CALCULUS
DEGREE IN SOFTWARE ENGINEERING
EXERCISES AND SOLUTIONS 5

1. We want to make a closed box, whose base is a square, so that the area of its surface measures A cm^2. Find the dimensions of the box that make the maximum volume.

   Solution:

   We have to build a parallelepiped. The lengths of the sides of the base are x and the height is labeled y. Thus, the volume V is

   $$V = x^2\, y$$

   and the area of the surface is

   $$A = 2x^2 + 4xy$$

   From this expression, we can isolate $y = \dfrac{A - 2x^2}{4x}$, and the volume can be written as a function of a single variable

   $$V = x\,(A - 2x^2)/4$$

   Now, the problem is clearly set out, we have to maximize V on a given domain. Since $x > 0$ and $y > 0$, the domain is $(0, \sqrt{A/2})$. We close it $[0, \sqrt{A/2}]$, including the minimum points, where the volume is zero and given that the function is continuous, there must be a maximum value at a critical point. We take the first derivative

   $$V' = (A - 6x^2)/4 = 0$$

   then, the solution is

   $$x = \sqrt{A/6}$$

   It is the only critical point and then the function has to attain its maximum value there. For this value of x, $y = \dfrac{A - A/3}{4\sqrt{A/6}} = \sqrt{A/6}$ and

   $$V_{max} = A/6\,\sqrt{A/6}$$

   The parallelepiped is cube.

2. Solve the same problem for an open box.

   Solution:

   The problem is quite similar. But, now the area of the surface is

   $$A = x^2 + 4xy$$

   We isolate

   $$y = \frac{A - x^2}{4x}$$

   and the volume is

   $$V = x\,(A - x^2)/4$$

   we must maximize this function on the domain $[0, \sqrt{A}]$. The solution for the maximum point is the only critical point

   $$V' = A - 3x^2 = 0$$

   $$x = \sqrt{A/3}$$

   the corresponding value of y is

   $$y = \frac{\sqrt{A/3}}{2}$$

   and the final solution is

   $$V_{max} = A/6\,\sqrt{A/3}$$

   As you have seen, in this problem there is no symmetry in x and y and the solution for the parallelepiped with the maximum volume is not a cube.

3. Given a circle of radius R, find the sides of the rectangle with the largest area that can be inscribed in the circle.

   Solution:

   If we call (x, y) the coordinates of the vertex of the inscribed rectangle in the first quadrant, we can work out the formula for the area

   $$A = 4\,x\,y$$

there is a constraint relating the coordinates, since this vertex is on the circle of radius R

$$x^2 + y^2 = R^2$$

Isolating y and writing the formula for the area as a function of x, we obtain

$$A = 4\,x\,\sqrt{R^2 - x^2}$$

The function is defined on the interval $[0, R]$.

Since the function is continuous, there will be a minimum value (0 at the endpoints) and a maximum value, in this case at the only critical point. We find the critical point

$$A' = 4(\sqrt{R^2 - x^2} - \frac{x^2}{\sqrt{R^2 - x^2}}) = 0$$

Solving for x

$$x = R/\sqrt{2}$$

and substituting it in the expression for y

$$y = R/\sqrt{2}$$

Finally, the maximum area is

$$A_{max} = 2\,R^2$$

and the rectangle is a square. Could you figure it out just by using symmetry considerations.

4. Solve the same problem for an ellipse of semiaxes a and b.

   Now, the coordinates of the vertex are (x, y), but they must satisfy the equation of an ellipse

$$x^2/a^2 + y^2/b^2 = 1$$

   We isolate y

$$y = b\,\sqrt{1 - x^2/a^2} = b/a\,\sqrt{a^2 - x^2}$$

   and write the area as a function of x

$$A = 4\,\frac{b\,x\,\sqrt{a^2 - x^2}}{a}$$

The domain is $[0, \sqrt{a}]$ and we will find the absolute maximum at the critical point. Taking the derivative and equating to zero (we omit the multiplicative constant $4b/a$)

$$\sqrt{a^2 - x^2} - \frac{x^2}{\sqrt{a^2 - x^2}} = 0$$

and the solution is

$$x = \frac{a}{\sqrt{2}}$$

Substituting this value in the expression for y

$$y = \frac{b}{\sqrt{2}}$$

and the maximum area is

$$A_{max} = 2\,a\,b$$

5. Calculate the length of the sides of an isosceles triangle with the largest area whose perimeter measures L meters.

   Solution:

   If we call x half the length of the base and y the length of any of the equal sides, we can write the height of the triagle as

$$h = \sqrt{y^2 - x^2}$$

   the area is

$$A = x\sqrt{y^2 - x^2}$$

   and the perimeter

$$L = 2x + 2y$$

   Isolating y and substituting it in the expression for the area, we find

$$A = \frac{1}{2}\,x\,\sqrt{L^2 - 4Lx}$$

The domain is $[0, L/4]$ and the critical point is found as the solution of

$$\sqrt{L^2 - 4Lx} - \frac{2Lx}{\sqrt{L^2 - 4Lx}} = 0$$

$$x = L/6$$

the sides are

$$2x = L/3$$

and

$$y = L/3$$

the area is

$$A_{max} = L^2/(12\sqrt{3})$$

the triangle is equilateral.

6. Find the volume of the largest right circular cone that can be inscribed in a sphere of radius 3.

   Solution:

   If we call x the distance from the center of the sphere to the base of the cone, the height of the cone will be $3 + x$ and the radius of its base $\sqrt{9 - x^2}$. Therefore, the formula for the volume of the cone is

$$V = \pi/3\ (9 - x^2)\,(x + 3)$$

   with $x \in [0, 3]$. We take the derivative of V to find the critical points

$$V' = -3x^2 - 6x + 9 = 0$$

   where we have omitted the factor $\pi/3$.

