Practice 5. Multivariable functions: Graphing. Partial derivatives.

1. Symbolic expressions involving several variables

Symbolic expressions that depend on more than one variable are constructed similarly to those of a single variable, therefore we must define — with `syms` — each of the variables involved in the expression.

To define the function $f(x,y) = xe^{-x^2-y^2}$ we must write:

```
>> syms x y
>> f=x*exp(-x^2-y^2)
```

To calculate the function at a given point, we write the command `subs`, as in previous practices:

`subs(S,x,a)`	In the symbolic expression `S`, substitutes the variable `a` for `x`.
`subs(S,{x,y},{a,b})`	In the symbolic expression `S`, substitutes the variables `a,b` for `x,y`.

To calculate the function $f(x,y) = xe^{-x^2-y^2}$ at the point $(1,2)$:

```
>> subs(f,{x,y},{1,2})
```

2. 3D graphs

To plot points and curves in 3D we carry out a command similar to `plot`, `plot3`:

`plot3(x,y,z)`	Draws the point with coordinates (x,y,z). If x, y, z are vectors of the same size, this command plots the lines joining the given points.

For example:

```
>> plot3(2,3,1)                    % plots the point (2,3,1)
>> plot3(2,3,1,'*r')               % plots the point (2,3,1) with a red asterisk
>> plot3([2,3,4],[3,0,-1],[1,-2,5])       % lines joining (2,3,1),(3,0,-2) y (4,-1,5)
>> plot3([2,3,4],[3,0,-1],[1,-2,5],'g*')  % draws (2,3,1),(3,0,-2) and (4,-1,5) with
green* at the points.
>> plot3([2,3,4],[3,0,-1],[1,-2,5],'g*-') % also draws the lines
```

Each time a command is used to draw: `ezplot`, `plot`, `plot3`,... Matlab creates and activates a graphic window to which the name Figure n is assigned.

Sometimes it is interesting to draw two or more functions on the same window or to open several graphic windows:

`hold on`	Activates the `hold` command, and from that moment on all the new graphs are added to the last open window
`hold off`	Deactivates the `hold` command.
`figure(n)`	Selects the graphic window `Figure n` as an active window; if it does not exist, it is created.
`close all`	Closes all the graphic windows.
`subplot(m,n,p)`	Divides the graphic window in a table of $m \times n$ subwindows and places the graph in the p-th from left to right and from top to bottom.

2.1. Example

Plot, with the command `plot3`, the circular helix defined by the equations:

$$\begin{cases} x = \cos t \\ y = \sin t \qquad\qquad t \in [0, 10\pi]. \\ z = t \end{cases}$$

Solution:

```
>> t=linspace(0,10*pi,1000);
>> x=cos(t);
>> y=sin(t);
>> z=t;
>> plot3(x,y,z)
```

Repeat the exercise using only 20 points with the command `linspace` and look at the plot obtained.

2.2. A surface step by step. The `meshgrid` command

Our goal is to plot a function that depends on two variables, for example $f(x,y) = x + y$, whose domain is the whole $\mathbb{R}^2$. Given that the computer has a limited processing capacity, we have to decide on which part of the domain we are going to plot the function. In our case, we will graph the function on the rectangle of $\mathbb{R}^2$:

$$D = [-1, 2] \times [2, 6] = \{(x,y) \in \mathbb{R}^2 / -1 \leq x \leq 2,\ 2 \leq y \leq 6\}$$

To be able to represent the part of the graph of f on D

$$graph(f) = \{(x,y,z) \in \mathbb{R}^3 / (x,y) \in D,\ z = f(x,y)\}$$

we must discretize the domain D, i.e., choose a finite set of points on which we work. We perform it in the following way: From two vectors `x` and `y`, the grid defined by two matrices X e Y is generated with the command `[X Y]=meshgrid(x,y)` .

Once we have constructed the grid, we calculate the values that the function of two variables f takes on that grid with $Z = f(X,Y)$. Then, we graph the function with the command `surf(X,Y,Z)` . For example,

```
>> x=[-1 0 1 2] , y=[2 4 6]
x =
    -1     0     1     2
y =
     2     4     6
```

```
>> [X,Y]=meshgrid(x,y)
X =
    -1     0     1     2
    -1     0     1     2
    -1     0     1     2
Y =
     2     2     2     2
     4     4     4     4
     6     6     6     6
>> Z=X+Y
Z =
     1     2     3     4
     3     4     5     6
     5     6     7     8
>> surf(X,Y,Z)  % draws the plane z=x+y
```

To understand what the `meshgrid` command does, we draw the grid points with the `plot` command:

```
>> plot(X,Y,'.r')
```

We can obtain different presentations of the surface with the instructions:

