, 2/5\, x^{5/2}\Big|_0^4 = 128\pi/5$$

In general these solids will have a hole (an example would be a doughnut) and the revolution will be similar to that of a merry-go-round.

We will do some exercises for this type of solids in Exercises 9.