1. Change of Basis

Exercise 1 *Consider B y B' basis of $\mathbb{R}_2[X]$:*

$$B = \{x, 1+x^2, x+x^2\}$$

$$B' = \{1, 1+x, x^{2\}}$$

Compute the change of basis matrix from B to B' and express the polynomial $p(x) = 4-2x-x^2$ in B'.

Exercise 2 *Let B and B' be two basis in $\mathbb{R}_2[X]$:*

$$B = \{x, 1+x, 1-x+x^2\}$$

$$B' = \{p_1, p_2, p_3\}$$

and $P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ -1 & 1 & 1 \end{pmatrix}$ *the change of basis matrix from B to B'. Compute B'.*

Exercise 3 *Now consider B and B' two basis in $\mathbb{R}_2[X]$:*

$$B = \{x, 1+x, 1-x+x^2\}$$

$$B' = \{p_1, p_2, p_3\}$$

and $P = \begin{pmatrix} 1 & 2 & 1 \\ -1 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$ *the change of basis matrix from B' to B. Find out B'.*

Exercise 4 *Consider the following basis in $\mathbb{R}_2[x]$:*
$\mathscr{B}_C = \{1, x, x^2\}$ *and* $\mathscr{B} = \{1+x+x^2, -1, 2x+x^2\}$.

1. *Compute the change of basis matrix P_1 from $\mathscr{B}_C$ to $\mathscr{B}$.*
2. *Give the coordinates of $p(x) = 1-x-x^2$ in $\mathscr{B}$.*
3. *Find out the polynomial $r(x)$ having the coordinates $(1,2,3)_{\mathscr{B}}$.*
4. *Now, consider $\mathscr{B}' = \{h_1(x), h_2(x), h_3(x)\}$ an other basis in$\mathbb{R}_2[x]$ and* $M = \begin{pmatrix} 3 & -1 & 2 \\ 5 & 4 & 1 \\ -2 & 0 & 0 \end{pmatrix}$ *the change of basis matrix from $\mathscr{B}$ to $\mathscr{B}'$. Compute $h_1(x)$, $h_2(x)$ and $h_3(x)$.*

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2. Sum, Intersection and Supplementary Space

Exercise 5 *Considering the following linear subspaces in* $\mathbb{R}_3[x]$*:*

$$U = \{a_0 + a_1x + a_2x^2 + a_3x^3 \in \mathbb{R}_3[x] \;/ a_0 - a_3 = 0,\ a_0 + 2a_1 = 0\}$$
$$V = \left\langle -1 - x - x^2 - 2x^3,\ 1 + 4x + 2x^2 + 4x^3,\ -2 + x - x^2 - 2x^3 \right\rangle.$$

a) Compute the basis and the dimensions of U *and* V*.*

b) Compute the basis and the dimensions of $U+V$ *and* $U \cap V$*.*

c) Is $U \bigoplus V = \mathbb{R}_3[x]$*?*

d) Find out a basis of the supplementary space of U*.*

Exercise 6 *Consider the linear subspaces* V *and* W *in* $\mathbb{R}^3$

$$a)\quad V = \{(x_1, x_2, x_3) \mid x_1 = \alpha + \beta\,,\ x_2 = \beta\,,\ x_3 = \alpha + 2\beta\}$$
$$b)\quad W = \{(y_1, y_2, y_3) \mid y_1 - y_2 + 2y_3 = 0\}$$

Compute:

a) Basis and dimensions of V *and* W*.*

b) Basis and dimensions of $V+W$ *and* $V \cap W$*.*

e) Is the vector $\vec{v} = (2,3,5)$ *in* $V+W$ *? If so, give its coordinates in the basis of* $V+W$*.*

Exercise 7 *Compute the sum and the intersection of* V_1 *and* V_2*, subspaces in* $\mathscr{M}_2(\mathbb{R})$*, generated by:*

$$S_1 = \left\{ \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \right\}, \qquad S_2 = \left\{ \begin{pmatrix} 2 & -1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 3 & 7 \end{pmatrix} \right\}.$$

Exercise 8 *Let* V *and* W *be two linear spaces in* $\mathbb{R}^3$*. Prove in each case that* $\mathbb{R}^3 = V \bigoplus W$

a) $V = \{(x,y,z) \in \mathbb{R}^3 \mid x = y = z\}$ *y* $W = \{(0,y,z) \in \mathbb{R}^3 \mid y,z \in \mathbb{R}^3\}$

b) $V = \{(x,y,z) \in \mathbb{R}^3 \mid x + y + z = 0\}$ *y* $W = \{(t, 2t, 3t) \in \mathbb{R}^3 \mid t \in \mathbb{R}^3\}$

Exercise 9 *Consider the linear spaces in* $\mathbb{R}^3$*:*

$$V : \left\{ \begin{array}{rcl} x & = & \alpha + \beta \\ y & = & \beta + \gamma \\ z & = & \alpha + 2\beta + \gamma \end{array} \right.$$
$$W : x - y + 2z = 0$$

Compute

a) *Basis of* V, W, $V+W$ *and* $V\bigcap W$.

b) *Implicit equations of* $V\bigcap W$.

c) *Basis and dimension of the supplementary space of* $V+W$.