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S3-Euclidean Spaces

Design of scalar products III

Linear Algebra

Ingeniería del Software-Universidad de Oviedo

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Class-23 October 2020

Orthogonal projections



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Orthogonal projections \longleftrightarrow Least-Squares

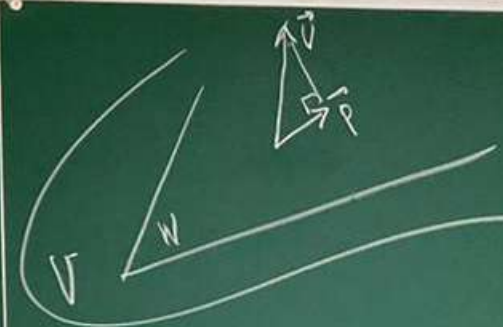
V euclidean space \longrightarrow finite dimension / $\mathbb{R}^n \longrightarrow \dim V = n$
infinite dimension $C^0[a,b] \longrightarrow \dim V = \infty$

W euclidean subspace of V of finite dimension. $\longrightarrow \mathcal{H} \longrightarrow$ plane in \mathbb{R}^n
 $\longrightarrow P_n(x) [a,b] \longrightarrow$ space of polynomials with degree $\leq n$ $\dim P_n(x) = n+1$

$\vec{v} \in V$ $\xrightarrow{\text{finding}}$ the orthogonal projection onto W

$$\begin{aligned} \vec{p} &= \text{Proj}_W(\vec{v}) \\ W &= \langle \vec{b}_1, \vec{b}_2, \dots, \vec{b}_s \rangle \\ &= \alpha_1 \vec{b}_1 + \alpha_2 \vec{b}_2 + \dots + \alpha_s \vec{b}_s \end{aligned}$$

$$\vec{b}_i \in V$$



Normal Equations

$$B^T B \vec{\alpha} = B^T \vec{v}$$

$$B = [\vec{b}_1(\cdot) \dots \vec{b}_s(\cdot)]$$

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Orthogonal projections

$$\vec{\alpha} = (B^T B)^{-1} B^T \vec{v}$$

$$\vec{p} = B \cdot \vec{\alpha}$$

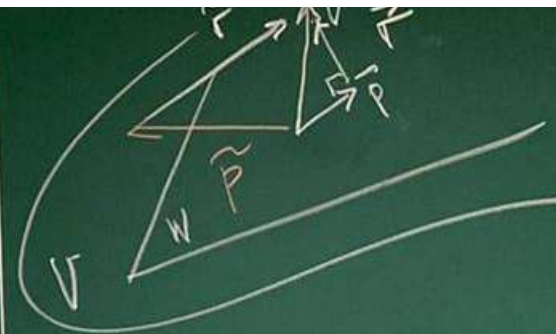
Residual $\vec{r} = \vec{v} - \vec{p} \in W^\perp$

Interest Approximating $\vec{v} \in V$
by $\vec{p} = \text{Proj}_W^\perp(\vec{v})$ which is the closest.

element in W to \vec{v} with respect to \bullet
 $\|\vec{r}\|_\bullet$ is minimum

$$\begin{aligned} \vec{p} &= \text{Proj}_W^\perp(\vec{v}) \\ W &= \langle \vec{b}_1, \vec{b}_2, \dots, \vec{b}_s \rangle \\ &= \alpha_1 \vec{b}_1 + \alpha_2 \vec{b}_2 + \dots + \alpha_s \vec{b}_s \end{aligned}$$

$$\vec{b}_i \in V$$



Normal Equations

$$B^T B \vec{\alpha} = B^T \vec{v}$$

$$B = \begin{bmatrix} \vec{b}_1(\cdot) & \dots & \vec{b}_s(\cdot) \\ \vdots & & \vdots \end{bmatrix}$$

