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S3-Euclidean Spaces
Part II

Linear Algebra

Ingeniería del Software-Universidad de Oviedo

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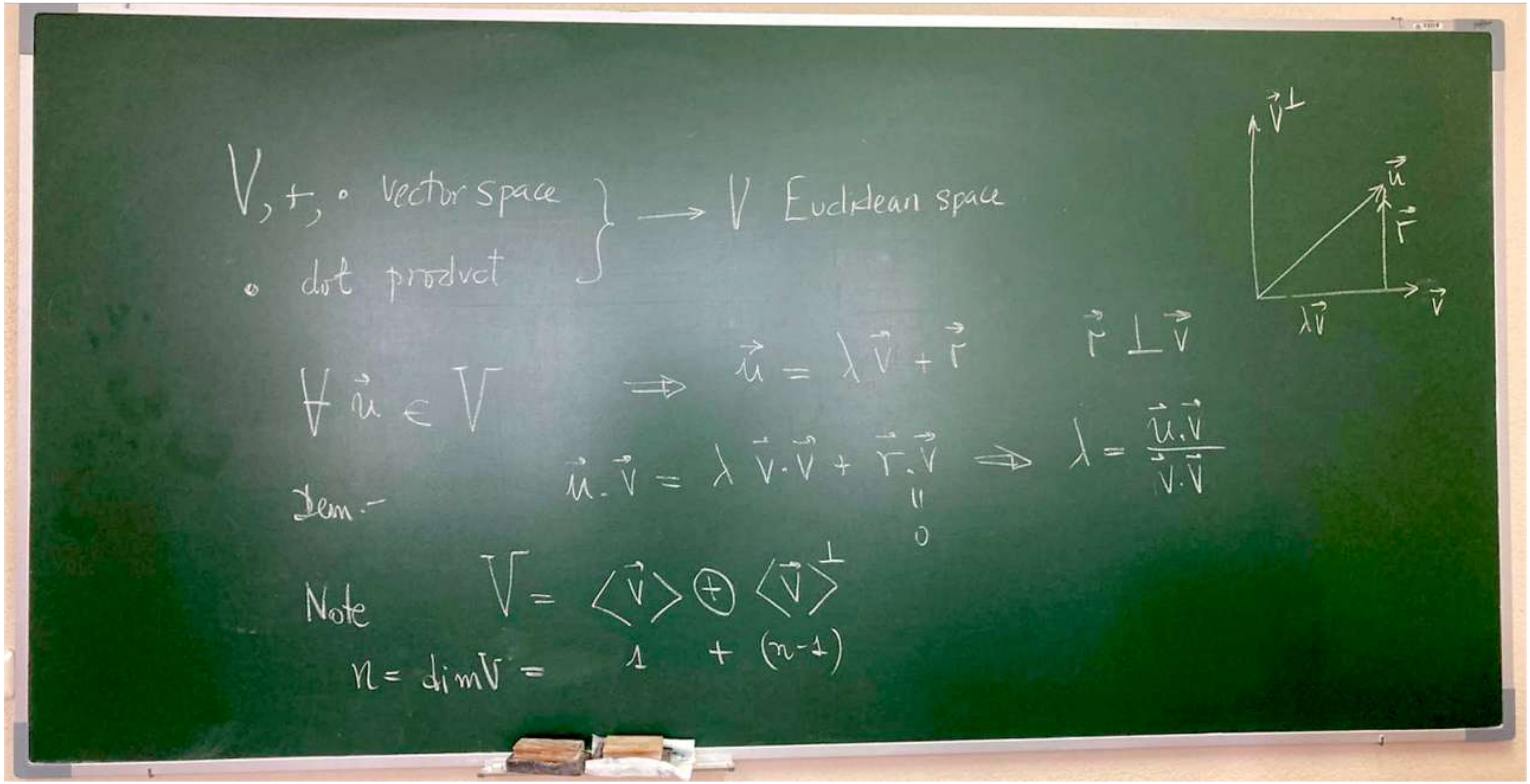
Orthogonality

- **Orthogonality:** Two vectors **u, v** are orthogonal if $\mathbf{u}.\mathbf{v} = 0$.
- **Unit vector**: $\dfrac{\mathbf{u}}{||\mathbf{u}||}$
- **Orthogonal basis:**
  - $\mathbf{V} = \langle \mathbf{v_1}, \mathbf{v_2}, \dots, \mathbf{v_n} \rangle$
  - $\mathbf{v_1}.\mathbf{v_2} = \mathbf{0}.$
- **Theorem:** Any vector in an Euclidean project any non-null vector $\mathbf{u}$, can be decomposed in a vector in the direction of $\mathbf{v}$ and another which is orthogonal to $\mathbf{v}$.

