$$\begin{tikzcd} T: V_{B_{st}} \arrow[r, "A"] \arrow[d, "P"'] & V_{B_{st}} \arrow[d, "P"] \\ B \arrow[r, "D"] & B \end{tikzcd}$$

Diagonalizing $A$ (or $T$) =

$$\boxed{D = P^{-1}AP}$$

$\exists B$ where the matrix of $T$ is $D$?

$B$ is formed eigenvectors

$$D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_p \end{pmatrix}$$

$\exists B$? → yes (check cond.) → $\boxed{A = PDP^{-1}}$

→ No

## Conditions for Diagonalization

$T: V \to V$

$\dim V = n$

$V$ l.s. on $K$

$K = \mathbb{R}$ or $\mathbb{C}$

$T$ is diagonalizable $\iff$

(1) $\exists \lambda_1, \lambda_2, \dots, \lambda_p \in K$ :

$$p(\lambda) = (\lambda_1 - \lambda)^{m_1} (\lambda_2 - \lambda)^{m_2} \cdots (\lambda_p - \lambda)^{m_p}$$

$$m_1 + m_2 + \cdots + m_p = n$$

(2) $\dim S(\lambda_i) = m_i, \ \forall i = 1, \dots, p$

EX: $A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

$\boxed{1}$ $p(\lambda) = |A - \lambda I| = \begin{vmatrix} -\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & -\lambda \end{vmatrix} = \begin{vmatrix} -\lambda & 0 & 1-\lambda^2 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & 0 \end{vmatrix} =$

$\lambda$ $\qquad (3,1)$

$$= (-1)^{3+1} 1 \cdot (1-\lambda)(1-\lambda^2) = -(1-\lambda)^2 \underbrace{(1+\lambda)}_{=0} \Rightarrow \begin{cases} \lambda_1 = -1 & m_1 = 1 \\ \lambda_2 = 1 & m_2 = 2 \end{cases}$$

$\boxed{2}$ $\cdot\ S(\lambda_1) = S(-1)$ : $(A + I)X = 0 \Rightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\Rightarrow \begin{cases} x + z = 0 \\ 2y = 0 \end{cases} \Rightarrow \begin{cases} x = \alpha \\ y = 0 \\ z = -\alpha \end{cases}, \ \alpha \in \mathbb{R} \Rightarrow \boxed{B_{S(-1)} = \{ \vec{v}_1 = (1, 0, -1) \}}$$

$\cdot\ S(\lambda_2) = S(1)$ : $(A - I)X = 0 \Rightarrow \begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \iff$

$\cdot\ S(\lambda_2)=\underline{S(1)}:\ (A-I)X=0 \Rightarrow \begin{pmatrix}-1&0&1\\0&0&0\\1&0&-1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix} \Leftrightarrow$

$x-z=0 \Rightarrow \begin{cases}x=\alpha\\y=\beta\\z=\alpha\end{cases}\quad \alpha,\beta\in\mathbb{R} \Rightarrow \boxed{B_{S(1)}=\{\bar v_2=(1,0,1);\ \bar v_3=(0,1,0)\}}$

$\dim S(-1)=1=m_1$ yes!
$\dim S(1)=2=m_2$ yes! $\quad\Rightarrow$ we have (2)

$\Rightarrow$ A is diagonalizable matrix

$\boxed{3} \Rightarrow \exists B=B_{S(-1)}\cup B_{S(1)}=\{\bar v_1,\bar v_2,\bar v_3\}$ basis of $\mathbb{R}^3$

where the matrix of T is $D=\begin{pmatrix}-1&0&0\\0&1&0\\0&0&1\end{pmatrix}$

**Ex 2** $\quad \lambda_1\quad m_1=2$
$\lambda_2\quad m_2=1$

$\bar v_1,\bar v_2\in S(\lambda_1)\Rightarrow \begin{cases}T(\bar v_1)=\lambda_1\bar v_1\\T(\bar v_2)=\lambda_1\bar v_2\end{cases}$ $\qquad T: V_B \longrightarrow V_B$

