CALCULUS
DEGREE IN SOFTWARE ENGINEERING
CHAPTER 24. EXTREME VALUES OF MULTIVARIABLE FUNCTIONS

In this chapter we will discuss how to find the extreme values of a multivariable function, emphasizing the case of functions of two variables for which we will give general criteria and show examples. We start with a brief discussion of topological definitions.

TOPOLOGICAL DEFINITIONS

Informally, Topology (science of place) studies how elements of a set relate spatially to each other. We will begin with the definition of interior point and boundary point in $\mathbb{R}^2$.

A point (x_0, y_0) of a region D in $\mathbb{R}^2$ is called **an interior point of D** if there is a disk centered at (x_0, y_0) that lies entirely in D.

A point (x_0, y_0) of a region D in $\mathbb{R}^2$ is called **a boundary point of D** if every disk centered at (x_0, y_0) contains points that lie outside of D and points that lie in D.

If you draw a region D of the XY-plane bounded by a curve, the points on the curve are boundary points of D, they need not belong to D- which could include the curve or not-. The points of D which are not on the curve are interior points.

A region is **open** if it consists entirely of **interior points**. A region is **closed** if it contains all its **boundary points**. The region D enclosed by a curve will be open if the curve (the boundary of D) is not included in D and closed if it is included. The curve itself is a closed region.

Finally, the region is bounded if it is inside a disk of finite radius.

For $\mathbb{R}^3$, we can write similar definitions replacing disk by ball (open ball, interior of a sphere, closed ball, interior plus spherical surface). Thus, we can write the following definitions:

A point (x_0, y_0, z_0) of a region D in $\mathbb{R}^3$ is called **an interior point of D** if there is a ball centered at (x_0, y_0, z_0) that lies entirely in D.

A point (x_0, y_0, z_0) of a region D in $\mathbb{R}^3$ is called **a boundary point of D** if every ball centered at (x_0, y_0) contains points that lie outside of D and points that lie in D.

If you draw a region D in space bounded by a surface, the points on the surface are boundary points of D, they need not belong to D- which could include the surface or not-. The points of D which are not on the surface are interior points.

A region is **open** if it consists entirely of **interior points**. A region is **closed** if it contains all its **boundary points**. The region D enclosed by a surface will be open if the surface (the boundary of D) is not included in D and closed if it is included. The surface itself is a closed region.

Finally, the region is bounded if it is inside a ball of finite radius.

We will use these defintions when we have to handle maxima and minima of multi-variable functions.

DEFINITION OF LOCAL MAXIMA AND MINIMA. DERIVATIVE TEST

Let $f(x, y)$ be defined on a region D containing the point (x_0, y_0). Then
1. $f(x_0, y_0)$ is a local maximum value of f if $f(x_0, y_0) \geq f(x, y)$ for all domain points in an open disk centered at (x_0, y_0).
2. $f(x_0, y_0)$ is a local minimum value of f if $f(x_0, y_0) \leq f(x, y)$ for all domain points in an open disk centered at (x_0, y_0).

As you can see the definition is the same as the one for single variable functions; we compare the value of f at (x_0, y_0) with that of nearby points. We can figure out the graph of the function as a range of mountains (surface) and the maxima (maximum values) as mountain peaks whereas the minima will be valley low points. The definition would be similar in $\mathbb{R}^3$, replacing disk by ball. It is geometrically clear that at the extremum values, if the tangent place to the graph is defined, it has to be horizontal. This is expressed in the following theorem.

THEOREM. FIRST DERIVATIVE TEST FOR LOCAL EXTREME VALUES

If $f(x, y)$ has a local maximum or minimum values at an interior point (x_0, y_0) of its domain and if the first partial derivatives exist there, then $\dfrac{\partial f}{\partial x}(x_0, y_0) = 0$ and $\dfrac{\partial f}{\partial y}(x_0, y_0) = 0$.

Proof:

We can define, $f(x, y_0) = F(x)$. If f has a local extremum (maximum or minimum) at (x_0, y_0), then $F'(x_0) = 0 = \dfrac{\partial f}{\partial x}(x_0, y_0)$. Besides, we can also define
$f(x_0, y) = G(y)$ and if f has a local extremum at (x_0, y_0), then $G'(y_0) = 0 = \dfrac{\partial f}{\partial y}(x_0, y_0)$.