Orthogonal projections



Orthogonal projections

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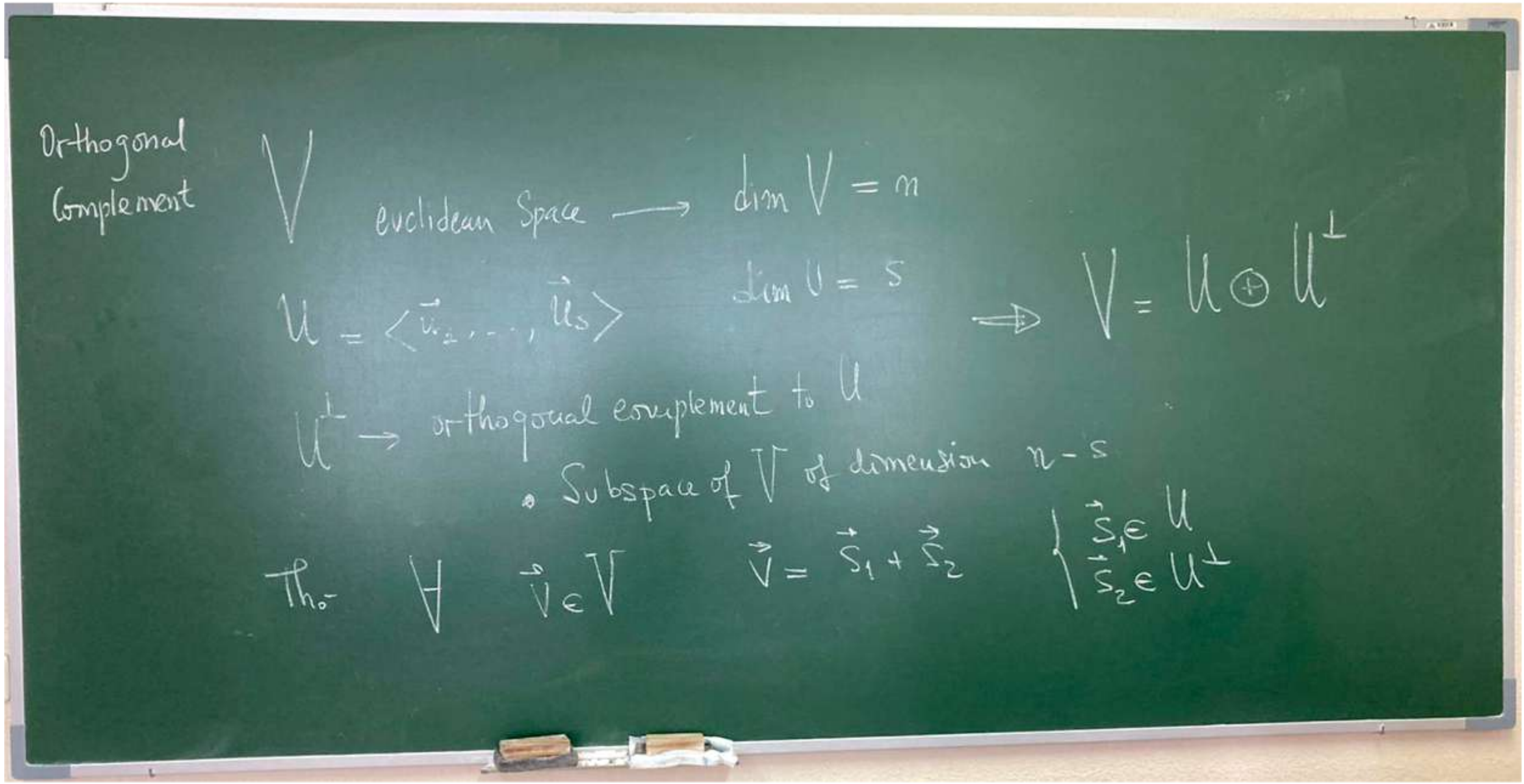

