CALCULUS
DEGREE IN SOFTWARE ENGINEERING
EXERCISES 8. THE DEFINITE INTEGRAL. AREAS

1. Calculate the area of the region bounded by the curves $y = x^2$ and $y = x^4$ in the first quadrant.

   We draw the region: the curves intersect at $x = 0$ and $x = 1$, $y = x^2$ is above whereas $y = x^4$ is below. Then,

$$Area = \int_0^1 (x^2 - x^4)\, dx = (x^3/3 - x^5/5)\Big|_0^1 = 1/3 - 1/5 = 2/15$$

2. Find the area of the region in the first quadrant bounded above by $y = \sqrt{x}$ and below by the x-axis and the line $y = x - 2$

   According to Figure 1, we have to divide the region into two parts, the first one with x running from 0 to 2 with $y = \sqrt{x}$ above and $y = 0$ below and the second one with x from 2 to 4 and bounded above by $y = \sqrt{x}$ and below by $y = x - 2$. Therefore, the area is

$$Area = \int_0^2 \sqrt{x}\, dx + \int_2^4 (\sqrt{x} - x + 2)\, dx$$

   Integrating

$$Area = 2x^{3/2}/3\Big|_0^2 + (2x^{3/2}/3 - x^2/2 + 2x)\Big|_2^4$$

$$Area = 16/3 - 8 + 2 + 8 - 4 = 16/3 - 2 = 10/3$$

   If we had calculated the area between $y = \sqrt{x}$ and $y = x - 2$, the integral would be

$$Area = \int_0^4 (\sqrt{x} - x + 2)\, dx$$

$$Area = (2x^{3/2}/3 - x^2/2 + 2x)\Big|_0^4 = 16/3 - 8 + 8 = 16/3$$

   that is, the sum of the previous area and the area of the triangle in the fourth cuadrant (see Figure 2).

