CALCULUS
DEGREE IN SOFTWARE ENGINEERING
EXERCISES AND SOLUTIONS 4

1. Find the extreme values (absolute and local) of the functions and where they occur

   (a) $y = x^2 - 6x + 7$
   (b) $y = x(4 - x)^3$
   (c) $Y = x^2 + \dfrac{2}{x}$
   (d) $y = \sqrt{2x - x^2}$

   Solutions:

   a) $y' = 2x - 6$. $x = 3$ is the only critical point. Since $y'' = 2$, the function has a local minimum at that point. Besides, given that y'' is always positive, the local minimum is an absolute minimum. The value of the absolute minimum is $y(3) = -2$.

   b) The derivative is $y' = (4-x)^3 - 3x(4-x)^2 = (4-x)^2(4-4x)$. The critical points are $x = 4$ and $x = 1$. The second derivative is $y'' = 4(4-x)(3x-6)$. $y''(1) < 0$ and $x = 1$ is a local maximum point. $y''(4) = 0$, but we can use that the first derivative does not change sign at $x = 4$. There is neither maximum nor minimum at that point.The function is increasing on $(-\infty, 1)$ and decreasing on $(1, \infty)$. This implies that it has an absolute maximum at $x = 1$, $y(1) = 27$.

   c) $y' = 2x - 2/x^2$, $x = 1$ is the only critical point. $y'' = 2 + 4/x^2$, $y''(1) = 6$. The function has a local minimum at $x = 1$. This minimum, $y(1) = 3$, is absolute on the interval $(0, \infty)$, but not on the whole domain (all points except $x = 0$).

   d) Here, we have to study the domain. It is the interval $[0, 2]$. The only critical point is $x = 1$. Since the function is increasing on $[0, 1]$ and decreasing on $[1, 2]$ (check the sign of the derivative), it has a local (absolute) maximum at that point, y(1)=1. At the endpoints the function attains its absolute minimum value $y(2) = y(0) = 0$. Since it is continuous on a closed interval $[a, b]$, it had to attain its extreme values on that interval.

2. Show that these functions have exactly one zero in the given interval

(a) $f(x) = x^4 + 3x + 1 \quad [-2, -1]$

(b) $f(x) = x^3 + \dfrac{3}{x^2} + 7 \quad (-\infty, 0)$

(c) $f(x) = \tan x - \cot x - x \quad (0, \frac{\pi}{2})$

(d) $f(x) = \dfrac{1}{1-x} + \sqrt{1+x} - 3 \quad (-1, 1)$

Solutions:

a) $f(-2) = 11$, $f(-1) = -1$. There is a change of sign and the function is continuous, thus, according th Bolzano's theorem, the function has a zero in $(-2, -1)$. The derivative is $f' = 4x^3 + 3$ and it is never zero in that interval. In conclusion, according to Rolle's theorem, it has only one zero in $(-2, -1)$. However, it could have another zero outside that interval. In fact, it has another zero.

b) This function tends to $-\infty$ at $-\infty$ and approaches ∞ at $x = 0$. So, it must have a zero in $(-\infty, 0)$. The derivative is $f' = 3x^2 - 9/x^3$ and it is always positive on $(-\infty, 0)$. In conclusion, there is only one zero in that interval. Note that we use Bolzano's and Rolle's theorems.

c) The function is continuous on $(0, \pi/2)$ and tends to $-\infty$ at $x = 0$ and to ∞ at $x = \pi/2$, its derivative is $f' = (1+\tan^2 x)+(1+\cot^2 x)-1 = \tan^2 x+\cot^2 x+1$, that is, it is always positive. Therefore, there is only one zero.

d) The function is negative $x = -1$ and tends to ∞ at $x = 1$. So, there is at least one zero in $(-1, 1)$. The derivative is

$$f' = \frac{1}{(1-x)^2} + \frac{1}{2\sqrt{1+x}}$$

and it is always positive. Thus, there is only one zero in the given interval.

3. Find the value or values of c that satisfy $f(b) - f(a) = f'(c)(b - a)$ for the following functions and intervals

(a) $x^{\frac{2}{3}} \quad [0, 1]$

(b) $x + \dfrac{1}{x} \quad [0.5, 2]$

(c) $x^{\frac{2}{3}} \quad [-1, 1]$

(d) $x^{\frac{4}{5}} \quad [0, 1]$

In one case you cannot apply the Mean Value Theorem. In which case and why not?

