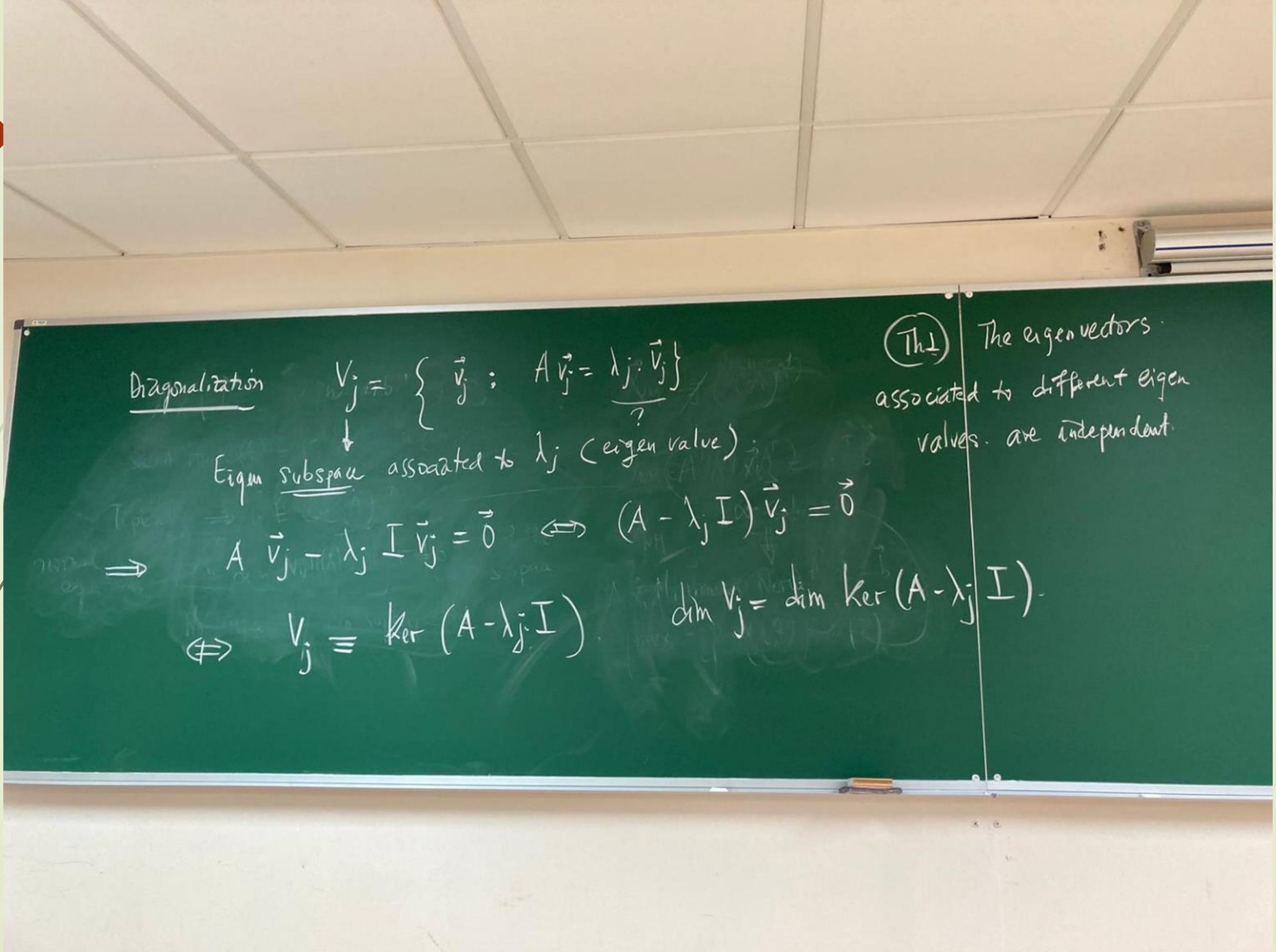
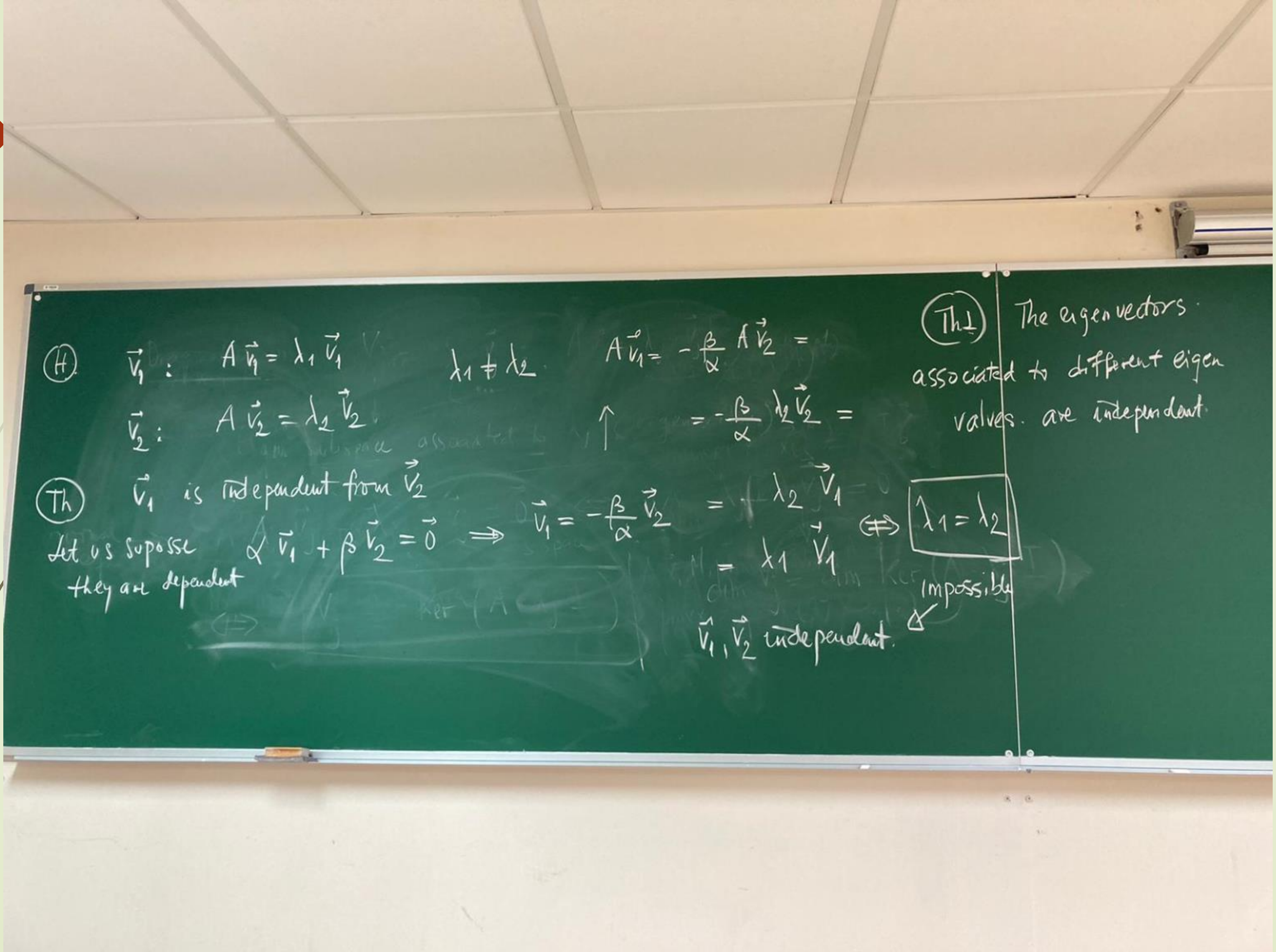
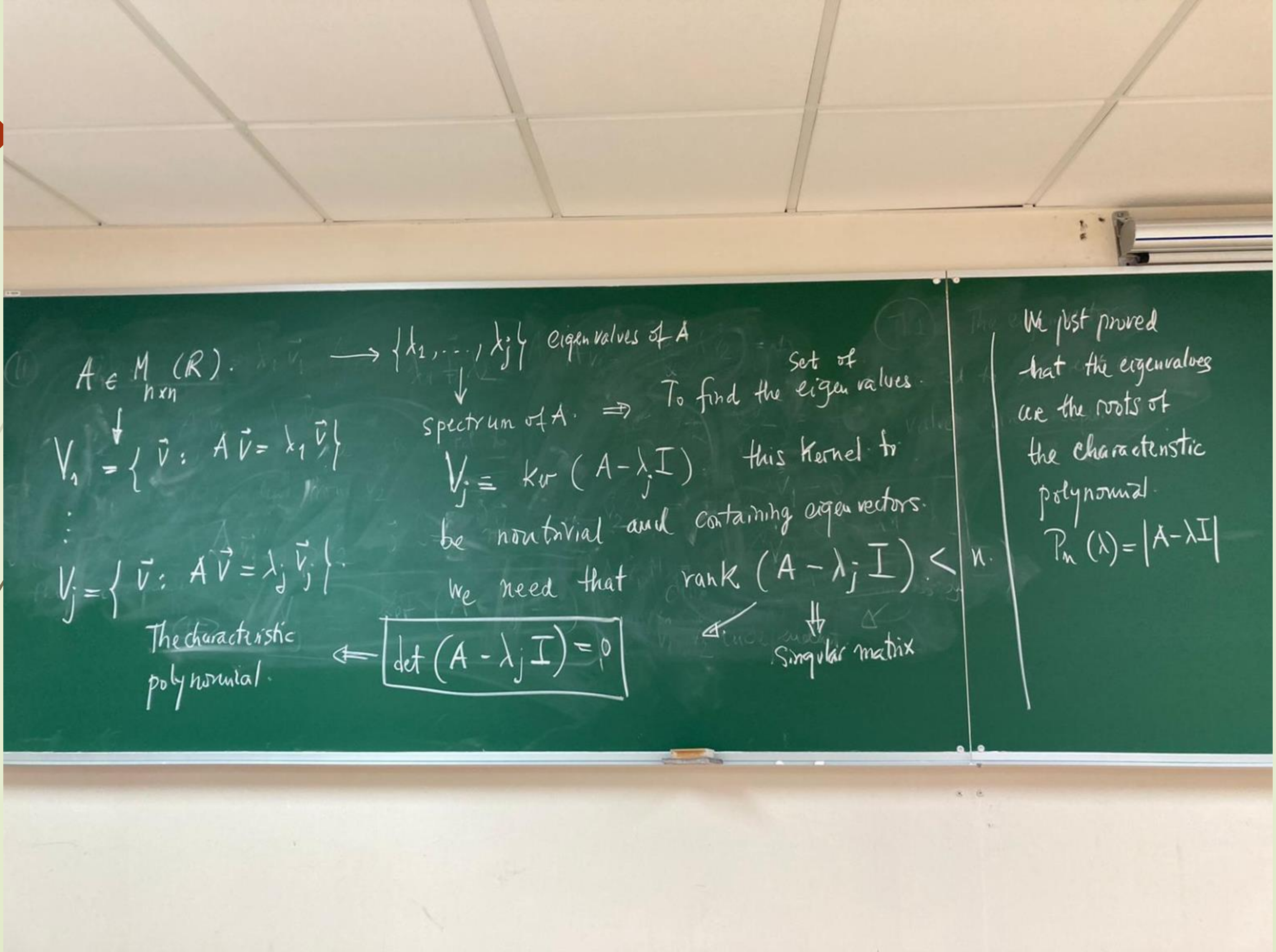
