CALCULUS
DEGREE IN SOFTWARE ENGINEERING
CHAPTER 17. THE DEFINITE INTEGRAL. DEFINITION AND PROPERTIES

The definite integral is the key tool in calculus for defining and calculating concepts that are essential in geometry and science. To mention just a few: areas, volumes, lengths of curved paths, the work done by a force,.... First, we will present the definite integral in relation to the definition and calculation of the area of a plane figure. Later, we will explore the properties of the definite integral and connect it with the derivative and the indefinite integral (Chapter 18). Finally, we will show how to calculate areas and volumes by means of integrals (Exercises 8).

DEFINITION OF THE DEFINITE INTEGRAL WITH UPPER AND LOWER SUMS

Imagine that we want to calculate the area of a region on the plane. Assume that it is the region bounded by the graph of a non-negative function $y = f(x)$, the vertical lines $x = a$, $x = b$ and the x-axis. An example is shown in Figure 5.1 (Presentation 6, The definite integral, power point file). We also assume, for the sake of simplicity, that the function is continuous. This region, called D, can then be defined as

$$D = \{(x, y) \in \mathbb{R}^2 / x \in [a, b], 0 \leq y \leq f(x)\}$$

That is, all the points (x, y) on the plane with x in the closed and bounded interval $[a, b]$ and y between 0 and the graph of $y = f(x)$. Now, we will use a few things we know about areas: 1) the area of a rectangle is base times height, 2) if $S \subseteq D \subseteq T$, then Area(S)$\leq$ Area(D)$\leq$ Area(T), where the area is a unique non-negative number assigned to the region. Of course, a much more precise definition is possible, but this will suffice for our needs.

Now, we will try to approximate the area of D by using rectangles inscribed in the region. We define a partition of the interval $[a, b]$ as the set of numbers (points between a and b)

$$P = \{x_0, x_1, x_2, ..., x_n\}$$

with $x_0 = a$, $x_n = b$ and $x_0 < x_1 < x_2 < .... < x_n$. A partition consists of $n + 1$ points chosen freely between a and b, but in increasing order and including the endpoints. The interesting idea is that a partition gives rise to the division of the interval $[a, b]$ into n subintervals

$$[x_0, x_1], [x_1, x_2], ..., [x_{n-1}, x_n]$$

and each subinterval will be used as the base of a rectangle inscribed in D. To do this, we only need to choose rectangles whose height is the minimum value m_k of the function on each subinterval. The existence of this minimum is guaranteed because the function is continuous. At any rate, the construction of the rectangles is also possible for bounded functions, but in this case we have to use the greatest lower bound of the function on each subinterval. Now, we proceed to add up the areas of all the inscribed rectangles. This sum is called a lower sum of the function with respect to the partition P and is usually written as

$$L(f,P) = \sum_{k=1}^{n} m_k \left(x_k - x_{k-1} \right)$$

We have multiplied height m_k times base $\Delta x_k = (x_k - x_{k-1})$. The sum of the areas of these inscribed rectangles is obviously less than or equal to the area of D. As you can see in Figure 5.4 of the corresponding power point presentation (Presentation 6), if you include more and more points in the partition- you make it finer- the lower sums approach closer and closer the real area, because there are smaller and smaller gaps between D and the collection of rectangles. We could imagine that by taking finer partitions, the lower sums will approach the real area in the limit.
To see this more clearly, we could define upper sums just by choosing circumscribed rectangles, that is, taking the maximum M_k of the function for each subinterval generated by the partition. See Figure 5.4 (bottom part). Again, if the function is bounded, we take the lowest upper bound. The upper sum is

$$U(f,P) = \sum_{k=1}^{n} M_k \left(x_k - x_{k-1} \right)$$

It is clear that for any partition

$$L(f,P) \leq A(D) \leq U(f,P)$$

The area is like the ham in a sandwich between the slices of bread (upper and lower sums). If we take finer partitions, there is only one number (the area) between all the lower and upper sums.
Of course, the uniqueness of the area can be proved rigourosly by using that the function is continuous. The proof is a little subtle because it involves a property called uniform continuity that all continuous functions on a closed and bounded interval have. The idea is that the minima and maxima involved in the lower and upper sums can be made as close as we want and that implies that upper and lower sums converge to the same number (the area or definite integral of the function) as we consider finer and finer partitions. All this will be written in a more precise language with de definition of Riemann sums.
And with this definition we proceed. If we want to calculate the area of a region D, we could construct lower or upper sums and take the limit of these sums as we consider finer and finer partitions- we could make the length of the longest subinterval tend to zero- . However, it would be simpler to consider the value of the function at any point c_k on each subinterval. We could, for instance, choose the midpoint. Thus, we define the so-called Riemann sums

$$S(f,P) = \sum_{k=1}^{n} f(c_k)\,(x_k - x_{k-1})$$

It is obvious that any Riemann sum will have a value between the corresponding lower and upper sums

$$L(f,P) \leq S(f,P) \leq U(f,P)$$

If the function is integrable, that is, if the lower and upper sums converge to the same number (the definite integral), the Riemann sums will also converge to that number.
In the following, we define the norm of a partition and then, the definite integral in terms of Riemann sums.
The norm of a partition $P = \{x_0, x_1, ..., x_n\}$, $||P||$, is the length of the longest subinterval generated by the partition