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Orthogonal projections

$$\vec{\alpha} = (B^T B)^{-1} B^T \vec{v}$$

$$\vec{p} = B \cdot \vec{\alpha}$$

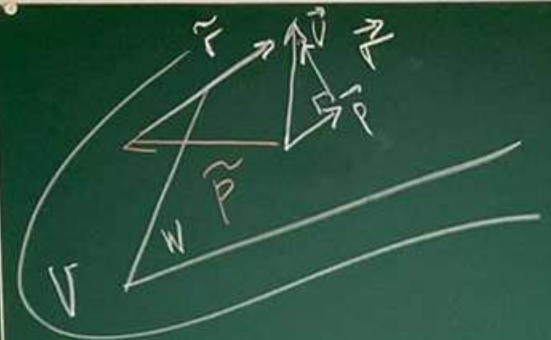
$$\vec{r} = \vec{v} - \vec{p} \in W^\perp$$

Residual

Interest Approximating $\vec{v} \in V$
by $\vec{p} = \underset{W}{\text{Proj}}^\perp(\vec{v})$ which is the closest

element in W to \vec{v} with respect to $\|\vec{r}\|_2$ is minimum

Conclusion \vec{v} is optimally approximated by \vec{p} with relative error

$$E_{\text{rel}} = \frac{\|\vec{r}\|_2}{\|\vec{v}\|_2} \cdot 100 (\%)$$


Normal Equations

$$B^T B \vec{\alpha} = B^T \vec{v}$$

$$B = \begin{bmatrix} \vec{b}_1(\cdot) & \dots & \vec{b}_s(\cdot) \end{bmatrix}$$

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Inversion of $B^T B$

$$B^T B = \begin{pmatrix} \vec{b}_1 \cdot \vec{b}_1 & \vec{b}_1 \cdot \vec{b}_2 & \dots & \vec{b}_1 \cdot \vec{b}_s \\ \vdots & \vdots & \ddots & \vdots \\ \vec{b}_s \cdot \vec{b}_1 & \vec{b}_s \cdot \vec{b}_2 & \dots & \vec{b}_s \cdot \vec{b}_s \end{pmatrix}$$

↓

Full matrix if $B = \{\vec{b}_1, \dots, \vec{b}_s\}$
is standard basis set of W

It would be great if B was
DIAGONAL

$B^T B$ diagonal $\Leftrightarrow \vec{b}_i \cdot \vec{b}_j = 0 \quad \forall i \neq j$

B is orthogonal basis

$$B^T B = \begin{pmatrix} \|\vec{b}_1\|^2 & 0 & \dots & 0 \\ 0 & \|\vec{b}_2\|^2 & & \\ \vdots & & \ddots & \\ 0 & & & \|\vec{b}_s\|^2 \end{pmatrix} \in M_{s \times s}(\mathbb{R})$$

$$(B^T B)^{-1} = \begin{pmatrix} \frac{1}{\|\vec{b}_1\|^2} & & 0 \\ & \ddots & \\ 0 & & \frac{1}{\|\vec{b}_s\|^2} \end{pmatrix}$$

$$\vec{x} = (B^T B)^{-1} B^T \vec{v} = (B^T B)^{-1} \begin{pmatrix} \vec{v} \cdot \vec{b}_1 \\ \vec{v} \cdot \vec{b}_2 \\ \vdots \\ \vec{v} \cdot \vec{b}_s \end{pmatrix} = \left(\frac{\vec{v} \cdot \vec{b}_1}{\|\vec{b}_1\|^2}, \frac{\vec{v} \cdot \vec{b}_2}{\|\vec{b}_2\|^2}, \dots, \frac{\vec{v} \cdot \vec{b}_s}{\|\vec{b}_s\|^2} \right)_B$$

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Inversion of $B^T B$

B orthogonal

$$\vec{\alpha} = \sum_{k=1}^s \left(\frac{\vec{v} \cdot \vec{b}_k}{\|\vec{b}_k\|^2} \right) \cdot \vec{b}_k$$

B is orthonormal

$$\vec{b}_i \cdot \vec{b}_j = \delta_{ij}$$

$$= \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$$

$\|\vec{b}_i\|=1$

Conclusion

$$\vec{\alpha} = \sum_{k=1}^s \underbrace{(\vec{v} \cdot \vec{b}_k)}_{\text{dot}(\vec{v}, \vec{b}_k)} \cdot \vec{b}_k$$

We are interested in
Working with Orthogonal
/ Orthonormal Basis sets.

