CALCULUS
DEGREE IN SOFTWARE ENGINEERING
EXERCISES 11. PARTIAL DERIVATIVES. DIFFERENTIATION

1. Find the gradient at each point for the following scalar fields:

a) $x^2 + y^2 \sin(xy)$ b) $e^x \cos y$ c) $x^2y^3z^4$

d) $\dfrac{xy}{x^2+y^2+5}$ e) $x^3e^{x^2+y^2}$ f) $\ln(x^2+2y^2-3z^2)$

g) $x^{(y^z)}$ h) $\dfrac{x^2y^3}{x^2+y^4}$ i) $e^{x+y^2}\cos(x+y)$

j) $x^{(y^{z^2})}$ k) $\dfrac{x^2y^5}{x+y}$ l) $sin(x^3y^2z^4)$

The solutions are obtained by using the basic rules of differentiation. In g) and j) we can use logarithmic differentiation to make the calculations more straightforward.

(a) $\vec{\bigtriangledown} f = (2x + y^3\cos(xy), 2y\sin(xy) + xy^2\cos(xy))$

(b) $\vec{\bigtriangledown} f = (e^x\cos y, -e^x \sin y)$

(c) $\vec{\bigtriangledown} f = (2xy^3z^4, 3x^2y^2z^4, 4x^2y^3z^3)$

(d) $\vec{\bigtriangledown} f = \Big(\dfrac{y(y^2-x^2+5)}{(x^2+y^2+5)^2}, \dfrac{x(x^2-y^2+5)}{(x^2+y^2+5)^2}\Big)$

(e) $\vec{\bigtriangledown} f = (3x^2e^{x^2+y^2} + 2x^4e^{x^2+y^2}, 2x^3ye^{x^2+y^2})$

(f) $\vec{\bigtriangledown} f = \Big(\dfrac{2x}{x^2+2y^2-3z^2}, \dfrac{4y}{x^2+2y^2-3z^2}, \dfrac{-6z}{x^2+2y^2-3z^2}\Big)$

(g) $\vec{\bigtriangledown} f = (x^{y^z-1}y^z, x^{y^z}zy^{z-1}\ln x, x^{y^z}y^z\ln x \ln y)$

(h) $\vec{\bigtriangledown} f = \Big(\dfrac{2xy^7}{(x^2+y^4)^2}, \dfrac{3x^4y^2-x^2y^6}{(x^2+y^4)^2}\Big)$

(i) $\vec{\bigtriangledown} f = ((cos(x+y) - \sin(x+y))e^{x+y^2}, (2ycos(x+y) - \sin(x+y))e^{x+y^2})$

(j) $\vec{\bigtriangledown} f = (x^{y^{z^2}-1}y^{z^2}, x^{y^{z^2}}z^2y^{z^2-1}\ln x, 2zx^{y^{z^2}}y^{z^2}\ln x \ln y)$

(k) $\vec{\bigtriangledown} f = \Big(\dfrac{x^2y^5+2xy^6}{(x+y)^2}, \dfrac{5x^3y^4+4x^2y^5}{(x+y)^2}\Big)$

(l) $\vec{\bigtriangledown} f = (3x^2cos(x^3y^2z^4), 2ycos(x^3y^2z^4), 4z^3cos(x^3y^2z^4))$

2. Calculate the directional derivatives of the following scalar fields at the given points and in the indicated directions:

   (a) $f(x,y,z) = x^2+2y^2+3z^2$ at $a = (1,0,0)$ in the direction of $\vec{v} = \vec{i}-\vec{j}+2\vec{k}$

   (b) $g(x,y,z) = \left(\frac{x}{y}\right)^z$ at $a = (1,1,1)$ in the direction of $\vec{v} = 2\vec{i}+\vec{j}-\vec{k}$

   (c) $h(x,y) = \sin^{-1}(\frac{y}{x})$ at $a = (2,1)$ in the direction of $\vec{v} = \vec{i}+3\vec{j}$

   (a) First, we compute the gradient at any point, $\vec{\nabla} f = (2x, 4y, 6z)$. At $(1,0,0)$ it will be $(2,0,0)$, the unit vector in the direction of $\vec{v}$ is $\vec{u} = (1,1,2)/\sqrt{6}$ and finally, we write the directional derivative

$$(D_{\vec{u}} f)_{P_0} = \vec{\nabla} f(1,0,0).\vec{u} = 2/\sqrt{6}$$

   (b) In this case the gradient is

$$\vec{\nabla} g = (zy^{-z}x^{z-1}, -zx^z y^{-z-1}, (x/y)^z \ln{(x/y)})$$

   At $(1,1,1)$

$$\vec{\nabla} g(1,1,1) = (1,-1,0))$$

   and

$$(D_{\vec{u}} g)_{P_0} = \vec{\nabla} g(1,1,1).\,\vec{u} = (1,-1,0).(2,1,-1)/\sqrt{6} = 1/\sqrt{6}$$

   (c)

$$\vec{\nabla} h = \Big(\frac{-y}{|x|\sqrt{x^2-y^2}}, \frac{|x|}{x\sqrt{x^2-y^2}}\Big)$$

$$\vec{\nabla} h(2,1) = (-1/2\sqrt{3}, 1/\sqrt{3})$$

$$(D_{\vec{u}} h)_{P_0} = (-1/2\sqrt{3}, 1/\sqrt{3}).(1,3)/\sqrt{10} = \sqrt{5}/2\sqrt{6}$$

3. Captain Alexandra has problems near the sunlit side of Mercury. The temperature of the spacecraft, when she is at the position (x, y, z) is $T(x,y,z) = e^{-x^2-2y^2-3z^2}$.
   Currently, she is situated at (1, 1, 1). In which direction must she travel to produce the fastest decrease of temperature?