Solutions:

a) We have to solve $1 - 0 = f'(c)(1-0)$, that is, $f'(c) = 1$. Since $f'(c) = 2/3\, c^{-1/3}$, $c = (3/2)^{(-3)} = 8/27$. Note that f' is not well defined at $x = 0$. However, since f is continuous on $[0,1]$ and differentiable on $(0,1)$, we can apply the Mean Value Theorem.

b) In this case $2.5 - 2.5 = f'(c)(1.5)$ and $f'(c) = 0 = 1 - 1/c^2$. Then, $c = 1$.

c) $(1-1) = f'(c)(2)$, $\quad 2/3\, c^{(-1/3)} = 0$. But, the derivative is never zero, because it is not well defined at $x = 0$. We cannot apply the Mean value Theorem.

d) $1 = 4/5\, c^{-1/5}$, $\quad c = (5/4)^{-5}$

4. Plot the function $f(x) = \ln(x^2 - 3x + 2)$, studying previously its domain, asymptotes, the intervals on which f is increasing and on which f is decreasing and its concavity.

Solution: $D = (-\infty, 1) \bigcup (2, \infty)$.
Vertical asymptotes at $x = 1$ and $x = 2$ with the function approaching $-\infty$.
$f'(x) = \dfrac{2x-3}{x^2-3x+2}$. Then, it is decreasing on $(-\infty, 1)$ and increasing on $(2, \infty)$.

The second derivative $f''(x) = \dfrac{-2x^2+6x-5}{(x^2-3x+2)^2}$ is always negative. The graph is concave down on its domain. No points of inflection or critical points.

5. Given the function $f(x) = \dfrac{x}{e^x - 1}$

   (a) Find its domain and symmetries. Classify its discontinuities.

   (b) Calculate its asymptotes.

   (c) Prove that $x = 0$ is the only real solution of the equation $-xe^x + e^x - 1 = 0$.

   (d) On what intervals is f increasing or decreasing ?

   (e) Sketch the curve.

   Solutions:

   a) $D = \{x/x \neq 0\}$. No symmetries. It has a removable discontinuity at $x = 0$. The left-hand and right-hand limits at $x = 0$ are both 1.

   b) Horizontal asymptote $y = 0$ at $x = \infty$. Oblique asymptote $y = -x$ at $x = -\infty$.

   c) We calculate the first derivative of the function $g(x) = -xe^x + e^x - 1$ and study its sign. $g'(x) = -x\, e^x$. It is positive for $x < 0$ and negative for $x > 0$. $g(x)$ is increasing on $(-\infty, 0)$ and decreasing on $(0, \infty)$. Therefore, it is always negative except at $x = 0$ where it is zero.

   d)
$$f'(x) = \frac{-xe^x + e^x - 1}{(e^x - 1)^2}$$
   and according to the last item, it is always negative ($y'(0) = -1/2$, check it). Thus, the function is always decreasing.

   e) Now, we plot the graph with Matlab. We see that it is always concave up. Though checking this is quite easy when we sketch the curve, even by hand, proving it with the second derivative is a little tricky.

6. Sketch the graph of $f(x) = \dfrac{x^2}{1 + x^2}$, studying the intervals on which it is increasing or decreasing , calculating its maxima and minima, analyzing its concavity, asymptotes, etc.

   Solution:

   a) $f'(x) = \dfrac{2x}{(1 + x^2)^2}$. It is decreasing on $(-\infty, 0)$ and increasing on $(0, \infty)$.

   b) It has an absolute minimum at $x = 0$. The second derivative is $f''(x) = \dfrac{(2 - 6x^2)}{(1 + x^2)^3}$. It is concave up on $(-1/\sqrt{3}, 1/\sqrt{3})$ and concave down on $(-\infty, -1/\sqrt{3})$

and $(1/\sqrt{3}, \infty)$. The points on the graph with $x = -1/\sqrt{3}$ or $x = 1/\sqrt{3}$ are points of inflection.

c) The function is even and has a horizontal asymptote $y = 1$ at $x = -\infty$ and $x = \infty$. We plot the graph in Figure 3.

7. Plot the function $f(x) = x^4 - 2x^2$, studying the intervals on which it is increasing or decreasing , calculating its maxima and minima, analyzing its concavity, asymptotes, etc.

   Solutions:

   a) $f'(x) = 4x^3 - 4x$. Increasing on $(-1, 0)$ and $(1, \infty)$, decreasing on $(-\infty, -1)$ and $(0, 1)$.
   b) Local maximum at $x = 0$, absolute minimum at $x = -1$ and $x = 1$. $f''(x) = 12x^2 - 4$. It is concave down on $(-1/\sqrt{3}, 1/\sqrt{3})$ and concave up on $(-\infty, -1/\sqrt{3})$ and $(1/\sqrt{3}, \infty)$. It is even and has no asymptotes. We show the graph in Figure 4.