In consequence, a necessary condition for having a local extremum at an interior point is

$$\vec{\nabla} f(x_0, y_0) = 0$$

An interior point where the gradient is null or where one or both first partial derivatives do not exist is a **critical point**. For finding local maxima and minima at interior points, we must search for critical points.
This can be generalized for functions of n variables: the gradient at the local extremum point must be the null vector. Of course, if the point is interior and all the first partial derivatives exist.

EXAMPLE 1

We consider $f(x,y) = x^2 + y^2$. The function has first partial derivatives on its domain $\mathbb{R}^2$, all the points of the domain are interior points. Then, the local maxima and minima must be at critical points

$$\frac{\partial f}{\partial x} = 2x = 0$$

and

$$\frac{\partial f}{\partial x} = 2y = 0$$

The only point where these conditions hold is $(0,0)$. In this case, the point is the only critical point and the function has a local (also absolute) minimum at that point.
Now, if we consider $f = x^2 + y^2$, restricted to the closed disk $D = \{(x,y) \in \mathbb{R}^2/x^2+y^2 \leq 1\}$, the only local (absolute) maximum value is 1 attained at all points of the circle $x^2 + y^2 = 1$. But, these points are not interior. Thus, boundary points have to be studied apart if they are involved in the search for extreme values. The situation is the same as that of the endpoints of an interval.

EXAMPLE 2

Let $f(x,y) = \sqrt{x^2 + y^2}$. What are the local extreme values of f. All the points of the domain are interior points. We look for the critical points

$$\frac{\partial f}{\partial x} = \frac{x}{\sqrt{x^2 + y^2}} = 0$$

and

$$\frac{\partial f}{\partial x} = \frac{y}{\sqrt{x^2 + y^2}} = 0$$

However, the partial derivatives are not defined at the only critical point $(0,0)$ and the minimum value is 0. The tangent plane is not horizontal at that point, simply because it does not exist.
An interior local extremum point must be a critical point, but a critical point is not always an extremum point. For instance, $f(x,y) = x^2 - y^2$ has a critical point at $(0,0)$, but $f(0,y) = -y^2$ and $f(x,0) = x^2$. This means that around the origin, there are points where the function is positive and points where it is negative and this happens for every disk centered at $(0,0)$. Therefore, the point is not a local

extremum point. The point $(x_0, y_0, f(x_0, y_0))$ where (x_0, y_0) is a critical point and the function does not attain a local extremum value is called a saddle point of the surface (generalization of inflection point).

DEFINITION OF SADDLE POINT

A differentiable function $f(x, y)$ has a **saddle point** at a critical point (x_0, y_0) if in every open disk centered at (x_0, y_0) there are domain points (x, y) where $f(x, y) > f(x_0, y_0)$ and domain points (x, y) where $f(x, y) < f(x_0, y_0)$. The corresponding point $(x_0, y_0, f(x_0, y_0))$ on the surface $z = f(x, y)$ is called a saddle point of the surface.

SECOND DERIVATIVE TEST FOR LOCAL EXTREME VALUES

It is clear that only with the first derivative we cannot classify the critical points as extremum points or saddle points. As was the case with functions of a single variable, we have to use the second-order derivatives. To justify this method we introduce Taylor's Formula for functions of two variables; we write it till order two, approximating the function by a polynomial in x and y of degree one

TAYLOR'S THEOREM

If $f(x, y)$ and its second-order partial derivatives are continuous throughout an open disk D centered at (x_0, y_0). Then, throughout D

$$f(x_0 + h, y_0 + k) = f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0)\, h + \frac{\partial f}{\partial y}(x_0, y_0)\, k +$$

$$1/2\left(\frac{\partial^2 f}{\partial x^2}(x_0 + ch, y_0 + ck)\, h^2 + 2\frac{\partial^2 f}{\partial x \partial y}(x_0 + ch, y_0 + ck)\, hk + \frac{\partial^2 f}{\partial y^2}(x_0 + ch, y_0 + ck)\, k^2\right)$$

where $(x_0 + ch, y_0 + ck)$ is a point on the line segment joining (x_0, y_0) and $(x_0 + h, y_0 + k)$, that is, $0 < c < 1$.

