CALCULUS
DEGREE IN SOFTWARE ENGINEERING
CHAPTER 20. MULTIVARIABLE CALCULUS. AN INTRODUCTION

We have studied real functions of a single variable till now. However, in the real world most processes depend on several independent variables: the pressure of a gas will depend on its volume and temperature, the gravitational potencial created by a given mass will change with the three spatial coordinates and so on. Therefore, it is logical to define real functions of several variables, also called multivariable functions. These function will go from $\mathbb{R}^2$ to $\mathbb{R}$, from $\mathbb{R}^3$ to $\mathbb{R}$ and, in general, from $\mathbb{R}^n$ to $\mathbb{R}$ with n any natural number. We begin with the definition of a two-variable function

$$\begin{aligned} f : \mathbb{R}^2 &\longrightarrow \mathbb{R} \\ (x, y) &\longrightarrow z = f(x, y) \end{aligned}$$

DEFINITIONS

A real function of two variables is a rule which assigns a unique real number to each element of $\mathbb{R}^2$, that is, to each ordered pair of real numbers.
The domain of the function, D, is the largest subset of $\mathbb{R}^2$ where the function is defined, the set of all input elements

$$D = \{(x, y) \in \mathbb{R}^2 / z = f(x, y) \in \mathbb{R}\}$$

The range of the function, $f(D)$, is the set of all output elements

$$f(D) = \{z = f(x, y) \in \mathbb{R} / (x, y) \in D\}$$

x and y are the independent or input variables and z the dependent or output variable. Mathematically, we will use this notation, though, of course, the names of the independent and dependent variables adapt themselves to the scientific problem.
We can costruct two-variable functions by combining the functions that we know and using two instead of one variable. For instance

$$f(x, y) = \sin(x^2 + y^3)$$

$$g(x, y) = \frac{xy}{x^2 + y^2}$$

$$h(x, y) = \ln(\cos xy)$$

EXAMPLES: DOMAIN. RANGE. GRAPH

According to the ideal gas law, $P = nRT/V$, where n is the number of moles of gas, P is its pressure, V its volume, R the ideal gas constant and T its temperature. We will write

$$\begin{aligned} P : \mathbb{R}^2 &\longrightarrow \mathbb{R} \\ (T, V) &\longrightarrow P = nRT/V \end{aligned}$$

Mathematically, the function is well defined if $V \neq 0$, but physically both T, absolute temperature, and V must be positive.
Now, we will show the domain and range of a very simple two-variable function

$$\begin{aligned} f : \mathbb{R}^2 &\longrightarrow \mathbb{R} \\ (x, y) &\longrightarrow z = \sqrt{1 - x^2 - y^2} \end{aligned}$$

For z to be real, $x^2 + y^2$ must be less or equal to 1.

$$x^2 + y^2 \leq 1$$

The domain will be a closed disk of radius 1 centered at the origin.

$$D = \{(x, y) \in \mathbb{R}^2 / x^2 + y^2 \leq 1\}$$

If we analyze the range, we find that z goes from 1 at the origin to 0 on the circle of radius 1, then

$$f(D) = [0, 1]$$

If we choose $\ln(1 - x^2 - y^2)$ as our example function, the domain will be

$$D = \{(x, y) \in \mathbb{R}^2 / x^2 + y^2 < 1\}$$

an open disk of radius 1 centered at the origin and

$$f(D) = (-\infty, 0]$$

Now, we could ask ourselves: what is the graph of a function of two variables? If we remember the definition of graph for a single variable function

$$graph(f) = \{(x, y) \in \mathbb{R}^2 / y = f(x),\ x \in D\}$$

that is, a curve on the plane. The natural extension for a two-variable function will be

$$graph(f) = \{(x, y, z) \in \mathbb{R}^3 / z = f(x, y), (x, y) \in D\}$$

a surface in space. What is the graph of our example function $z = \sqrt{1 - x^2 - y^2}$?

It will be the surface of equation $z = \sqrt{1 - x^2 - y^2}$ or $x^2 + y^2 + z^2 = 1$ with $z \geq 0$, the semisphere of radius 1 above the xy-plane, see Figure 1. In this case, the graph is the semisphere in space, the domain is the unit closed disk centered at the origin on the xy-plane and the range the closed interval $[0, 1]$ on the z-axis. We show other surfaces in the following figures. They are plotted with the MATLAB commands ezsurf or meshgrid.

