



Universidad de Oviedo  
*Universidá d'Uviéu*  
*University of Oviedo*

# S3-Euclidean Spaces Least-Squares

Linear Algebra

Ingeniería del Software-Universidad de Oviedo

Juan Luis Fernández Martínez

Class-30 October 2020

# Least Squares



Universidad de Oviedo  
Universidá d'Oviéu  
University of Oviedo

Least-squares in  $C^0[a, b]$

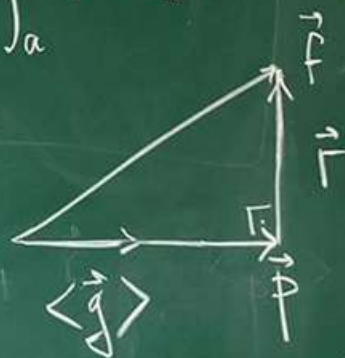
$\vec{f}, \vec{g} \in C^0[a, b]$  vector space  
↓  
Euclidean space

$$\vec{p} = \text{Proj}_{\langle \vec{g} \rangle}(\vec{f})$$

$$\vec{p} = \lambda \vec{g} + \vec{r}$$

$$\vec{r} \perp \langle \vec{g} \rangle$$

$$\vec{f} \cdot \vec{g} = \int_a^b f(x) \cdot g(x) dx \quad (\text{scalar product})$$



$$\vec{f} \cdot \vec{g} = \lambda \vec{g} \cdot \vec{g} + \vec{r} \cdot \vec{g}$$

$$\vec{r} \cdot \vec{g} = 0$$

$$\lambda = \frac{\vec{f} \cdot \vec{g}}{\vec{g} \cdot \vec{g}}$$

$\vec{p} = \lambda \vec{g}$   
 $\vec{p}$  is the function  
parallel to  $\vec{g} = g(x)$   
which is closer to  
 $\vec{f} = f(x)$ .

# Least Squares



Universidad de Oviedo  
Universidà d'Oviéu  
University of Oviedo

Least-squares in  $C^0[a,b]$

$\vec{f}, \vec{g} \in C^0[a,b]$  Vector space

$$\vec{f} \cdot \vec{g} = \int_a^b f(x) \cdot g(x) dx \quad (\text{scalar product})$$

$\infty$  dimension  
polynomials.

Subspaces of finite dimension

$$P_n(x) = \langle 1, x, \dots, x^n \rangle \quad x \in [a,b]$$

Space of Lagrange

$$T_n(x) = \langle 1, \cos x, \sin x, \cos 2x, \sin 2x, \dots, \cos nx, \sin nx \rangle$$

$$\cos nx, \sin nx$$

Space of trigonometric  
polynomials

Fourier Polynomials.



# Least Squares



Universidad de Oviedo  
Universidá d'Oviéu  
University of Oviedo

Least-squares in  $C^0[a,b]$

$\vec{f}, \vec{g} \in C^0[a,b]$  Vector space

$$\vec{f} \cdot \vec{g} = \int_a^b f(x) \cdot g(x) dx \quad (\text{scalar product})$$

$$f(x) = \ln(x) \quad x \in [1, e]$$

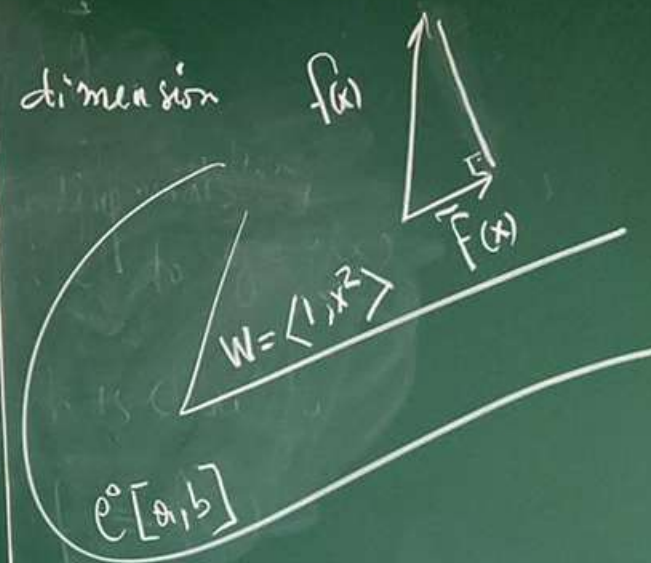
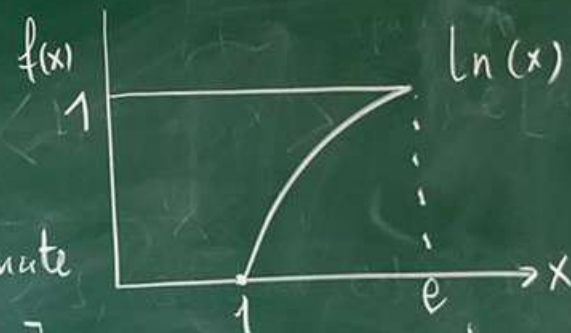
$$W = \langle 1, x^2 \rangle$$

