CALCULUS
DEGREE IN SOFTWARE ENGINEERING
CHAPTER 10. ROLLE'S THEOREM. MEAN VALUE THEOREM. APPLICATIONS.

In this chapter we use derivatives to locate and identify extreme (maximum or minimum) values of a function. We also apply them to determine where a function is increasing or decreasing. All this can be achieved by using some basic results, such as the First Derivative Theorem for Local Extreme Values, Rolle´s Theorem and the Mean Value Theorem.
First of all, we must define local extreme values:

DEFINITION OF LOCAL EXTREME VALUES (MAXIMUM OR MINIMUM)

A function f has a local maximum value at an interior point c of its domain D if $f(x) \leq f(c)$ for all $x \in D$ lying in some open interval containing c.
A function f has a local minimum value at an interior point c of its domain D if $f(x) \geq f(c)$ for all $x \in D$ lying in some open interval containing c.

Note that the word local in the definition is essential. We compare the value of f at c with its value at nearby points or, more precisely, in a local neigbourhood of the point. The definition also refers to an interior point. If we consider a closed interval $[a, b]$, the function attains a local maximum (minimum) at a if $f(x) \leq f(a)$ ($f(x) \geq f(a)$) for all x in some half-open interval $[a, a+\delta)$. At b, we could write a similar definition with x in $(b-\delta, b]$.

EXAMPLE

As a simple example, we show the graph of $f(x) = x^3/3 - 3x^2 + 8x + 2$. The function has two local extreme values. A local maximum at $x = 2$, and a local minimum at $x = 4$. Neither of them is an absolute extremum, as the function's range is $(-\infty, \infty)$. We have used MATLAB to plot the function. We can see that the slope of the function at the points where the local extreme values are attained is zero. Is this a general feature ?, what are the conditions for this to be true ? The answer is in the next theorem.

THE FIRST DERIVATIVE THEOREM FOR LOCAL EXTREME VALUES

If f has a local extreme value at an interior point c of its domain, and if f' is defined at c, then $f'(c) = 0$

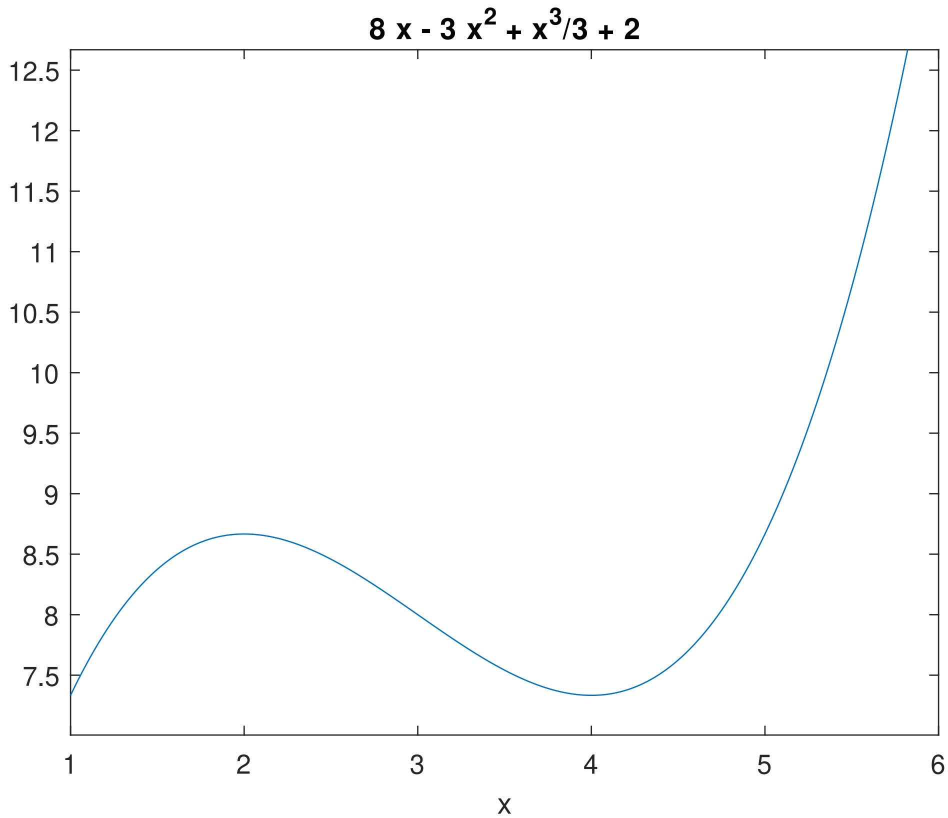

Figure 1: Local Extreme Values

The proof is quite simple. Consider that the function has a local minimum at c. Since f' is defined, we can write

