CALCULUS
DEGREE IN SOFTWARE ENGINEERING
CHAPTER 18. THE FUNDAMENTAL THEOREM OF CALCULUS

The Fundamental Theorem of Calculus is a key result that will enable us to calculate definite integrals by using primitives, connecting definite and indefinite integrals and relating, in this way, the definite integral to differentiation.
Our goal in this chapter is to present the statement of this theorem and its proof.
As a necessary step towards the proof, we introduce:

THE MEAN VALUE THEOREM FOR DEFINITE INTEGRALS

If f is continuous on $[a, b]$, then at some point c in $[a, b]$,

$$f(c) = \frac{\int_a^b f(x)\,dx}{b-a}$$

The proof makes use of the min-max property of the definite integral, i. e.

$$min\, f\,.(b-a) \leq \int_a^b f(x)\,dx \leq max\, f\,.(b-a)$$

Dividing by $(b-a)$, the definite integral is a number between the minimum and the maximum of the function on $[a, b]$ and according to the Intermediate Value Theorem (see Chapter 7), there is a number c in $[a, b]$, such that $f(c)$ is equal to any number between the minimum and the maximum, in particular, to the integral.
We have proved that a continuous function takes on its average value at a certain point.
Then, we can present the Fundamental Theorem of Calculus

THE FUNDAMENTAL THEOREM OF CALCULUS (PART 1)

If f is continuous on $[a, b]$, then $F(x) = \int_a^x f(t)\,dt$ is continuous on $[a, b]$, differentiable on (a, b) and its derivative is $f(x)$

Proof:
In (a, b) the definition of $F'(x)$ is

$$F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h}$$

and in terms of integrals

$$F'(x) = \lim_{h \to 0} \frac{\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt}{h}$$

By using the additivity property, we can write

$$F'(x) = \lim_{h \to 0} \frac{\int_x^{x+h} f(t)\,dt}{h}$$

The integral divided by h, according to the Mean Value Theorem for Definite Integrals, must be

$$\frac{\int_x^{x+h} f(t)\,dt}{h} = f(c_h)$$

where c_h is a point between x and $x+h$. You can check that the Mean Value Theorem holds for h negative too. Finally, by taking limits in both sides

$$F'(x) = f(x)$$

taking into account that f is continuous. For a and b the derivative is equal to the function in the sense of right and left derivative respectively, and since F is differentiable it is also continuous. That is the end of the proof.
The theorem can be interpreted graphically as saying that the derivative of the area under the curve (for a non-negative function) between a and x is the ordinate at x. So, the derivative of the definite integral- considered as a function of x- is the integrand.

THE FUNDAMENTAL THEOREM OF CALCULUS (PART 2)

If f is continuous at every point in $[a,b]$ and G is any antiderivative of f on $[a,b]$, then

$$\int_a^b f(x)\,dx = G(b) - G(a)$$

Proof:
By definition

$$\int_a^b f(x)\,dx = F(b)$$

with F as defined in the previous demonstration. Besides, $F(a) = 0$ and therefore

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

However, F is a particular primitive of f, but any other primitive G will satisfy

$$F(x) = G(x) + C$$

Substituting in the previous formula

$$\int_a^b f(x)\,dx = G(b) + C - G(a) - C = G(b) - G(a)$$

This formula is very useful indeed, since we only need to know any primitive of the integrand to calculate the definite integral, in particular the area under a curve. For instance, we want to compute the area under the parabola $y = x^2$ with $0 \leq x \leq 1$. This is

$$A = \int_0^1 x^2\,dx$$

and $G(x) = x^3/3$ is a simple primitive of x^2. Then,

$$A = x^3/3\,(1) - x^3/3\,(0) = 1/3$$

The primitive at the upper limit minus the same primitive at the lower limit. This method can be used for any calculation of a definite integral.
The result is given in the units of area that we are employing. For instance, if we measure x and y in meters, the area would be in square meters