Orthogonal complement

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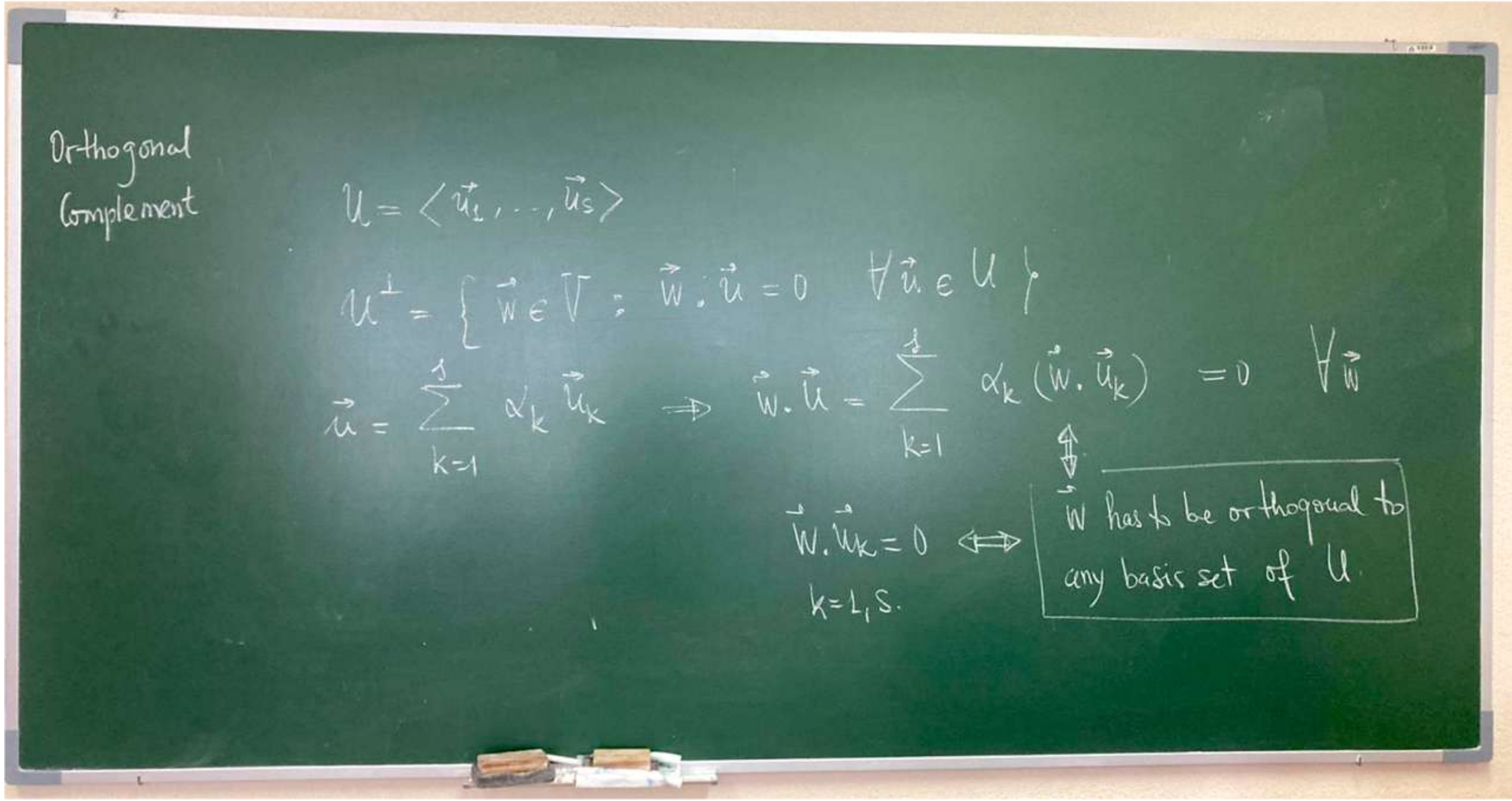
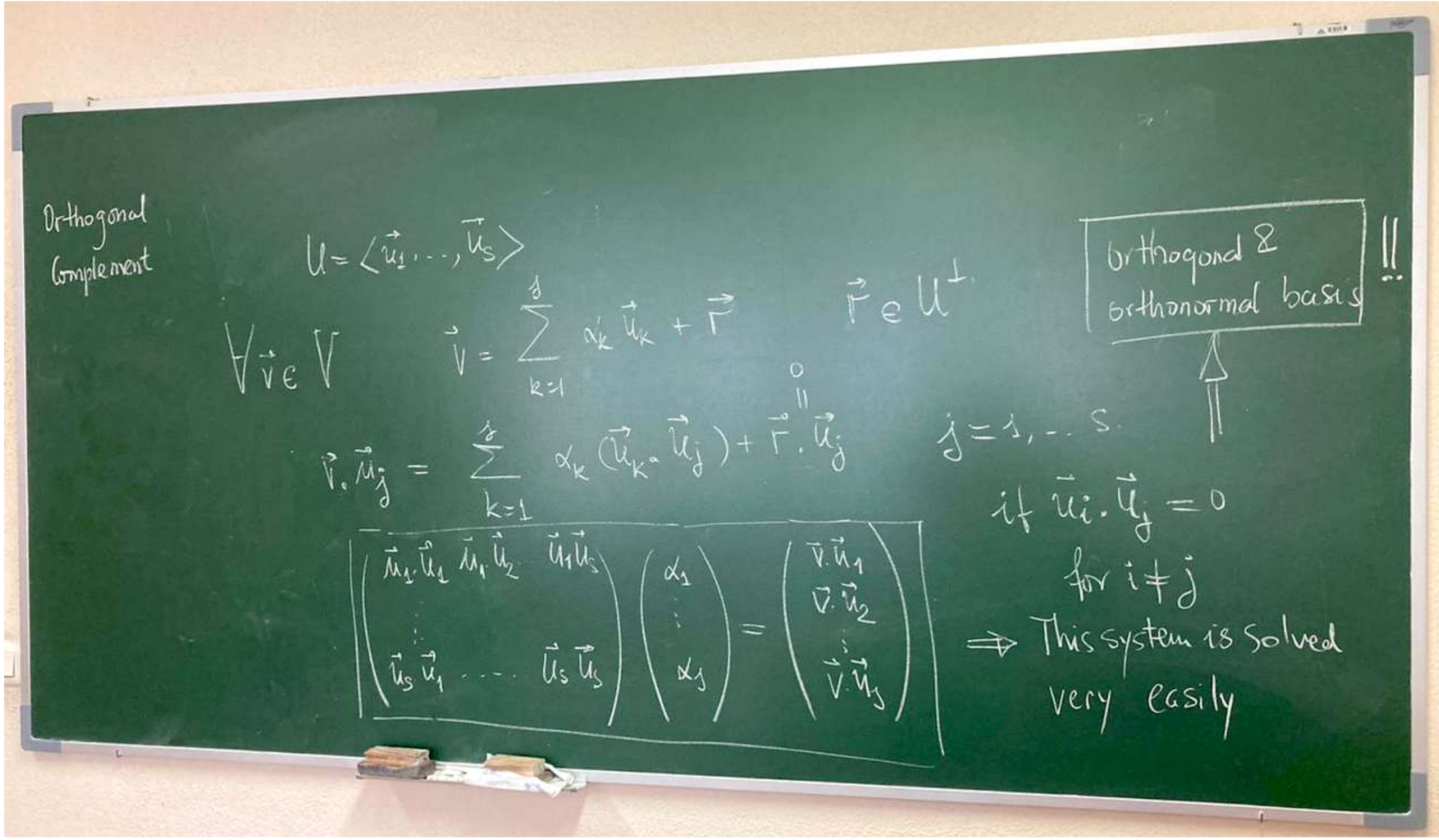


