CALCULUS
DEGREE IN SOFTWARE ENGINEERING
EXERCISES 9. VOLUMES

In these exercises, we will apply the different methods introduced in Chapter 19 for finding the volume of solids

CROSS-SECTIONS

1. Find the volume of the ellipsoid

$$x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$$

We consider cross-sections prependicular to the z-axis, the intersection of a plane with z constant and the ellipsoid is an ellipse whose equation is

$$x^2/a^2 + y^2/b^2 = 1 - z^2/c^2$$

We can write this equation as

$$\frac{x^2}{a^2(1 - z^2/c^2)} + \frac{y^2}{b^2(1 - z^2/c^2)} = 1$$

The area of this ellipse is π times the product of the semiaxes

$$A(z) = \pi a\, b\,(1 - z^2/c^2)$$

and in consequence, the volume is

$$V = \int_{-c}^{c} \pi a\, b\,(1 - z^2/c^2)\, dz = \pi a\, b(z - \frac{z^3}{3c^2})\Big|_{-c}^{c} = 4\pi abc/3$$

If you assume that $a = b = c$, a sphere, the volume is $V = 4\pi a^3/3$ as expected.

In Figure 1 we can see an ellipsoid with $a = 3$, $b = 6$, $c = 9$.

2. Find the volume of a square pyramid with height $h = 3$ and a base of side $a = 3$, see Figure 6.5 in Presentation 7.

It is clear that the cross-sections are squares with $A(x) = x^2$, we only have to integrate to calculate the volume.

$$V = \int_0^3 x^2\, dx = \frac{x^3}{3}\Big|_0^3 = 9 = \frac{A_{base} h}{3}$$

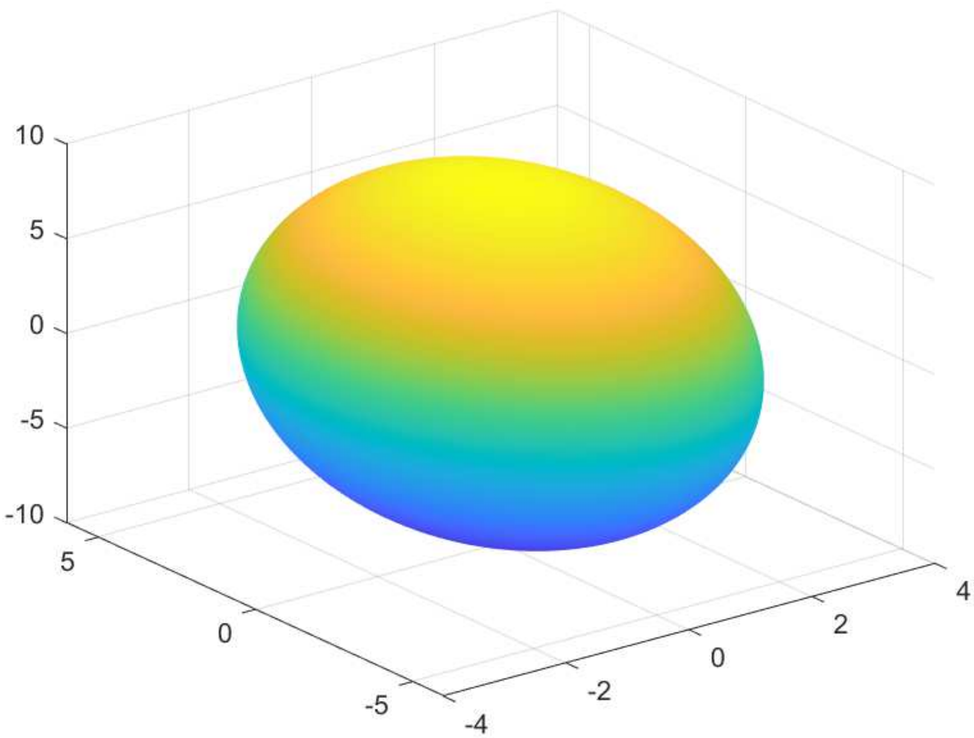

Figure 1: Exercise 1

3. Find the volume of the wedge of Figure 6.6 (Presentation 7)

   According to the figure, the cross-sections with x constant are rectangles whose base rests (lies) on the semicircle $y = \sqrt{9-x^2}$. Then, the area of a typical cross-section is

$$A(x) = 2x\sqrt{9-x^2}$$

   and the volume of the wedge will be

$$V = \int_0^3 2x\sqrt{9-x^2}\,dx = -\frac{2(9-x^2)^{3/2}}{3}\Big|_0^3 = 18.$$

SOLIDS OF REVOLUTION

4. Calculate the volume of a circular cone of height h and a circular base of radius R.

   The cone is generated by a line $y = mx$ that passes through the origin and the point (R, h). Thus, the equation of the line is $y = Rx/h$ and the limits of integration are 0 and h.