$$||P|| = max\{\Delta x_1, \Delta x_2, ..., \Delta x_k, ..., \Delta x_n\}$$

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DEFINITION OF THE DEFINITE INTEGRAL IN TERMS OF RIEMANN SUMS

Let $f(x)$ be a function defined on a closed and bounded interval $[a,b]$. We say that a number J is the definite integral of f over $[a,b]$ and that J is the limit of the Riemann sums

$$S(f,P) = \sum_{k=1}^{n} f(c_k)\,\Delta x_k$$

if the following condition is satisfied:
Given any number $\epsilon > 0$ there is a corresponding number $\delta > 0$ such that for every partition $P = \{x_0, x_1, ..., x_n\}$ of $[a,b]$ with $||P|| < \delta$ and any choice of c_k in $[x_{k-1}, x_k]$, we have

$$|\sum_{k=1}^{n} f(c_k)\,\Delta x_k - J| < \epsilon$$

According to the reasoning given above, it is clear that a continuous function on $[a,b]$ will be integrable, that is, there will be a definite integral J of f on $[a,b]$. Even a piecewise continuous function will also be integrable, as expressed in the following theorem.

THEOREM. INTEGRABILITY OF CONTINUOUS FUNCTIONS

If a function f is continuous over the interval $[a, b]$ or if it has at most finitely many jump dicontinuities there (piecewise continuous), then the definite integral exists and f is integrable on $[a, b]$.

The notation for the definite integral is

$$J = \int_a^b f(x)\, dx$$

where a is the lower integration limit, b the upper integration limit, $f(x)$ is the integrand and dx the differential of x. x is the variable of integration and it is a dummy variable, in the sense that it can be substituted by any other variable, that is

$$\int_a^b f(x)dx = \int_a^b f(t)\, dt$$

the name of the variable is irrelevant, the definite integral is a number that only depends on the limits and the function.
Probably you have noticed the identity of symbols and names used for the definite and indefinite integral, all this will be explained by means of the Fundamental Theorem of Calculus.
Maybe you have wondered which functions are not integrable. Unbounded functions can be non-integrable since their values go to infinity and can produce an infinite value for the integral. Examples of bounded functions that are not integrable are less common. The typical example is a function on any interval $[a, b]$ which takes on the value 1 if x is irrational and 0 if x is rational. In this case, the lower sums are always zero and the upper sums are $(b - a)$, why ?
If the function $f(x)$ is non-negative, the definite integral can be interpreted as the area under the graph of the function. However, the function can be positive or negative at different points and the integral will also be perfectly defined, then it will not be the area but it will have a different interpretation. In the next section, we will see the main properties of the definite integral.

PROPERTIES OF DEFINITE INTEGRALS

1. Order of integration
$$\int_b^a f(x)dx = -\int_a^b f(x)\, dx$$
2. Zero width interval
$$\int_a^a f(x)\, dx = 0$$
3. Constant Multiple
$$\int_a^b kf(x)\, dx = k\int_a^b f(x)\, dx$$

4. Sum and Difference

$$\int_a^b (f(x)\pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$$

5. Additivity

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$

6. Max-Min inequality

   If f has a minimum value m and a maximum value M on $[a,b]$,what is always true if f is continuous on $[a,b]$, then

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$$

7. Domination

   If $f(x) \geq g(x)$ on $[a,b]$, then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$

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The meaning and the intuitive proofs of these properties can be seen in the power point presentation of the definite integral, Figure 5.11.
As we have seen before, the area under the graph of a non-negative function is the definite integral, but what is the area bounded by the graph of a general function and the x-axis between a and b? In this case the negative ordinates would be added with a negative sign if we calculate the integral. However, we want to add them up as positive ordinates since the area has to be positive. We manage to do this by integrating the absolute value of the function.

$$Area = \int_a^b |f(x)|\,dx$$

In the same way, if we want to work out the area between the graph of two functions $f(x)$ and $g(x)$ with $a \leq x \leq b$, the right formula is

$$Area = \int_a^b |f(x) - g(x)|\,dx$$

The positive value of the area is then guaranteed.
Another interesting application of the definite integral is the calculation of the average or mean of a function on an interval $[a,b]$. If you want to calculate the average weight of a group of cats, you just have to add their weights and divide by the number of animals. With a function, we find a problem: we have infinitely many points in an interval. Nevertheless, we can consider a partition of the interval consisting of $N+1$ points equally separated. Then, we take the midpoint of the N subintervals

generated by the partition. If these points are called c_k, the average value of f over this set of points is

$$\sum_{k=1}^{N} f(c_k)\,/N$$

or

$$\frac{\sum_{k=1}^{N} f(c_k)\,\Delta x}{(b-a)}$$

with Δx the step between succesive points. If we consider finer and finer partitions, increasing N- considering the mean with more and more points- , and finally making it tend to infinity, we can define the average as the limit of Riemann sums divided by the length of the interval, that is

$$\frac{\int_a^b f(x)\,dx}{(b-a)}$$

This formula is useful to calculate means and we will also apply it for the proof of the Fundamental Theorem of Calculus in the next chapter.