Of course, this formula can be used to approximate the function in a neighbourhood of the point by a linear expression and set a bound to the error expressed in terms of second-order partial derivatives. In our case, we are specially interested in classifying the critical points. Let us write the formula assuming that (x_0, y_0) is a critical point

$$f(x_0 + h, y_0 + k) = f(x_0, y_0) + 1/2\left(\frac{\partial^2 f}{\partial x^2}(x_0 + ch, y_0 + ck)\, h^2 + 2\frac{\partial^2 f}{\partial x \partial y}(x_0 + ch, y_0 + ck)\, hk + \frac{\partial^2 f}{\partial y^2}(x_0 + ch, y_0 + ck)\, k^2\right)$$

This formula can be written in terms of matrices

$$f(x_0+h, y_0+k) = f(x_0, y_0) + 1/2\Big(v\, Hf(x_0+ch, y_0+ck)\, v^t\Big)$$

Here, $v = (h\ k)$, v^t its transpose and Hf is the so-called Hessian matrix calculated at $(x_0 + ch, y_0 + ck)$.

$$Hf = \begin{bmatrix} \dfrac{\partial^2 f}{\partial x^2} & \dfrac{\partial^2 f}{\partial x \partial y} \\ \\ \dfrac{\partial^2 f}{\partial x \partial y} & \dfrac{\partial^2 f}{\partial y^2} \end{bmatrix}$$

It is clear that the sign of $f(x_0 + h, y_0 + k) - f(x_0, y_0)$ is the sign of $v\, Hf(x_0 + ch, y_0 + ck)\, v^t$ and since the second-order derivatives are continuous, it is determined by the sign of $v\, Hf(x_0, y_0)\, v^t$. If we consider points as close as we like to (x_0, y_0), this sign just depends on the matrix $Hf(x_0, y_0)$. If all the eigenvalues (two) of the matrix are positive, the matrix is positive definite and the sign of $v\, Hf(x_0, y_0)\, v^t$ is always positive, whereby the function f has a local minimum at the critical point (x_0, y_0). If the two eigenvalues are negative, the matrix is negative definite and $v\, Hf(x_0, y_0)\, v^t$ is negative, whereby f has a local maximum at the critical point. If the eigenvalues have different sign, the matrix is indefinite and the sign of $v\, Hf(x_0, y_0)\, v^t$ could be positive or negative, then the critical point is a saddle point. Finally, if the determinant of the Hessian matrix is zero, we cannot say anything about the nature of the point. The criterion based on the eigenvalues can be simplified and is equivalent to the following:

THEOREM: SECOND DERIVATIVE TEST FOR LOCAL EXTREME VALUES

Suppose that $f(x, y)$ and its first and second-order partial derivatives are continuous thoughout a disk centered at (x_0, y_0) and (x_0, y_0) is a critical point. Then

1. If $\dfrac{\partial^2 f}{\partial x^2}(x_0, y_0)$ is positive and the determinant of the Hessian is positive as well, then f has a local minimum at (x_0, y_0).

2. If $\dfrac{\partial^2 f}{\partial x^2}(x_0, y_0)$ is negative and the determinant of the Hessian, $|Hf(x_0, y_0)|$, is positive, then f has a local maximum at $(x_0, , y_0)$.

3. If the determinant of the Hessian is negative, then f has a saddle point at (x_0, y_0).

4. If the determinant of the Hessian is zero, the test is inconclusive and we must find another way to determine the behaviour of f at the critical point.

EXAMPLES

For $f(x,y) = x^2 + y^2$, the only critical point is $(0,0)$. At this point the Hessian matrix is

$$Hf = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

and $\dfrac{\partial^2 f}{\partial x^2}(0,0) = 2, \quad |Hf(0,0)| = 4$. Therefore, f has a local minimum at $(0,0)$. This minimum is absolute.
For $f(x,y) = -(x^2 + y^2)$, the only critical point is $(0,0)$. At this point the Hessian matrix is

$$Hf = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$$

and $\dfrac{\partial^2 f}{\partial x^2}(0,0) = -2, \quad |Hf(0,0)| = 4$. Therefore, f has a local maximum at $(0,0)$. This maximum is absolute.
For $f(x,y) = x^2 - y^2$, the only critical point is $(0,0)$. At this point the Hessian matrix is

$$Hf = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$$

and $\dfrac{\partial^2 f}{\partial x^2}(0,0) = 2, \quad |Hf(0,0)| = -4$. Therefore, f has a saddle point at $(0,0)$.
Finally, for $f(x,y) = x^3 + y^3$ the only critical point is $(0,0)$. At this point the Hessian matrix is

$$Hf = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

the test is inconclusive, but if we take $f(x,0) = x^3$, we can see that the function is both greater than zero and less than zero in any neighbourhood of the origen. The origen is a saddle point.