   If we rotate this line around the x-axis, we generate a cone whose volume is , according to the disk method

$$V = \pi \int_0^h \frac{R^2x^2}{h^2}\,dx = \pi \frac{R^2x^3}{3h^2}\Big|_0^h = \pi \frac{R^2h}{3}$$

   A well known formula: base area times height divided by three.

   What happens if we rotate the region between the line $y = Rx/h$, the x-axis and $x = h$ around the y-axis ? We must apply the shell method now and

$$V_y = 2\pi \int_0^h xRx/h\,dx = 2\pi \frac{Rx^3}{3h}\Big|_0^h = \frac{2\pi Rh^2}{3}$$

   .

   Try to figure out the solid !!

5. The region bounded by the the two branches of of the curve $(y-x)^2 = x^3$ and the line $x = 1$ is rotated about the x-axis. Find the volume of the generated solid.

   First, we draw the region taking into account that $x \geq 0$ and the upper and lower branches have equations $y = x + x^{3/2}$ and $y = x - x^{3/2}$ respectively (see Figure 2). The volume will be calculated with the washer method

$$V = \pi \int_0^1 [(x + x^{3/2})^2 - (x - x^{3/2})^2]\, dx = 4\pi \int_0^1 x^{5/2}\, dx = 8\pi x^{7/2}/7\Big|_0^1 = 8\pi/7$$

6. Consider the same region and revolve it about the y-axis. What is the volume of the generated solid ?

   Now, we apply the shell method

   $$V = 2\pi \int_0^1 x[(x + x^{3/2}) - (x - x^{3/2})]\, dx = 4\pi \int_0^1 x^{5/2}\, dx = 8\pi x^{7/2}/7\Big|_0^1 = 8\pi/7$$

   It is the same result. Why ?

7. Revolve the same region about the axis $x = 1$. What is the volume now ?

   We use the shell method, but instead of x, we have to multiply $f(x) - g(x)$ times $1 - x$- the distance to the rotation axis- in the integrand . Therefore

   $$V = 2\pi \int_0^1 (1-x)[(x+x^{3/2})-(x-x^{3/2})]\, dx = 4\pi \int_0^1 (x^{3/2}-x^{5/2})\, dx = 4\pi(2x^{5/2}/5-2x^{7/2}/7)\Big|_0^1$$

   $$V = 8\pi(1/5 - 1/7) = 16\pi/35$$

8. The region enclosed by the curve $y = 1 + x^2$ and the line $y = 3 - x$ is rotated around the x-axis to generate a solid. Find its volume.

   We apply the washer method again. First, we calculate the points at which both graphs intersect, by equating the corresponding functions

   $$1 + x^2 = 3 - x$$

   The solutions are $x = -2$ and $x = 1$

   Drawing the region (see Figure 3), we observe that $y = 3-x$ is above $y = 1+x^2$ on the interval $[-2, 1]$. Therefore, the volume is

   $$V = \pi \int_{-2}^1 [(3-x)^2-(1+x^2)^2]\, dx = \pi \int_{-2}^1 (9-6x+x^2-1-2x^2-x^4)\, dx = \pi \int_{-2}^1 (8-6x-x^2-x^4)\, dx$$