8. Sketch the graph of $g(x) = \dfrac{x^2 - 4}{x^3}$, following the same steps as in the previous exercises.

   Solution:

   a) $f'(x) = \dfrac{12 - x^2}{x^4}$. The function is increasing on $(-2\sqrt{3}, 2\sqrt{3})$ and decreasing on the complement set.

   b) Local minimum at $x = -2\sqrt{3}$ and local maximum at $x = 2\sqrt{3}$.

   c) $f''(x) = \dfrac{2x^2 - 24}{x^5}$. Concave down on $(-\infty, -2\sqrt{6})$ and $(0, 2\sqrt{6})$, concave up on the complement set.

   d) It is odd and has a vertical asymptote at $x = 0$ and a horizontal asymptote $y = 0$ at $x = -\infty$ and $x = \infty$.

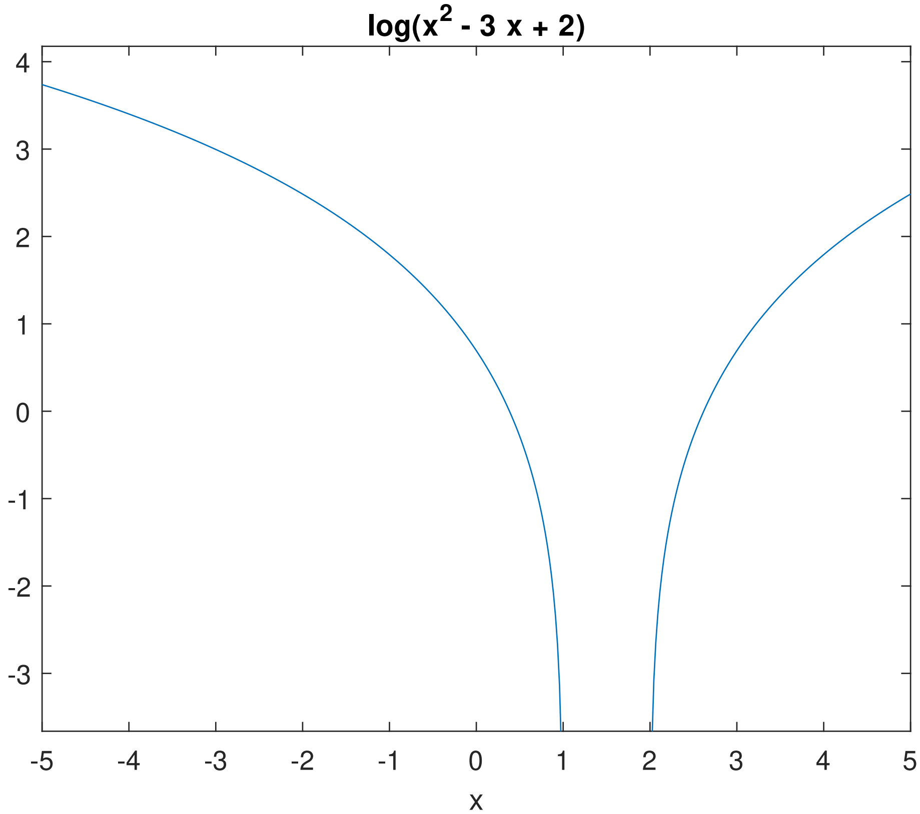

