CALCULUS
DEGREE IN SOFTWARE ENGINEERING
CHAPTER 9. DIFFERENTIATION RULES. L'HÔPITAL'S RULE.

This chapter introduces the basic rules that allow us to differentiate polynomials, rational functions and their combinations in a direct way, without having to take limits each time. We also show how to calculate the derivatives of exponentials, trigonometric functions and their inverse functions. Finally, we use derivatives to find the limits of some indeterminate forms, introducing L'Hôpital's Rule.

DERIVATIVE OF A SUM, DIFFERENCE, PRODUCT AND QUOTIENT

If f and g are differentiable at a point x, the following properties hold:

1) $f+g$ is differentiable at x and $(f+g)'(x) = f'(x) + g'(x)$
2) $f-g$ is differentiable at x and $(f-g)'(x) = f'(x) - g'(x)$
3) If c is a constant, (cf) is differentiable at x and $(cf)'(x) = cf'(x)$
4) $f.g$ is differentiable at x and $(f.g)'(x) = f'(x).g(x) + f(x).g'(x)$. (Leibniz's Rule)

5) f/g is differentiable at x and $(f/g)'(x) = \dfrac{f'(x).g(x) - f(x).g'(x)}{g^2(x)}$, provided $g(x) \neq 0$.

The proofs of the first properties are elementary as they can be done by simply using the definition of derivative. Then, we will prove 4)
Proof of 4)

According to the definition

$$(f.g)'(x) = \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

Now, we add and subtract $f(x+h)g(x)$ and write, grouping the suitable terms together

$$(f.g)'(x) = \lim_{h\to 0} \frac{f(x+h)\left[g(x+h) - g(x)\right] + \left[f(x+h) - f(x)\right]g(x)}{h}$$

Calculating the corresponding limits and taking into account that f must be continuous at x, we complete the proof.
Try to demonstrate 5) . Hint: prove $(1/g)'(x) = -g'(x)/g^2(x)$ and use 4).
In order to take derivatives of composite functions, we need another important result:
THE CHAIN RULE

THE CHAIN RULE

If f(u) is differentiable at the point $u = g(x)$ and $g(x)$ is differentiable at x, then the composite function $(f \circ g)(x) = f(g(x))$ is differentiable at x and $(f \circ g)'(x) = f'(g(x))\, g'(x)$.

Though there is a simple and intuitive proof of this result in the case $\Delta u \neq 0$ in a neighbourhood of x, the general proof is more complicated. The first proof can be seen in a Khan Academy video (however, they present it as a general proof). The second proof, more complicated, can be read in the the English Wikipedia corresponding article or in a book (Thomas' book page 170) .
In the following, we will take the derivatives of some basic functions, by using the differentiation rules:

EXAMPLE 1. POLYNOMIALS. RATIONAL FUNCTIONS

$$(5x^4 + 3x^3 - 8x^2 + 6x + 8)' = 20x^3 + 9x^2 - 16x + 6$$

This could be used as the general method of differentiating a polynomial, we apply $(x^n)' = nx^{n-1}$ and the corresponding rules for the derivative of sums, differences, and the rule for differentiating a constant times a function. For a rational function, we apply the rule for differentiating a quotient

$$\left(\frac{x^2 - 1}{x^2 + 1}\right)' = \left(\frac{2x(x^2 + 1) - 2x(x^2 - 1)}{(x^2 + 1)^2}\right) = \frac{4x}{(x^2 + 1)^2}$$

EXAMPLE 2. TRIGONOMETRIC FUNCTIONS

$$(\sin x)' = \cos x$$

. We will use the definition:

$$(\sin x)' = \lim_{h \to 0} \frac{\sin(x + h) - \sin x}{h}$$

and the formula for the sine of the sum

$$(\sin x)' = \lim_{h \to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h}$$

$$(\sin x)' = \lim_{h \to 0} \frac{\sin x[\cos h - 1] + \cos x \sin h}{h} = \cos x$$

The limit of the first addend is zero (Why ? Hint: multiply numerator and denominator by $\cos h + 1$). The limit of the second is $\cos x$. We have used $\lim\limits_{h \to 0} \sin h/h = 1$.

From this, we can find $(\cos x)' = -\sin x$, since $\cos x = \sin(x + \pi/2)$, $(\cos x)' = \cos(x + \pi/2) = -\sin x$. We have applied the Chain Rule and basic properties of the sine and cosine.
The derivative of the tangent can also be easily obtained

$$(\tan x)' = \left(\frac{\sin x}{\cos x}\right)' = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$$

In the same way, we can prove:

$$(\cot x)' = -\csc^2 x$$

$$(\sec x)' = \sec x \tan x$$

$$(\csc x)' = -\csc x \cot x$$

In this example, we have found the derivatives of the basic trigonometric functions.