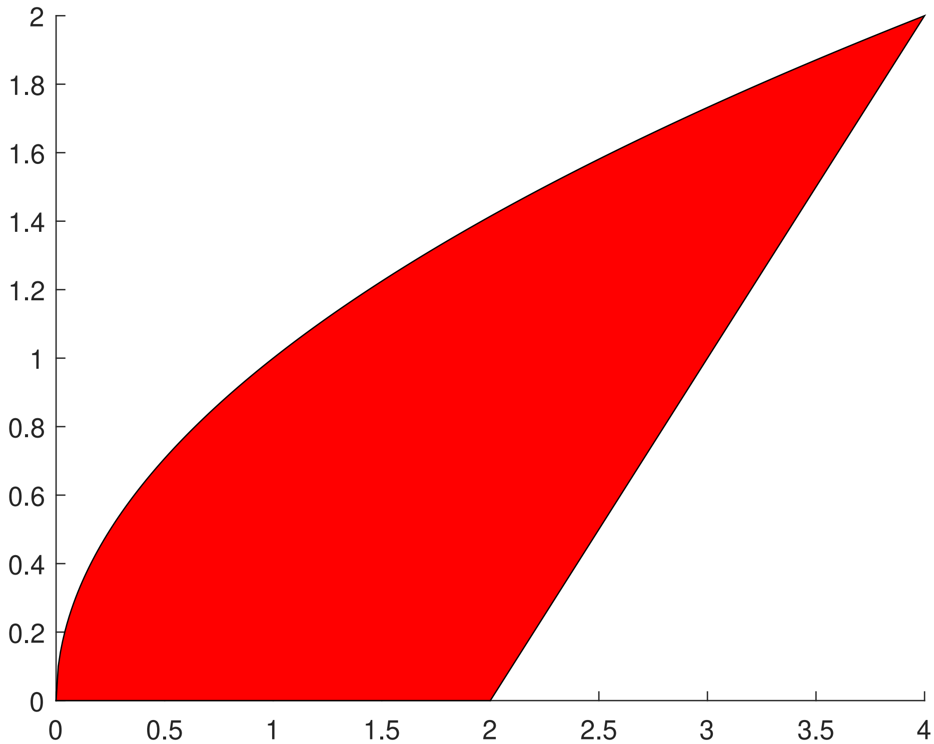

Figure 1: Exercise 2

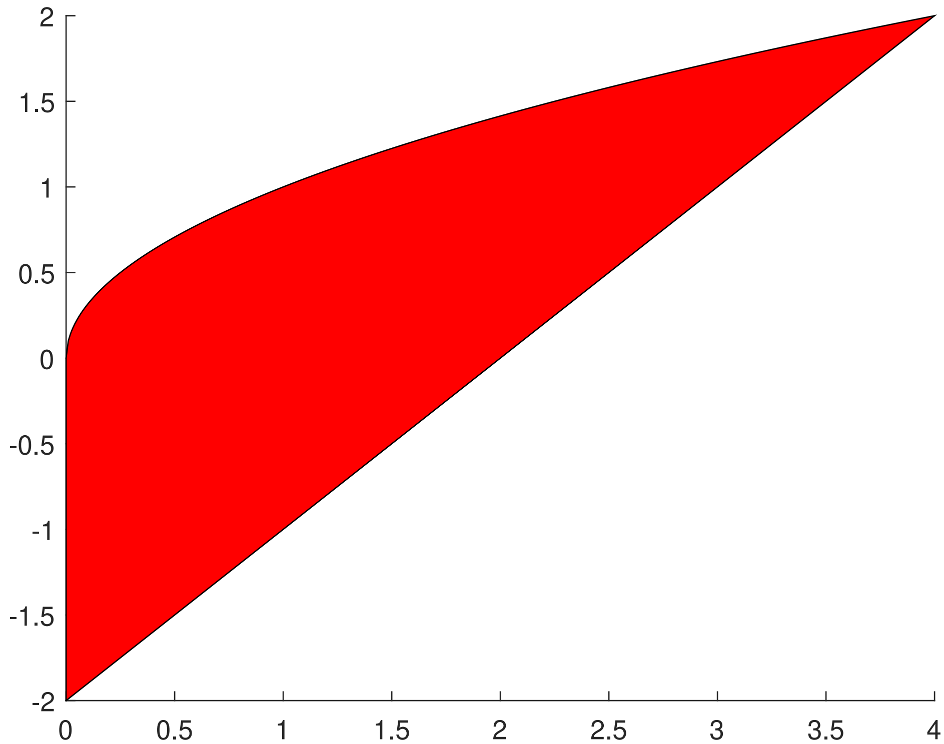

Figure 2: Exercise 2b

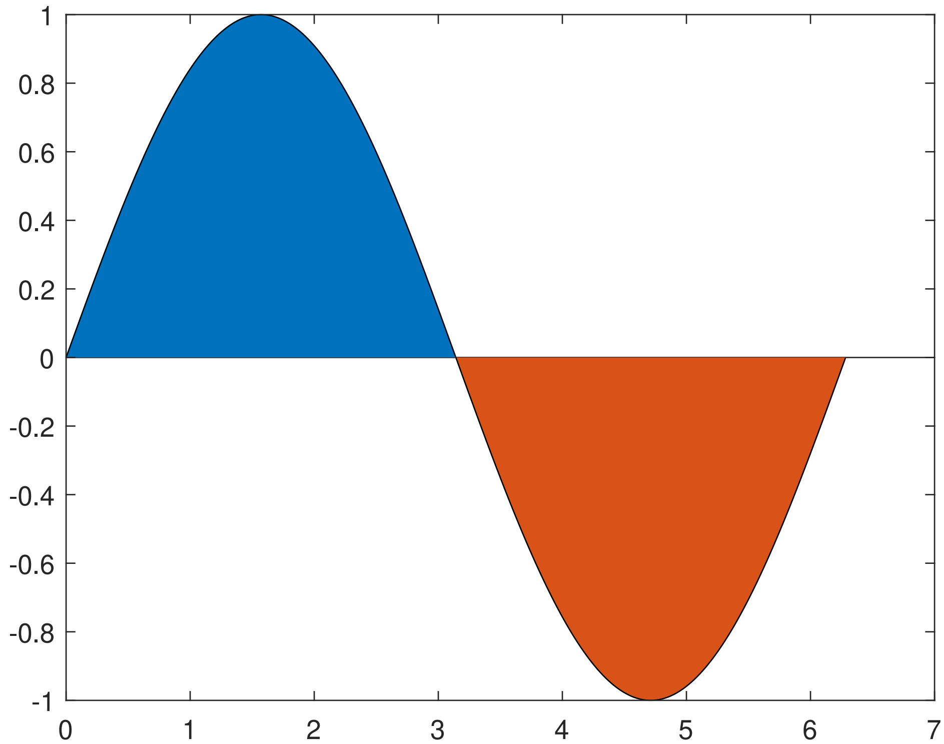

Figure 3: Exercise 3

3. Find the area enclosed by the curve $y = \sin x$ and the x-axis with $0 \leq x \leq 2\pi$.

   We draw the region (Figure 3) and notice that the area can be found as

$$Area = \int_0^{2\pi} |\sin x|\, dx$$

   Splitting the integral into two parts

$$Area = \int_0^{\pi} \sin x\, dx + \int_{\pi}^{2\pi} -\sin x\, dx$$

   Integrating