→ subspace to approximate  $\ln(x)$  in  $[1, e]$

$$\tilde{f}(x) \in W \Rightarrow \tilde{f}(x) = a_0 \cdot 1 + a_1 \cdot x^2$$

⇒ Continuous Least-squares

$$\tilde{f}(x) \equiv \text{Proj}_W(f(x))$$



# Least Squares



Universidad de Oviedo  
Universidat d'Oviéu  
University of Oviedo

Least-squares in  $C^0[a,b]$

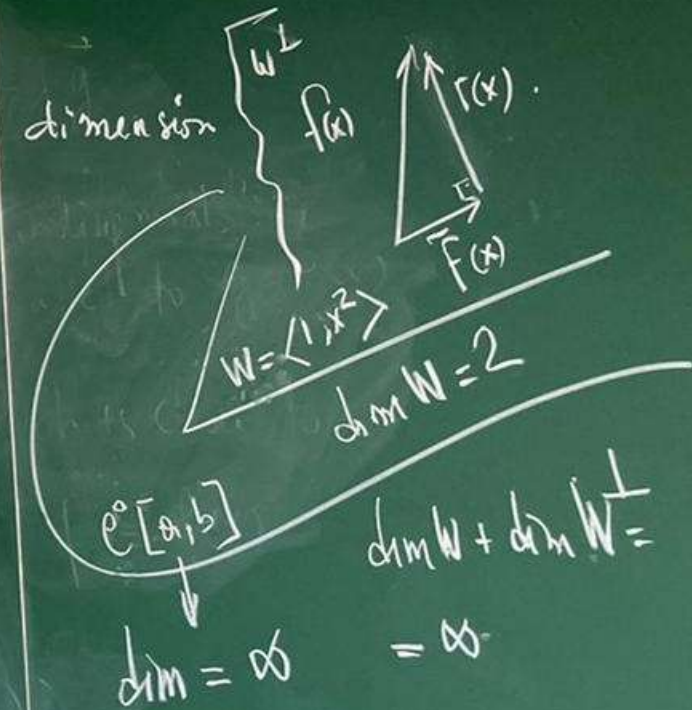
$$r(x) = f(x) - \tilde{f}(x) = \ln(x) - a_0 \cdot 1 - a_1 \cdot x^2 \quad (\text{scalar product}) \quad \infty$$

$$r(x) \in W^\perp$$

subspace of  $C^0[a,b]$  of dimension  $\infty$

$$\Rightarrow r(x) \perp W \Leftrightarrow \begin{cases} r(x) \perp 1 \\ r(x) \perp x^2 \end{cases} \quad \int_1^e r(x) \cdot 1 = 0$$

$$\begin{cases} \vec{r}(x) \cdot \vec{1} = 0 \\ \vec{r}(x) \cdot \vec{x^2} = 0 \end{cases} \quad \text{Normal equations.}$$





# Least Squares



Universidad de Oviedo  
Universidá d'Oviéu  
University of Oviedo

## Least-squares in $C^0[a,b]$

Normal equations

$$r(x) \cdot 1 = 0 \Rightarrow$$

$\Rightarrow$

$$f(x) \cdot 1 - a_0 \cdot 1 \cdot 1 - a_1 \cdot 1 \cdot x^2 = 0$$

$$r(x) \cdot x^2 = 0 \Rightarrow$$

$\Rightarrow$

$$f(x) \cdot x^2 - a_0 x^2 \cdot 1 - a_1 x^2 \cdot x^2 = 0$$

$$\Rightarrow \begin{pmatrix} 1 \cdot 1 & 1 \cdot x^2 \\ x^2 \cdot 1 & x^2 \cdot x^2 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = \begin{pmatrix} f(x) \cdot 1 \\ f(x) \cdot x^2 \end{pmatrix}$$

$$f(x) \cdot 1 = \int_1^e \ln x \, dx$$

(Scalar product)

