1. Linear Subspaces

Exercise 1 *Prove that the following sets are linear subspaces, or not.*

a) $U_1 = \left\{(x,y) \in \mathbb{R}^2 \mid x+y=1\right\}$

b) $U_2 = \left\{(x,y) \in \mathbb{R}^2 \mid x+2y=0\right\}$

c) $U_3 = \left\{(x,y) \in \mathbb{R}^2 \mid x^2+y=0\right\}$

d) $U_4 = \left\{(x,y) \in \mathbb{R}^2 \mid x-3y=0 \text{ con } y \geq 0\right\}$

e) $U_5 = \{p(x) = ax^2+bx+c \in \mathbb{R}_2[x] \mid a-2b+c=0\}$

f) $U_6 = \left\{\begin{pmatrix} x & y \\ z & t \end{pmatrix} \in \mathscr{M}_2(\mathbb{R}) \mid x+t=0,\ \ y-z^2=0\right\}$

g) $U_7 = \{(x,y,0) \mid x,y \in \mathbb{R}\}$

h) $U_8 = \{(x,y,z) \in \mathbb{R}^3 \mid x+y-z=0, \quad x^2+y^2=0\}$

i) $U_9 = \{(x,y,z) \in \mathbb{R}^3 \mid 2x+3y+z=0\}$

j) $U_{10} = \{(x,y,z) \in \mathbb{R}^3 \mid 2x+3y+z=1\}$

Exercise 2 *The set of symmetric matrices of order n is a linear subspace of $\mathscr{M}_n(\mathbb{R})$.*

Exercise 3 *The set of of polynomial functions with degree $\leq n$*

$$P_n = \{p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n\}$$

is a linear subspace of $\mathscr{C}^0$, also called the Lagrange subspace.

Exercise 4 *Check if the following set*

$$U = \{a+bx^2 \mid a,b \in \mathbb{R}\}$$

is a linear subspace in $\mathbb{R}_3[x]$, .

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2. Linear combination, linear independence, spanned space

Exercise 5 *Let us consider the following vectors system in* $\mathbb{R}^4$*:*

$$S = \{(1,1,1,1),(1,-1,0,2),(0,0,2,-1)\}$$

Check if $\vec{u} = (1,2,3,4)$ *and* $\vec{v} = (2,0,3,2)$ *belong to* $\langle S\rangle$*, and write them as a linear combination of the vectors of S, if possible.*

Exercise 6 *Which of these sets of* $\mathbb{R}^3$ *are linearly independent?*

- $S_1 = \{(1,1,1),(1,0,1),(0,1,0),(-1,1,1)\}$.
- $S_2 = \{(1,1,1),(0,1,1)\}$
- $S_3 = \{(1,1,1),(1,0,1),(0,0,0)\}$
- $S_4 = \{(1,1,1),(0,1,1),(0,0,1)\}$

Exercise 7 *Consider* $S = \{\vec{v}_1 = (1,0,-1), \vec{v}_2 = (1,1,-2), \vec{v}_3 = (1,-2,1)\}$ *a vector system in* $\mathbb{R}^3$.

a) Check if S is linearly dependent or independent.

b) Find a basis of $\langle S\rangle$ *and its dimension.*

Exercise 8 *Consider in* $\mathbb{R}^3$*:*

$$S_1 = \{\vec{v}_1 = (1,0,2), \vec{v}_2 = (0,1,1)\} \quad y$$

$$S_2 = \{\vec{v}_1 = (1,1,3), \vec{v}_2 = (1,-1,1), \vec{v}_3 = (2,-3,1)\}$$

Check if $S_1 \sim S_2$.

Exercise 9 *Consider* $S = \{\vec{v}_1 = (1,-1,1,-1), \vec{v}_2 = (0,0,1,-1), \vec{v}_3 = (3,2,1,0)\} \subset \mathbb{R}^4$.

a) Is $\vec{u} = (1,-1,1,5)$ *a linear combination of the vectors in S?*

b) The same question for the vector $\vec{v} = (1,4,2,-1)$.

Exercise 10 *Compute* $a,b \in \mathbb{R}$ *such as the following vector system in* $\mathscr{M}_2(\mathbb{R})$ *is linearly independent, and give the rank of S in this case:*

$$S = \left\{ \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix}, \begin{pmatrix} 5 & -3 \\ 2 & -1 \end{pmatrix}, \begin{pmatrix} -11 & a \\ 4 & b \end{pmatrix} \right\}$$

Exercise 11 *Consider S in* $\mathbb{R}_2[x]$*, with* $a,b \in \mathbb{R}$ *and* $a \neq b$*.*

$$S = \left\{1 + ax + ax^2, 1 + bx + bx^2, x^2\right\}.$$

¿Is S linearly independent? Compute its rank.

Exercise 12 *Compute the rank of S for* $\alpha \in \mathbb{R}$*.*

$$S = \{\vec{v}_1 = (1, 1, \alpha),\ \vec{v}_2 = (1, \alpha, 1),\ \vec{v}_3 = (\alpha, 1, 1)\}.$$
