CALCULUS
DEGREE IN SOFTWARE ENGINEERING
CHAPTER 22. PARTIAL DERIVATIVES.

In this chapter we will present the concept of partial derivative for multivable functions, giving examples of functions of two and three variables.

PARTIAL DERIVATIVE OF A TWO-VARIABLE FUNCTION

A point (x_0, y_0) of the domain D of a two-variable function is called **an interior point of D** if there is an open disk centered at (x_0, y_0) and contained in D. The geometric idea is that (x_0, y_0) is in the interior and not on the boundary of D- which will be defined with precision later-. From an interior point we can move in any direction and remain inside the domain, what allows us to define the partial derivatives of a function $f(x, y)$ at (x_0, y_0). A set whose points are all interior points is called an open set.

DEFINITION

The partial derivative of a function $f(x, y)$ with respect to x at an interior point (x_0, y_0) of its domain is

$$\frac{\partial f}{\partial x}(x_0, y_0) = \lim_{h\to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}$$

The partial derivative of a function $f(x, y)$ with respect to y at an interior point (x_0, y_0) of its domain is

$$\frac{\partial f}{\partial y}(x_0, y_0) = \lim_{h\to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h}$$

In most of the practical cases we do not need to calculate the limit; we just take the derivative with respect to x or y holding the other variable constant. This means that we will use the differentiation rules studied in Chapter 9.
Clearly, the concept is a generalization of the derivative of a single variable function: for the partial derivative with respect to x, we take the variation of the function when we move along a direction parallel to the x-axis and divide this variation by the change of x, calculating the limit of this quotient as the change of x goes to zero, that is, we find the local rate of change in the direction of the x-axis. For the partial derivative with respect to y, we find the local rate of change in the direction of the y-axis.
We can also give a geometrical interpretation of partial derivatives. If we consider the intersection of the surface $z = f(x, y)$ with the plane $y = y_0$, we obtain a curve

$z = f(x, y_0)$ on that plane. The partial derivative $\frac{\partial f}{\partial x}(x_0, y_0)$ is the slope of that curve at $x = x_0$, the equation of the corresponding tangent line is

$$z - f(x_0, y_0) = \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0)$$

See Figure 14.15 in Presentation 9 for an example.
If we intersect the surface $z = f(x, y)$ with the plane $x = x_0$, we obtain a curve $z = f(x_0, y)$ on that plane. The partial derivative $\frac{\partial f}{\partial y}(x_0, y_0)$ is the slope of that curve at $y = y_0$, the equation of the corresponding tangent line is

$$z - f(x_0, y_0) = \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0)$$

See Figure 14.16 for an example of this geometrical interpretation of $\frac{\partial f}{\partial y}(x_0, y_0)$.
In Figure 14.17 we can observe the plane generated by both tangent lines. It is logically called the tangent plane and as a generalization of tangent lines, it represents the best linear approximation of the surface locally when we approach (x_0, y_0). The equation of the tangent plane can be found taking into account that it contains both tangent lines and reads

$$z - f(x_0, y_0) = \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0)$$

Now, we will see a few examples of how to calculate partial derivatives by using the differentiation rules. In the examples we will call (x, y) a generic point where we calculate the partial derivatives of the function.

EXAMPLES

1.
$$f(x, y) = x^2 y\, e^{x+y^2}$$

$$\frac{\partial f}{\partial x}\,(x, y) = 2xy\, e^{x+y^2} + x^2 y\, e^{x+y^2} = xy(2 + x)\, e^{x+y^2}$$

$$\frac{\partial f}{\partial y}\,(x, y) = x^2\, e^{x+y^2} + x^2 y.2y\, e^{x+y^2} = x^2(1 + 2y^2)\, e^{x+y^2}$$

At a point, for instance $(1, 1)$, the values of the partial derivatives are

$$\frac{\partial f}{\partial x}\,(1, 1) = 3\, e^2$$

$$\frac{\partial f}{\partial y}\,(1, 1) = 3\, e^2$$

Imagine we consider the surface $z = x^2 y\, e^{x+y^2}$ and the intersection of this surface with the plane $y = 1$, we obtain a curve on that plane , $z = x^2\, e^{x+1}$, the number $\dfrac{\partial f}{\partial x}\,(1, 1) = 3\, e^2$ is the slope of the given curve at $x = 1$. If we now intersect the surface with the plane $x = 1$, we have another curve, $z = y\, e^{1+y^2}$, on the plane $x = 1$, the number $\dfrac{\partial f}{\partial y}\,(1, 1) = 3\, e^2$ is the slope of this curve at $y = 1$. In general, the partial derivatives are the slopes of the intersection curves of the surface $z = f(x, y)$ with the planes $y = y_0$, partial derivative with respect to x, and $x = x_0$, partial derivative with respect to y, at the point (x_0, y_0). We can draw the tangent line to the first curve, it will have the following equation