   $$V = \pi\Big(8x - 3x^2 - x^3/3 - x^5/5\Big)\Big|_{-2}^1$$

   $$V = \pi\Big(8-3-1/3-1/5-(-16-12+8/3+32/5)\Big) = 33-3-33/5 = 117\pi/5$$

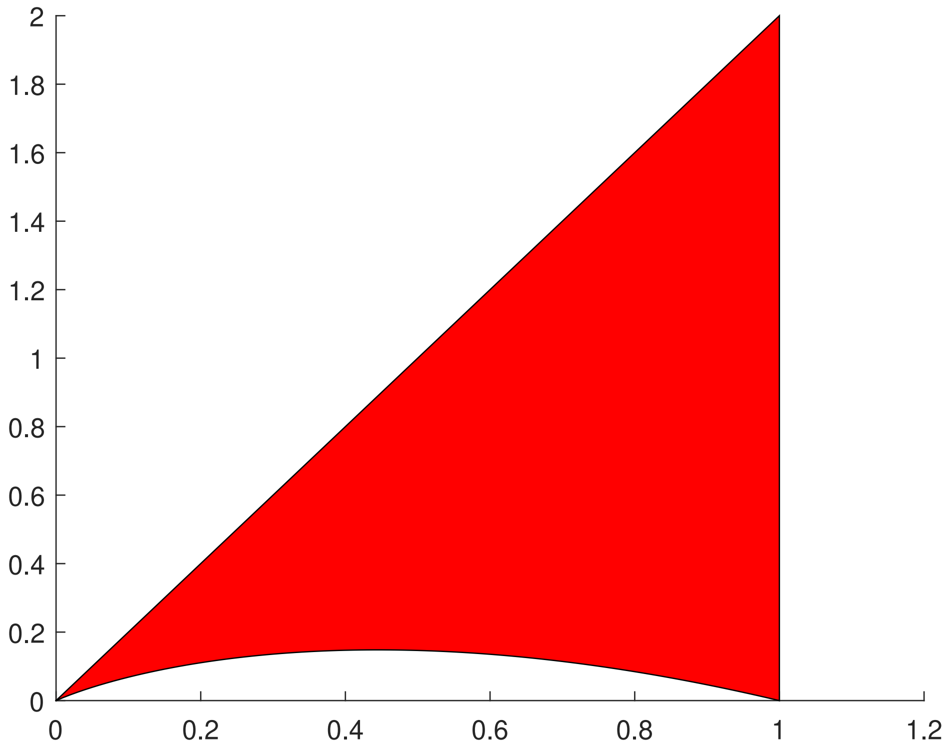

Figure 2: Exercise 5

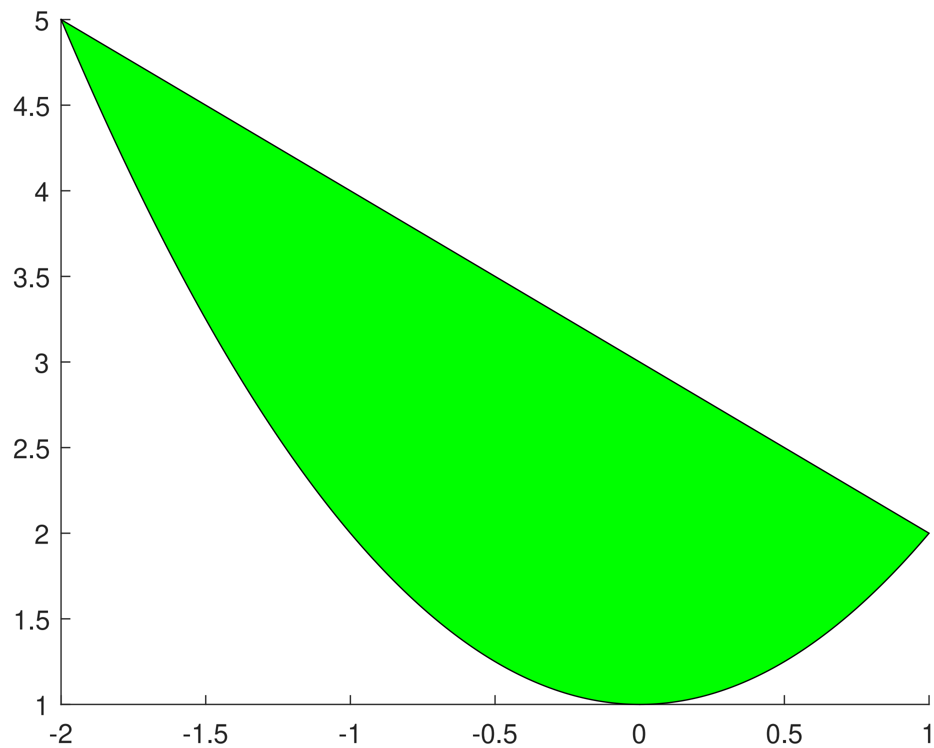

Figure 3: Exercise 8

9. What happens if we revolve the part of this region in the first quadrant about the y-axis. What is the volume of the generated solid ?

