CALCULUS
DEGREE IN SOFTWARE ENGINEERING
CHAPTER 21. MULTIVARIABLE FUNCTIONS. LIMITS AND CONTINUITY

In this chapter we will extend the concepts of limit and continuity to multivariable functions. We will start with the case of a two-variable function.

LIMIT OF A TWO-VARIABLE REAL FUNCTION

Given a real function of two variables

$$\begin{aligned} f : D \subseteq \mathbb{R}^2 &\longrightarrow \mathbb{R} \\ (x, y) &\longrightarrow z = f(x, y) \end{aligned}$$

We say that the limit of the function $f(x, y)$ as (x, y) approaches a limit point of D ,(x_0, y_0), is a real number, L, if the values of $f(x, y)$ lie arbitrarily close to L for all points (x, y) sufficiently close to (x_0, y_0). We write

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L$$

Now, we are going to express this sentence in a more precise algebraic way:

DEFINITION:

If, for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all (x, y) in the domain of f,

$$|f(x, y) - L| < \epsilon$$

whenever

$$0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta,$$

we say that the limit of $f(x, y)$ at (x_0, y_0) is L.
That is, we have used the idea of limit taking into account the definition of Euclid distance in $\mathbb{R}^2$. Hereunder,we will list the basic properties of limits.

THE LIMIT LAWS

1) The limit is unique
If $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = L$ and $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = M$, then $L = M$

If $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = L$ and $\lim\limits_{(x,y)\to(x_0,y_0)} g(x,y) = M$

2) Sum Rule

$$\lim_{(x,y)\to(x_0,y_0)} (f(x,y) + g(x,y)) = L + M$$

3) Difference Rule

$$\lim_{(x,y)\to(x_0,y_0)} (f(x,y) - g(x,y)) = L - M$$

4) Constant Multiple Rule

$$\lim_{(x,y)\to(x_0,y_0)} k.f(x,y) = k.L$$

5) Product Rule

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y).g(x,y) = L.M$$

6) Quotient Rule

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y)/g(x,y) = L/M\,,\, M \neq 0$$

7) Power Rule

$$\lim_{(x,y)\to(x_0,y_0)} (f(x,y))^n = L^n$$

with n a positive integer

8) Root Rule

$$\lim_{(x,y)\to(x_0,y_0)} (f(x,y))^{1/n} = L^{1/n}$$

with n a positive integer. If n is even, we must assume that $L > 0$.
All these rules are akin to those presented in Chapter 4 and the proofs are quite similar, since we use the basic properties of the absolute value. Next, we will show a few examples of limits:

EXAMPLES

$$\lim_{(x,y)\to(0,1)} \frac{x - xy + 3}{x^2y + 5xy - y^3}$$

We apply the limit laws- sum, difference, product and quotient-

$$\lim_{(x,y)\to(0,1)} \frac{x - xy + 3}{x^2y + 5xy - y^3} = -3/1 = -3$$

the value of the function at the point; this will happen for all polynomial and rational functions with non-zero denominator.

$$\lim_{(x,y)\to(2,2)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}} = 0/0$$

This is an indeterminate form, thus, we multiply both numerator and denominator by the conjugate of the denominator. The new expression is

$$\lim_{(x,y)\to(2,2)} \frac{x(x-y)(\sqrt{x} + \sqrt{y})}{x - y} = 4\sqrt{2}$$

where we have canceled out $(x - y)$

$$\lim_{(x,y)\to(1,1)} \frac{x^2 - y^2}{x - y} = \lim_{(x,y)\to(1,1)} \frac{(x-y)(x+y)}{x - y} = 2$$

CONTINUITY

DEFINITION

A function $f(x, y)$ is continuous at a point (x_0, y_0) if

1. f is defined at (x_0, y_0)
2. $\lim_{(x,y)\to(x_0,y_0)} f(x, y)$ exists
3. $\lim_{(x,y)\to(x_0,y_0)} f(x, y) = f(x_0, y_0)$

i.e., the function is defined at (x_0, y_0) and the limit at that point exists and is equal to the value of the function.

PROPERTIES OF CONTINUITY

The properties of a continuous function at a point can be derived from the limit laws and read

If $f(x, y)$ and $g(x, y)$ are continuous at (x_0, y_0), then

1. $f+g$ is continuous at (x_0, y_0)
2. $f-g$ is continuous at (x_0, y_0)
3. kf is continuous at (x_0, y_0)
4. $f.g$ is continuous at (x_0, y_0)
5. f/g is continuous at (x_0, y_0) provided $f(x_0, y_0) \neq 0$
6. f^n is continuous at (x_0, y_0)
7. $f^{1/n}$ is continuous at (x_0, y_0) provided it is well defined at the point.

