Practice 4: Orthogonality, Projections, Least Squares.

0.1. Orthonormal Basis

Function	Output
`orth(A)`	Computes the orthonormal basis of the linear space generated by the columns of A, using the dot product.

Example 1 *For a given basis $\mathscr{B}$, obtain the orthonormal basis,*

$$\mathscr{B} = \{\vec{e}_1 = (-1,-2,0,3), \vec{e}_2 = (-3,5,-5,3), \vec{e}_3 = (0,-1,0,-3), \vec{e}_4 = (3,-3,4,-5)\}$$

using the scalar product defined by the matrix:

$$G = \begin{pmatrix} 6 & -2 & 4 & 2 \\ -2 & 5 & 0 & -2 \\ 4 & 0 & 5 & 0 \\ 2 & -2 & 0 & 2 \end{pmatrix}$$

We will use Gram Schmidt, thus starting with a basis $\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$, we want to obtain an orthogonal basis $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$,

$$\vec{v}_k = \vec{e}_k - \alpha_1 \vec{v}_1 - \alpha_2 \vec{v}_2 - \cdots - \alpha_{k-1} \vec{v}_{k-1}, \quad k = 1, \ldots, n$$

with

$$\alpha_i = \frac{\vec{e}_p \cdot \vec{v}_i}{\vec{v}_i \cdot \vec{v}_i}, \quad i = 1, \ldots, k-1$$

In our example we have:

```
»e1=[-1,-2,0,3] ,e2=[-3,5,-5,3] ,e3=[0,-1,0,-3],e4=[3,-3,4,-5]
»g=[6 -2 4 2;-2 5 0 -2;4 0 5 0;2 -2 0 2]
```

Applying Gram Schmidt:

```
» v1=e1
» v12=v1*g*v1'

» alfa1=(e2*g*v1')/v12
» v2=e2-alfa1*v1
» v22=v2*g*v2'

» beta1=(e3*g*v1')/v12; beta2=(e3*g*v2')/v22
» v3=e3-beta1*v1-beta2*v2;
» v32=v3*g*v3';
```

```
» gama1=(e4*g*v1')/v12, gama2=(e4*g*v2')/v22, gama3=(e4*g*v3')/v32
» v4=e4-gama1*v1-gama2*v2-gama3*v3
» v42=v4*g*v4'
```

Now we normalize the vectors and we obtain the orthonormal basis:

```
» u1=v1/sqrt(v12); u2=v2/sqrt(v22); u3=v3/sqrt(v32);
u4=v4/sqrt(v42);
```

0.2. Orthogonal Projection

Función	**Salida**
`int(f,x,a,b)`	definite integral `f` of `x` from `a` to `b`.

Example 2 *Compute the orthogonal projection P of the function $f(x) = \sin(\pi x)$ defined in the interval $[-1,1]$ on the linear space $\mathbb{R}_3[x]$, considering the scalar product*

$$f \cdot g = \int_{-1}^{1} f(x)\, g(x)\, dx$$

The vector $\sin(\pi x) - P$ will be orthogonal to $\mathbb{R}_3[x]$, thus $\sin(\pi x) - P$ is orthogonal to a basis of $\mathbb{R}_3[x]$:

$$\left\{\begin{array}{l} (\sin(\pi x) - P)\cdot 1 = 0 \\ (\sin(\pi x) - P)\cdot x = 0 \\ (\sin(\pi x) - P)\cdot x^2 = 0 \\ (\sin(\pi x) - P)\cdot x^3 = 0 \end{array}\right. \leftrightarrow \left\{\begin{array}{l} \sin(\pi x)\cdot 1 = P\cdot 1 \\ \sin(\pi x)\cdot x = P\cdot x \\ \sin(\pi x)\cdot x^2 = P\cdot x^2 \\ \sin(\pi x)\cdot x^3 = P\cdot x^3 \end{array}\right.$$

Taking into account that $P \in \mathbb{R}_3[x]$, we have $P = a_0 1 + a_1 x + a_2 x^2 + a_3 x^3$ and we substitute P:

$$\left\{\begin{array}{l} \left(a_0 1 + a_1 x + a_2 x^2 + a_3 x^3\right)\cdot 1 = sen(\pi x)\cdot 1 \\ \left(a_0 1 + a_1 x + a_2 x^2 + a_3 x^3\right)\cdot x = sen(\pi x)\cdot x \\ \left(a_0 1 + a_1 x + a_2 x^2 + a_3 x^3\right)\cdot x^2 = sen(\pi x)\cdot x^2 \\ \left(a_0 1 + a_1 x + a_2 x^2 + a_3 x^3\right)\cdot x^3 = sen(\pi x)\cdot x^3 \end{array}\right. \Longrightarrow$$

$$\begin{pmatrix} 1\cdot 1 & 1\cdot x & 1\cdot x^2 & 1\cdot x^3 \\ 1\cdot x & x\cdot x & x\cdot x^2 & x\cdot x^3 \\ 1\cdot x^2 & x\cdot x^2 & x^2\cdot x^2 & x^2\cdot x^3 \\ 1\cdot x^3 & x\cdot x^3 & x^2\cdot x^3 & x^3\cdot x^3 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} \sin(\pi x)\cdot 1 \\ \sin(\pi x)\cdot x \\ \sin(\pi x)\cdot x^2 \\ \sin(\pi x)\cdot x^3 \end{pmatrix}$$

```
» syms x
» v=[1 x x^2 x^3]; % the basis of R3[X]
» H=v.'*v  % Matriz simbólica de Gramm
» G=int(H,x,-1,1) %Matriz de Gramm
» D=v.'*sin(pi*x) % Matriz simbólica segundo miembro
» B=int(D,x,-1 ,1) % Matriz numérica segundo miembro
» s=G\B
```

```
» P=s'*v'
» xnum=-1:0.01:1;
» y=sin(pi*xnum);
» P=subs(P,x,xnum)
» plot(xnum,y,'b',xnum,P,'m')
```