$$f'_+(c) = \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c}$$

The numerator is positive or zero in some open interval containing c, the denominator is positive. Therefore, the quotient must be positive or zero and the limit cannot be negative. Why ?
If we write the left-hand derivative

$$f'_-(c) = \lim_{x \to c^-} \frac{f(x) - f(c)}{x - c}$$

the numerator is still positive or zero, but the denominator is negative, so that the quotient is negative or zero. In conclusion, the limit cannot be positive. But, the left-hand derivative and the right-hand derivative must be equal, one cannot be positive and the other cannot be negative. The derivative exists and cannot be either positive or negative: THE DERIVATIVE MUST BE ZERO. The proof for a local maximum is the same. Try to do it.

This theorem leads us to a classification of the possible points where a function has a local (it could be global) extreme value.
We have to look for the interior points where the derivative is zero or it is not defined and for the endpoints of the domain where the previous result cannot be applied.

DEFINITION OF CRITICAL POINT

An interior point of the domain of a function f where f' is zero or undefined is a critical point of f

We can say that critical points are the first suspects in the search for local maxima or minima. If they are not guilty, we must investigate the endpoints of the domain. However, it is possible that a function does not have local extrema. I remind you that, according to the Extreme Value Theorem, it will have absolute extrema (and then local, Why?) if it is continuous on a closed interval $[a, b]$.
x^3 is a function without local extrema on its domain $(-\infty, \infty)$. Of course, if we select an interval such as $[0, 1]$, it has a local maximum value (global) at $x = 1$ and a local (global) minimum value at $x = 0$. Is this true on $(0, 1)$?
Our next result will pave the way for the proof of the Mean Value Theorem and will also have some interesing applications.

ROLLE'S THEOREM

If a real-valued function f is continuous on a closed interval $[a, b]$, differentiable on the open interval (a, b), and $f(a) = f(b)$, then there exists at least one c in the open interval (a, b) such that

$$f'(c) = 0$$

The proof is very easy. Since f is continuous on $[a, b]$, it must attain an absolute maximum value and an absolute minimum value on that interval. If any of the extreme values is attained at an interior point c, this must be a critical point and $f'(c) = 0$. If the function has the maximum and minimum values at the endpoints, maximum and minimum must be equal, since $f(a) = f(b)$, the function is then constant and $f' = 0$ at any interior point.

EXAMPLE

Given the function $f(x) = x^3 + 3x + 1$, prove that it has only one zero.

We apply Bolzano's theorem after locating a change of sign, $f(0) = 1$, $f(-1) = -3$: there is at least one zero in (-1,0). To prove that there is only one, we take the derivative, $f'(x) = 3x^2 + 3$, and check that it is never zero. Now, we apply Rolle's theorem: if the function had two zeros, c_1 and c_2, there should be an intermediate point where the derivative should be zero, but that is impossible. So, there is only one zero.
Next, we state and prove the Mean Value Theorem

THE MEAN VALUE THEOREM

If $f(x)$ is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then there exists a point c in (a, b) such that the tangent at c is parallel to the secant line through the endpoints $(a, f(a))$, $(b, f(b))$, that is,

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Proof:

We define

$$g(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}\,(x - a)$$

This is the difference between the function and the secant joining $(a, f(a))$ and $(b, f(b))$. It is clear that $g(a) = 0$ and $g(b) = 0$. Therefore, if we apply Rolle's Theorem with

$$g'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}$$

g' must be zero at a certain c in (a, b) and

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

.

The geometrical interpretation of the theorem is clear in its statement. An example of a physical interpretation is given in the definition of average velocity of a moving body during a certain interval of time: displacement divided by time elapsed. Given the conditions of the Mean Value Theorem, quite reasonable in practical cases, the instantaneous velocity (the derivative) is equal to the average velocity at least once. We will see a few examples of the Mean Value Theorem in the exercises. This theorem can be generalized and is then called Cauchy's Mean Value Theorem (see Thomas' page 401 or search for it on the Internet). This generalization is used for a general proof of L'Hôpital's Rule.
Now, we will write two corollaries (simple consequences) of the Mean Value Theorem. These corollaries have very important applications as we will soon see.

COROLLARY ONE

If $f'(x) = 0$ at each point of an open interval (a, b), then $f(x) = C$ for all x in (a, b), where C is a constant.

Proof: We write the Mean Value Theorem for any two points x_1 and x_2 in (a, b)

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

.

Since $f'(c) = 0$, $f(x_1) = f(x_2)$, and this is true for any pair of points. Then, the function is constant in the interval.