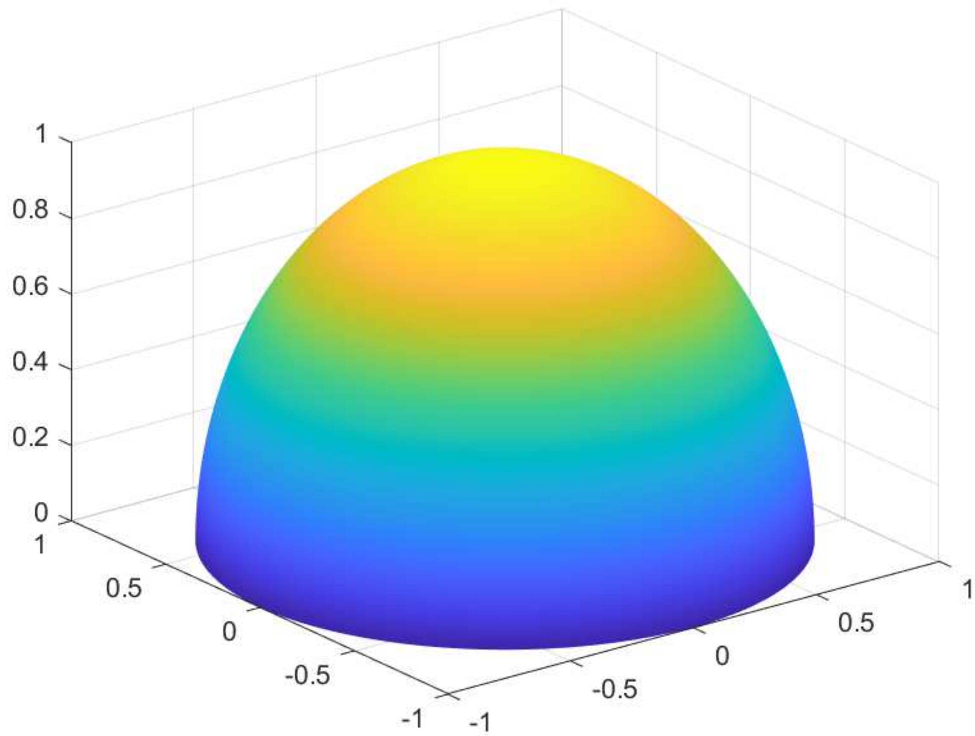

Figure 1: Example 1

LEVEL CURVES

Another important concept in the plotting of surfaces (graphs of two-variable functions) is that of level curves. A level curve of a function $f(x, y))$, L_c, is the set of points of the domain where the function takes on a constant value c.

$$L_c = \{(x, y) \in D/f(x, y) = c\}$$

Imagine a function $T(x, y)$ with T the temperature at different points on a plane. The curves $T(x, y) = c$ are isotherms that join those points with the same temperature c, as on a weather map.
For the function $z = x^2 + y^2$, the level curves $x^2 + y^2 = c$ are circles centered at the origin with radius $\sqrt{c}$; the graph is a circular paraboloid (see Figure 2). The point $(0, 0)$ is a degenerate level curve $c = 0$ consisting of a point. If $z = x^2 - y^2$, the level curves are hyperbolas $x^2 - y^2 = c$, except for $c = 0$, where we obtain the lines $y = x$, $y = -x$, as the corresponding level curve; the graph is a hyperbolic paraboloid (see Figure 4).
If we lift the level curves on the graph, we obtain the contour curves C_c with

