CALCULUS
DEGREE IN SOFTWARE ENGINEERING
CHAPTER 5. LIMITS AT INFINITY. INFINITE LIMITS.

LIMITS AT INFINITY

In this section we study the behaviour of a function when the independent variable becomes increasingly large. The independent variable can be as large as we please, since the set of real numbers is not bounded. What happens to a given function if x approaches ∞, infinity, that is, if it is as large as we want or $-\infty$, minus infinity, that is, it is as large as we want in absolute value, but its sign is negative ? First, we will show a simple example and then, we will give a precise definition of the limits at ∞ and $-\infty$.

EXAMPLE

What is the limit of $f(x) = \dfrac{x^2}{x^2+1}$ as x approaches ∞ or $-\infty$? We could calculate the value of the function for larger and larger values of x and check that it approaches 1, regardless of the sign. We could also divide numerator and denominator by x^2 and see that the term $1/x^2$ tends to zero, as x approaches ∞, making the limit 1. The same happens for $-\infty$. Moreover, the function is even, $f(-x) = f(x)$, and the behaviour is independent of the sign. We can write

$$\lim_{x \to \infty} \frac{x^2}{x^2+1} = 1$$

and

$$\lim_{x \to -\infty} \frac{x^2}{x^2+1} = 1$$

you can check, by plotting the graph, that the function approaches 1 at $-\infty$, being always less than 1, decreases, reaching the minimum value 0 at $x = 0$ and then grows, approaching 1 again at ∞. The graph is symmetric with respect to the y-axis. In this case, the function has a finite limit at ∞ and $-\infty$. Now, we write the precise definition.

DEFINITION OF LIMIT AT INFINITY

We say that $f(x)$ has the limit L as x approaches infinity and write $\lim\limits_{x \to \infty} f(x) = L$, if, for every number $\epsilon > 0$, there exists a corresponding number M such that for all x

if

$$x > M$$

then

$$|f(x) - L| < \epsilon$$

The idea is very simple, as x grows without bound, $f(x)$ approaches a given finite number L. The definition for the limit at $-\infty$ is similar. In this case, we say that $f(x)$ has the limit L as x approaches minus infinity and write $\lim_{x \to -\infty} f(x) = L$, if, for every number $\epsilon > 0$, there exists a corresponding number N such that for all x

if

$$x < N$$

then

$$|f(x) - L| < \epsilon$$

You can prove that

$$\lim_{x \to \infty} 1/x = 0$$

, by choosing any $\epsilon > 0$ and the corresponding $M = 1/\epsilon$. For all x, if $x > 1/\epsilon = M$ then $1/x < \epsilon$. It is also true that

$$\lim_{x \to -\infty} 1/x = 0$$

Try to choose the suitable N. The limit laws can be proved and used in the case of limits at ∞ and $-\infty$. We will see some examples in the exercises.
In our example, the graph approached the horizontal line $y = 1$, without touching it, we say that $y = 1$ is a horizontal asymptote of $f(x) = \dfrac{x^2}{x^2 + 1}$ both at ∞ and $-\infty$. The precise definition of horizontal aymptote is the following:

DEFINITION OF HORIZONTAL ASYMPTOTE

A line $y = b$ is a horizontal asymptote of the graph of the function $y = f(x)$ if either $\lim_{x \to \infty} f(x) = b$ or $\lim_{x \to -\infty} f(x) = b$. In the first case the asymptote is at ∞, in the second at $-\infty$.

The word asymptote comes from ancient Greek, meaning " not falling together", that is, the asymptote and the graph do not meet. Though this is true in many typical examples, it is not always true. For instance

$$\lim_{x \to \infty} \sin x/x = 0$$

You can check this by observing that $\sin x$ is bounded and if we divide it by numbers larger and larger, the quotient is as small as we want. Therefore $y = 0$ is a horizontal asympote of this function at ∞. However, the function intersects the line $y = 0$ at infinitely many points on the real line. Graph and function do not fall together, but they often meet. Be careful, remember that $\lim_{x \to 0} \sin x/x = 1$, not 0. We finish this section by stating a result that we have just applied and will be very useful in applications

