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S3-Euclidean Spaces

Design of scalar products II

Linear Algebra

Ingeniería del Software-Universidad de Oviedo

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Scalar product in $C^0_{[a,b]}$



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Scalar product in $C^0[a,b]$ and in $M_{m \times n}(\mathbb{R})$.

$$\vec{f}, \vec{g} \in C^0[a,b] \Rightarrow \vec{f} \cdot \vec{g} \stackrel{\text{def}}{=} \int_a^b f(x)g(x) dx$$

$$\|\vec{f}\|_2^2 \stackrel{\text{def}}{=} \vec{f} \cdot \vec{f} \Rightarrow \|\vec{f}\|_2 = \sqrt{\underbrace{\int_a^b f^2(x) dx}_0}$$

$\int_a^b f^2(x) dx > 0$ and
it's only 0 if $f(x) = 0(x)$

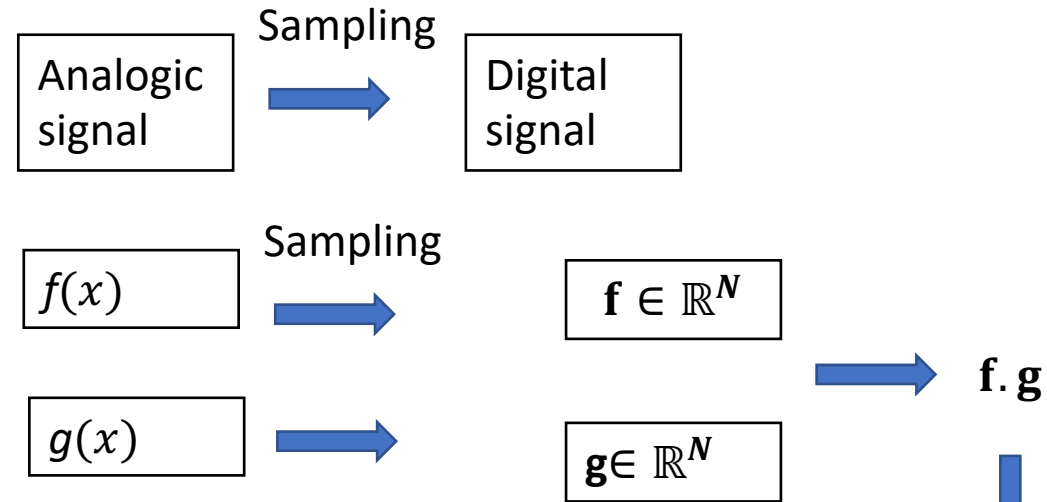
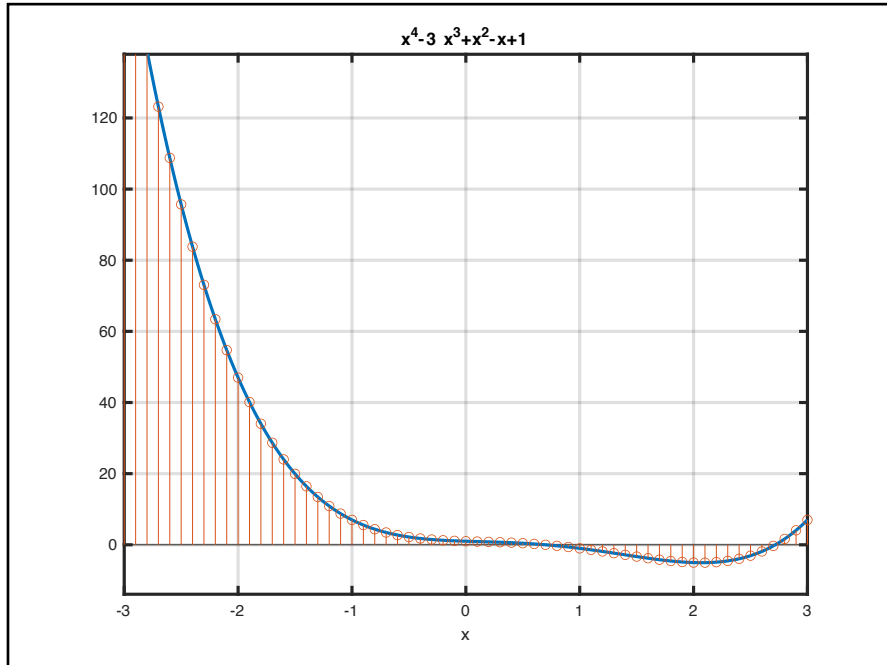
$$\cos(\vec{f}, \vec{g}) = \frac{\vec{f} \cdot \vec{g}}{\|\vec{f}\| \cdot \|\vec{g}\|}$$

Physically if f is an analogic
signal, then its norm
 $\|\vec{f}\|_2$ is its energy in $[a,b]$
↓
content of information

Scalar product in $C_{[a,b]}^0$



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When the sampling interval is approaching to 0