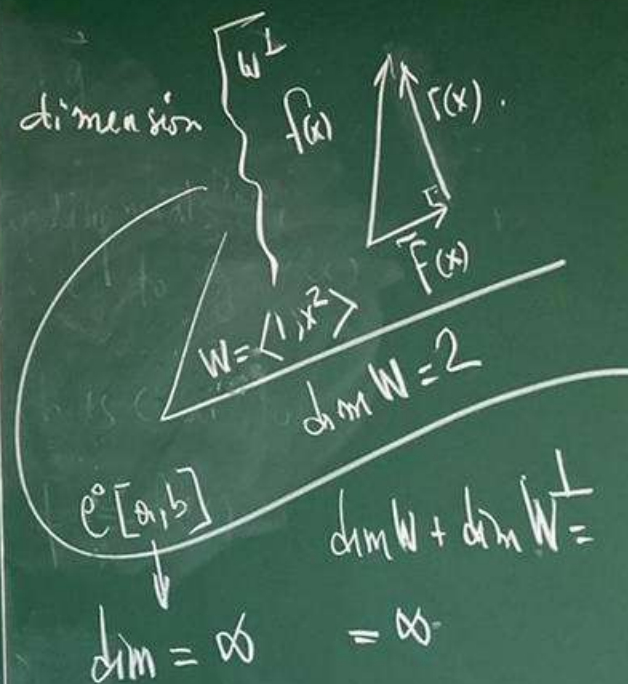
$\infty$

$$r(x) \perp 1$$

$$r(x) \perp x^2$$

$$1 \cdot 1 = \int_1^e 1 \cdot 1 \, dx = e - 1$$

$$x^2 \cdot x^2 = \int_1^e x^4 \, dx = \frac{e^5 - 1}{5}$$



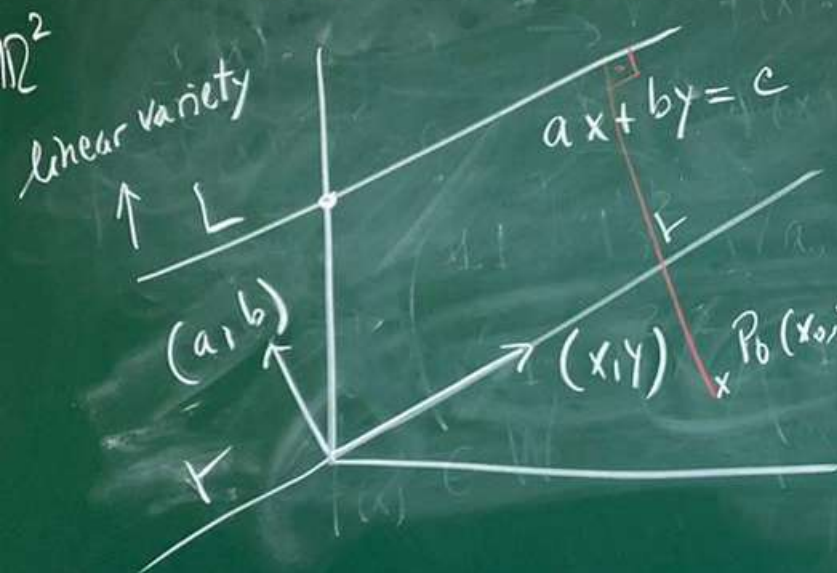
# Least Squares



Universidad de Oviedo  
Universidà d'Uviéu  
University of Oviedo

## Euclidean Geometry in $\mathbb{R}^2$ and $\mathbb{R}^3$

- Distance from a point to a plane ( $\mathbb{R}^3$ )
- to straight line ( $\mathbb{R}^2, \mathbb{R}^3$ )



