CALCULUS
DEGREE IN SOFTWARE ENGINEERING
EXERCISES AND SOLUTIONS 7.

INTEGRATION OF TRIGONOMETRIC FUNCTIONS.

Calculate the following integrals

1.
$$\int \sin^6 x \, \cos^7 x \, dx$$

Solution:

The integrand is odd in $\cos x$, then $t = \sin x$ is a suitable substitution. With this change, the integral becomes

$$\int t^6 (1 - t^2)^3 \, dt$$

We have taken into account that $\cos^2 x = (1 - t^2)$ and $dt = \cos x \, dx$. We expand Newton's binomial and write

$$\int t^6 (1 - t^2)^3 \, dt = \int t^6 (1 - 3t^2 + 3t^4 - t^6) \, dt = t^7/7 - t^9/3 + 3t^{11}/11 - t^{13}/13 + C$$

and, changing back to the original variable, the final solution is

$$\sin^7 x/7 - \sin^9 x/3 + 3 \sin^{11} x/11 - \sin^{13} x/13 + C$$

2.
$$\int \sin^9 x \, \cos^5 x \, dx$$

Now, the suitable change is $t = \cos x$, since the integrand is odd in $\sin x$. Then, the integral can be written

$$\int \sin^9 x \, \cos^5 x \, dx = -\int t^5 (1 - t^2)^4 \, dt = -\int t^5 (t^8 - 4t^6 + 6t^4 - 4t^2 + 1) \, dt$$

and integrating the last expression, the solution is

$$I = -t^{14}/14 + 4t^{12}/12 - 6t^{10}/10 + 4t^8/8 - t^6/6 + C$$

Reversing the substitution

$$I = -\cos^{14} x/14 + \cos^{12} x/3 - 3\cos^{10} x/5 + \cos^8 x/2 - \cos^6 x/6 + C$$

3.

$$\int \sec^5 x \, dx$$

Solution:

Though the integrand is odd in $\cos x$, it is simpler to integrate by parts, defining

$$u = \sec^3 x, \; dv = \sec^2 x \, dx$$

Hence

$$du = 3\sec^3 x \; \tan x$$
$$v = \tan x$$

and

$$\int \sec^5 x \, dx = \sec^3 x \tan x - \int 3\sec^3 x(\sec^2 x - 1) \, dx$$

where we have used $\tan^2 x = \sec^2 x - 1$. Isolating the original expression in this equation, we find

$$4\int \sec^5 x \, dx = \sec^3 x \tan x + \int 3\sec^3 x \, dx$$

We look up the indefinite integral of $\sec^3 x$ worked out in Chapter 15 and finally write

$$\int \sec^5 x \, dx = \sec^3 x \tan x/4 + 3[\sec x \tan x + \ln|\sec x + \tan x|]/8 + C$$

INTEGRATION OF IRRATIONAL FUNCTIONS.

4.
$$\int \sqrt{3-x}\,dx$$

with the change $t = \sqrt{3-x}$, the integral becomes

$$\int -t.2t\,dt$$

since $3 - x = t^2$ and $dx = -2tdt$.

Integrating now

$$\int -t.2t\,dt = -2t^3/3 + C$$

and reversing the substitution

$$\int \sqrt{3-x}\,dx = -2(3-x)^{3/2}/3 + C$$

5.
$$\int \frac{x^{1/2}}{4(1+x^{3/4})}\,dx$$

the least common multiple of the denominators in the fractional exponents is 4. So, $x = t^4$ is the most suitable change of variable. The new integral is

$$\int \frac{t^5}{(1+t^3)}\,dt$$

Dividing the numerator by the denominator of the rational function

$$\int \frac{t^5}{(1+t^3)}\,dt = \int \left(t^2 - \frac{t^2}{(1+t^3)}\right)dt = t^3/3 - \ln|1+t^3|/3 + C$$

Writing the solution in terms of the original variable

$$t^3/3 - \ln|1+t^3|/3 + C = x^{3/4}/3 - \ln|1+x^{3/4}|/3 + C$$

6.
$$\int \sqrt{1-e^x}\,dx$$

This exercise involves a combination of irrational and exponential functions. However, it is easy to see that the substitution $\sqrt{1-e^x} = t$ works very well. If we write $1 - e^x = t^2$ and differentiate $2tdt = -e^x dx$ or

$$dx = \frac{2tdt}{t^2-1}$$

Thus,

$$\int \sqrt{1-e^x}\,dx = \int \frac{2t^2dt}{t^2-1}\,dt$$

Dividing and integrating

$$\int \frac{2t^2dt}{t^2-1}\,dt = 2\int \left(1+\frac{1}{t^2-1}\right)dt = 2t + \ln\left|\frac{t-1}{t+1}\right| + C$$

Finally, the solution is

$$2\sqrt{1-e^x} + \ln\left|\frac{\sqrt{1-e^x}-1}{\sqrt{1-e^x}+1}\right| + C$$

TRIGONOMETRIC CHANGES.

7.

$$\int \frac{dx}{\sqrt{4-9x^2}}$$

Solution:

we can write this integral as

$$\int \frac{dx}{3\sqrt{4/9-x^2}}$$

now, it is clear that the change $x = 2/3\sin t$ will do the job. Applying it

$$\int \frac{dx}{3\sqrt{4/9-x^2}} = \int \frac{dt}{3} = \arcsin t/3 + C = \frac{1}{3}\arcsin(3x/2) + C$$

where we have replaced t by the original variable, $t = \arcsin(3x/2)$

8.

$$\int \frac{dx}{x\sqrt{1+x^2}}$$

Solution:

we try the substitution $x = \tan t$. The integral changes to

$$\int \frac{dx}{x\sqrt{1+x^2}} = \int \frac{\sec t\, dt}{\tan t} = \int \csc t\, dt = -\ln|\csc t + \cot t| + C$$

Now, we must remember that if $x = \tan t$, $\cot t = 1/x$ and $\csc t = \dfrac{\sqrt{1+x^2}}{x}$ and we find the solution

$$\int \frac{dx}{x\sqrt{1+x^2}} = -\ln\left|\frac{1+\sqrt{1+x^2}}{x}\right| + C$$

9.

$$\int \frac{x^2\, dx}{\sqrt{x^2-4}}$$

Solution:

the most convenient substitution is $x = 2\sec t$, with $dx = 2\sec t\, \tan t$. Applying the Substitution Rule, the integral reads

$$4\int \sec^3 t\, dt$$

where we have taken into account that

$$sec^2 t - 1 = \tan^2 t$$

and canceled the tangents.

Now, we use our knowledge of this integral (Chapter 15) and write

$$4\int \sec^3 t\, dt = 2\sec t\, \tan t + 2\ln|\sec t + \tan t| + C$$

Since $\sec t = x/2$ and $\tan t = \sqrt{x^2/4 - 1}$, we can finally obtain the indefinite integral in terms of the original variable

$$\int \frac{x^2\, dx}{\sqrt{x^2-4}} = \frac{x\sqrt{x^2-4}}{2} + 2\ln|x + \sqrt{x^2-4}| + C$$

Try to justify the last simplifications we have done.