COROLLARY TWO

Suppose that f is continuous on $[a, b]$ and differentiable on (a, b).
If $f'(x) > 0$ at each point in (a, b), then $f(x)$ is increasing on $[a, b]$.
If $f'(x) < 0$ at each point in (a, b), then $f(x)$ is decreasing on $[a, b]$.

Proof: We write the Mean Value Theorem for any two points x_1 and x_2 in $[a, b]$, with $x_2 > x_1$

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

.

If $f'(c) > 0$, $f(x_2) > f(x_1)$, and the function is increasing. If $f'(c) < 0$, $f(x_2) < f(x_1)$, and the function is decreasing.

Intuitively, this corresponds to the idea of positive slope for an increasing function, negative slope for a decreasing function. Note that if a function is increasing (or decreasing) on an interval, the derivative can be zero at some points.
We will finish this chapter with an example of the application of Corollary Two

EXAMPLE

Determine the intervals on which $f(x) = x^3 - 12x - 5$ is increasing/ decreasing.
We take the derivative $f'(x) = 3x^2 - 12 = 3(x+2)(x-2)$. f is increasing on $(-\infty, -2)$ and $(2, \infty)$ $(f' > 0)$, f is decreasing on $(-2, 2)$ $(f' < 0)$. Though we will use open intervals (as a convention), we could have written that f is decreasing on $[-2, 2]$. We cannot say that a function is increasing or decreasing at a single point (we need an interval). The function has two critical points, $x = -2$ and $x = 2$. The first is a maximum point, since the function's derivative changes from positive to negative (the function changes from increasing to decreasing). The second is a minimum point, since the function's derivative changes from negative to positive (the function changes from decreasing to increasing). This can be used as a general criterion for classifying critical points, provided f' is defined at every point in some interval containing c, the critical point, except possibly at c itself and f is continuous on the interval.
If f' does not change sign at c, there is neither maximum nor minimum value at c. Recall that the critical point is an interior point, this result cannot be applied to endpoints.
We finish the chapter with the graph of the function (Figure 2)

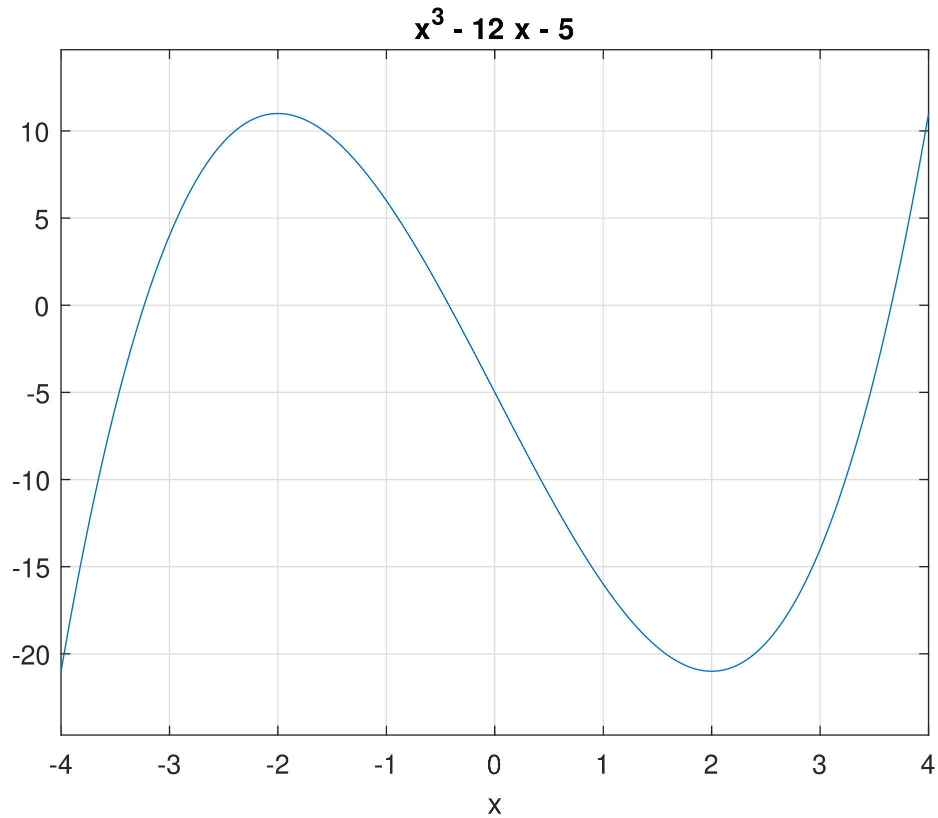

Figure 2: INCREASING, DECREASING