Any $\vec{B} \xrightarrow{\text{Gram-Schmidt}} \vec{O}$ orthogonal

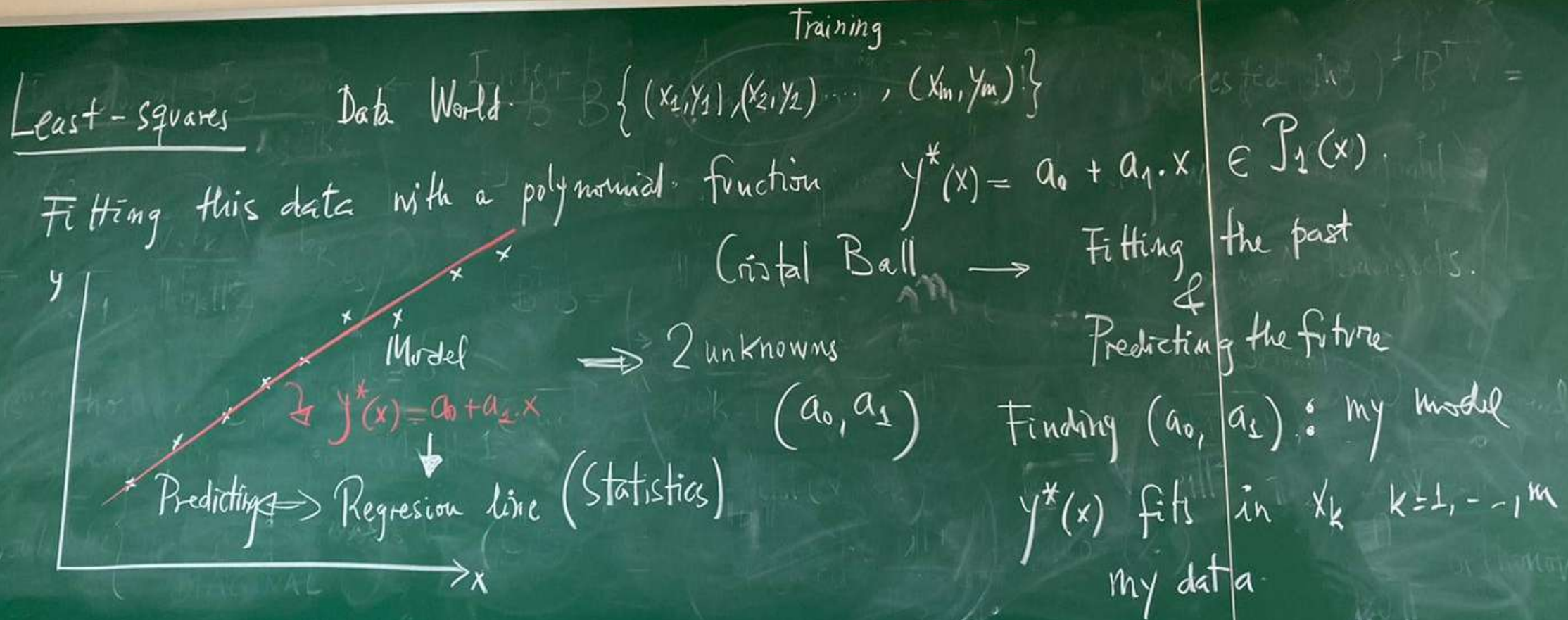
$\vec{O} \downarrow$

$\frac{\vec{O}}{\|\vec{O}\|}$ orthonormal

Least Squares



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Least Squares



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Least-squares

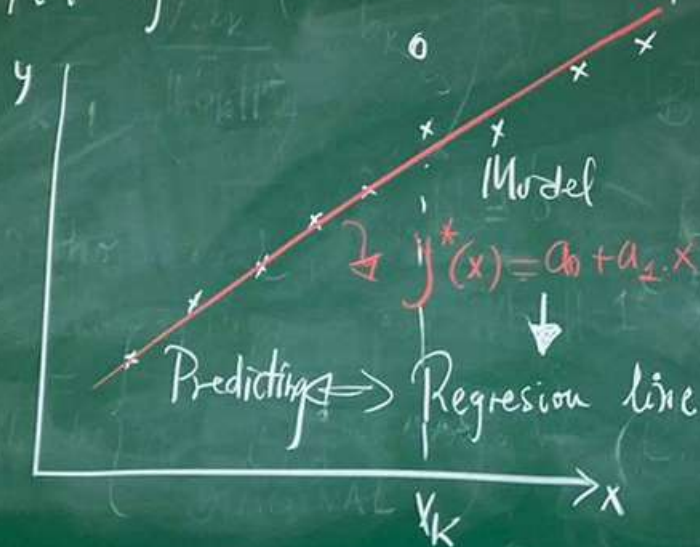
Data World: $\{(x_1, y_1), (x_2, y_2), \dots, (x_m, y_m)\}$

