**CALCULUS**
**DEGREE IN SOFTWARE ENGINEERING**
**EXERCISES AND SOLUTIONS 1**

1. Find the domains of the following real functions

   (a) $y = \sqrt{1-x^2}$

   The graph of this function is a semicircle centered at the origin of coordinates with $y \geq 0$. For $x$ to be in the domain, $x^2 \leq 1$, or, equivalently, $|x| \leq 1$. Therefore, the domain is the interval $D = [-1, 1]$. The range is the interval $f(D) = [0, 1]$.

   (b) y=$(1-x^2)^{1/3}$

   In this case, the radicand can be positive, negative or zero and the function is well defined because the denominator of the exponent is odd. Then, $D = \mathbb{R}$, and since the maximum possible value of the funcion is $y = 1$, the range is $f(D) = (-\infty, 1]$

   (c) $y = \sqrt{x^2 - 5x - 6}$

   We have to factorize the radicand, by finding the roots, $-1$, and $6$ . Then $x^2 - 5x - 6 = (x+1)(x-6)$. Analyzing the behaviour of the radicand in the intervals $(-\infty, -1]$, $[-1, 6]$ and $[6, \infty)$, we see that it is non-negative in $D = (-\infty, -1] \cup [6, \infty]$, what defines the domain. You can easily check, by graphing the function, that $f(D) = [0, \infty)$

   (d) $y = \sqrt{-x^2 + 6x - 11}$

   As before, we find the zeros of the radicand. In this case, the roots are complex numbers: $3 + \sqrt{2}i, 3 - \sqrt{2}i$ . This means that the graph of the radicand never intersects the $x$-axis and it is always positive or negative. By substituting $x = 0$ in the radicand, we check that it is always negative. Therefore, the function is not defined as a real-valued function and the domain is the empty set $D = \emptyset$.

   (e) $y = \sqrt{(x+2)(x-2)(x-6)}$

   We could have written the expanded polynomial and found the roots. Try to do it. However, as it is factorized, we just have to analyze the intervals on which the radicand keeps the same sign:$(-\infty, -2]$, $[-2, 2]$, $[2, 6]$ and $[6, \infty)$. Multiplying the signs of the factors, we find that the radicand is non-negative in $[-2,2] \cup [6,\infty)$. Check that $f(D) = [0,\infty)$.

(f) $y = \sqrt{(x-1)(x-3)(x-5)(x-7)}$

Here, the radicand is positive when all the factors are positive, all are negative or finally, when two are positive and two negative. Including the zeros of the radicand, the domain is $D = (-\infty,1] \cup [3,5] \cup [7,\infty)$. Change the sign of the radicand and try to find the domain of the new function.

(g) $y = \dfrac{1}{x^2-9}$

This is a rational function, that is, the quotient of two polynomials. The domain is the set of points at which the denominator is different from zero. For this function

$$D = \{x \in \mathbb{R} / x \neq 3 \, and \, x \neq -3\}$$

The study of the range of this function is more complicated than in the previous cases. In general, it will be necessary to plot the function to study the range. We will do this in future exercises.

(h) $y = \dfrac{1}{\sqrt{|x|-x}}$

We take into account the definition of the absolute value: if $x \geq 0$, $|x| = x$, if $x < 0$, $|x| = -x$. Then, for $x \geq 0$, the denominator is zero and the function is not defined. For $x < 0$, $y = \dfrac{1}{\sqrt{-2x}}$ , and the function is well defined. Then, $D = \{x \in \mathbb{R} / x < 0\}$.

(i) $y = \dfrac{1}{\sqrt{x^2-x}}$

Factorizing the radicand in the denominator, $x(x-1)$, this is only positive in $D = (-\infty,0) \cup (1,\infty)$. That is the domain. Take care, the intervals must be open in this case.

(j) $y = \dfrac{1}{\sqrt{x^2-|x|}}$

It is a little more complicated than the exercise above. For $x > 0$, the function is

$$y = \frac{1}{\sqrt{x^2-x}}$$

and the domain is $(1, \infty)$. For $x < 0$, the function is

$$y = \frac{1}{\sqrt{x^2 + x}}$$

and the domain is $(-\infty, -1)$. Putting together both parts of the domain,

$$D = (-\infty, -1) \cup (1, \infty)$$

(k) $y = \arcsin \dfrac{x}{1 + x}$

The argument of the arcsine must always be in $[-1, 1]$. Then

$$-1 \le \frac{x}{1 + x} \le 1.$$

Of course, $x \ne -1$, and if we multiply the inequality by $1 + x$, we obtain

$$-(1 + x) \le x \le 1 + x$$

provided that $x > -1$. From the first inequality, we derive $x \ge -1/2$. The second is always true. If $x < -1$, multiplying by $1 + x$, we find

