CALCULUS
DEGREE IN COMPUTER SOFTWARE ENGINEERING
CHAPTER 1. FUNCTIONS. DOMAIN. RANGE. REAL NUMBERS.

Functions are a basic tool for describing the real world in mathematical terms. They connect one magnitude to another or several magnitudes with other different variables.

EXAMPLE 1

The temperature at which water boils depends on elevation (height above sea level). We say that it is a function of elevation. If T stands for temperature and h for height, then $T = T(h)$, h is called the independent or input variable, T is the dependent or output variable. As we ascend, T decreases, and for each value of h, we obtain a **unique** value of T. This uniqueness is essential for the definition of function.
At sea level $T = 100^{\text{o}}$ C, but on Mount Everest $T = 71^{\text{o}}$ C, this is due to the lower pressure. Taking into account that the function is roughly linear, you could try to figure it out.

EXAMPLE 2

The pressure of an ideal gas at constant temperature is a function of its volume. According to Boyle's law, $P = \dfrac{k}{V}$, where k is a constant. Therefore, the pressure of the gas, P, is the dependent (output) variable and is obtained as a function of the independent (input) variable, its volume V, applying Boyle's law. In general, if we deal with real numbers, as in these examples, we can write

$$\begin{aligned} f : \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longrightarrow y = f(x) \end{aligned}$$

This means that the function f assigns a unique real number $y = f(x)$ to each real number x. Of course, the function can be restricted to a subset of the real numbers, the domain of the function, and the output values define another subset, the range.

EXAMPLE 3

Finally, we will show another example: the volume of a cylinder is a function of its height, h, and of the radius of its base, r.

$$V = \pi r^2 h \tag{1}$$

In this case, we have two independent variables, r and h.

$$\begin{aligned} f : \mathbb{R}^2 &\longrightarrow \mathbb{R} \\ (r,h) &\longrightarrow V = f(r,h) = \pi r^2 h \end{aligned}$$

and the function goes from the set of ordered pairs of real numbers $\mathbb{R}^2$ to the set of real numbers $\mathbb{R}$. In general, we can define functions from general sets to general sets: sets of pairs, triples, etc. of real numbers, sets of complex numbers or even sets of people, cars, telephones, etc. The essential point in the definition of function is the **uniqueness**. Now, we are ready for the following definition

DEFINITION 1

A function $f : D \longrightarrow Y$ from a set D to a set Y is a rule which assigns a unique element of Y to each element of D.

$$\begin{aligned} f : D &\longrightarrow Y \\ x &\longrightarrow y = f(x) \end{aligned}$$

We must realize that, in general, x is an element of D, not necessarily a real number. A rule means a formula, a table, a graph, or any method we have to associate a unique element of Y to each element of D. D is called the domain of the function and the set of all output elements, the images of the elements of the domain, is called the range of the function. Be careful, in general the range is not identical to Y. We define the range of the function, $f(D)$, as

$$f(D) = \{f(x) \in Y / x \in D\}$$

This means: the set of all elements $f(x)$ belonging to Y, such that x belongs to D. $f(D)$ is included in Y, $f(D) \subseteq Y$. Y is also called the codomain. For instance, imagine that D consists of three people and Y of four houses. We can define a function by assigning a house to each person. In this case, one of the houses will be left out of the range. At any rate, the function is perfectly defined. If we assign two houses to the same person, it would not be a function. However, two people could share the same house and the function is well defined. The **Uniqueness** of the image associated with each input element is the key feature of functions.
In the following and for most of our course, we will deal with real-valued functions of a real variable, also called functions of a single variable. That is,

$$f : \mathbb{R} \longrightarrow \mathbb{R}$$

The set of real numbers $\mathbb{R}$ can be introduced in an axiomatic way. An axiom is a statement or proposition which is regarded as being established, accepted or self-evidently true. Three types of axioms define real numbers: algebraic, order axioms and the axiom of completeness.

The algebraic properties involved in the first-type axioms have to do with two basic operations: addition and multiplication. You can look up these axioms in Appendix

6 of Thomas' Calculus twelfth edition. They are just the commutative and associative properties of addition and multiplication, the existence of a neutral element for both operations- 0 for addition, 1 for multiplication-the existence of an inverse element for both operations, except for 0 in the case of multiplication and, finally, the distributive property of multiplication over addition. All these properties look quite obvious to you, but you can easily check that subtraction and exponentiation, two common operations, are neither associative nor commutative.
These algebraic properties make the set of real numbers a **field**. Now, we need to order the real numbers. The order axioms are:

O1: For any real numbers a and b, either $a \leq b$ (a is less than or equal to b) or $a \geq b$ (a is greater than or equal to b)
O2: If $a \leq b$ and $a \geq b$ then $a = b$
O3: If $a \leq b$ and $b \leq c$ then $a \leq c$
O4: If $a \leq b$ then $a + c \leq b + c$
O5: If $a \leq b$ and $0 \leq c$ then $ac \leq bc$

Of course, $a + b$ means 'a plus b' and ab means 'a times b'. These properties allow us to order the real numbers and place them on the real line (where less than means to the left of and greater than means to the right of). We say that the real numbers are an **ordered field**. From these basic properties, we can prove that if $a \leq b$ and $c < 0$ then $ac \geq bc$, that is, an inequality changes if we multiply it by a negative number.
To complete the real numbers, we need another axiom that enables us to include numbers such as: $\sqrt{2}$, π, e, i.e. the irrational numbers. Numbers that satisfy the previous properties but cannot be expressed as the quotient of integers. To introduce our final axiom, we need another definition

DEFINITION 2

A number M is an upper bound for a set of real numbers if all numbers in the set are smaller than or equal to M. M is a least upper bound if it is the smallest upper bound.