If we consider functions of n variables, the local extrema must be at critical points and we can use the Hessian matrix to classify these points. The comments made about eigenvalues and the definiteness of the Hessian matrix are also valid for functions of n variables and we can develop classification criteria based on the determinants of the square submatrices along the diagonal.

ABSOLUTE EXTREMA

DEFINITIONS

We say that a function $f(x, y)$ has an absolute maximum at a point (x_0, y_0) of its domain if $f(x_0, y_0) \geq f(x, y)$ for all (x, y) in D.

We say that a function $f(x, y)$ has an absolute minimum at a point (x_0, y_0) of its domain if $f(x_0, y_0) \leq f(x, y)$ for all (x, y) in D.

An absolute extremum is an absolute maximum or an absolute minimum. The value of an absolute extremum is unique, though it can be attained at different points.

Does a function always have absolute extrema? The answer is no, and the existence will depend on the domain where we consider that function.

EXAMPLE

The simple function $f(x, y) = x^2 + y^2$ has an absolute minimum, zero, and no absolute maximum on its natural domain $\mathbb{R}^2$. But if we restrict the domain to the closed disk $x^2 + y^2 \leq 1$, the absolute maximum is 1, attained at every point of the circle $x^2 + y^2 = 1$. Besides, if the domain is the open disk $x^2 + y^2 < 1$, there is no absolute maximum. It seems that the existence of absolute extrema depends, as was the case for single variable functions, of whether the domain is open or closed. The general sufficient condition is given in the following theorem

THEOREM. THE EXTREME VALUE THEOREM FOR FUNCTIONS OF TWO VARIABLES

If a function $z = f(x, y)$ is continuous on a closed and bounded region of $\mathbb{R}^2$, then it attains its absolute maximum and its absolute minimum on that region.

The theorem can be extended, with a similar statement, to functions of three or more variables. How do we search for the absolute extrema if the premises of the theorem hold?

1. We search for the critical points, that is, the interior points where the partial derivatives are zero or at least, one of them does not exist. At these points there can be local maxima or minima (for having an absolute extremum at a point, the function must have a local extremum at that point).

2. We analyze the behaviour of the boundary points, trying to find boundary points where the function attains local maxima or minima.

3. We list the points selected in 1 and 2 and compare the values of the function at those points, the maximum and minimum values of f in the list will be the absolute maximum and the absolute minimum of the function.

EXAMPLE

Find the absolute extrema of $f(x, y) = x^4 + y^4$ on $x^2 + y^2 \leq 1$.
The domain is closed and bounded, thus we can be sure there is an absolute maximum and an absolute minimum.
1. For the interior points both partial derivatives must be zero.

$$\frac{\partial f}{\partial x} = 4x^3 = 0$$

$$\frac{\partial f}{\partial x} = 4y^3 = 0$$

Then, $P_1 = (0, 0)$ is the only critical point.

2. We consider the boundary of the closed disk, $x^2 + y^2 = 1$. In this case, we substitute $y^2 = 1 - x^2$ in the function, obtaining

$$g(x) = x^4 + (1 - x^2)^2 = 2x^4 - 2x^2 + 1$$

defined on $[-1, 1]$. We find the possible extrema of this single variable function

$$g'(x) = 8x^3 - 4x = 0$$

The critical points of $g(x)$ are $x = 0$, $x = -1/\sqrt{2}$ and $x = 1/\sqrt{2}$. Besides, we have to consider the endpoints $x = -1$, $x = 1$. Finally, we have the following candidates among the boundary points

$P_2 = (0, 1)$, $P_3 = (0, -1)$, $P_4 = (1/\sqrt{2}, 1/\sqrt{2})$, $P_5 = (1/\sqrt{2}, -1/\sqrt{2})$,
$P_6 = (-1/\sqrt{2}, 1/\sqrt{2})$, $P_7 = (-1/\sqrt{2}, -1/\sqrt{2})$, $P_8 = (1, 0)$, $P_9 = (-1, 0)$.

3. We compare the values of f at the points of this long list.
We find that the maximum value is 1 reached at $(1, 0)$, $(-1, 0)$, $(0, 1)$ and $(0, -1)$ and the minimum value is 0, attained at $(0, 0)$. The other points are the points where the function has an absolute minimum on $x^2 + y^2 = 1$, but its value, $1/2$, is greater than $f(0, 0) = 0$.

As we have seen, the procedure can be long and tedious but it is quite mechanical. However, in some cases it is not easy to write the function $f(x, y)$ on the boundary and another method, Lagrange multipliers, has to be applied. For now, we will not deal with this this method.