   Now we use the shell method and the formula is

$$V = 2\pi \int_0^1 x[(3-x)-(1+x^2)]\,dx = 2\pi \int_0^1 x[2-x-x^2]\,dx = 2\pi \int_0^1 (2x-x^2-x^3)\,dx$$

$$V = 2\pi\Big(x^2 - x^3/3 - x^4/4\Big)\Big|_0^1 = 2\pi(1 - 1/3 - 1/4) = 5\pi/6$$

   Finally , if we rotate the part in the second quadrant around the y-axis, the formula is

$$V = 2\pi \int_{-2}^0 -x[(3-x)-(1+x^2)]\,dx = 2\pi \int_{-2}^0 -x[2-x-x^2]\,dx = 2\pi \int_{-2}^0 (-2x+x^2+x^3)\,dx$$

$$V = 2\pi\Big(-x^2 + x^3/3 + x^4/4\Big)\Big|_{-2}^0 = 2\pi(4 + 8/3 - 4) = 16\pi/3$$

Notice that we had to multiply by $-x$ since the distance to the rotation axis has to be positive.

10. Let $y = \sqrt{x}$ and $y = \sqrt{2 - x^2}$. Draw the region bounded by the graphs of these two functions and the x-axis.

    We solve

$$\sqrt{x} = \sqrt{2 - x^2}$$

    The only solution is $x = 1$. The region will be bounded by the x-axis and $y = \sqrt{x}$ from $x = 0$ to $x = 1$ and by the x-axis and $y = \sqrt{2 - x^2}$ from $x = 1$ to $x = 2$ (see Figure 4).

    (a) Find the volume of the solid generated by rotating this region around the x-axis.

    The integral (disk method) must be split into two parts.

$$V = \pi \int_0^1 x\,dx + \pi \int_1^{\sqrt{2}} (2 - x^2)\,dx$$

$$V = \pi x^2/2\Big|_0^1 + \pi(2x - x^3/3)\Big|_1^{\sqrt{2}} = \pi(1/2 + 2\sqrt{2} - 2\sqrt{2}/3 - 2 + 1/3)$$

$$V = \pi(4\sqrt{2}/3 - 7/6) = 0.7189...$$

    (b) Calculate the volume obtained when the same region is revolved about the y-axis

    Now, we apply the shell method

$$V = 2\pi \int_0^1 x\sqrt{x}\,dx + 2\pi \int_1^{\sqrt{2}} x\sqrt{2 - x^2}\,dx$$

$$V = 4\pi x^{5/2}/5\Big|_0^1 - 2\pi(2 - x^2)^{3/2}/3\Big|_1^{\sqrt{2}}$$

$$V = \pi(4/5 + 2/3) = 22\pi/15$$

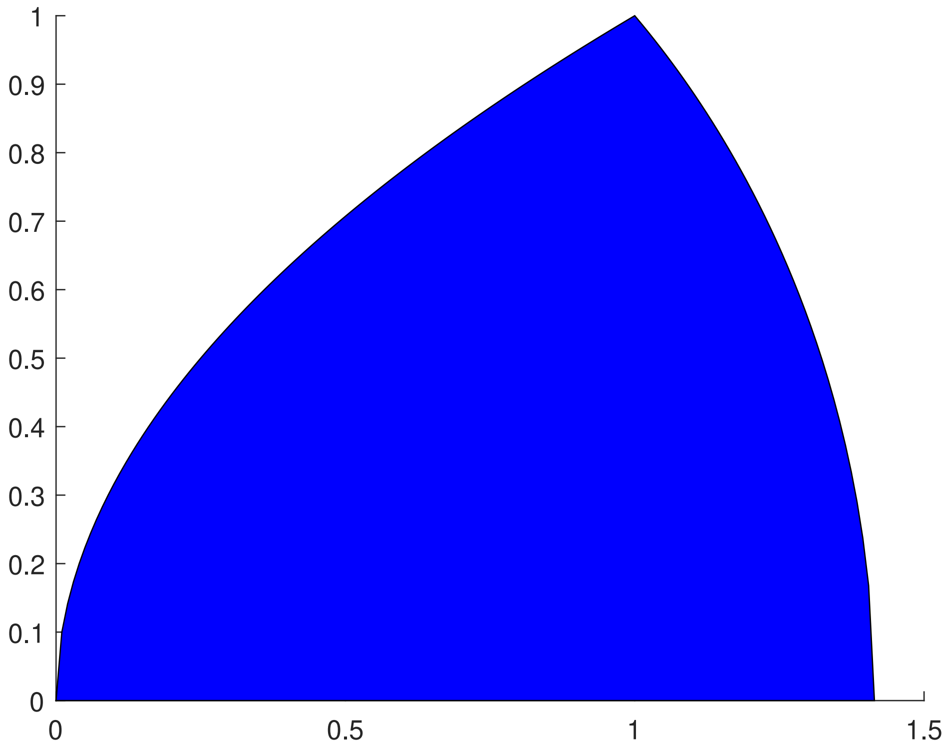

Figure 4: Exercise 10