`surf(X,Y,Z), shading interp`	Draws the surface $z = f(x, y)$ with a smooth transition of colours.
`surfc(X,Y,Z)`	The same as `surf`, adding the level curves of the surface projected on the plane xy.
`mesh(X,Y,Z)`	Draws the surface $z = f(x, y)$ with coloured lines and white surface.
`meshc(X,Y,Z)`	The same as `mesh`, adding the level curves of the surface projected on the plane xy.
`contour(X,Y,Z)`	Draws the level curves of the surface. The number of curves is selected automatically.
`contour(X,Y,Z,n)`	As `contour`, but drawing n curves.
`contour3(X,Y,Z)`	Draws the level curves on the surface. The number of curves is selected automatically.
`contour3(X,Y,Z,n)`	As `contour3`, but drawing n curves.

Observe the plots obtained for the above plane with the following commands:

```
>> figure(2), surfc(X,Y,Z), shading interp
>> figure(3), mesh(X,Y,Z)
>> figure(4), meshc(X,Y,Z)
>> figure(5), contour(X,Y,Z)
>> figure(6), contour3(X,Y,Z)
```

2.3. Example

Plot the paraboloid of equation $z = x^2 + y^2$, with $-2 \leq x \leq 2$, $-3 \leq y \leq 3$.

In the other figures, draw 100 level curves on the plane and plot 100 level curves on the surface . Draw different plots carrying out the above commands.

Solution:

```
>> clear                                          % clears all the variables
>> close all                                      % clears all the graphic windows
>> x=-2:0.1:2; y=-3:0.1:3; [x,y]=meshgrid(x,y);
>> z=x.^2+y.^2;                            % we operate with the matrices element by element
>> surf(x,y,z), shading interp
>> figure(2), contour(x,y,z,100)     %  draws 100 level curves on the plane
>> figure(3), contour3(x,y,z,100)    %  draws 100 level curves on the surface
```

2.4. Exercise

Plot the functions

$$f(x,y) = \frac{xy}{x^2+y^2} \quad with \quad -2 \le x \le 2, \quad -2 \le y \le 2.$$

$$g(x,y) = \frac{x^2 y}{x^4+y^2} \quad with \quad -4 \le x \le 4, \quad -4 \le y \le 4.$$

$$h(x,y) = y\frac{x^2-y^2}{x^2+y^2} \quad with \quad -4 \le x \le 4, \quad -4 \le y \le 4.$$

Rotate the images to observe the behaviour of the functions at the point $(x,y) = (0,0)$.

2.5. Example

Plot the following functions on the given domain. Use `subplot(2,3,p)` to draw each function and below, 20 level curves on the plane.

(a) $z = x\,y,$ in $[-1,1] \times [-1,1]$.

(b) $z = x^2 - y^2,$ in $[-1,1] \times [-1,1]$.

(c) $z = \sqrt{x^2+y^2},$ in $[-1,1] \times [-1,1]$.

Solution:

```
>> x=linspace(-1,1,30); y=x; [x,y]=meshgrid(x,y);
>> z=x.*y; subplot(2,3,1), surf(x,y,z), shading interp
>> subplot(2,3,4), contour(x,y,z,20)
>> z=x.^2-y.^2; subplot(2,3,2), surf(x,y,z), shading interp
>> subplot(2,3,5), contour(x,y,z,20)
>> z=sqrt(x.^2+y.^2); subplot(2,3,3), surf(x,y,z), shading interp
>> subplot(2,3,6), contour(x,y,z,20)
```

2.6. Example

Plot the graph and the level curves of the function $z = \ x \cdot e^{-(x^2+y^2)}$ on the region $[-2,2] \times [-2,2]$.

Plot in another figure: on the left, the surfaces, in the center, the level curves on the surface, and, on the right, the level curves.

Solution:

```
>> clear
>> close all
>> x=-2:0.1:2; y=x; [x,y]=meshgrid(x,y);
>> z=x.*exp(-(x.^2+y.^2));
>> surfc(x,y,z), shading interp
>> figure(2), subplot(1,3,1), surf(x,y,z), shading interp
>> subplot(1,3,2), contour3(x,y,z)
>> subplot(1,3,3), contour(x,y,z)
```

3. Partial derivatives

To calculate the partial derivatives of a multivariable function f defined by a symbolic expression, MATLAB has the command `diff`:

`diff(f,x)`	Calculates the partial derivative of f with respect to the variable x.
`diff(f,x,n)`	Calculates the n-th partial derivative of f with respect to the variable x.