$$Area = -\cos x\Big|_0^{\pi} + \cos x\Big|_{\pi}^{2\pi} = -(-1-1) + (1+1) = 4$$

   Therefore, the complete wave of the sine function will give rise to an area equal to 4 and half a wave to an area equal to 2.

4. Calculate the area of the region bounded by the graphs of $y = \sin x$, $y = \cos x$ and the x-axis, with $0 \leq x \leq \pi/2$.

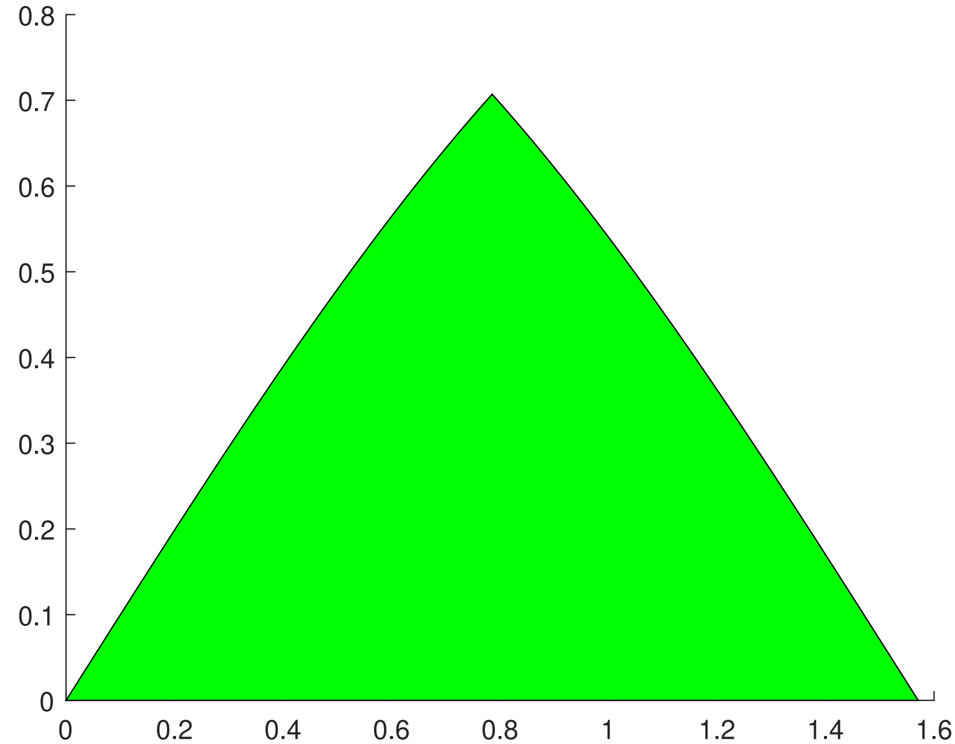

Figure 4: Exercise 4

We draw the region: on the left, the region is bounded by $y = \sin x$ and on the right by $y = \cos x$, then