Training

V

$\equiv \mathbb{R}^m$

Fitting this data with a polynomial function



$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix}$$

$$\vec{y}^{obs} = \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix} \in \mathbb{R}^m$$

$$a_0 \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}_{\mathbb{R}^m} + a_1 \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix} = \vec{y}^*$$

$W \Rightarrow$ the Subspace where I want to project.

$$= \begin{pmatrix} a_0 + a_1 \cdot x_1 \\ a_0 + a_1 \cdot x_2 \\ \vdots \\ a_0 + a_1 \cdot x_m \end{pmatrix} \in \mathbb{R}^m$$

Least squares



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Least-squares Training Data World $\{(x_1, y_1), (x_2, y_2), \dots, (x_m, y_m)\}$

Fitting this data with a polynomial function

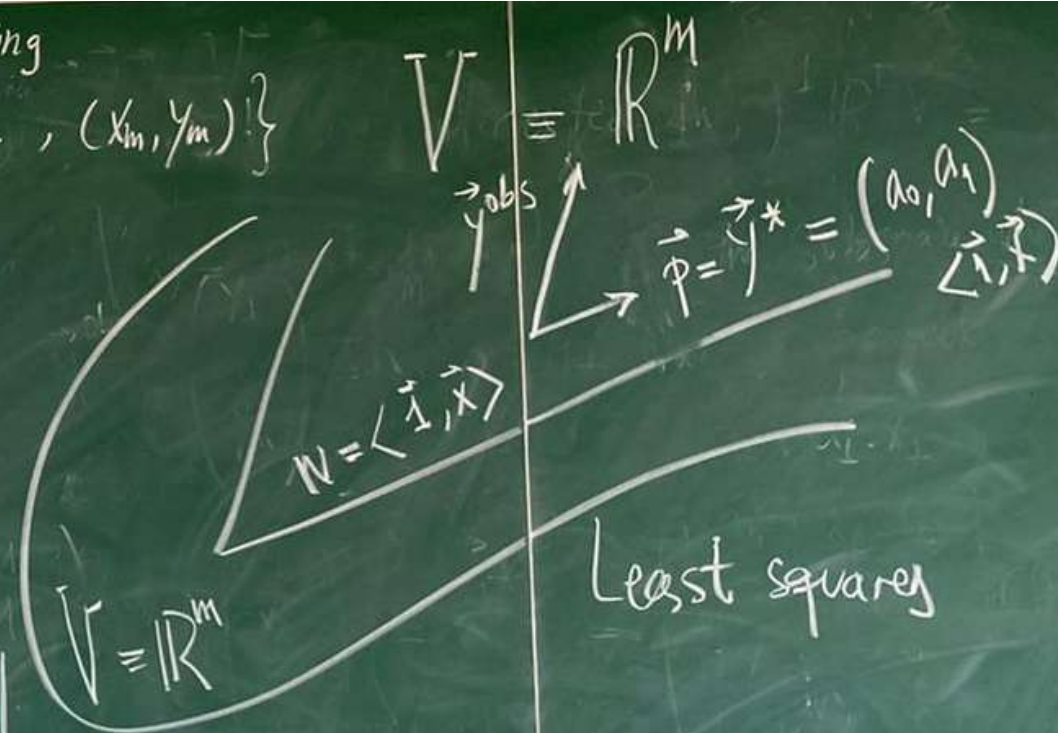
$$\vec{y}^* = a_0 \vec{1}_{\mathbb{R}^m} + a_1 \cdot \vec{x} \in \langle \vec{1}, \vec{x} \rangle$$

W is $\langle \vec{1}, \vec{x} \rangle$, is a subspace of dimension

2 in \mathbb{R}^m .

