1. Scalar Product

Exercise 1 *Consider the following scalar product in* $\mathbb{R}^2$*:*

$$(x_1,x_2)\cdot(y_1,y_2) = x_1y_1 - x_2y_1 - x_1y_2 + 4x_2y_2$$

Compute the Gram matrix and the associated quadratic form. Compute $(1,0)^2$ *and* $(-1,1)^2$.

Exercise 2 *Compute the Gram matrix and the scalar product associated to the following quadratic form:*

$$q(x_1,x_2) = 3{x_1}^2 - x_1x_2 + 2{x_2}^2$$

Compute $(1,2)\cdot(-1,1)$.

Exercise 3 *a) Compute the Gram matrix respect of the standard basis of* $\mathbb{R}^3$*, associated to the scalar product:*

$$\vec{u}\cdot\vec{v} = x_1y_1 + x_1y_3 + 2x_2y_2 - x_2y_3 + x_3y_1 - x_3y_2 + 2x_3y_3$$

b) Using the matrix form ($\vec{u}\cdot\vec{v} = X^tGY$)*, compute* $(1,2,3)\cdot(1,0,-1)$.

Exercise 4 *a) Compute the Gram matrix respect of the standard base of* $\mathbb{R}^3$ *associated to the scalar product:*

$$(x_1,x_2)\cdot(y_1,y_2) = x_1y_1 + x_1y_2 + x_2y_1 + 2x_2y_2 + 3x_3y_3 - x_1y_3 - x_3y_1$$

(taking into account that $g_{ij} =$ *is the coefficient of* x_iy_j *in the scalar product)*

b) Compute the Gram matrix respect of the standard base of $\mathbb{R}^3$ *associated to the quadratic form:*

$$q(x_1,x_2,x_3) = 2{x_1}^2 + {x_2}^2 + {x_3}^2 - 2x_1x_2 + x_1x_3$$

(taking into account that the elements in the diagonal $g_{ii} =$ *are the coefficients of* ${x_i}^2$ *and the rest* $g_{ij} = \frac{1}{2}$*coeff* x_ix_j*, with* $i \neq j$*)*

Exercise 5 *Obtain the Gram matrix in each case, and decide which of them is actually a scalar product:*

a) $\vec{u}\cdot\vec{v} = x_1y_1 - x_2y_2 - x_1y_2 + 4x_2y_2$*, in* $\mathbb{R}^2$

b) $\vec{v}^2 = 2x^2 + 2xy + 2y^2 - 2yz + 3z^2$ *in* $\mathbb{R}^3$.

Exercise 6 *We consider the following scalar product in* $\mathbb{R}_2[X]$*:*

$$p \cdot q = \int_0^1 p(x)\, g(x)\, dx\,.$$

a) Compute the Gram matrix referred to the standard basis $\mathbb{R}_2[X]$.

b) Compute the angle between the polynomials $p_1 = -x + 2x^2$ *y* $p_2 = 1 + x + x^2$.

Exercise 7 *Considering the scalar product in* $\mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow \mathbb{R}$

$$\vec{u} \cdot \vec{v} = x_1y_1 + (a-1)x_1y_2 + x_1y_3 + 2ax_2y_2 - x_2y_3 + x_3y_1 - x_3y_2 + bx_3y_3,$$

dónde a y b son parámetros reales y los vectores están expresados en la base canónica de $\mathbb{R}^3$.

a) Obtain the Gram matrix and compute the real values of a and b which give us a scalar product.

b) Choosing a pair of values, compute the Gram matrix in $B = \{(1,1,1),(0,1,1),(0,0,1)\}$.

c) Compute $\|(1,0,1)\|$.