EXAMPLE 3. EXPONENTIAL FUNCTIONS

We write the derivative of $f(x) = e^x$

$$\lim_{h \to 0} \frac{e^{x+h} - e^x}{h} = \lim_{h \to 0} \frac{e^x(e^h - 1)}{h}$$

By the definition of the number $e = 2.718281828....$

$$\lim_{h \to 0} \frac{e^h - 1}{h} = 1$$

and

$$(e^x)' = e^x$$

Besides

$$a^x = e^{x \ln a}$$

and applying the Chain Rule

$$(a^x)' = e^{x \ln a} \, \ln a = a^x \, \ln a$$

Then, $(a^x)' = a^x$ if and only if $a = e$

EXAMPLE 4. INVERSE FUNCTIONS

If $f(x)$ is one-to-one on an interval I, we can define its inverse function $f^{-1}(x)$. If we compose these functions

$$(f \circ f^{-1})(x) = x$$

and take the derivative, we obtain

$$(f \circ f^{-1})'(x) = f'(f^{-1}(x))(f^{-1})'(x) = 1$$

Isolating the derivative of the inverse function

$$(f^{-1})'(x) = \frac{1}{f'(f^{-}1(x))}$$

This is true, provided $f'(f^{-1}(x)) \neq 0$.

As a first application of this important result, we will differentiate $f^{-1}(x) = \ln x$, defined as the inverse function of $f(x) = e^x$

$$(\ln x)' = \frac{1}{f'(f^{-1}(x))} = \frac{1}{e^{\ln x}} = \frac{1}{x}$$

In fact, we could have defined the natural logarithm as $\ln x = \int_1^x 1/t\, dt$ and later, the exponential function e^x as the inverse of the natural logarithm. In this way, we could deduce the basic properties of these functions. However, we have not studied integration yet and we have to resign ourselves to our former approach. We can also differentiate x^p, with p any real number

$$(x^p)' = (e^{p \ln x})' = (e^{p \ln x})\, p/x = p\, x^{p-1}$$

provided the function and the derivative are well defined (this is not the case for $x = 0$ and $p < 1$). Finally, we can calculate the derivatives of some inverse trigonometric functions. For instance

$$(\arcsin x)' = \frac{1}{f'(f^{-1}(x))} = \frac{1}{\cos(\arcsin x)} = \frac{1}{\sqrt{1-x^2}}$$

Please, note that if the sine of an angle is x, its cosine is $\sqrt{1-x^2}$. The sign is positive, given the range of the arcsine function.
In a similar way,

$$(\arctan x)' = \frac{1}{f'(f^{-1}(x))} = \frac{1}{1+\tan^2(\arctan x)} = \frac{1}{1+x^2}$$

With these basic derivatives, we can calculate all the derivatives of elementary functions by using the DIFFERENTIATION RULES and the CHAIN RULE.

In the corresponding exercises, we will use these techniques. We will finish this chapter with the introduction of L'Hôpital's Rule, a powerful method to solve indeterminate forms based on the application of derivatives. L'Hôpital (1661-1704) was a French mathematician who published the rule that bears his name in the first Calculus textbook "Analyse des infiniment petits pour l'intelligence des lignes courbes" (1696), though the method was invented by Johann Bernoulli (1667-1748), a Swiss mathematician, one of the most important mathematicians in history.

Let us state L'Hôpital's (or Bernoulli's) Rule for the case 0/0:

L'HÔPITAL'S RULE

Suppose that $f(a) = g(a) = 0$, that f and g are differentiable on an open interval I containing a, and that $g'(x) \neq 0$ on I if $x \neq a$. Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

assuming that the limit on the right side of this equation exists.

Given that

$$\lim_{x \to a} \frac{(f(x) - f(a))/(x - a)}{(g(x) - g(a))/(x - a)} = \frac{f'(a)}{g'(a)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

This could be a a proof of L'Hôpital's Rule, but in these equalities, we have assumed the continuity of f' and g' at a, what does not appear in the statement of the Rule. For the general proof, we need Cauchy's Mean Value Theorem, that we will study in the next chapter.

Apart from the indeterminate form 0/0, the rule can be extended to ∞/∞ and to other indeterminate forms as we will see in the Exercise sheet.

AN EXAMPLE OF L'HÔPITAL'S RULE

$$\lim_{x \to 0} \frac{3x - \sin x}{x} = \lim_{x \to 0} \frac{3 - \cos x}{1} = 2$$

First, we check that we have an indeterminate form 0/0. As all the necessary conditions for applying L'Hôpital's Rule hold, we take the derivatives of numerator and denominator, finding the limit.