CALCULUS
DEGREE IN SOFTWARE ENGINEERING
CHAPTER 12. APPLIED OPTIMIZATION. TAYLOR'S THEOREM.

In this chapter we apply our knowledge about maxima and minima to practical problems such as: what are the dimensions of a rectangle with fixed perimeter having maximum area ? What are the dimensions for the least expensive cylindrical can of a given volume ? How many items should be produced for the most profitable production run ?, and so on.
We must maximize or minimize a real-valued function of a single variable (the multivariable case will be studied in a future chapter). We search for the best (optimal) value of a given function, the absolute maximum or the absolute minimum. Because of this, the field is called applied optimization and is one of the most important branches of applied mathematics with applications to physics, engineering, biology, economics, etc.
First of all, we set out a scheme to solve an optimization problem. The scheme consists of several steps:

SOLVING APPLIED OPTIMIZATION PROBLEMS

1) Read the problem till you understand it. What is given ? What is the quantity to be optimized ?

2) Draw a picture. Label any part that can be important to the problem.

3) Introduce variables. Try to express the function you want to optimize as a single variable function. In typical cases, you will have two variables related by a constraint. This will allow you to isolate one of them as a function of the other and express the goal function (the one you have to optimize) as a function of a single variable.

4) Identify the domain of this function. Search for the critical points and endpoints of the domain. They will be the candidates for the points where the function attains its maximum or minimum values.

5) Use the first and second derivatives to identify and classify the critical points. If the function is continuous on a closed interval $[a, b]$, there will always be an absolute maximum and an absolute minimum. In other cases, we will have to think a little more: completing the interval with the endpoints or analyzing, for instance by using the second derivative on the domain, if the local extremum is also global. Now, we present a very simple example

EXAMPLE

What are the dimensions of a rectangle with a fixed parameter having maximum area ?

First, we draw the rectangle and call x and y the lengths of the sides. The area is the function we want to maximize and can be written

$$A = x\,y$$

We also know that the perimeter p is fixed. Its value is

$$p = 2x + 2y$$

From this constraint, we isolate y as a function of x,

$$y = \frac{p - 2x}{2}$$

and substituting y in the formula for the area, we finally write

$$A = \frac{x(p - 2x)}{2}$$

This function is defined on the interval $(0, p/2)$, since x and y must be both positive. We include the endpoints (that would include the unphysical solutions $x = 0$ and $y = 0$, but enables us to work with a continuous function on a closed interval). There will then be a maximum area and a minimum area.
We find the critical points $A' = (p - 4x)/2 = 0, \quad x = p/4$. Calculating the corresponding y from the constraint, $y = p/4$. The maximum area is attained for these values and it is

$$A = p^2/16$$

The minimum area is $A = 0$, attained at the endpoints and corresponds to a line (no rectangle). As you see, the rectangle of maximum area with a fixed perimeter is a square, what could be figured out because of symmetry considerations, since x and y play the same role in the functions involved.
This is a very simple example, but illustrates the way we have to solve these problems. We will solve several optimization problems in Exercises 5.
Next, we present Taylor's theorem, a theorem which enables us to approximate very general functions by polynomials in the neighbourhood of a point. We will write the general theorem and prove Taylor's formula with an approximation of order one. This formula is very useful in numerical analysis, physics, engineering, etc.

TAYLOR'S THEOREM

If f and its first n derivatives f', f'', ...,$f^{(n)}$ are continuous on a closed interval between a and b, and $f^{(n)}$ is differentiable on the open interval between a and b, then there exists a number c between a and b such that

$$f(b) = f(a)+f'(a)(b-a)+f''(a)(b-a)^2/2+........+f^{(n)}(a)(b-a)^n/n!+f^{(n+1)}(c)(b-a)^{(n+1)}/(n+1)!$$

This is called Taylor's formula of order n, the last term is called the remainder of Taylor's polynomial (order n) and the other terms make up the Taylor polynomial of order n, that is the polynomial of degree n that approximates $f(x)$ on the interval with an error given by the remainder. For $n = 1$, we can write Taylor's formula of order 1 and prove this formula

$$f(b) = f(a) + f'(a)(b - a) + f''(c)(b - a)^2/2$$

Proof:

We define

$$P_1(x) = f(a) + f'(a)(x - a)$$

.
We can check that

$$P_1(a) = f(a)$$

and

$$P_1'(a) = f'(a)$$

Now, we define

$$\Phi(x) = P_1(x) + K(x - a)^2$$

It is clear that

$$\Phi(a) = f(a)$$

and

$$\Phi'(a) = f'(a)$$

We will choose K such that $f(b) = \Phi(b)$. This leads to

$$K = \frac{f(b) - P_1(b)}{(b - a)^2}$$

Now, we write the difference between the function and the approximation $\Phi(x)$ (bear in mind that this approximation matches the function at the endpoints)

$$F(x) = f(x) - \Phi(x)$$

We see that

$$F(a) = F(b) = 0$$

Using Rolle's theorem, there must be a point d between a and b at which $F'(d) = 0$. Besides, $F'(a) = f'(a) - \Phi'(a) = 0$, We apply Rolle's theorem again for a point c, now between a and d.

$$\Phi''(x) = 2K$$

as can be easily checked. Then

$$F''(c) = 0 = f''(c) - \Phi''(c) = f''(c) - 2K$$

Therefore

$$K = f''(c)/2$$

Substituting in the previous formula for K

$$f''(c)/2 = \frac{f(b) - P_1(b)}{(b-a)^2}$$

and isolating $f(b)$, we finish the proof

$$f(b) = P_1(b) + f''(c)(b-a)^2/2$$

, that is

$$f(b) = f(a) + f'(a)(b-a) + f''(c)(b-a)^2/2$$

Though the proof is a little long, we can realize that we have approximated $f(x)$ by a quadratic function $\Phi(x)$: the tangent line at $x = a$, $P_1(x)$, plus a quadratic part $K(x-a)^2$ chosen so that this approximation intersects $f(x)$ at the endpoints and has the same derivative $f'(a)$ at $x = a$. The rest of the proof is an application of Rolle's theorem twice.

Note that the Mean Value Theorem is Taylor's theorem with $n = 0$, that is with a linear remainder. We also want to remark that though, normally, we think of the interval $[a, b]$, the proof is also valid for $[b, a]$, that is $b < a$.

The theorem bears the name of Brook Taylor, an English mathematician (1685-1731).

The theorem is valid for general functions, provided they have as many derivatives as required, and can be used in general intervals. However, if we want to approximate a function by a polynomial of low degree, we have to choose a neighbourhood of a and check that the remainder is bounded on that neighbourhood. In this way, we can set a bound to the error made when we substitute the polynomial for the real function.

Next, we will see a simple example of this. We will show another examples in Exercises 5. Taylor's formula is called MacLaurin's formula when we work at $x = 0$.

TAYLOR'S FORMULA OF ORDER ONE AND TWO FOR THE EXPONENTIAL FUNCTION AROUND THE ORIGIN.

The derivatives of $y = e^x$ are $y' = e^x$, $\;y'' = e^x$ and so on, and their value at $x = 0$ is 1. This means that we can write MacLaurin's formula (we substitute x for b in the expression) as

$$e^x = 1 + x + x^2/2 + ... + x^n/n! + e^c\, x^{n+1}/(n+1)!$$

For the order-one formula we obtain

$$e^x = 1 + x + e^c\, x^2/2$$

Then, e^x can be approximated by $1 + x$, and the error is $e^c\, x^2/2$. The previous formula is always true, but the error is only small if $|x| < 1$. In this case, the error is bounded by $e\, x^2/2$. For instance if $x = 0.1$, the approximate value of $e^{0.1}$ is

$$e^{0.1} \approx 1.1$$

and the error is less than $0.01\, e/2$. In fact, it will be of order 0.005, why? , and the real value is $e^{0.1} = 1.10517....$
Now, we will use the second order approximation

$$e^x = 1 + x + x^2/2 + e^c x^3/3!$$

for calculating $e^{0.01}$

$$e^{0.01} \approx 1.01 + 0.0001/2 = 1.01005$$

and the error will be bounded by $e\, 10^{-6}/6$, being of order $10^{-6}/6$. The real value is

$$e^{0.01} = 1.010050167....$$

If x is negative, we can use the same expressions. For instance, if $x = -0.1$ and we use MacLaurin's Formula of order one, we can write

$$e^{-0.1} \approx 1 - 0.1 = 0.9$$

with an error bound 0.01/2. The real error bound will be a little smaller, why? and the real value is

$$e^{-0.1} = 0.904837.....$$

We finish here by remarking the interest of these formulas for approximating general functions by simple ones (polynomials) and giving bounds of the errors. These approximations are very common in physics.