For instance, the number $M = 1$ is an upper bound of the set of negative numbers. However, it is not the least upper bound, we have to concede that honour to $M = 0$. The least upper bound is also called the supremum.
If we work with rational numbers, the set of numbers whose square is less than two has upper bounds, for instance, 1.5 is an upper bound. The least upper bound is the number x such that $x^2 = 2$, we call this number $x = \sqrt{2}$. It can be proved (see for instance " A proof that the square root of 2 is irrational" on the internet) that this number cannot be expressed as a fraction of integers, it is not rational. However, we want to include this type of numbers in the set of real numbers because we need to operate with them. For that we use the last axiom

AXIOM OF COMPLETENESS

Every non-empty set of real numbers which has an upper bound has a least upper bound.

We will delve deeper into the usefulness of this axiom in future chapters. Now, I will sketch a brief summary of the real numbers from a historical and practical point of view. First, we start with the natural numbers: 1, 2, 3 and so on. We use them from childhood and all human cultures understand them to a greater or lesser extent. If we wish to solve the following equation,

$$x + 5 = 5$$

we need the number $x = 0$, 0 means 'nothing' and though the idea of nothing in absolute terms can be very tricky, it is quite simple in relative terms. For instance: what is the number of elephants in this room ? , normally the answer will be zero. Besides, the number zero is essential in our decimal numeral system, the Hindu-Arabic numeral system, created by Indian mathematicians before the 6th century CE and spread throughout the West by the Arabs in the following centuries. An interesting document in which these numbers appear is the Codex Vigilianus (Spain, 976). The practical importance of the numeral positional system that includes zero as one of the basic digits is amazing. When we write 103, we are saying: one hundred, zero tens and three units. if we used a blank space instead of zero, the notation would be confusing. This numeral system allows the performance of mathematical operations in a simple way. If we want to solve

$$x + 3 = 1$$

we need negative numbers too. They are used to indicate debts or the floors below the ground, for instance, when we take the lift to go to the garage. However, ancient Greek mathematicians thought that negative solutions of equations were meaningless, it was when Arab and Italian mathematicians (for instance, Fibonacci) used them to count debts as negative quantities of money in the Middle Ages, that they became more common. The set of natural numbers $\mathbb{N}$ extended with zero and the negative numbers makes up the set of integers $\mathbb{Z}$. Obviously, $\mathbb{N}$ stands for natural, not so obviously, $\mathbb{Z}$ comes from Zahl, number in German. If we want to solve

$$3x = 2$$

we need rational numbers, ratios, that we construct by dividing integers. We denote the set of rational numbers $\mathbb{Q}$, from quotient. We finish by trying to solve

$$x^2 = 2$$

the solution, $\sqrt{2}$, is not a rational number and appears as the length of the hypotenuse of a right isosceles triangle whose sides are 1 unit of length. It seems that the fact that this number is not rational came as a shock to Pythagorean mathematicians in the 5th century BCE. If we try to divide the leg of a right isosceles triangle

into n equal parts and then completely cover the hypotenuse with m of these pieces of equal size, it happens that the task is impossible. Of course, Greek mathematicians could approximate $\sqrt{2}$ quite well by rationals, 99/70 or 665857/470832 are two such approximations (Check the error), but $\sqrt{2}$ is not rational, is "irrational" and is the typical example of one of the numbers that complete the real numbers. There is a story, maybe a legend, about a real Greek mathematician, Hippasus of Metapontum, who was travelling by boat and was thrown into the sea by his colleagues when he talked about the existence of irrational numbers. He drowned. Mathematics was a question of death and life for Pythagoreans.
The union of the sets of rational and irrational numbers makes up the set of real numbers, $\mathbb{R}$.
For real-valued functions of a single real variable, the domain is the largest set on which the function is defined, that is

$$D = \{x \in \mathbb{R}/y = f(x) \in \mathbb{R}\}$$

phisically, the domain can be further restricted. For example, if $P = k/V$, V must be positive. However, in the problems presented in Exercises 1, we will just consider the mathematical domain. The range is, as defined before, $f(D)$. When we plot a real function of a real variable, the graph is a curve on the plane or, more precisely, the set

$$graph = \{(x, f(x)) \in \mathbb{R}^2/x \in D\}$$

The domain and the range of f will be the projection of the graph on the x-axis and y-axis, respectively.