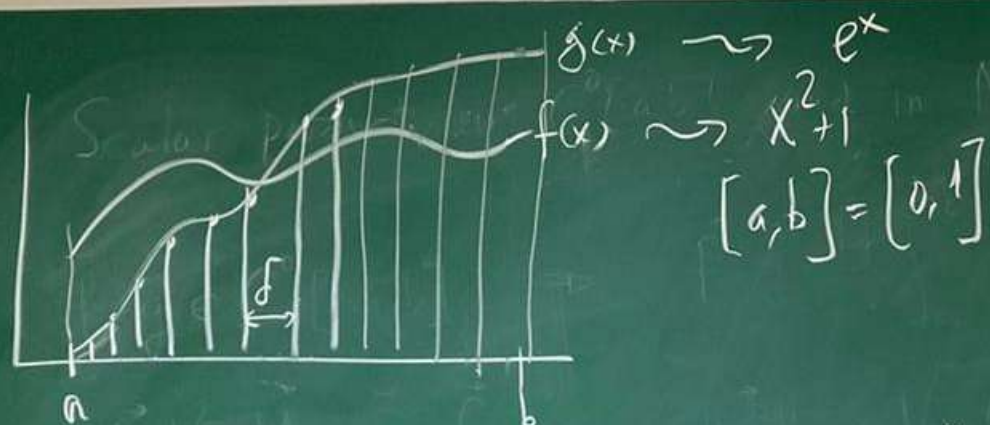
$$\langle f, g \rangle = \int_a^b f(x)g(x)dx$$

$$\cos \langle \mathbf{f}, \mathbf{g} \rangle = \frac{\mathbf{f} \cdot \mathbf{g}}{\|\mathbf{f}\| \|\mathbf{g}\|} = \frac{\int_a^b f(x)g(x)dx}{\sqrt{\int_a^b f^2(x)dx} \sqrt{\int_a^b g^2(x)dx}}$$

Scalar product in $C^0_{[a,b]}$



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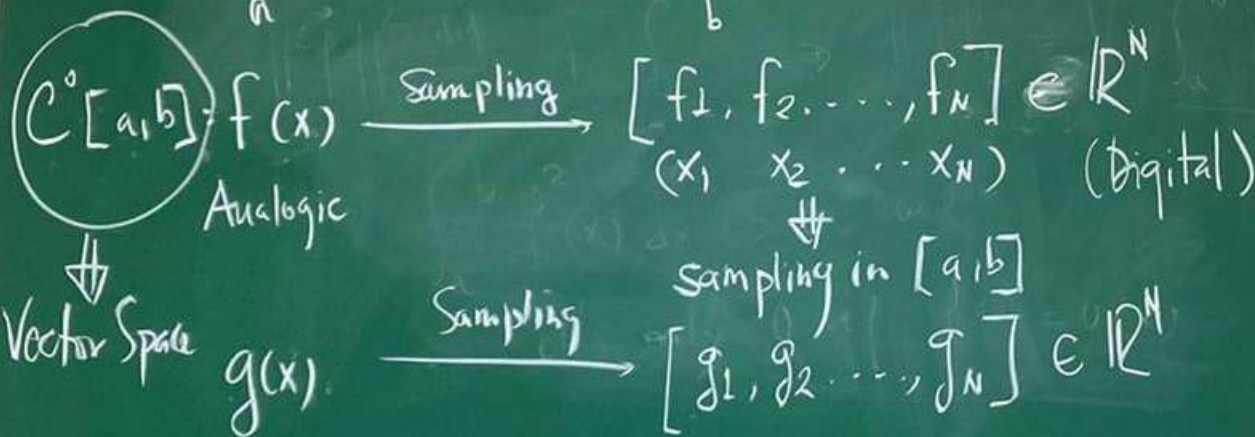
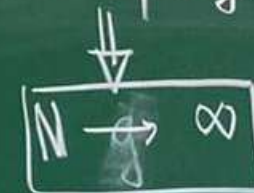
$$\vec{f} \cdot \vec{g} = \int_a^b f(x) \cdot g(x) \cdot dx$$

\mathbb{R}^N (digital)

$$\vec{f} \cdot \vec{g} = \sum_{k=1}^N f_k \cdot g_k$$

Euclidean Scalar Product in \mathbb{R}^N

Δ Sampling (δ) $\Rightarrow 0$



Entrywise Scalar Product in $M_{m \times n}(\mathbb{R})$



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$$M_{m \times n}(\mathbb{R}) \quad A, B \in M_{m \times n}(\mathbb{R})$$

I want to define $\vec{A} \cdot \vec{B}$ for $\vec{A}(\cdot) \cdot \vec{B}(\cdot)$ Columnwise

I know that $M_{m \times n}(\mathbb{R})$
is a vector space of dimension $m \times n$

↓
Euclidean Vector Space of Matrices

Entrywise scalar product.
pixel by pixel scalar product.