$$Area = \int_0^{\pi/4} \sin x \, dx + \int_{\pi/4}^{\pi/2} \cos x \, dx$$

Integrating

$$Area = -\cos x\Big|_0^{\pi/4} dx + \sin x\Big|_{\pi/4}^{\pi/2} = -(1/\sqrt{2} - 1) + (1 - 1/\sqrt{2}) = 2 - \sqrt{2} = 0.585786..$$

5. Find the area of a parabolic segment of base $2R$ and height h.

   We draw the parabolic segment with the base on the x-axis and the height on the y-axis. The equation of the symmetric parabola is $y = a + bx^2$. We obtain a and b by using that the parabola passes through the points $(0, h)$ and $(R, 0)$. Then $a = h$, $b = -h/R^2$. The area is found by computig a definite integral. Since the region is symmetric with respect to the y-axis, we can find the area as 2 times the area in the first quadrant.

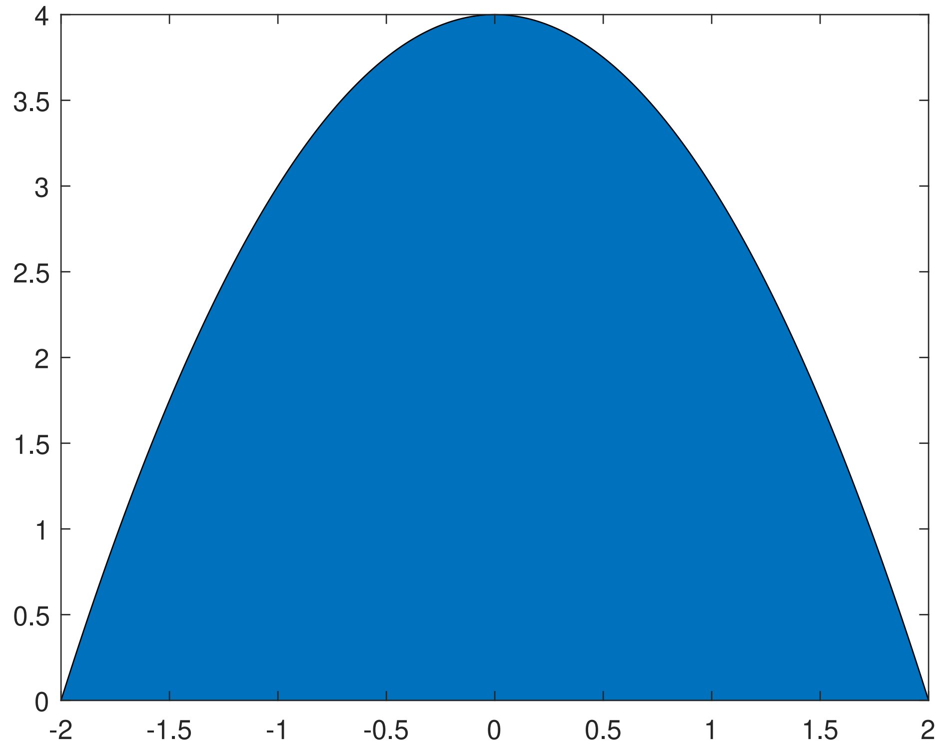

Figure 5: Exercise 5, with $R = 2$, $h = 4$

$$Area = \int_{-R}^{R} h(1 - x^2/R^2)\, dx = 2\int_{0}^{R} h(1 - x^2/R^2)\, dx$$

$$Area = 2h(x - \frac{x^3}{3R^2})\Big|_0^R = 2h(R - R/3) = 4hR/3$$

that is, 2/3 of base times height, a result known to Archimedes in the third century BC, obtained by filling the region with rectangles, the method of exhaustion, a precedent of integration.