$$z - e^2 = 3\, e^2(x - 1)$$

with $y = 1$

The equation of the tangent line to the second curve will be

$$z - e^2 = 3\, e^2(y - 1)$$

with $x = 1$

Both tangent lines determine a plane

$$z - e^2 = 3\, e^2(x - 1) + 3\, e^2(y - 1)$$

You can check that it contains both tangent lines and passes through $(1, 1, e^2)$, the point on the surface. This plane is the tangent plane to the surface at $(1, 1, e^2)$.

2.

$$f(x, y) = \frac{xy^3}{x^2 + y^2}$$

The partial derivatives are

$$\frac{\partial f}{\partial x}\,(x, y) = \frac{y^5 - x^2y^3}{(x^2 + y^2)^2}$$

$$\frac{\partial f}{\partial y}\,(x, y) = \frac{3x^2y^2 + xy^4}{(x^2 + y^2)^2}$$

and at (1,2)

$$\frac{\partial f}{\partial x}\,(1, 2) = 24/25$$

$$\frac{\partial f}{\partial y}(1,2) = 28/25$$

You can write the equation of the respective tangent lines and finally, that of the tangent plane at $(1, 2, 8/5)$

$$z - 8/5 = 24/25(x-1) + 28/25(y-2)$$

3.

$$z = \sqrt{x^2+y^2}$$

$$\frac{\partial f}{\partial x}(x,y) = \frac{x}{\sqrt{x^2+y^2}}$$

$$\frac{\partial f}{\partial y}(x,y) = \frac{y}{\sqrt{x^2+y^2}}$$

However, these partial derivatives are not well defined at $(0, 0)$. At that point we must use the definition

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0}\frac{f(0+h,0)-f(0,0)}{h} = \lim_{h\to 0}\frac{|h|}{h}$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{h\to 0}\frac{f(0,0+h)-f(0,0)}{h} = \lim_{h\to 0}\frac{|h|}{h}$$

In both cases the limit does not exist. The situation is similar to that of $y = |x|$ at $x = 0$. In fact, the surface $z = \sqrt{x^2+y^2}$ is a cone generated by the line $z = y$ rotating around the z-axis. The point $(0, 0, 0)$ is the vertex of the cone, where the tangent plane is not well defined. See the cone in Figure 1.

4.

$$P = nRT/V$$

This is the ideal gas equation and we can calculate

$$\frac{\partial P}{\partial V} = -\frac{nRT}{V^2}$$

$$\frac{\partial P}{\partial T} = \frac{nR}{V}$$

We can interpret these formulas as saying that the pressure of the gas decreases as you increase its volume and fix its temperature, negative derivative, and increases as you increase its temperature keeping a constant volume, positive derivative.