Element-wise
Schur Scalar Product

Rowwise

$$\boxed{\vec{A} \cdot \vec{B}}$$

$$A = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

$$\vec{A} \cdot \vec{B} = (1, -1, 0, 1) \cdot \begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix} =$$

$$= 1 - 2 + 1 = 0$$

$$\vec{A} \cdot \vec{B} = (1 \ 0 \ -1 \ 1) \begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \end{pmatrix} = 0$$

Entrywise Scalar Product in $M_{m \times n}(\mathbb{R})$



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$$M_{m \times n}(\mathbb{R}) \quad A, B \in M_{m \times n}(\mathbb{R})$$

More Generally $\vec{A} \cdot \vec{B} = \vec{a}^T \underset{\substack{\uparrow \\ \text{Gram matrix}}}{G} \vec{b}$

$$A \longrightarrow A(:,) = \vec{a}$$

$$B \longrightarrow B(:,) = \vec{b}$$

$M_{m \times n}$

$\mathbb{R}^{m \times n}$



Gram matrix
in $\mathbb{R}^{m \times n}$

$$\vec{a}^T \text{ is } A(:,)^T \text{ (row)}$$

$$\vec{b} \text{ is } B(:,)$$

$$\vec{A} \cdot \vec{B} = \vec{a}^T \underset{\substack{\uparrow \\ \text{Weight Matrix}}}{G} \vec{b}$$

Weight Matrix

By default.

$$(G = I_{m \times n})$$

$$\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

Cauchy-Schwartz



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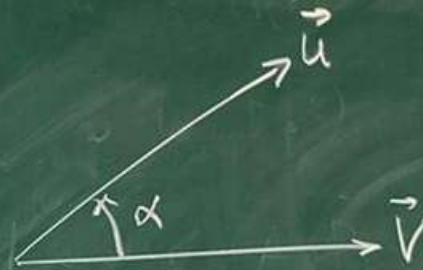
V , • Euclidean Space

$$\vec{u}, \vec{v} \in V$$

$$\cos(\vec{u}, \vec{v}) \stackrel{\text{def}}{=} \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \cdot \|\vec{v}\|} \leq 1$$

$$V (\mathbb{R}^n, C^0[a,b], M_{m \times n}(\mathbb{R}))$$

This works because of the Cauchy Schwartz inequality.



$$|\cos \alpha| \leq 1 \text{ Restriction}$$

The cosine definition is well-posed

$$|\vec{u} \cdot \vec{v}| \leq \|\vec{u}\| \cdot \|\vec{v}\| \Rightarrow$$

$$\left| \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \cdot \|\vec{v}\|} \right| \leq 1$$

$\cos \alpha$

Orthogonality



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$$d(\vec{u}, \vec{v}) = \|\vec{u} - \vec{v}\|$$

distance between \vec{u} and \vec{v}

\vec{u} is independent from \vec{v}

Orthogonality

$\vec{u} \perp \vec{v}$
according to

\Leftrightarrow

$$\vec{u} \cdot \vec{v} = 0$$

\Leftrightarrow

Orthogonality
Perpendicularity
Independence

Originality
Innovation

$\vec{u} \parallel \vec{v}$

\Leftrightarrow

$$\vec{u} \in \langle \vec{v} \rangle \Leftrightarrow \vec{u} = \lambda \vec{v}$$

Parallelism

Redundancy

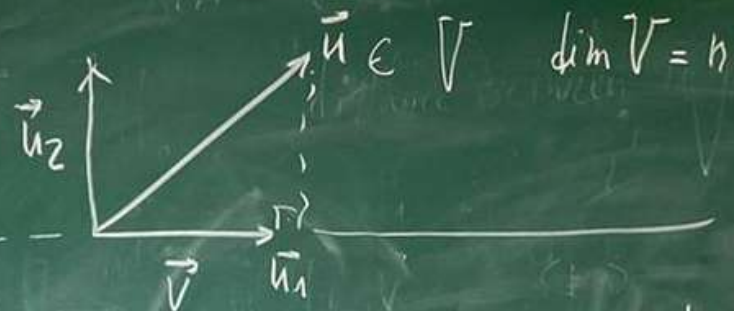
Dependency

Mediocrity

$$\vec{u} \cdot \vec{v} = (\lambda \vec{v}) \cdot \vec{v} = \lambda \|\vec{v}\|^2$$

\parallel

Orthogonality



$$\langle \vec{v} \rangle \quad \dim \langle \vec{v} \rangle = 1$$

\vec{u}_1 is the orthogonal projection of \vec{u} onto $\langle \vec{v} \rangle$

$\vec{u}_2 \in \langle \vec{v} \rangle^\perp$ orthogonal supplement to $\langle \vec{v} \rangle$
complement

$$\vec{u} = \vec{u}_1 + \vec{u}_2$$

$$\vec{u}_1 = \lambda \vec{v} \quad \text{and} \quad \vec{u}_2 \cdot \vec{v} = 0$$

$$\vec{u} = \lambda \vec{v} + \vec{u}_2$$

\vec{u}_2 orthogonal to $\langle \vec{v} \rangle$.

$$\vec{u} \cdot \vec{v} = \lambda \vec{v} \cdot \vec{v} + \underbrace{\vec{u}_2 \cdot \vec{v}}_{=0}$$

$$\Rightarrow \lambda = \frac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}} = \frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|^2}$$

$$\vec{u}_1 = \lambda \vec{v}$$

$$\vec{u}_2 = \vec{u} - \vec{u}_1$$

Cq d