CALCULUS
DEGREE IN SOFTWARE ENGINEERING
CHAPTER 4. LIMITS. THE LIMIT LAWS.

The concept of limit is one of the most important and fruitful in Calculus, leading in a natural way to essential ideas such as differentiation and integration. To introduce the concept, for real functions of a single real variable, the following intuitive example is shown. We want to calculate the limit as x approaches 2 of a given rational function

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2}$$

We mean: Does the rational function $\dfrac{x^2-4}{x-2}$ approach a certain number ?, if we consider values of x as close as we please to 2. We could carry out the calculation of the function for values of x very close to 2, $x = 1.99, 1.999, ..., 2.0001, 2.00001, ...$ and so on. We could, as would be the case for continuous functions, calculate the function at $x = 2$. However, in this case the function is not defined at that point. Nevertheless, we can write the fraction as $\dfrac{(x-2)(x+2)}{x-2}$ and simplifying, the function is $x + 2$, provided that $x \neq 2$. At any rate, the concept of limit will be defined assuming that x is very close to the point at which we are calculating it, but does not take on that value. We study a neighbourhood of the point, but not the point itself. With the previous simplification, it is very easy to see that the limit is 4 in our example.
Our " intuitive" definition of limit could then be written as:

DEFINITION 1

Suppose that f is defined on an open interval $(x_0 - d, x_0 + d)$, except maybe at x_0 itself. If $f(x)$ is arbitrarily close to L, a real number, for all x sufficiently close to x_0, we say that f approaches the limit L as x approaches x_0.

We also say that the limit of f as x tends to x_0 is L. We have considered that the function is defined, at least, on an open interval around x_0, this is not necessary, as we will see later. At any rate, the function must be defined at points as close to x_0 as we wish. Technically, we say that x_0 is an accumulation point of the domain of f. The function need not be defined at x_0. Whether it is defined or not at this point is irrelevant for the definition of limit. We are also using the word "close" in an intuitive way. It is only natural to define the distance between two points on the real line as $d(x_0, x_1) = |x_0 - x_1|$. With these basic ideas we can write a precise definition of limit.

DEFINITION 2

Let $f(x)$ be defined on an open interval about x_0, except possibly at x_0 itself. We say that the limit of $f(x)$ as x approaches x_0 is the number L and write $\lim_{x \to x_0} f(x) = L$ if, for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$, such that for all x
If

$$0 < |x - x_0| < \delta$$

then

$$|f(x) - L| < \epsilon$$

As written in the definition, the value of δ will depend on the value of ϵ. Apart from being precise, this definition allows us to prove the basic limit laws, that is, how limits behave with respect to the basic operations: addition, subtraction, multiplication, division,....

THE LIMIT LAWS

If $\lim_{x \to x_0} f(x) = L$ and $\lim_{x \to x_0} g(x) = M$, then

Sum Rule

$$\lim_{x \to x_0} (f(x) + g(x)) = L + M$$

Difference Rule

$$\lim_{x \to x_0} (f(x) - g(x)) = L - M$$

Constant Multiple Rule

$$\lim_{x \to x_0} k.f(x) = k.L$$

Product Rule

$$\lim_{x \to x_0} f(x).g(x) = L.M$$

Quotient Rule

$$\lim_{x \to x_0} f(x)/g(x) = L/M \, , M \neq 0$$

Power Rule

$$\lim_{x \to x_0} (f(x))^n = L^n$$

with n a positive integer

Root Rule

$$\lim_{x \to x_0} (f(x))^{1/n} = L^{1/n}$$

with n a positive integer. If n is even, we must assume that $L > 0$.
All these properties look very reasonable, but they should be proved and they are proved by using our precise definition of limit. For instance, a simple proof of the Sum Rule can be seen in Thomas' Calculus twelfth edition, page 62. It uses a basic property of the absolute value: the triangle inequality, that is, $|a + b| \leq |a| + |b|$.
By using the limit laws, we can prove that the limit of a polynomial function at any point is the value of the polynomial at that point. The same is true for rational functions, provided the denominator is not zero. We can also use the limit laws for calculating the limits of powers and roots. Now, we will state an important result that will be used in some proofs and exercises.

THE SANDWICH THEOREM

Suppose that $g(x) \leq f(x) \leq h(x)$ for all x in some open interval containing x_0 , except possibly at x_0. Suppose also that

$$\lim_{x \to x_0} g(x) = \lim_{x \to x_0} h(x) = L$$

then

$$\lim_{x \to x_0} f(x) = L$$

The name of the theorem is very suitable. An application of the sandwich theorem enables us to prove

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

The angle must be expressed in radians. See a proof in the Khan Academy video of the limit of $\sin x/x$ as x tends to 0. In the video, the theorem is called the squeeze theorem.

Other important result is that if $\lim_{x \to x_0} f(x) > 0$, then $f(x)$ will be positive in a neighbourhood of x_0, except maybe at x_0. If the limit is negative, the function will be negative in the neighbourhood. There is a continuity of the sign.
Now, we will study one-sided limits

ONE-SIDED LIMITS

Till now, we have assumed that the function is defined on $(x_0 - d, x_0 + d)$, on the left and the right of x_0, except maybe at x_0. It can happen that the function is only defined on one side, left or right, and, at any rate, we can define the limit. In this case, we say that the limit is a one-sided limit. If we approach the point from the left, we call the limit a left-hand limit and we can define it in the usual way: We say that the limit of $f(x)$ as x approaches x_0 from the left is the number L_1

and write $\lim\limits_{x \to x_0^-} f(x) = L_1$ if, for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$, such that for all x
If
$$0 < x_0 - x < \delta$$
then
$$|f(x) - L_1| < \epsilon$$
Clearly, we just consider values of x less than x_0. The right-hand limit is defined in a similar way: we say that the limit of $f(x)$ as x approaches x_0 from the right is the number L_2 and write $\lim\limits_{x \to x_0^+} f(x) = L_2$ if, for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$, such that for all x
If
$$0 < x - x_0 < \delta$$
then
$$|f(x) - L_2| < \epsilon$$
Now, we take $x > x_0$. Since the limit at a point has to be unique, if the function is defined both on the left and right of the point, the limit exists if and only if $L_1 = L_2$, that is, if left-hand limit and right-hand limit are the same.

EXAMPLE 1

Let us calculate
$$\lim_{x \to 0} \frac{x}{|x|}$$
.
The limit from the right is 1, as x is positive, but the limit from the left is -1 as x is negative. The function is not defined at 0 but that is irrelevant. The limit at $x = 0$ does not exist, as left-hand and right-hand limits differ. This function is the step function, a typical example of a function with a jump discontinuity.

EXAMPLE 2

In this case the function is quite common $y = \sqrt{4 - x^2}$. The domain of the function is $[-2, 2]$ and at $x = 2$, it is not defined on the right, so we can only define the left-hand limit that is 0, obviously. At $x = -2$, the function is not defined on the left of the point and we can only define the right-hand limit , 0, again. The limit exists in both cases, but only as right-hand or left-hand limit. The graph is a semicircle.