THEOREM: CHANGE OF VARIABLE IN A DEFINITE INTEGRAL

In the next problem we will use the formula for the change of variable in a definite integral. Now, we are going to state the formula as a theorem.

Statement: If g' is continuous on the interval $[a, b]$ and f is continuous on the range of $g(x) = u$, then

$$\int_a^b f(g(x))\, g'(x)\, dx = \int_{g(a)}^{g(b)} f(u)\, du$$

Proof:

$$\int_a^b f(g(x))\, g'(x)\, dx = F(g(x))|_a^b = F(g(b) - F(g(a)) = F(u)|_{u=g(a)}^{u=g(b))} = \int_{g(a)}^{g(b)} f(u)\, du$$

6. Find the area of a circle (disk) of radius a.

   The equation of the semicircle with $y \geq 0$ is $y = \sqrt{a^2 - x^2}$. The total area will be

$$Area = 2\int_{-a}^{a} \sqrt{a^2 - x^2}\, dx = 4\int_0^a \sqrt{a^2 - x^2}\, dx$$

   We make the change $x = a \sin t$, $dx = a \cos t$, then the new limits in t are $t = 0$ for $x = 0$ and $t = \pi/2$ for $x = a$. We obtain

$$Area = 4\int_0^{\pi/2} a^2 \cos^2 t\, dt$$

$$\int_0^{\pi/2} a^2 \cos^2 t\, dt = a^2 \int_0^{\pi/2} (1 + \cos 2t)/2\, dt = a^2\pi/4$$

   Finally,

$$Area = \pi a^2$$

   the well known formula for the area of a circle.

7. Find the area of an ellipse of semiaxes a and b.

   The equation of the ellipse is

$$x^2/a^2 + y^2/b^2 = 1$$

   The formula for the area A will be

$$A = 2\int_{-a}^{a} b/a \sqrt{a^2 - x^2}\, dx = 2b/a \int_{-a}^{a} \sqrt{a^2 - x^2}\, dx = \frac{2b\pi a^2}{2a} = \pi ab$$

   where we have used the result of the previous problem. This is the region bounded by the ellipse with $a = 2$ and $b = 3$

