CALCULUS
DEGREE IN SOFTWARE ENGINEERING
CHAPTER 16. INTEGRATION OF IRRATIONAL FUNCTIONS. TRIGONOMETRIC CHANGES.

INTEGRATION OF IRRATIONAL FUNCTIONS.

We call irrational functions to those that involve fractional powers of x. In this section, we will deal with integrands that can be expressed as

$$R(x, x^{p_1/q_1}, x^{p_2/q_2}, ...x^{p_n/q_n})$$

where R is a rational function whose arguments are x and some fractional powers of x. If we calculate the least common multiple (lcm) of the denominators

$$q = lcm(q_1, q_2, ..., q_n)$$

and make the substitution $x = t^q$, the new integral will always be that of a rational function of t and we can solve it by using the methods proposed in Chapter 15. Let us see an example

EXAMPLE.

$$\int \frac{x^{1/2}\, dx}{1 + x^{1/3}}$$

Since $\mathrm{lcm}(2, 3) = 6$, we make the change $x = t^6$. Then $dx = 6t^5\, dt$ and the integral becomes

$$\int \frac{t^3 .\, 6t^5\, dt}{1 + t^2}$$

Our rational function is improper and we have to divide

$$\int \frac{t^3 .\, 6t^5\, dt}{1 + t^2} = 6 \int \left(t^6 - t^4 + t^2 - 1 + \frac{1}{1 + t^2}\right) dt$$

and the final indefinite integral I is

$$I = 6(t^7/7 - t^5/5 + t^3/3 - t + \arctan t) + C = 6(x^{7/6}/7 - x^{5/6}/5 + x^{1/2}/3 - x^{1/6} + \arctan x^{1/6}) + C$$

In general, we could integrate an expression such as

$$R(x, (ax + b)^{p_1/q_1}, (ax + b)^{p_2/q_2}, ...(ax + b)^{p_n/q_n})$$

with the change

$$(ax + b) = t^q$$

, converting the integrand into a rational function of t.

INTEGRATION OF RATIONAL FUNCTIONS of e^x

In this case, the substitution $t = e^x$ will make the integral a rational function of t

EXAMPLE

$$\int \frac{e^x\,dx}{1+e^{2x}}$$

$t = e^x$ and then, $dt = e^x\,dx$. The new integral is

$$\int \frac{dt}{1+t^2} = \arctan t + C = \arctan e^x + C$$

TRIGONOMETRIC CHANGES

The so-called trigonometric changes can be applied to the following integrands, making the integration easier.

$$1)\; R(x, \sqrt{a^2 - x^2})$$

This is a rational function of its arguments, with a a positive constant, and the subtitution which changes the integrand into a suitable trigonometric function is

$$x = a\ \sin t$$

$$2)\; R(x, \sqrt{a^2 + x^2})$$

Now, the convenient change is

$$x = a\ \tan t$$

Finally,

$$3)\; R(x, \sqrt{x^2 - a^2})$$

with the corresponding substitution

$$x = a\ \sec t$$

We will do an exercise for each type of integrand, trying to illustrate why the changes work.

EXAMPLE. x=a sin(t)

$$\int \sqrt{1-x^2}\,dx$$

Clearly, the suitable change is $x=\sin t,\ dx=\cos t\,dt$
Making the substitution

$$\int \sqrt{1-x^2}\,dx = \int \sqrt{1-\sin^2 t}\ \cos t\,dt = \int \cos^2 t\,dt$$

where we choose t in $(-\pi/2, \pi/2)$, so that $\cos t$ is positive.
Now, we use

$$\int \cos^2 t\,dt = \int \frac{(1+\cos 2t)}{2}\,dt = t/2 + \frac{\sin 2t}{4} + C$$

Changing back to the original variable, the indefinite integral is

$$\frac{\arcsin x}{2} + \frac{x\sqrt{1-x^2}}{2} + C$$

This integral is very useful since it allows us to calculate the area of a circle.

EXAMPLE. x=a tan(t)

$$\int \sqrt{4+x^2}\,dx$$

With $x = 2\ \tan t$, the integral becomes

$$\int \sqrt{4+x^2}\,dx = 2\int \sqrt{4+4\tan^2 t}\ (1+\tan^2 t)\,dt = 4\int \sec^3 t\,dt$$

Now, we use our knowledge of $\int \sec^3 t\,dt$

$$4\int \sec^3 t\,dt = 2(\sec t\ \tan t + \ln|\sec t + \tan t|) + C =$$

and reversing the change, the indefinite integral is

$$\frac{x\sqrt{4+x^2}}{2} + 2\ln|x+\sqrt{4+x^2}| + C$$

where we have taken into account that $x/2 = \tan t$ and $\sec t = \dfrac{\sqrt{4+x^2}}{2}$

EXAMPLE. x=a sec(t)

$$\int \frac{dx}{\sqrt{x^2-16}}$$

With $x = 4\ \sec t$ and $dx = 4\ \sec t \tan t\,dt$, we find

$$\int \frac{dx}{\sqrt{x^2-16}} = \int \sec t \, dt = \ln|\sec t + \tan t| + C$$

and, in terms of the original variable,

$$\int \frac{dx}{\sqrt{x^2-16}} = \ln|x + \sqrt{x^2-16}| + C$$

We will show you a variety of problems, involving the different methods of integration in Exercises 6 and 7.