Orthogonal projections into a subspace



Bit of Practice

Considering in $\mathbb{R}^2$ the Gram matrix:

$$G = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

1. Finding the norm of $\mathbf{u} = (1,1)$.
2. The angle that forms $\mathbf{u} = (1,1)$ and $\mathbf{v} = (1,-1)$. Are they orthogonal?
3. Find one vector orthogonal to $\mathbf{u}$.

Considering in $\mathbb{R}^3$ the scalar product that in the canonic basis has the following Gram matrix:

$$G = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

1. Finding the angle that forms $\mathbf{u} = (1,0,0)$ and $\mathbf{v} = (0,1,0)$. Are they orthogonal?
2. Confirm that $\mathbf{u} \cdot \mathbf{v} = \|\boldsymbol{u}\|\|\mathbf{v}\| cos\langle \mathbf{u},\ \mathbf{v}\rangle$

Orthogonal projection

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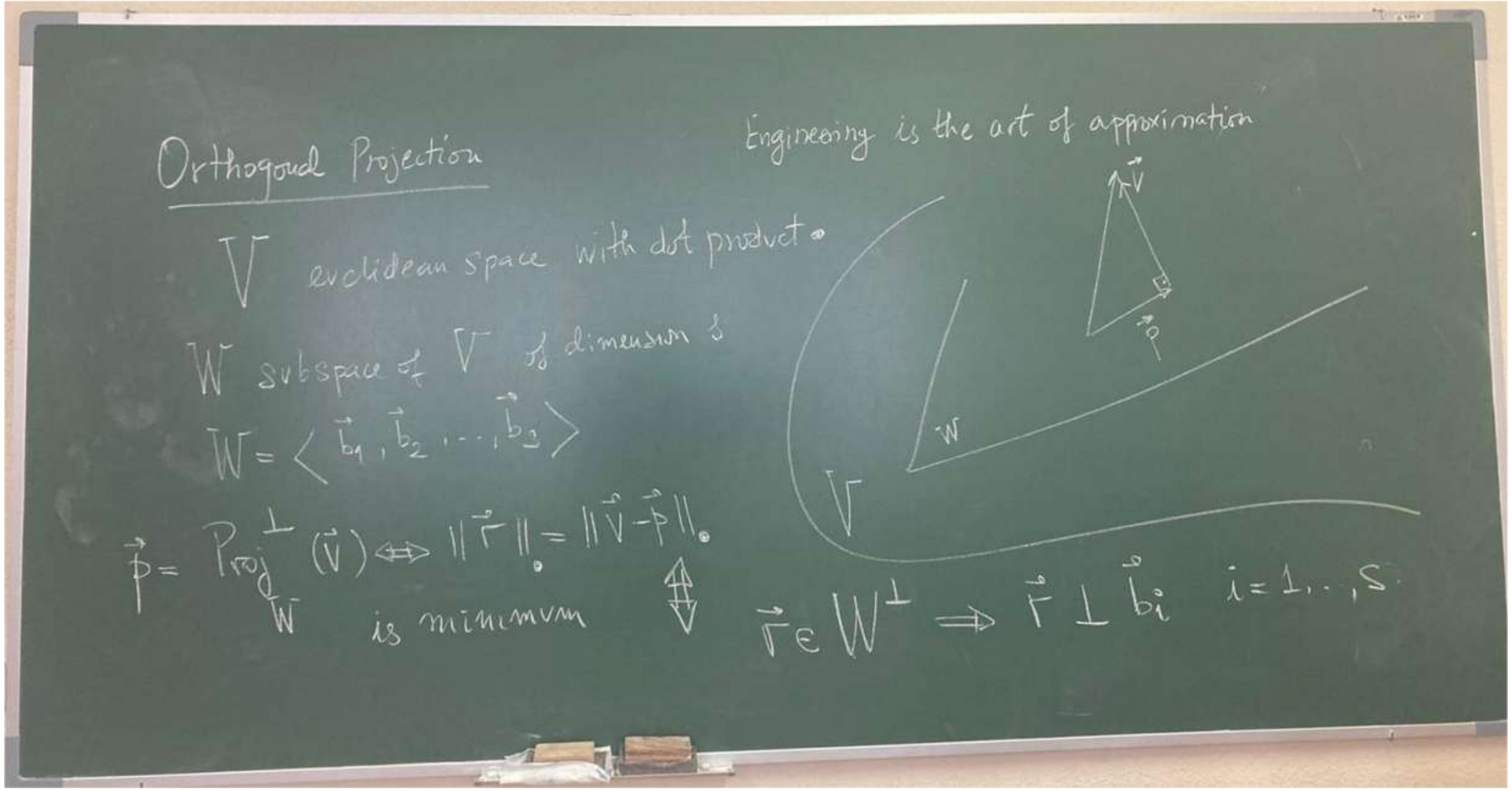