   The only positive solution is $x = 1$ and is the maximum point.

   Then, the length of the cone is 4 and its radius $\sqrt{8}$.

   Hence,

$$V = 32\pi/3$$

   Prove that for a general sphere with radius r, the solution is $x = r/3$ and the volume is

$$V = (32\pi/81)\,r^3$$

7. You have been asked to design a one-liter can shaped like a right circular cylinder. What dimensions will use the least material ?

   Solution:

   We will solve first for a general fixed volume V. We call h the height of the cylinder and r the radius of its base. Then, the volume is

$$V = \pi\, r^2\, h$$

   and the area of its surface is

$$A = 2\pi\, r^2 + 2\pi\, r\, h$$

   The phrase least material means that we want to make the surface area as small as possible for the given volume. We ignore the thickness of the material. Then, we isolate h from the expression for the volume

$$h = \frac{V}{\pi r^2}$$

   and substitute in the area

$$A = 2\pi r^2 + \frac{2V}{r}$$

   In this case, r is not defined on a bounded interval, but on $(0, \infty)$. At any rate, we will search for the critical points

$$A' = 4\pi r - \frac{2V}{r^2} = 0$$

   Solving this equation, we obtain

$$r = (V/2\pi)^{1/3}$$

   the only critical point. If we calculate the second derivative

$$A'' = 4\pi + \frac{4V}{r^3}$$

   we realize that it is always positive. This implies that the area has a global minimum at

$$r = (V/2\pi)^{1/3}.$$

   From the graph of the function, we deduce that it goes to ∞ at $r = 0$, then it decreases till it attains the global minimum. From that point on, it grows, tending to ∞ at ∞. The function is always concave up.

If we calculate h, we find $h = 2r$. In the case of a one-liter can

$$r \approx 5.42\, cm$$

and

$$h \approx 10.84\, cm$$

8. Write the Taylor expansion of the following functions at $x = 0$ with a remainder of order three (MacLaurin)

   (a) $\sin x$

   Solution:

   The general formula is :

   $$f(x) = f(0) + f'(0)\, x + f''(0)\, x^2/2 + f'''(0)\, x^3/3! + f^{iv}(c)\, x^4/4!$$

   For $f(x) = \sin x$, we obtain, after finding the derivatives, $f'(x) = \cos x$, $f''(x) = -\sin x$, $f'''(x) = -\cos x$, $f^{iv}(x) = \sin x$

   $$\sin x = x - x^3/6 + (\sin c)\, x^4/4!$$

   Taking into account that $f^{iv}(0) = 0$, we can use the fourth-order formula, since it gives us a better approximation

   $$\sin x = x - x^3/6 + (\cos c)\, x^5/5!$$

   Then, the error is bounded by

   $$x^5/120$$

   , what gives us a very good approximation if x is small

   $$\sin x \approx x - x^3/6$$

   For instance, if $x = 0.1$

   $$\sin 0.1 \approx 0.09983333..$$

   we know that the error is less than

   $$|Error| < 8.33\, 10^{-8}$$

   In fact, the real value is

   $$\sin 0.1 = 0.099833416646828..$$

   and the error estimator is very close to the real one, why ?

The error for $x = 1$ is bounded by $1/120$. So, we can estimate $\sin(1) \approx 1 - 1/6 = 0.83333...$ and the real value is

$$\sin(1) = 0.841470984807897$$

Note that we must work in radians and $1\,rad = 57.29^{\text{o}}$. We see that, even for a not very small angle, the approximation is not bad. This is due to the dividing factorial.

Finally, if we use a second-order approximation, which is equivalent to the first-order one

$$\sin x \approx x$$

the error is

$$-\cos c\, x^3/6$$

$$\sin 0.1 \approx 0.1$$

and

$$|Error| < 0.001/6 = 0.00016666....$$

the real value is

$$\sin 0.1 = 0.099833416646828..$$

and the real error is very close to the bound, basically because $\cos c$ is very close to 1 as c is between 0 and 0.1.

(b) $\cos x$

Solution: The MacLaurin's Formula is, in this case

$$\cos x = 1 - x^2/2 + \cos c\, x^4/24$$

For instance, if $x = 0.1$

$$\cos 0.1 \approx 0.995$$

with

$$|Error| < 10^{-4}/24 = 0.000004166...$$

and the real value is

$$\cos 0.1 = 0.995004165278026$$

In these examples, we have seen that we can use

$$\sin x \approx x - x^3/6$$

and

$$\cos x \approx 1 - x^2/2$$

as good approximations, but we always must take care of the errors by giving an appropriate error bound.

(c) $ln(1+x)$

Solution: In this case

$$\ln(1+x) = x - x^2/2 + x^3/3 - \frac{x^4}{4(1+c)^4}$$

Let us choose $x = 0.1$ to see how well the approximation works.

$$\ln 1.1 \approx 0.1 - 0.005 + 0.001/3 = 0.095333...$$

and the error bound is

$$|Error| < 0.0001/4$$

the real value is

$$\ln 1.1 = 0.095310179804325$$

What happens for $x = -0.1$?

Now, the approximation is

$$\ln 0.9 \approx -0.1 - 0.005 - 0.001/3 = -0.105333333333333$$

and the error bound

$$|Error| < \frac{10^{-4}}{4(0.9)^4} = 3.8\,10^{-5}$$

The real value is

$$\ln 0.9 = -0.105360515657826$$