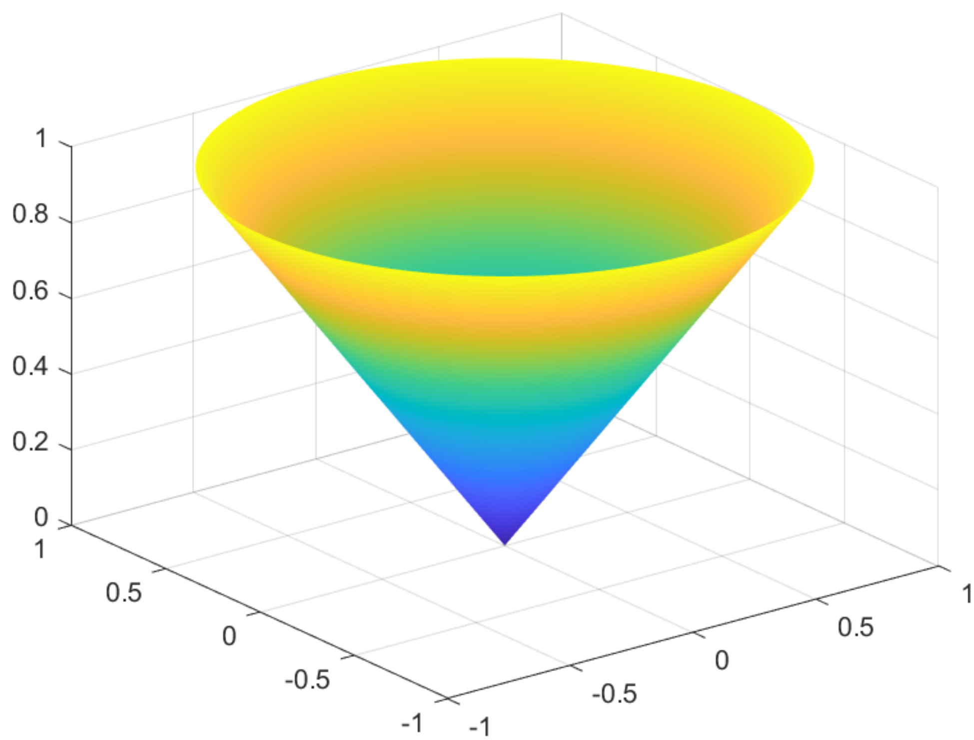

Figure 1: Example 1. $z = \sqrt{x^2 + y^2}$

FUNCTIONS OF MORE THAN TWO VARIABLES

For a function of more than two variables, its partial derivatives are defined in the same way as for a two-variable function: we take the derivative with respect to one of the variables while the other variables are held constant. The definition in terms of the limit is, for a three-variable function

$$\frac{\partial f}{\partial x}(x_0, y_0, z_0) = \lim_{h\to 0} \frac{f(x_0+h, y_0, z_0) - f(x_0, y_0, z_0)}{h}$$

$$\frac{\partial f}{\partial y}(x_0, y_0, z_0) = \lim_{h\to 0} \frac{f(x_0, y_0+h, z_0) - f(x_0, y_0, z_0)}{h}$$

$$\frac{\partial f}{\partial z}(x_0, y_0, z_0) = \lim_{h\to 0} \frac{f(x_0, y_0, z_0+h) - f(x_0, y_0, z_0)}{h}$$

For instance, if

$$f(x, y, z) = x^2 y^3 z^4$$

, then

$$\frac{\partial f}{\partial x}(x, y, z) = 2xy^3z^4$$

$$\frac{\partial f}{\partial y}(x, y, z) = 3x^2y^2z^4$$

$$\frac{\partial f}{\partial y}(x, y, z) = 4x^2y^3z^3$$

and if

$$f(x, y, z) = \frac{K}{\sqrt{x^2+y^2+z^2}}$$

,then

$$\frac{\partial f}{\partial x}(x, y, z) = -\frac{Kx}{(x^2+y^2+z^2)^{3/2}}$$

$$\frac{\partial f}{\partial y}(x, y, z) = -\frac{Ky}{(x^2+y^2+z^2)^{3/2}}$$

$$\frac{\partial f}{\partial y}(x, y, z) = -\frac{Kz}{(x^2+y^2+z^2)^{3/2}}$$

$f(x, y, z)$ could stand for the external potencial created by a homogenous spherical body. The corresponding gravitational field- a vector field- is minus the vector field formed by the partial derivatives,that is

$$\vec{E} = -\Big(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\Big)$$

where

$$\vec{\nabla} f = \Big(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\Big)$$

is called the gradient of f and $\vec{\nabla}$ is the nabla operator.
For a function of two variables

$$\vec{\nabla} f = \Big(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\Big)$$

and in general, the gradient of a function of n variables, $f(x_1, x_2, ..., x_n)$, is the vector field consisting of the n partial derivatives.

$$\vec{\nabla} f = \Big(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, ..., \frac{\partial f}{\partial x_n}\Big)$$

The concept of gradient is essential in physics.

SECOND-ORDER PARTIAL DERIVATIVES

If we differentiate a function of two variables $f(x, y)$ twice, we produce its second-order partial derivatives

$$\frac{\partial^2 f}{\partial x^2}, \quad \frac{\partial^2 f}{\partial y^2}, \quad \frac{\partial^2 f}{\partial x \partial y}, \quad \frac{\partial^2 f}{\partial y \partial x}$$

In the first case, we have differentiated twice with respect to x, holding y constant; in the second, we have differentiated twice with respect to y, holding x constant; in the third, we have differentiated once with respect to y, holding x constant, and a second time with respect to x, holding y constant; finally we have changed the order of differentiation in the fourth case.
All these derivatives extend the idea of second-order derivative of a single variable function.

$$\frac{\partial^2 f}{\partial x \partial y}, \quad \frac{\partial^2 f}{\partial y \partial x}$$

are called the mixed partial derivatives. These calculations can be also carried out for functions of n variables and for instance, in the three-variable case, we obtain, in principle, nine different second-order partial derivatives, just by combining two variables and taking into account the order of differentiation. However, the order of differentiation will not matter in practice as is stated in this theorem