V can be vector space of infinite dimension

W is always a subspace of finite dimension

Orthogonal projection- The normal equations

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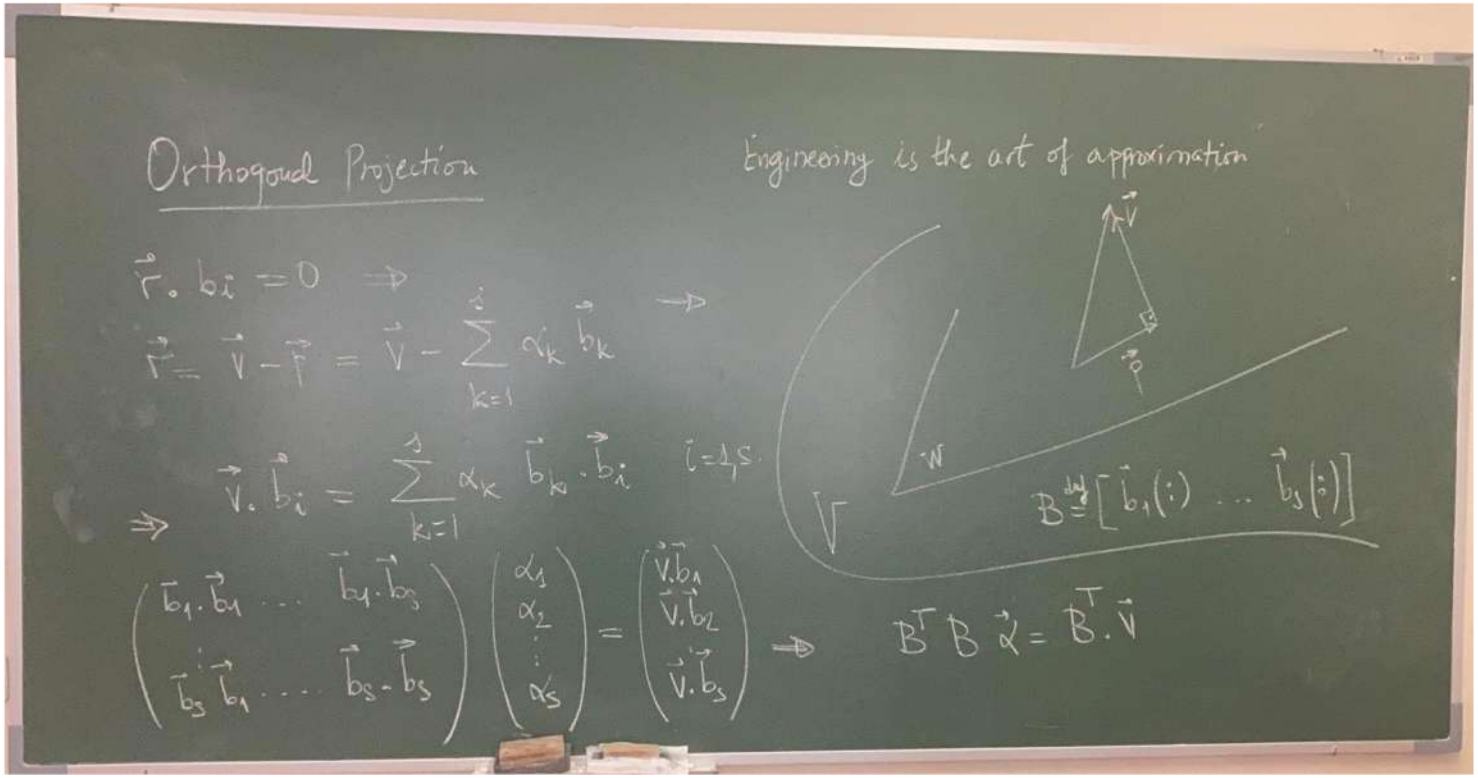

What happens is the basis set of **W** is orthogonal?

$$\alpha_k = \frac{\mathbf{v}.\,\mathbf{b}_k}{\|\mathbf{b}_k\|^2}$$

And if the basis set is orthonormal?

$$\alpha_k = \mathbf{v}.\,\mathbf{b}_k$$

Bit of Practice

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1. Finding the orthogonal projection of $f(x) = e^x, x \in [0,1]$ onto