   She has to move along the direction opposite to the gradient of the temperature. The gradient is

$$\vec{\nabla}T = e^{-x^2-2y^2-3z^2}(-2x, -4y, -6z)$$

At $(1,1,1)$

$$\vec{\nabla}T(1,1,) = e^{-6}\,(-2,-4,-6)$$

The direction of maximum decrease of the temperature is

$$\frac{-\vec{\nabla}T(1,1,1)}{|\vec{\nabla}T(1,1,1)|} = (1,2,3)/\sqrt{14}$$

4. Let $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ y $r = |\vec{r}|$. Prove that:

$$\nabla\left(\frac{1}{r}\right) = \frac{-\vec{r}}{r^3}$$

$$1/r = \frac{1}{\sqrt{x^2+y^2+z^2}}$$

It is easy to check that

$$\vec{\nabla}\Big(\frac{1}{\sqrt{x^2+y^2+z^2}}\Big) = (-x,-y,-z)/r^3$$

that is

$$\nabla\left(\frac{1}{r}\right) = \frac{-\vec{r}}{r^3}$$

apart from a constant, the gravitational field outside a homogeneous sphere.

5. Calculate the second-order partial derivatives and check that the mixed derivatives are equal:

a) $f(x,\ y) = \dfrac{2xy}{(x^2+y^2)}$ b) $f(x,\ y,\ z) = e^z + xe^{-y} + \frac{1}{x}$

c) $f(x,\ y) = \cos(xy^2)$ d) $f(x,\ y) = e^{-xy^2} + x^4y^3$

e) $f(x,\ y) = (\cos^2 x + e^{-y})^{-1}$ f) $f(x,\ y,\ z) = ze^{xy} + yz^3x^2$

Though long and cumbersome, this exercise is quite simple. You just have to use the typical differentiation rules. In all cases, the functions are C^2 on their domains. The mixed derivatives must then be equal.

(a)

$$\frac{\partial^2 f}{\partial x^2} = \frac{4x^3y - 12xy^3}{(x^2+y^2)^3}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{4xy^3 - 12x^3y}{(x^2+y^2)^3}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{12x^2y^2 - 2x^4 - 2y^4}{(x^2+y^2)^3}$$

(b) $\frac{\partial^2 f}{\partial x^2} = 2/x^3$, $\frac{\partial^2 f}{\partial y^2} = xe^{-y}$, $\frac{\partial^2 f}{\partial z^2} = e^z$, $\frac{\partial^2 f}{\partial x \partial y} = -e^{-y}$,$\frac{\partial^2 f}{\partial x \partial z} = 0$,$\frac{\partial^2 f}{\partial y \partial z} = 0$

(c) $\frac{\partial^2 f}{\partial x^2} = -y^4 \cos(xy^2)$, $\frac{\partial^2 f}{\partial y^2} = -4x^2y^2\cos(xy^2) - 2x\sin(xy^2)$, $\frac{\partial^2 f}{\partial x \partial y} = -2xy^3\cos(xy^2) - 2y\sin(xy^2)$

(d) $\frac{\partial^2 f}{\partial x^2} = y^4e^{-xy^2} + 12x^2y^3$, $\frac{\partial^2 f}{\partial y^2} = 4x^2y^2e^{-xy^2} - 2xe^{-xy^2} + 6x^4y$, $\frac{\partial^2 f}{\partial x \partial y} = 2xy^3e^{-xy^2} - 2ye^{-xy^2} + 12x^3y^2$

(e)

$$\frac{\partial^2 f}{\partial x^2} = 2(cos^2x + e^{-y})^{-3}\,\sin^2 2x + (cos^2x + e^{-y})^{-2} 2cos(2x)$$

$$\frac{\partial^2 f}{\partial y^2} = 2(cos^2x + e^{-y})^{-3}\,e^{-2y} - (cos^2x + e^{-y})^{-2}e^{-y}$$

$$\frac{\partial^2 f}{\partial x \partial y} = 2(cos^2x + e^{-y})^{-3}\,\sin 2x\, e^{-y}$$

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(f) $\frac{\partial^2 f}{\partial x^2} = y^2ze^{xy} + 2yz^3$, $\frac{\partial^2 f}{\partial y^2} = x^2ze^{xy}$, $\frac{\partial^2 f}{\partial z^2} = 6x^2yz$, $\frac{\partial^2 f}{\partial x \partial y} = ze^{xy} + xyze^{xy} + 2xz^3$, $\frac{\partial^2 f}{\partial x \partial z} = ye^{xy} + 6xyz^2$, $\frac{\partial^2 f}{\partial y \partial z} = xe^{xy} + 3x^2z^2$,