Finally, we can compose functions of two variables

$$\begin{aligned} f : \mathbb{R}^2 &\longrightarrow \mathbb{R} \\ (x, y) &\longrightarrow z = f(x, y) \end{aligned}$$

$$\begin{aligned} g : \mathbb{R} &\longrightarrow \mathbb{R} \\ z &\longrightarrow u = g(z) \end{aligned}$$

We can define the composite function $h(x, y) = (g \circ f)(x, y) = g(f(x, y))$ and h is continuous at (x_0, y_0) whenever f is continuous at (x_0, y_0) and g is continuous at $f(x_0, y_0)$. This is called **THE CONTINUITY OF THE COMPOSITE FUNCTION**

We show some examples of these properties below

1)

$$\ln\left(1 + x^2 + y^2\right)$$

This is the composition of the natural logarithm, a continuous function on its domain ,$(0, \infty)$, and $1+x^2+y^2$, continuous on $\mathbb{R}^2$ and always positive. Therefore, the function is continuous on $\mathbb{R}^2$. We simply say that the function is continuous, meaning that the function is continuous at all points of its domain. In order to find a limit at a point, we just calculate the function at that point.

2)

$$\cos \frac{xy}{x+1}$$

Its domain is

$$\mathbb{R}^2 \setminus \{(-1, y)\}$$

and the function is continuous on its domain. However, it is not continuous on $\mathbb{R}^2$. Given these properties, all elementary functions- polynomial, rational, exponential, logarithmic, trigonometric and inverse trigonometric,....-will be continuous. This means that the calculation of their limits amounts to the calculation of the function. Nevertheless, we can find more complicated situations.

For instance, what is the value of the following limit

$$\lim_{(x,y)\to(0,0)} \frac{x+y}{x-y}$$

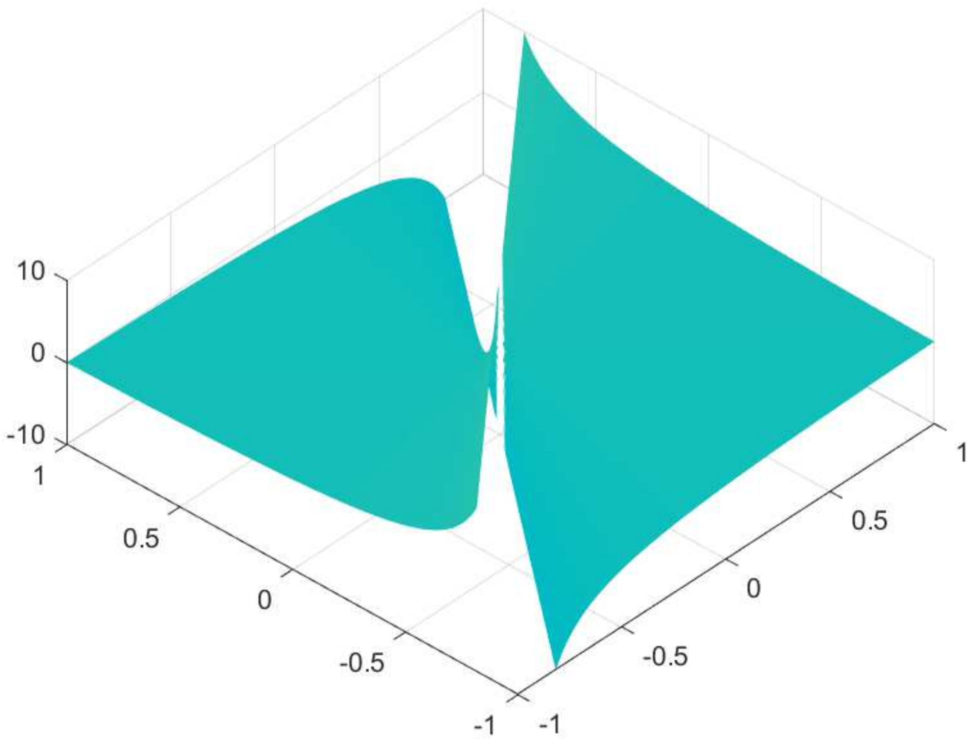

Figure 1: $\dfrac{x+y}{x-y}$

The question is legitimate since $(0,0)$ is a limit point of the domain of this function- we can consider points in the domain as close to $(0,0)$ as we like-. However, the function is not defined at $(0,0)$.
For calculating the limit, also called double limit, we consider the limit along all the lines passing through the origin, except $y = x$ which is not in the domain. If $y = mx$

$$f(x, mx) = \frac{x + mx}{x - mx} = \frac{1+m}{1-m}$$

The directional limit- limit along a line- is this quotient and depends on the slope of the line. Thus, as we approach $(0,0)$, the function approaches different values, what contradicts the uniqueness of the limit. In conclusion, the double limit does not exist and the function cannot be made continuous at $(0,0)$. We plot the function in Figure 1. Notice that it is not defined at $y = x$.
Many exercises will be worked out in Exercises 10.