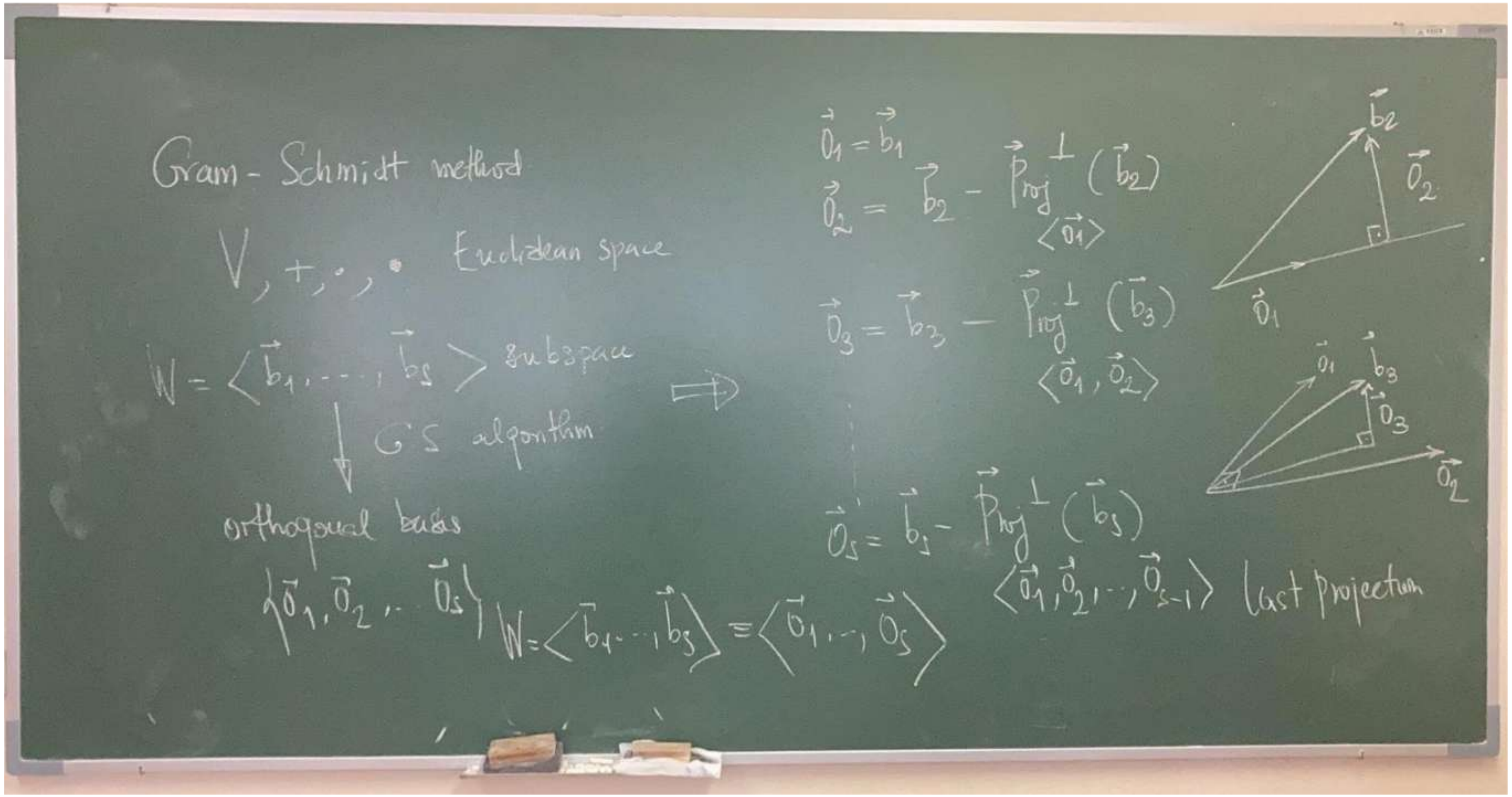
$$\mathbf{W} = \langle 1, x, x^2 \rangle.$$

2. Finding the orthogonal projection of $\mathbf{v}$ =(1,1,-2) onto

$$\mathbf{W} = \{(x, y, z): x + z = 0\}.$$

3. Finding the orthogonal projection of the matrix $A = \begin{pmatrix} 1 & 2 \\ 1 & -2 \end{pmatrix}$ into the subspace of symmetric matrices.
Do we obtain the same result using the column-wise and the row-wise scalar product?

Gram-Schmidt method



Bit of Practice

1. Finding an orthogonal basis $\mathbf{W} = \langle 1, x, x^2 \rangle, x \in [0,1]$ using the common scalar product defined in $C^0_{[0,1]}$

2. Finding an ortogonal bases of

$$\mathbf{W} = \{(x, y, z) : x + z = 0\},$$

using the scalar product in $\mathbb{R}^3$.

3. Finding an ortogonal basis of the subspace of symmetric matrices in $M_{3\times 3}(\mathbb{R})$ using the columnwise scalar product.