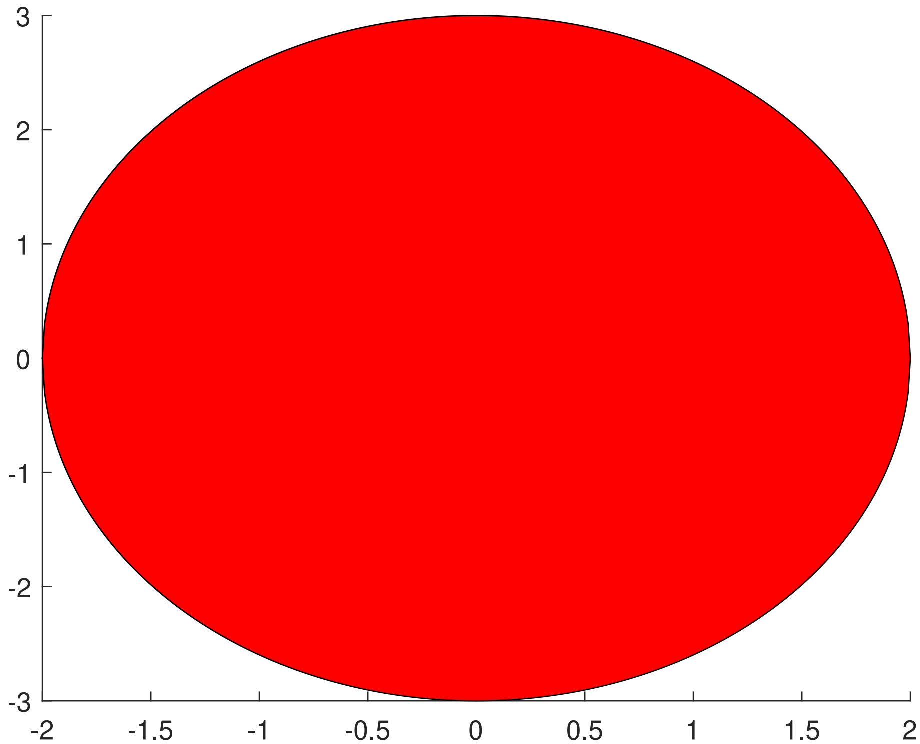

Figure 6: Exercise 7, with $a = 2$, $b = 3$

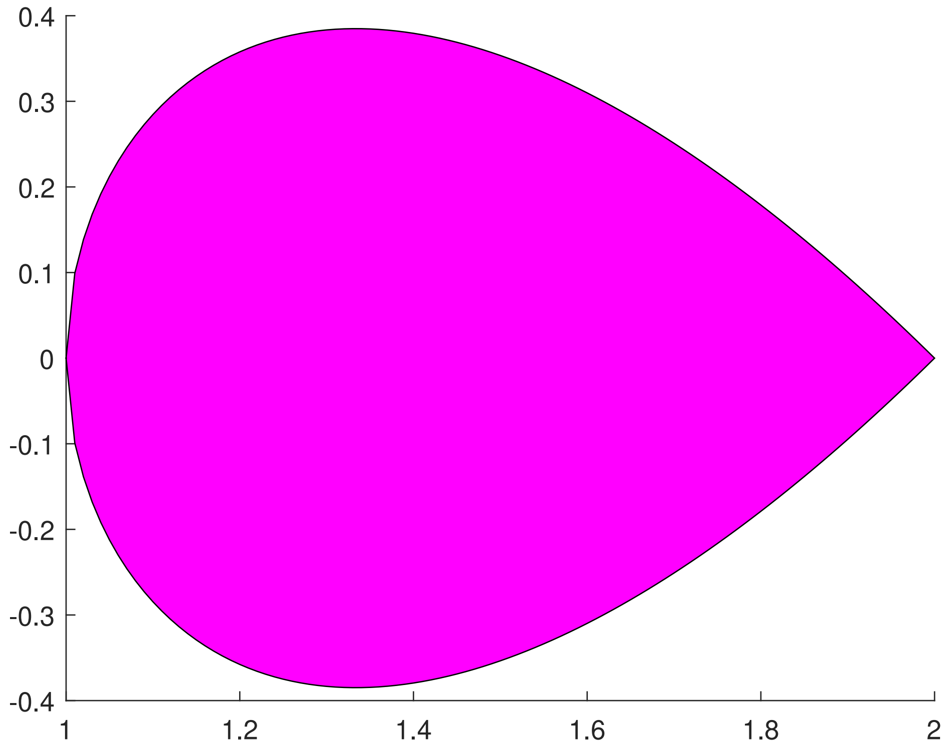

Figure 7: Exercise 8

8. Find the area of the region enclosed by the graph of $y^2 = (x-1)(x-2)^2$.

   We realize that $x \geq 1$ and $y = 0$ again at $x = 2$. We plot the graph and notice that the bounded region is between $x = 1$ and $x = 2$.

   The area can be computed with the following definite integral

$$A = 2\int_1^2 \sqrt{x-1}\,|x-2|\,dx = 2\int_1^2 \sqrt{x-1}\,(2-x)\,dx$$

   We make the substitution $\sqrt{x-1} = t$. Then, $dx = 2tdt$, for $x = 1$, $t = 0$ and for $x = 2$, $t = 1$. The new integral in the new variable is

$$A = 4\int_0^1 (1-t^2)\,t^2\,dt = 4\int_0^1 (t^2 - t^4)\,dt$$

$$A = 4(t^3/3 - t^5/5)\Big|_0^1 = 8/15$$

9. Find the area of the region enclosed by the hyperbola $x^2 - y^2 = 1$, the line $x = 2$ and the x-axis.

   We draw the region (Figure 8) and set out (write) the corresponding definite integral

$$Area = \int_1^2 \sqrt{x^2-1}\,dx$$

   A suitable change is $x = \sec t$, then $dx = \sec t \tan t$ and for $x = 1$ and $x = 2$, $t = 0$, $t = \pi/3$, respectively. The definite integral is written in the new variable now.

