CALCULUS
DEGREE IN SOFTWARE ENGINEERING
CHAPTER 8. DIFFERENTIATION.

Derivatives are essential in applied mathematics. They appear in physics, chemistry, biology, engineering and other sciences like economics. Derivatives represent the local rate of change of a dependent variable with respect to the independent variable: velocity is the derivative of space with respect to time, acceleration its second derivative. The variation of pressures, temperatures, energies... can be expressed in terms of derivatives, giving rise to differential equations, equations involving derivatives, which are ubiquitous in physics. Most physical laws can be encapsulated in one or several differential equations.
Our introduction to the concept of derivative of a function at a point will be canonical, using the geometrical idea of slope of a curve. When we calculate the derivative of a function, we can also say that we differentiate the function and the operation is called differentiation. We speak about differential calculus. As we will see, the concept derives from the calculation of quotients of differences.

DERIVATIVE AND SLOPE OF A CURVE

Imagine you draw the graph of a continuous function $y = f(x)$ from $\mathbb{R}$ to $\mathbb{R}$ and select an interior point x_0 of its domain. The corresponding point on the graph, P_0 , will have coordinates $(x_0, f(x_0))$. If we select a nearby point $x_0 + h_1$, we can mark the corresponding point of the graph $P_1(x_0 + h_1, f(x_0 + h_1))$. The line joining both points is called a secant line and has a slope

$$m_1 = \frac{f(x_0 + h_1) - f(x_0)}{h_1}$$

h_1 can be positive or negative. We draw an example in Figures 2. Now, we choose values h_1, h_2,......h_n, smaller and smaller in absolute value. In our graphical example, we choose these numbers positive. We have created a sequence of secant lines, joining points of the graph that are closer and closer. The slope of each secant line can be calculated according to the previous formula. For instance, if we take $P_k(x_0 + h_k, f(x_0 + h_k))$, the slope of the secant line will be

$$m_k = \frac{f(x_0 + h_k) - f(x_0)}{h_k}$$

If we make the change, h, in the independent variable tend to zero, we can calculate the following limit

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

If this limit $f'(x_0)$ exists, as a finite limit, we call it the derivative of the function $f(x)$ at x_0 and means the limit of the slope of secant lines joining points closer and closer to P_0. We can also draw a line that passes through P_0 with slope $f'(x_0)$, this line is the tangent line: $y - f(x_0) = f'(x_0)(x - x_0)$. $f'(x_0)$ is also the definition of the slope of the graph at P_0. It is the local quotient of the distance travelled upward (or downward) over the distance travelled horizontally when we move from P_0 to a nearby point along the curve. Therefore, we have several definitions and ideas for the derivative:

1) The slope of the graph of $y = f(x)$ at $x = x_0$.
2) The slope of the tangent to the curve $y = f(x)$ at $x = x_0$.
3) The derivative $f'(x_0)$ as the limit defined before.
4) The rate of change of $f(x)$ with respect to x at $x = x_0$, (instantaneous or local) rate of change.

We can define the derivative function $f'(x)$ at any interior point of the domain as the function whose value at x is the derivative at that point.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

The concept can be extended to the endpoints a and b of a closed interval $[a, b]$

$$f'_+(a) = \lim_{h\to 0^+} \frac{f(a+h) - f(a)}{h}$$

$$f'_-(b) = \lim_{h\to 0^-} \frac{f(b+h) - f(b)}{h}$$

We call these derivatives right-hand derivative ane left-hand derivative respectively. We can also say that a derivative is ∞ or $-\infty$ if the limit is infinite. In this case, the tangent line will be vertical.

EXAMPLES

1) Let us differentiate $y = f(x) = x^3$

$$f'(x) = \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h}$$

Using Newtons's binomial

$$f'(x) = \lim_{h\to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h} == \lim_{h\to 0} 3x^2 + 3xh + h^2 = 3x^2$$

This can be generalized for any n natural to $(x^n)' = nx^{n-1}$

2) If $f(x) = 1/x$, then $f'(x) = -1/x^2$. This can be easily proved

$$f'(x) = \lim_{h\to 0} \frac{1/(x+h) - 1/x}{h} == \lim_{h\to 0} -1/(x(x+h)) = -1/x^2$$

Of course, the function is not defined at $x = 0$ and the derivative is not defined there either.

3) $y = \sqrt{x}$

$$f'(x) = \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} = \lim_{h\to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

where we have multiplied numerator and denominator by the conjugate of the numerator and simplified the quotient. This derivative is well defined, as the function, for $x > 0$. For $x = 0$, the function is well defined and continuous (right-continuous), but the derivative is infinite, ∞. Geometrically, this means that the tangent to the curve at that point is a vertical line. As an exercise, you can calculate the derivative of $y = \sqrt{|x|}$. The solution is $y' = \dfrac{1}{2\sqrt{|x|}}$, if $x > 0$, and $y' = -\dfrac{1}{2\sqrt{|x|}}$, if $x < 0$. At $x = 0$, the right-hand derivative is ∞ and the left-hand derivative is $-\infty$. Grafically, we can observe a cusp. See the figure in Figures 2.
4) $y = f(x) = |x|$. The derivative is 1 if $x > 0$ and -1 if $x < 0$. At $x = 0$ is not defined, as it is 1 from the right and -1 from the left. The limit

$$f'(0) = \lim_{h\to 0} \frac{|h|}{h}$$

is 1 from the right and -1 from the left. There is a corner at $x = 0$.

Finally, we will state and prove an important property of functions which have derivatives, differentiable functions: they must be continuous. The reciprocal is not in general true. The absolute value function at $x = 0$ is an example: it is continuous and it is not differentiable, as left and right-derivatives are not equal.

THEOREM: If f is differentiable at $x = c$, then f is continuous at $x = c$

Proof: We write

$$f(c+h) = f(c) + \frac{f(c+h) - f(c)}{h}\, h$$

This looks like a different way of writing $f(c+h)$, but it is ingenious, because if we take limits on both sides

$$\lim_{h\to 0} f(c+h) = f(c) + \lim_{h\to 0} \frac{f(c+h) - f(c)}{h}\, h$$

Taking into account the definition of derivative, the limit on the right-hand side is 0 and

$$\lim_{h\to 0} f(c+h) = f(c)$$

This is just the defintion of continuous function at c.

If c is an interior point of the domain, the proof does mot need further clarifications. However, if it is an endpoint of a closed and bounded interval, the derivatives must be understood in the sense of right and left derivatives and the continuity in the same sense.