$\bar v_3\in S(\lambda_2) \Rightarrow T(\bar v_3)=\lambda_2\bar v_3$

$B=\{\bar v_1,\bar v_2,\bar v_3\}$ is basis?? check it

If yes $\mathcal{M}(T;\underline{B})=A=\begin{pmatrix}T(\bar v_1)_B & T(\bar v_2)_B & T(\bar v_3)_B\end{pmatrix}=\begin{pmatrix}\lambda_1\bar v_{1B} & \lambda_1\bar v_{2B} & \lambda_2\bar v_{3B}\end{pmatrix}$

$[\lambda_1\bar v_1]_B=(\lambda_1,0,0)$
$[\lambda_1\bar v_2]_B=(0,\lambda_1,0) \quad\Rightarrow\quad A=\begin{pmatrix}\lambda_1&&0\\&\lambda_1&\\0&&\lambda_2\end{pmatrix}$
$[\lambda_2\bar v_3]_B=(0,0,\lambda_2)$

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A symmetric matrix $\Rightarrow$ A is ~~always~~ <u>DIAGONAL</u> in $\mathbb{R}$

$\exists B$ <u>orthogonal</u> where D

is orthogonal where D

**Ex 6** $A = \begin{pmatrix} 4 & -4 & 0 \\ -4 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}$

$\boxed{1}$ $p(\lambda) = |A - \lambda I| = \begin{vmatrix} 4-\lambda & -4 & 0 \\ -4 & 4-\lambda & 0 \\ 0 & 0 & 4-\lambda \end{vmatrix} = (4-\lambda)\left[(4-\lambda)^2 - 4^2\right] =$

$= (4-\lambda)(-\lambda)(8-\lambda) \Rightarrow \begin{cases} \lambda_1 = 0 & m_1 = 1 \\ \lambda_2 = 4 & m_2 = 1 \\ \lambda_3 = 8 & m_3 = 1 \end{cases}$

$\boxed{2}$ • $S(0)$: $AX = 0 \Rightarrow \begin{pmatrix} 4 & -4 & 0 \\ -4 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} x = \alpha \\ y = \alpha \\ z = 0 \end{cases}$

$B_{S(0)} = \{\bar{v}_1 = (1,1,0)\}$

• $S(4)$: $(A - 4I)X = 0 \Rightarrow \begin{pmatrix} 0 & -4 & 0 \\ -4 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} x = 0 \\ y = 0 \\ z = \alpha \end{cases}, \alpha \in \mathbb{R}$

$B_{S(4)} = \{\bar{v}_2 = (0,0,1)\}$

• $S(8)$: $(A - 8I)X = 0 \Rightarrow \begin{pmatrix} -4 & -4 & 0 \\ -4 & -4 & 0 \\ 0 & 0 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} x = \alpha \\ y = -\alpha \\ z = 0 \end{cases}$

$B_{S(8)} = \{\bar{v}_3 = (1,-1,0)\}$

$\Rightarrow B = B_{S(0)} \cup B_{S(4)} \cup B_{S(8)} = \{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$ basis of $\mathbb{R}^3$ where the matrix of T is $D = \begin{pmatrix} 0 & & 0 \\ & 4 & \\ 0 & & 8 \end{pmatrix}$

$\boxed{T}$ A symm. (matr. of T) $\lambda, \mu$ eigenvalues of T $\lambda \neq \mu$ $\Bigg| \rightarrow S(\lambda) \perp S(\mu)$

$$\Rightarrow \quad S(0) \perp S(4) \perp S(8) \perp S(0)$$

$$\Rightarrow \quad \bar{v}_1 \perp \bar{v}_2 \perp \bar{v}_3 \perp \bar{v}_1$$

$$\Rightarrow \quad B \text{ is orthogonal} \xrightarrow[\frac{\bar{v}_k}{\|\bar{v}_k\|}]{\text{Normalize}} B' \text{ orthonormal}$$