$$ax + by = 0 \text{ (subspace in } \mathbb{R}^2)$$

$$(a, b) \cdot \begin{pmatrix} x \\ y \end{pmatrix} = 0$$

Euclidean Scalar Product in  $\mathbb{R}^2$

Conclusion  $(a, b)$  is the orthogonal direction to  $r$  and  $L$

$$\mathbb{R}^2 = r \oplus \langle (a, b) \rangle$$

$$(a, b) \perp (x, y) \in L$$

$$P_0(x_0, y_0) \notin L$$

finding the distance from  $P_0$  to  $L$

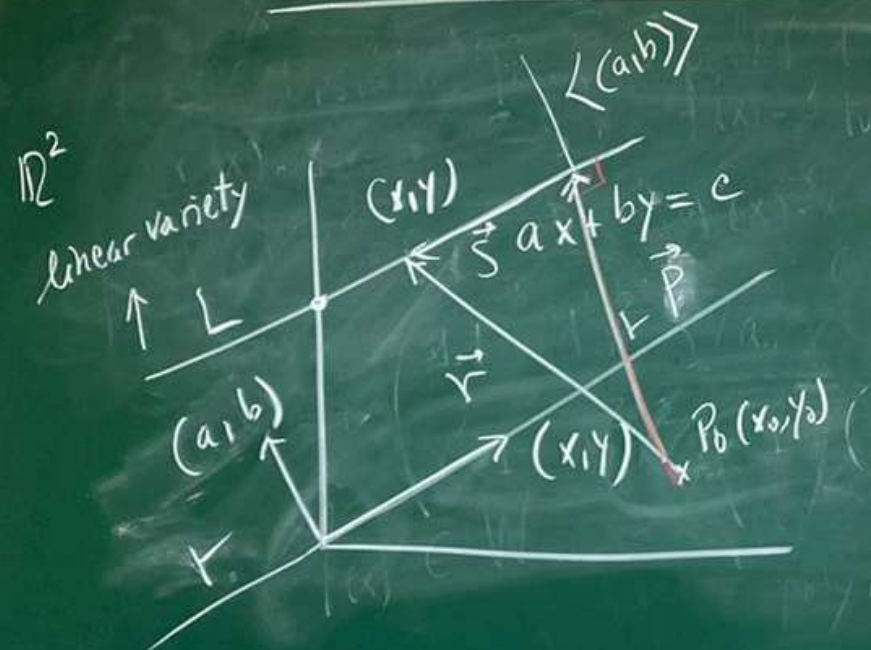


# Least Squares



Universidad de Oviedo  
Universidá d'Oviéu  
University of Oviedo

Euclidean Geometry in  $\mathbb{R}^2$  and  $\mathbb{R}^3$



$$(x, y) \in L \Rightarrow ax + by = c$$

$$\vec{r} = (x, y) - (x_0, y_0) = (x - x_0, y - y_0)$$

$$d = \|\vec{p}\|$$

$$\vec{p} = \text{Proj}_{\langle (a, b) \rangle}^\perp(\vec{r})$$

$$\vec{n} = (a, b)$$

$$\vec{r} = \vec{p} + \vec{s} \quad \vec{s} \perp \langle (a, b) \rangle$$

$$\vec{r} = \lambda \vec{n} + \vec{s}$$

$$\vec{r} \cdot \vec{n} = \lambda \vec{n} \cdot \vec{n} + 0$$

$$\lambda = \frac{\vec{r} \cdot \vec{n}}{\vec{n} \cdot \vec{n}}$$



# Least squares



Universidad de Oviedo  
Universidá d'Oviéu  
University of Oviedo

Euclidean Geometry in  $\mathbb{R}^2$  and  $\mathbb{R}^3$

$$(x, y) \in L \Rightarrow ax + by = c$$

$$\lambda = \frac{(x - x_0) \cdot a + (y - y_0) \cdot b}{a^2 + b^2}$$

$$\vec{p} = \lambda \cdot \vec{n}$$

$$d_{\min} = \|\vec{p}\| = |\lambda| \cdot \|\vec{n}\| =$$

$$\vec{n} = (a, b)$$

$$\frac{|c - ax_0 - by_0|}{\sqrt{a^2 + b^2}}$$

$$= \frac{ax + by = c \quad ax_0 + by_0 = c$$

$$= \frac{c - ax_0 - by_0}{a^2 + b^2}$$

formula

$$\text{Ex. } P_0(1, -1)$$

$$L \equiv 2x - y = 3 \quad d_{\min} =$$

$$\frac{|2 \cdot 1 - (-1) - 3|}{\sqrt{5}} = 0$$