$$C_c = \{(x, y, c) \in \mathbb{R}^3/f(x, y) = c\}$$

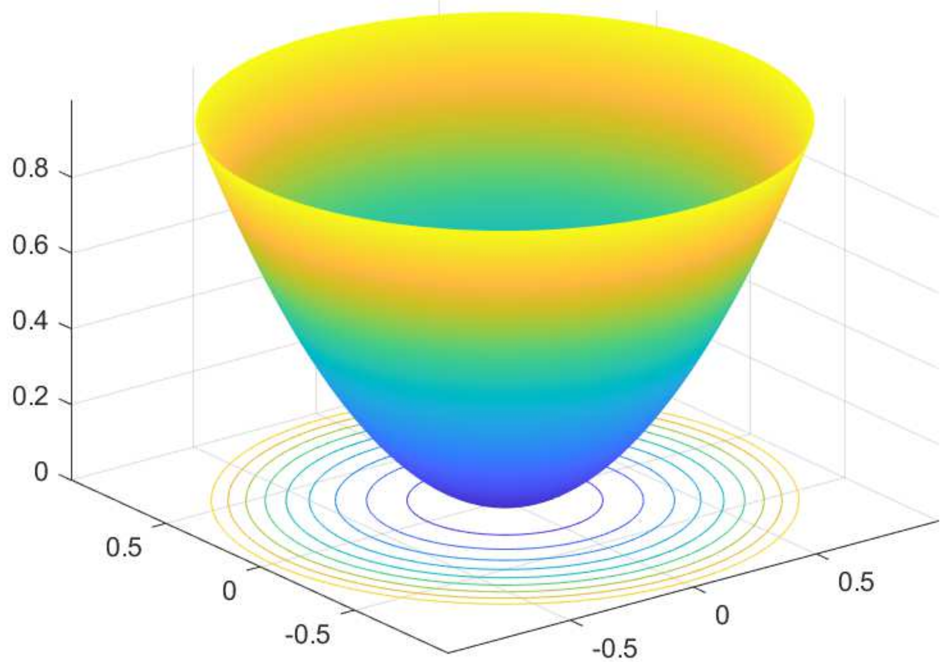

Figure 2: Paraboloid and level curves

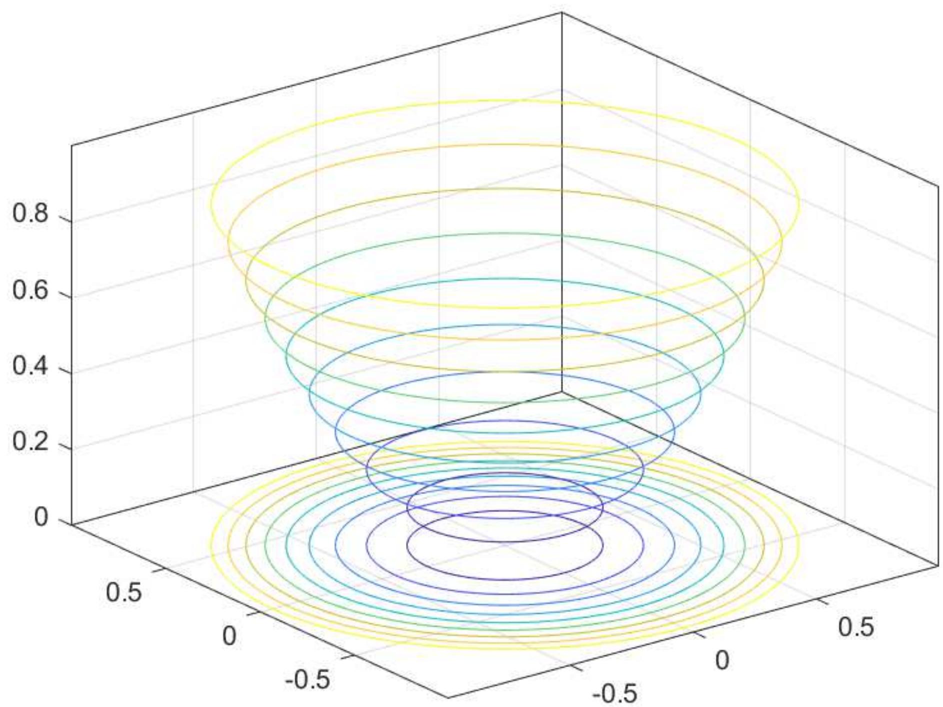

Figure 3: Contour and level curves

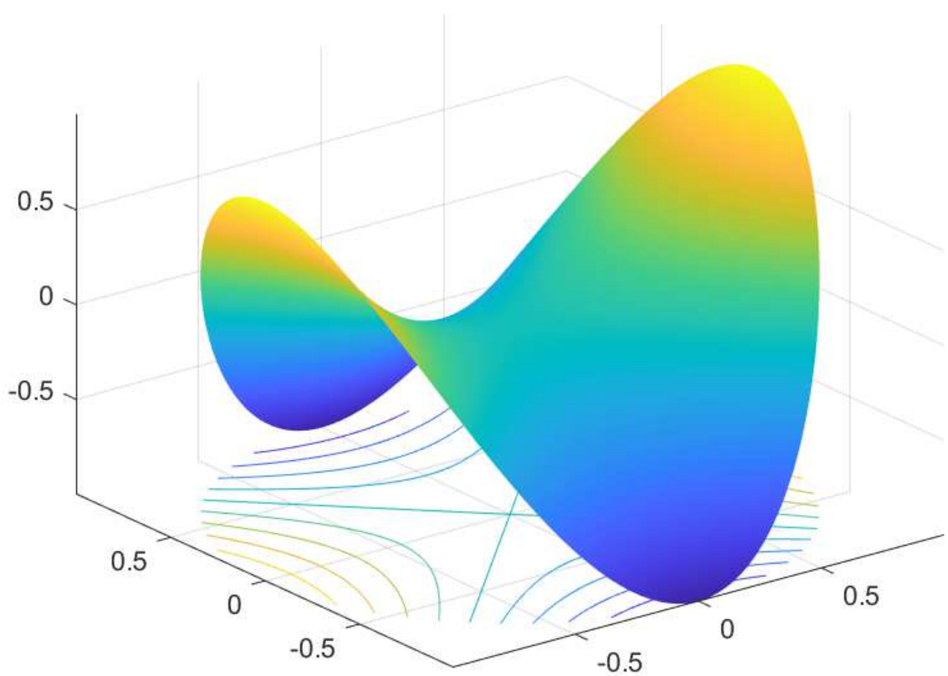

Figure 4: Hyperbolic paraboloid and level curves

that is, (x, y) have the same values as for the level curve L_c but $z = c$, the curve is on the surface, see Figure 3.

FUNCTIONS OF THREE VARIABLES

A function of three variables assigns a unique real number to each ordered triple of real numbers. The domain is $\mathbb{R}^3$ or a subset D of $\mathbb{R}^3$. The range is $\mathbb{R}$ or a subset $f(D)$ of $\mathbb{R}$. We will write

$$\begin{aligned} f : D \subseteq \mathbb{R}^3 &\longrightarrow \mathbb{R} \\ (x, y, z) &\longrightarrow u = f(x, y, z) \end{aligned}$$

For instance, we can associate a temperature to each point of space

$$\begin{aligned} f : \mathbb{R}^3 &\longrightarrow \mathbb{R} \\ (x, y, z) &\longrightarrow T = T(x, y, z) \end{aligned}$$

We shall give an example of domain and range for a three-variable function

$$\begin{aligned} f : D \subseteq \mathbb{R}^3 &\longrightarrow \mathbb{R} \\ (x, y, z) &\longrightarrow u = \sqrt{1 - x^2 - y^2 - z^2} \end{aligned}$$

D is the set of points satisfying $x^2+y^2+z^2 \leq 1$, that is, the interior of the unit sphere of radius 1 centered at the origin- including the spherical surface. In Mathematics, we call this set the unit closed ball. It is easy to check that the range is the interval $[0, 1]$.
How could we plot the graph of this function. The logical definition of the graph is

$$graph(f) = \{(x, y, z, u) \in \mathbb{R}^4 / u = f(x, y, z)\}$$