$$A = \int_0^{\pi/3} \sec t\ \tan^2 t\,dt = \int_0^{\pi/3} \sec t\,(\sec^2 t - 1)\,dt$$

   where we have used trigonometric identities. Finally, we integrate $\sec^3 t$ and $\sec t$, using the results of the trigonometric indefinite integrals

$$A = (\sec t \tan t - 1/2 \ln|\sec t + \tan t|)\Big|_0^{\pi/3} = \sqrt{3} - 1/2\ln(2+\sqrt{3}) = 1.07357..$$

   If we consider the region bounded by the hyperbola and the line $x = 2$, including the third quadrant, the result is $2A$

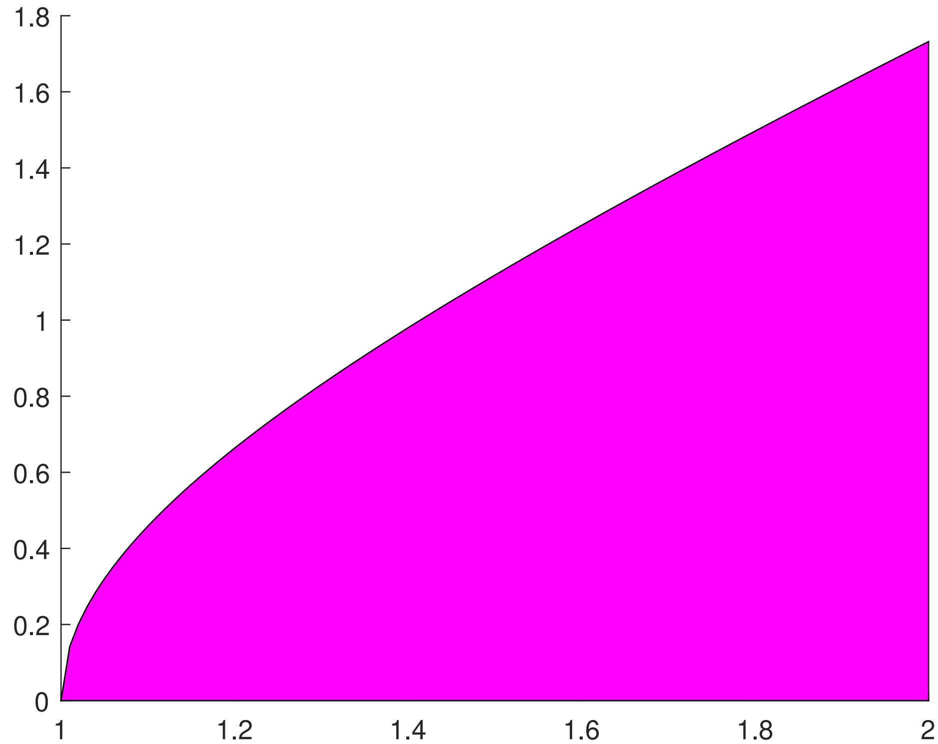

Figure 8: Exercise 9

10. Find the average of the function $\sin^2 x$ on the interval $[0, \pi]$.

According to the definition, the average or mean is calculated as

$$mean = \int_0^\pi \sin^2 x \, dx/\pi$$

and

$$\int_0^\pi \sin^2 x \, dx = \int_0^\pi (1 - \cos 2x)/2 \, dx = \pi/2$$

Finally, the average is equal to 1/2.

This means that if we generate random numbers $\{x_1, x_2, ..., x_n\}$ between 0 and π and calculate

$$\sum_{k=1}^{n} \sin^2 x_k/n$$

the result will approach 1/2 as we take more and more numbers. Check it with Matlab and the command rand.