LIMIT OF THE PRODUCT OF A BOUNDED FUNCTION AND A FUCTION WITH ZERO LIMIT

If $f(x)$ is bounded on $(x_0 - d, x_0 + d)$ and $\lim_{x \to x_0} g(x) = 0$, then $\lim_{x \to x_0} f(x)g(x) = 0$.
It is a simple consequence of the definition of limit. Bounded means that there exists a number M such that $|f(x)| < M$ on the given interval. That is, the absolute value of the function cannot grow without bound.
Before we start the study of infinite limits, I will make a short comment on the origin of the ∞ symbol.
John Wallis, a famous English mathematician, is credited with introducing the infinity symbol with its mathematical meaning in 1655 in his book "De sectionibus conicis". Wallis did not explain his choice of this symbol, but it has been conjectured to be a variant form of a Roman numeral for 1,000 (see Wikipedia in English about the infinity symbol), or a variant of the Greek letter ω (omega), the last letter in the Greek alphabet.
Till now, the limit has been a finite number. However, there are cases in which a function grows without bound as x approaches a given number. For instance, what is the limit as x approaches 0 of $1/x^2$? It is clear that the function does not approach a finite number, but grows without bound. In the sense of our previous definition, the limit does not exist, but, in a certain sense, we could say that the limit is ∞, that the function has an infinite limit at $x = 0$. Let us make this idea more precise

DEFINITION OF INFINITE LIMIT

We say that $f(x)$ approaches infinity as x approaches x_0 and write $\lim_{x \to x_0} f(x) = \infty$ if, for every positive real number B, there exists a corresponding $\delta > 0$ such that for all x, if

$$0 < |x - x_0| < \delta$$

then

$$f(x) > B$$

The definition for $-\infty$ is similar:
We say that $f(x)$ approaches minus infinity as x approaches x_0 and write $\lim_{x \to x_0} f(x) = -\infty$ if, for every negative real number $-B$, there exists a corresponding $\delta > 0$ such that for all x, if

$$0 < |x - x_0| < \delta$$

then

$$f(x) < -B$$

EXAMPLE

Let us consider the following limit

$$\lim_{x \to -2} \frac{x+3}{x+2}$$

Here, the limit will be different depending on from which side we approach $x = -2$. First, we calculate

$$\lim_{x \to -2^+} \frac{x+3}{x+2}$$

If we approach the point from the right, the numerator is positive, tending to 1 , and the denominator approaches 0 with positive sign, we can write 0^+. Therefore, the limit is ∞. If we calculate now

$$\lim_{x \to -2^-} \frac{x+3}{x+2}$$

the denominator tends to zero with negative sign, 0^-. Then, the limit is $-\infty$. The right-hand and left-hand limits are not the same. If we plot the graph of this function, we see a vertical asymptote, that is, the graph is as close as we please to the line $x = -2$, as x tends to -2 and $f(x)$ grows in absolute value, with positive sign on the right and negative sign on the left. The precise definition of vertical asymptote is written below

DEFINITION OF VERTICAL ASYMPTOTE

A line $x = a$ is a vertical asymptote of the graph of $y = f(x)$ if either

$$\lim_{x \to a^+} f(x) =^+_- \infty$$

or

$$\lim_{x \to a^-} f(x) =^+_- \infty$$

Vertical asymptotes are typical in rational functions, at points where the denominator is zero. We will see more examples in the chapter dedicated to graphs.
When we study infinite limits and limits at infinity, we can use the limit laws: adding, subtracting, multiplying, dividing,..... limits. However, there are cases in which we obtain expressions such as:

$$\infty - \infty$$

$$\infty . 0$$

$$\frac{\infty}{\infty}, \frac{0}{0}$$

These expressions are called indeterminate forms, because the limit is not known beforehand, that is, it will depend on the functions involved. We have other indeterminate forms, such as 0^0, 1^∞, that involve raising a function to another function. Most of the indeterminate forms will be dealt with by using L'Hopital's Rule in future chapters.