Unfortunately, we cannot plot points in $\mathbb{R}^4$. However, we could plot section-functions of two variables- by making one of the independent variables constant. For example, if we make $z = c$ in the previous function, we obtain

$$u = \sqrt{1 - c^2 - x^2 - y^2}$$

which is a semisphere $x^2 + y^2 + u^2 = 1 - c^2$ with $u \geq 0$. The same would happen if we make $x = c$ or $y = c$. A very interesting concept is the analogue of level curves for functions of three variables. We can define the level surfaces in the following way

$$S_c = \{(x, y, z) \in D / f(x, y, z) = c\}$$

Level surfaces consist of the points in space where the function takes on a constant value c. In our example, the level surfaces are the spheres

$$x^2 + y^2 + z^2 = 1 - c^2$$

Of course, c must be in $[0, 1]$. A typical example of level surfaces in Physics is that of equipotencial surfaces, where the potencial (gravitational or electrical) is constant.

The gravitational potential outside the Earth has well studied equipotential surfaces, spherical in a first approximation, but a deformed ellipsoid in practice.
Finally, we can define functions of four variables, with the fourth variable being for instance time and, in general, functions of n variables (scalar fields)

$$f : \mathbb{R}^n \longrightarrow \mathbb{R}$$
$$(x_1, x_2, ..., x_n) \longrightarrow y = f(x_1, x_2, ..., x_n)$$

and even fuctions from $\mathbb{R}^n$ to $\mathbb{R}^m$ (vector fields) with m and n natural numbers. An example could be a force (a vector with three components) acting at each point in space. This is a function from $\mathbb{R}^3$ to $\mathbb{R}^3$.