$$-(1 + x) \ge x \ge 1 + x$$

The second inequality is never true. To sum up, the domain is

$$D = [-1/2, \infty)$$

(l) $y = \arccos \dfrac{2x}{5 + x}$
Now,

$$-1 \le \frac{2x}{5 + x} \le 1$$

$x \ne -5$ and multiplying by $5 + x$

$$-(5 + x) \le 2x \le 5 + x$$

provided that $x > -5$. Hence, $x \in [-5/3, 5]$. If $x < -5$, we can write

$$-(5 + x) \ge 2x \ge 5 + x$$

, which is impossible, as $x \ge 5$ and $x < -5$ at the same time. Finally, the domain is

$$D = [-5/3, 5]$$

(m) $y = \arctan \dfrac{2x}{4+x}$

This is very simple, since the argument of the arctangent is any real number. Thus,

$$D = \{x \in \mathbb{R} / x \neq -4\}$$

In general, the argument of arcsine and arcosine must be in $[-1, 1]$ and that of arctangent and arccotangent can be any real number. What happens for arcsecant ?

(n) $y = arcsec \dfrac{3x}{x-5}$

Either $\dfrac{3x}{x-5} \leq -1$ or $\dfrac{3x}{x-5} \geq 1$. Besides, $x \neq 5$. In the first case, there are two possible subcases: if $x > 5$, when we multiply $\dfrac{3x}{x-5} \leq -1$ by $x-5$, we obtain $3x \leq -x+5$. Thus, $x \leq 5/4$. But this is not compatible with $x > 5$. If $x < 5$, when we multiply, we find $3x \geq -x+5$, that is $x \geq 5/4$. So, the interval $[5/4, 5)$ is included in the domain. The second case is $\dfrac{3x}{x-5} \geq 1$. If $x > 5$, when we multiply the previous inequality by $x-5$, we can write $3x \geq x-5$ and hence, $x \geq -5/2$. Then, the interval $(5, \infty)$ is also in the domain. Finally, if $x < 5$, $3x \leq x-5$ and $x \leq -5/2$. If we put all these results together, the domain is

$$D = (-\infty, -5/2] \cup [5/4, 5) \cup (5, \infty)$$

You can check that the domain of $\arccos \dfrac{3x}{x-5}$ is $D = [-5/2, 5/4]$, just the complement of the domain of the arcsecant, except for $x = 5$.

(o) $y = \dfrac{1}{\sqrt{|x^2+x-2|-3}}$

This is equivalent to $|x^2+x-2| > 3$. Then, either $x^2+x-2 > 3$ or $x^2+x-2 < -3$. In the first case, we solve $x^2+x-5 = 0$, whose roots are $x_1 = \dfrac{-1-\sqrt{21}}{2} = -2.79.....$ and $x_2 = \dfrac{-1+\sqrt{21}}{2} = 1.79......$ Therefore $(-\infty, x_1) \cup (x_2, \infty)$ is included in the domain. In the second case, $x^2+x+1 < 0$, the roots are complex and $x^2+x+1$ is always positive. Thus,

$$D = (-\infty, x_1) \cup (x_2, \infty)$$

2. Find the composite function of the following functions $f \circ g$ and $g \circ f$

| | | |
|---|---|---|
| I) | $f(x) = x + 5$ | $g(x) = x^2 - 3$ |
| II) | $f(x) = ln(x)$ | $g(x) = sinx$ |
| III) | $f(x) = x^2 + 9$ | $g(x) = \sqrt{x}$ |
| IV) | $f(x) = x^2 + 7$ | $g(x) = \dfrac{1}{x+5}$ |
| V) | $f(x) = x^3 + x$ | $g(x) = x^{1/3}$ |

Solutions

| | | |
|---|---|---|
| I) | $x^2 + 2$ | $x^2 + 10x + 22$ |
| II) | $\ln(\sin(x))$ | $\sin(\ln(x))$ |
| III) | $x + 9$ | $\sqrt{x^2 + 9}$ |
| IV) | $\dfrac{7x^2 + 70x + 176}{x^2 + 10x + 25}$ | $\dfrac{1}{x^2 + 12}$ |
| V) | $x^{1/3} + x$ | $(x^3 + x)^{1/3}$ |

The solutions are trivial. Carry out the composition to get a little practice. Analyze the domains in II)