We are going to calculate the partial derivatives of the function $f(x,y) = x^2 \cos y + xy^2$:

```
>> syms x y
>> f=x^2*cos(y)+x*y^2;
>> diff(f,x)
>> diff(f,y)
```

3.1. Example

Let $f(x,y) = \tan^{-1} \dfrac{y}{x}$. Calculate the value of the expression

$$E(x,y) = \frac{\partial^2 f(x,y)}{\partial x^2} + \frac{\partial^2 f(x,y)}{\partial y^2}\,.$$

We find the value of the second partial derivatives of f and we add them:

```
>> syms x y
>> f=atan(y/x);
>> E=diff(f,x,2)+diff(f,y,2)
E =
2*y/x^3/(1+y^2/x^2)-2*y^3/x^5/(1+y^2/x^2)^2-2/x^3/(1+y^2/x^2)^2*y
>> simplify(E)
```

3.2. Example

Let $f(x,y) = x^2 + \sin xy$. Calculate $\dfrac{\partial^3 f(x,y)}{\partial x^2 \partial y}$.

We calculate the third partial derivative:

```
>> syms x y
>> f=x^2+sin(x*y);
>> diff(diff(f,y),x,2)
ans =
-cos(x*y)*y^2*x-2*sin(x*y)*y
```

3.3. Gradient

We can obtain the gradient vector $\nabla f(x,y)$ of the above function at the point (x,y):

```
>> grad=[diff(f,x),diff(f,y)]
```

If we want to calculate the gradient of f at the point $(1,-2)$:

```
>> subs(grad,{x,y},{1,-2})
```

4. Directional derivatives

If f is differentiable at the point (x_0,y_0), the directional derivative of f at the point (x_0,y_0) in the direction of a unit vector $\vec{u}$ is defined by

$$f'_{\vec{u}}(x_0,y_0) = df(x_0,y_0)(\vec{u}) = \nabla f(x_0,y_0)\cdot\vec{u}.$$

If the vector is not a unit vector, we must *normalize* it; for this we can use the `norm` command that calculates the modulus of a vector.

4.1. Example

Calculate the directional derivative of the function

$$f(x,y) = \sqrt{x^2+2y^2}$$

at the point $(-1,2)$ in the direction of the vector $\vec{v} = (-2,3)$.

First, we calculate the gradient of f at the point $(-1,2)$:

```
>> syms x y
>> f=sqrt(x^2+2*y^2);
>> grad=[diff(f,x),diff(f,y)]
grad =
[ 1/(x^2+2*y^2)^(1/2)*x, 2/(x^2+2*y^2)^(1/2)*y]
>> gradp=subs(grad,{x,y},{-1,2})
gradp =
-0.3333    1.3333
```

Second, we normalize the vector $\vec{v}$:

```
>> v=[-2,3];
>> u=v/norm(v)
u =
-0.5547    0.8321
```

Finally, we calculate the directional derivative of f at $(-1,2)$ in the direction of $\vec{v} = (1,3)$. To calculate the scalar product of the gradient and the vector $\vec{u}$, we multiply the gradient (as a row vector) by $\vec{u}$ (as a column vector). To convert a row vector into a column vector(or viceversa) we must add (**'**) to the corresponding vector.

```
>> gradp*u'
ans =
1.2943
```

5. Exercises

1. Represent graphically the following surfaces with the corresponding level curves on the rectangle $[-4,4]\times[-4,4]$.

$$f(x,y)=\frac{8y}{1+x^2+y^2} \qquad f(x,y)=\ln\left(\frac{1+x^2}{1+y^2}\right) \qquad f(x,y)=\sin(x^2+y^2)$$

2. Let $f(x,y)=\cos(x+y^2)$. Calculate

$$\frac{\partial^2 f(x,y)}{\partial y\partial x}-\frac{\partial^3 f(x,y)}{\partial x\partial y^2}\,.$$

Sol.: $2(1-y)\cos(x+y^2)-4y^2\sin(x+y^2)$.

3. Find the directional derivative of the function $f(x,y)=x^3+y^4$ at the point (1,1) in the direction of the vector $\vec{v}=(-1,3)$.

Sol.: 2.8460

4. A team of oceanographers is developing a map of the sea to try to recover the wrecks (remains of ships that have sunk). By using the sonar they construct the model

$$f(x,y)=0.1(xy^2+x^2y) \qquad (x,y)\in[0,2]\times[0,2],$$

where x and y are the coordinates on the plane and f is the depth (all the distances are measured in kilometers).

a) Represent graphically the depth f on the given rectangle.

b) Represent graphically the seabed $(-f)$ on the given rectangle.

c) How deep is the wreck if it is at $(1,3/2)$? Represent this point on the plotted surface.

d) If we move the shipwreck to the east, will we raise it or will we sink it further ? What will happen if we move it to the south?

Sol.: *c*) 0.3750 *d*) It sinks in the east direction and it rises in the south direction.