THE MIXED DERIVATIVE THEOREM

If $f(x, y)$ and its partial derivatives $\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}, \dfrac{\partial^2 f}{\partial x \partial y}$ and $\dfrac{\partial^2 f}{\partial y \partial x}$ are defined throughout an open disk containing a point (x_0, y_0) and are all continuous at (x_0, y_0), then

$$\frac{\partial^2 f}{\partial x \partial y}(x_0, y_0) = \frac{\partial^2 f}{\partial y \partial x}(x_0, y_0)$$

This theorem can be generalized to the case of more variables or derivatives of higher order, implying that, given the condition of existence and continuity of the derivatives, the order of differentiation is irrelevant.
We will see two examples of these calculations:

EXAMPLE 1

$$f(x, y) = x^3 y^5$$

$$\frac{\partial f}{\partial x} = 3x^2 y^5 \ \ , \frac{\partial f}{\partial y} = 5x^3 y^4$$

$$\frac{\partial^2 f}{\partial y \partial x} = 15x^2 y^4 \ \ , \frac{\partial^2 f}{\partial x \partial y} = 15x^2 y^4$$

$$\frac{\partial^2 f}{\partial x^2} = 6xy^5 \ \ , \frac{\partial^2 f}{\partial y^2} = 20x^3 y^3$$

This function is continuous on $\mathbb{R}^2$, we write $f \in C^0(\mathbb{R}^2)$, its first-order partial derivatives are also continuous, $f \in C^1(\mathbb{R}^2)$ and its second-order partial derivatives are continuous too, $f \in C^2(\mathbb{R}^2)$, this implies, according to the previous theorem, that the order of differentiation does not matter and as we have checked

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = 15x^2 y^4$$

for all points in the domain, $\mathbb{R}^2$.

EXAMPLE 2

$$f(x, y, z) = (x^2 + y^2 + z^2)^{-1/2}$$

$$\frac{\partial f}{\partial x} = -x(x^2+y^2+z^2)^{-3/2} \ \ , \frac{\partial f}{\partial y} = -y(x^2+y^2+z^2)^{-3/2} \ \ , \frac{\partial f}{\partial z} = -z(x^2+y^2+z^2)^{-3/2}$$

$$\frac{\partial^2 f}{\partial x^2} = -(x^2 + y^2 + z^2)^{-3/2} + 3x^2(x^2 + y^2 + z^2)^{-5/2}$$

$$\frac{\partial^2 f}{\partial y^2} = -(x^2 + y^2 + z^2)^{-3/2} + 3y^2(x^2 + y^2 + z^2)^{-5/2}$$

$$\frac{\partial^2 f}{\partial z^2} = -(x^2 + y^2 + z^2)^{-3/2} + 3z^2(x^2 + y^2 + z^2)^{-5/2}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = 3xy(x^2 + y^2 + z^2)^{-5/2}$$

$$\frac{\partial^2 f}{\partial z \partial x} = \frac{\partial^2 f}{\partial x \partial z} = 3xz(x^2 + y^2 + z^2)^{-5/2}$$

$$\frac{\partial^2 f}{\partial y \partial z} = \frac{\partial^2 f}{\partial z \partial y} = 3yz(x^2 + y^2 + z^2)^{-5/2}$$

all these derivatives are defined and continuous on $\mathbb{R}^2 \setminus \{(0,0,0)\}$
you can check that

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$$

this is Laplace's equation, valid for the gravitational potencial in empty space. The operator obtained by the sum of these functions is called the Laplacian and written

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

PARTIAL DERIVATIVES AND CONTINUITY

If a function of a single variable had a derivative at a point, it was also continuous at that point. Then, a natural question is: Does the same happen for a multivariable function? The answer is no. We will show a counterexample. Let

$$f(x,y) = \begin{cases} \dfrac{xy^2}{x^2 + y^4} & \text{if} \quad (x,y) \neq (0,0) \\ \\ 0 & \text{if} \quad (x,y) = (0,0) \end{cases}$$

the function is not continuous at $(0,0)$, but are the partial derivatives defined ?

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h \to 0} 0/h = 0$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{h \to 0} \frac{f(0,h) - f(0,0)}{h} = \lim_{h \to 0} 0/h = 0$$

Thus, both partial derivatives exist but the function is not continuous. When the derivative of a single variable function existed, we said that it was differentiable and we could prove that it was continuous as well. In the next section we will present the concept of differentiability for a two-variable function, a concept which will require something else than the mere existence of derivatives and will imply continuity.