Finding \vec{p} : $\|\vec{y}^{\text{obs}} - \vec{y}^*\|$

is minimum.



Least squares

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Algorithm

$$\vec{y}^{obs} = \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix}$$

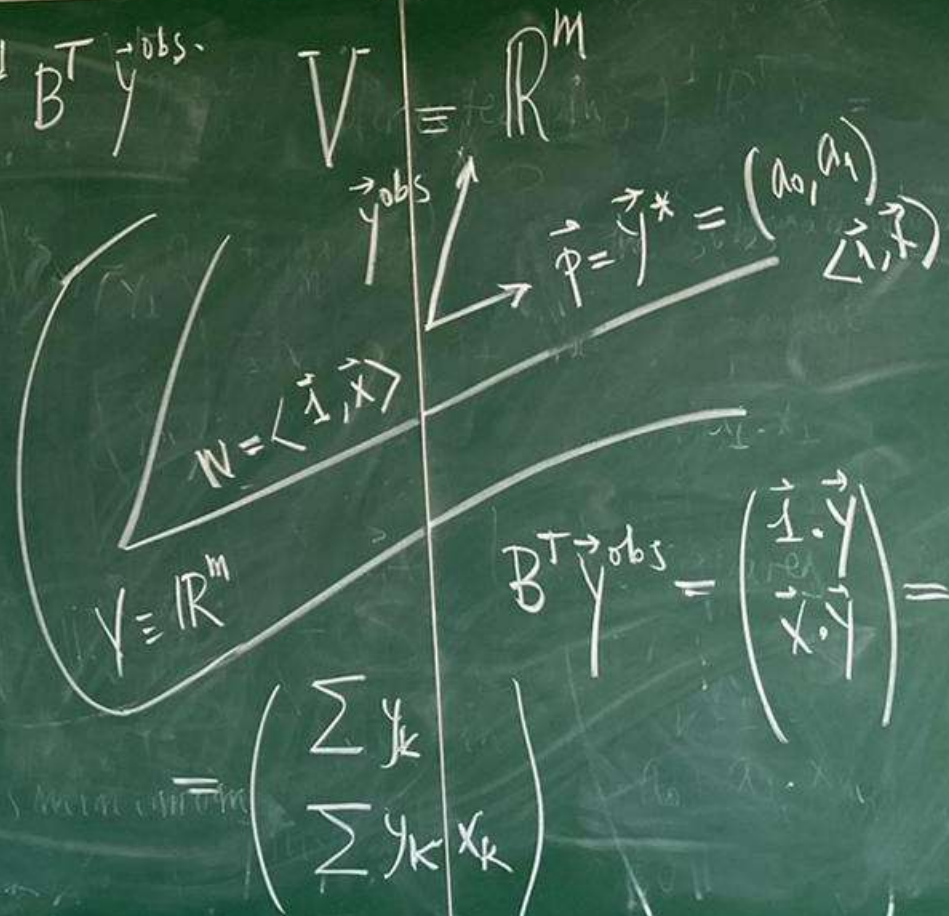
$$\vec{x} = (B^T B)^{-1} B^T \vec{y}^{obs}$$

$$W = \langle \vec{1}, \vec{x} \rangle$$

$\vec{b}_1 \quad \vec{b}_2$

$$B = \begin{bmatrix} \vec{1}^{(:)} & \vec{x}^{(:)} \end{bmatrix}$$

$$B^T B = \begin{bmatrix} \vec{1}^T \\ \vec{x}^T \end{bmatrix} \begin{bmatrix} \vec{1} & \vec{x} \end{bmatrix} = \begin{bmatrix} \vec{1} \cdot \vec{1} & \vec{1} \cdot \vec{x} \\ \vec{x} \cdot \vec{1} & \vec{x} \cdot \vec{x} \end{bmatrix} = \begin{bmatrix} m & \sum x_k \\ \sum x_k & \sum x_k^2 \end{bmatrix}$$



$$B^T \vec{y}^{obs} = \begin{pmatrix} \vec{1} \cdot \vec{y} \\ \vec{x} \cdot \vec{y} \end{pmatrix} =$$

$$= \begin{pmatrix} \sum y_k \\ \sum y_k x_k \end{pmatrix}$$