Figure 1: $y = \ln\left(x^2 - 3x + 2\right)$

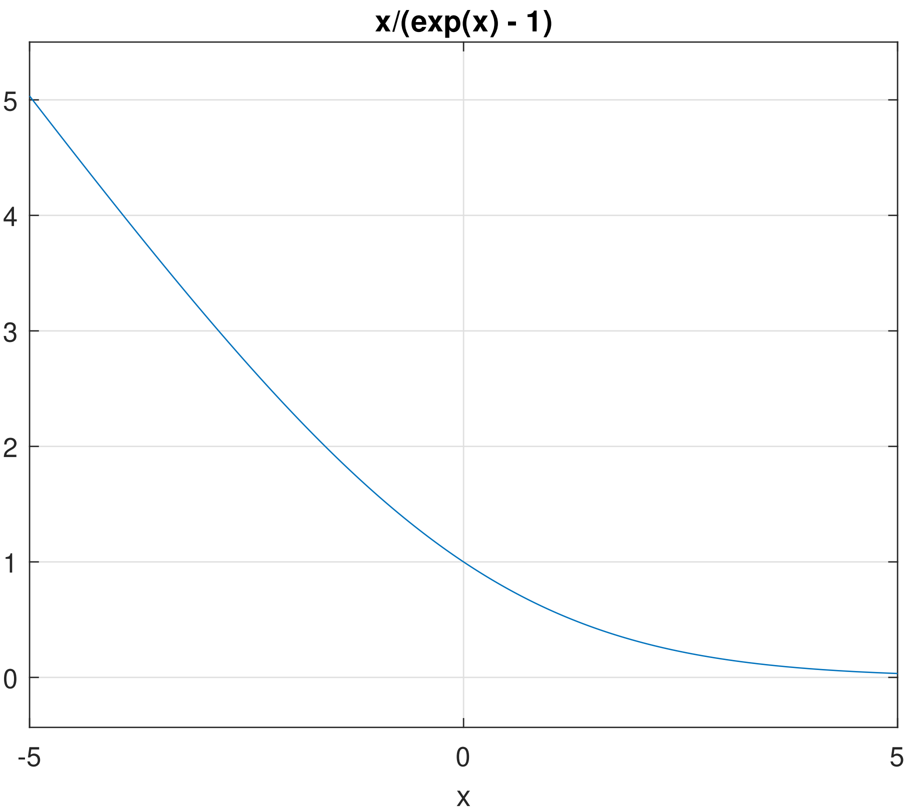

Figure 2: $y = \dfrac{x}{e^x - 1}$

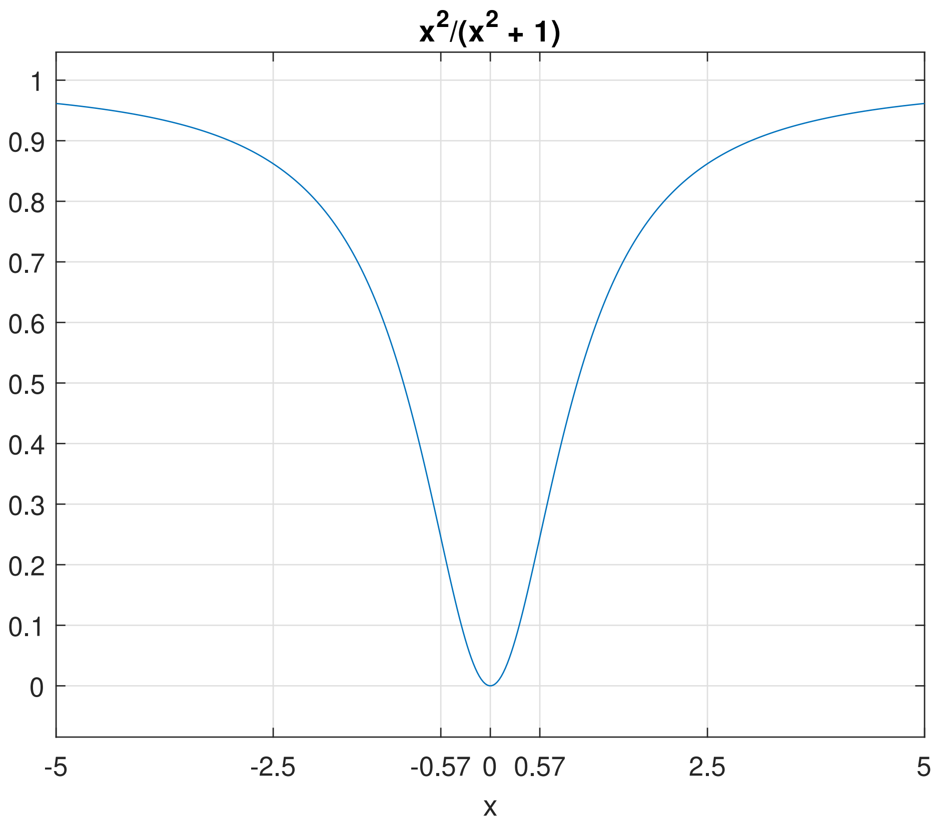

Figure 3: $y = \dfrac{x^2}{1+x^2}$

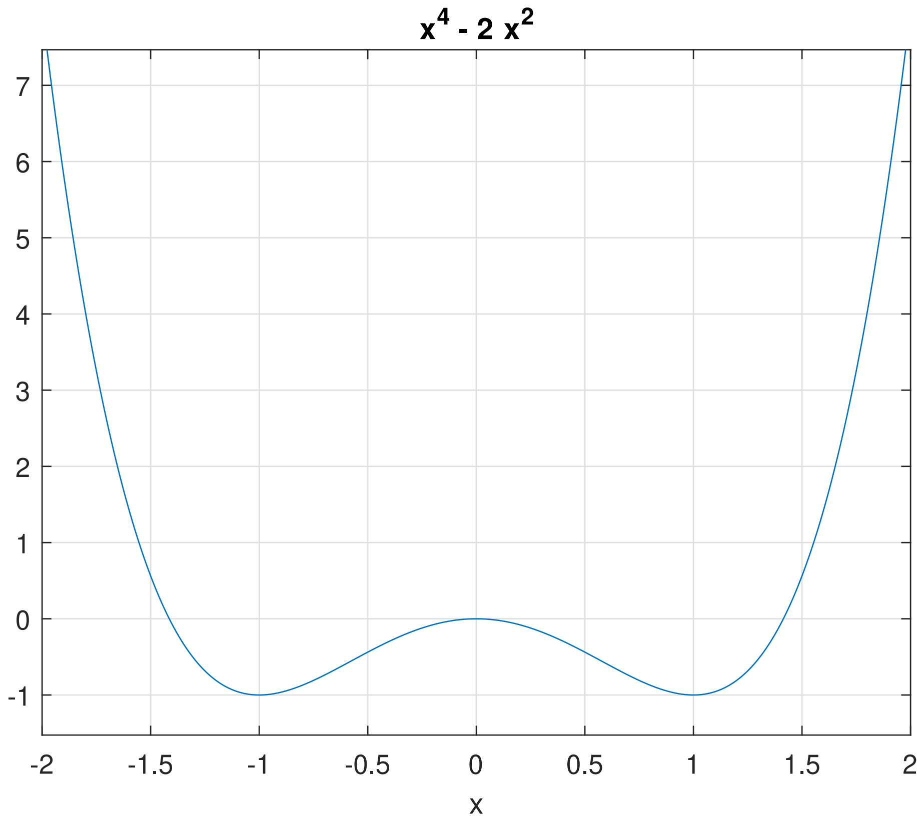

Figure 4: $y = x^4 - 2x^2$

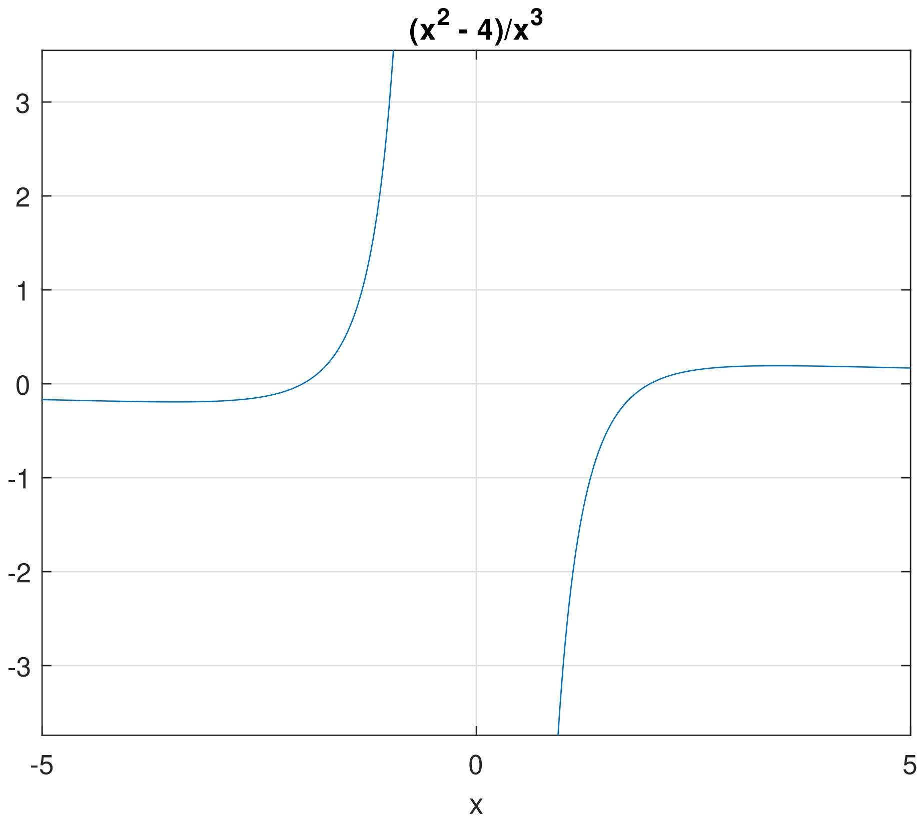

Figure 5: $y = \dfrac{x^2-4}{x^3}$