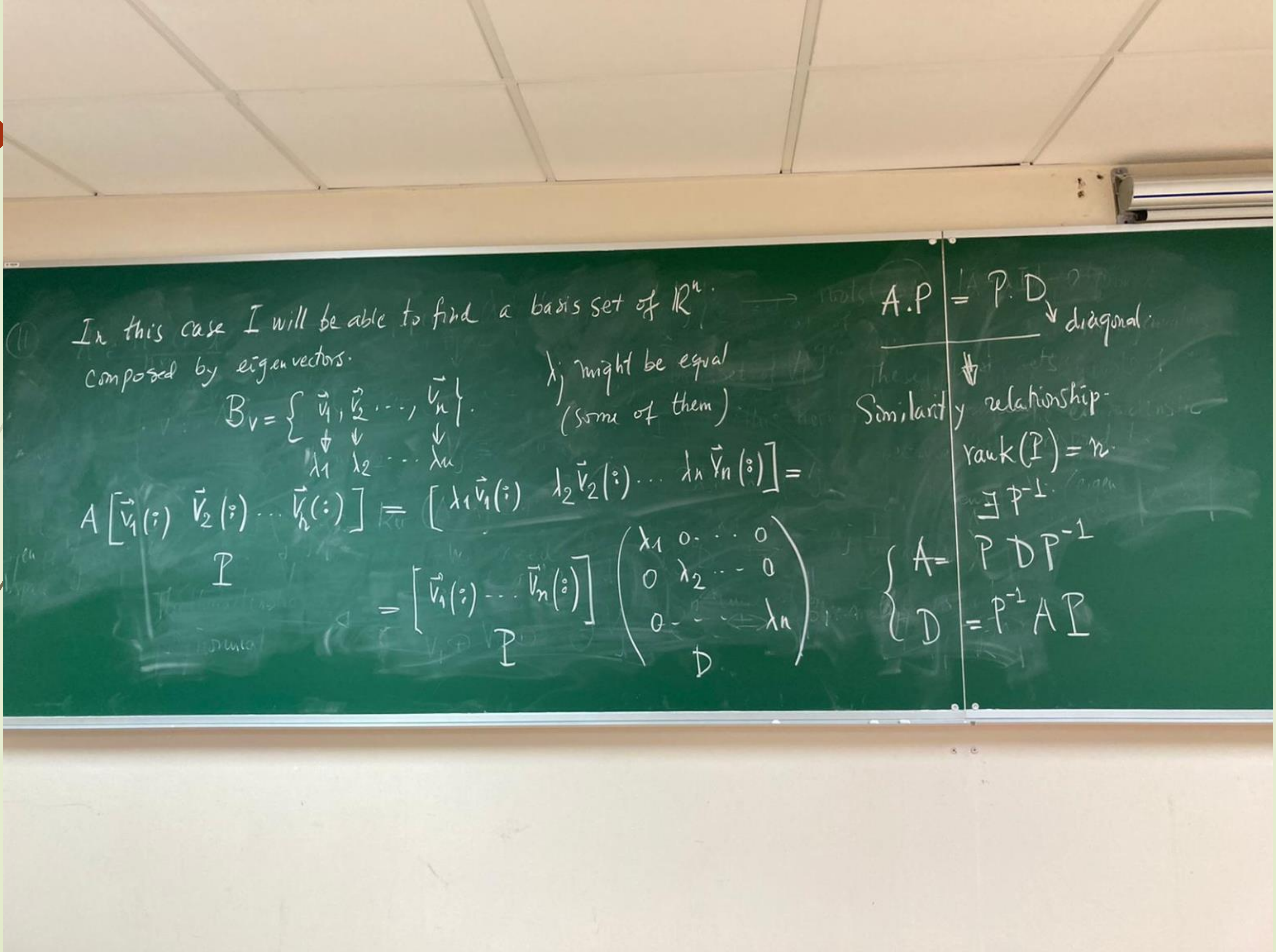
Diagonalization of Endomorphisms

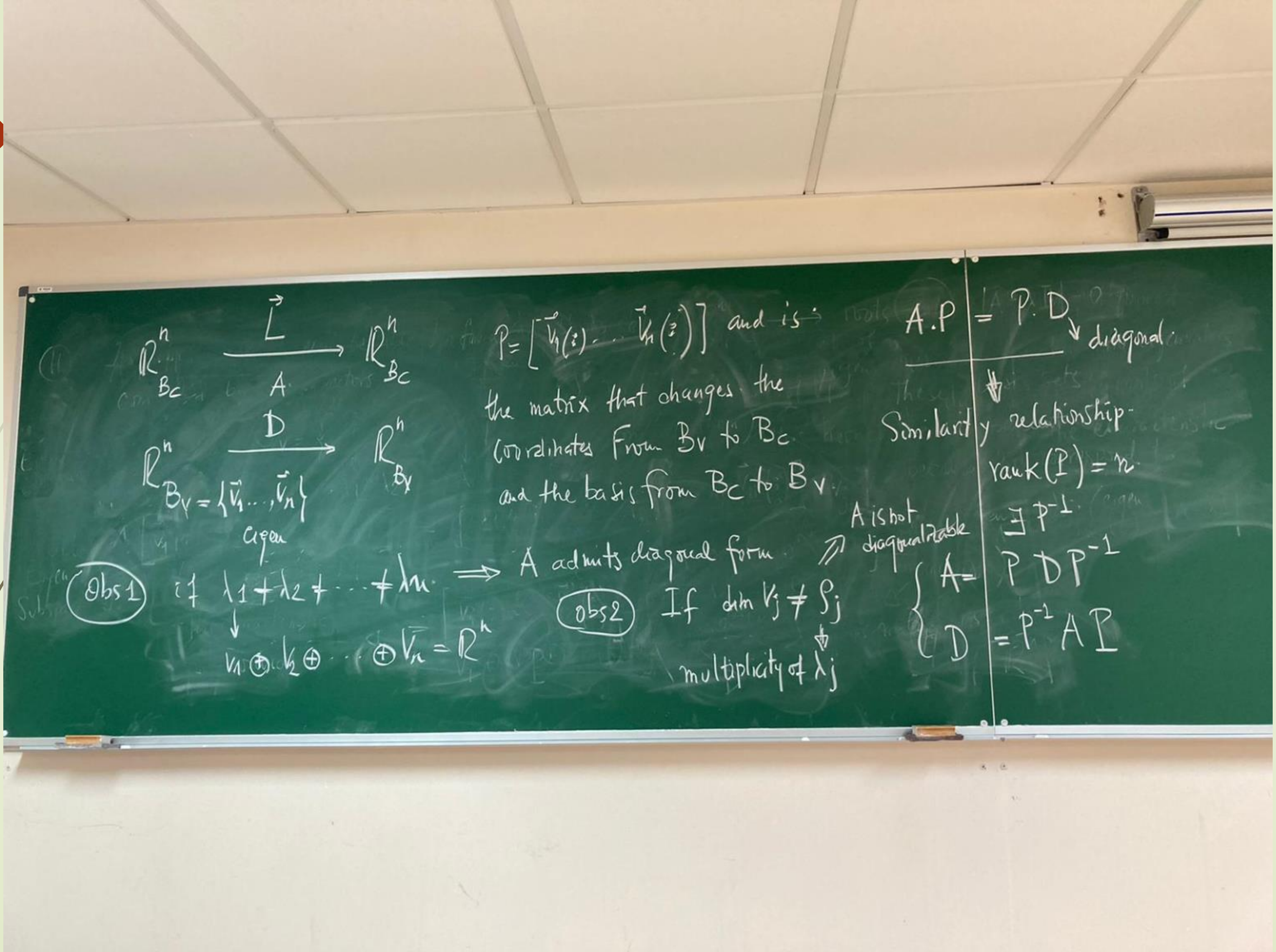
Session 2


Diagonalization
$V_j = \{ \vec{v}_j : A\vec{v}_j = \lambda_j \cdot \vec{v}_j \}$
?
Eigen subspace associated to λ_j (eigen value)
$\Rightarrow A\vec{v}_j - \lambda_j I \vec{v}_j = \vec{0} \Leftrightarrow (A - \lambda_j I)\vec{v}_j = \vec{0}$
$\Leftrightarrow V_j \equiv \operatorname{Ker}(A - \lambda_j I)$
$\dim V_j = \dim \operatorname{Ker}(A - \lambda_j I)$.
Th1 The eigenvectors associated to different eigen values are independent.

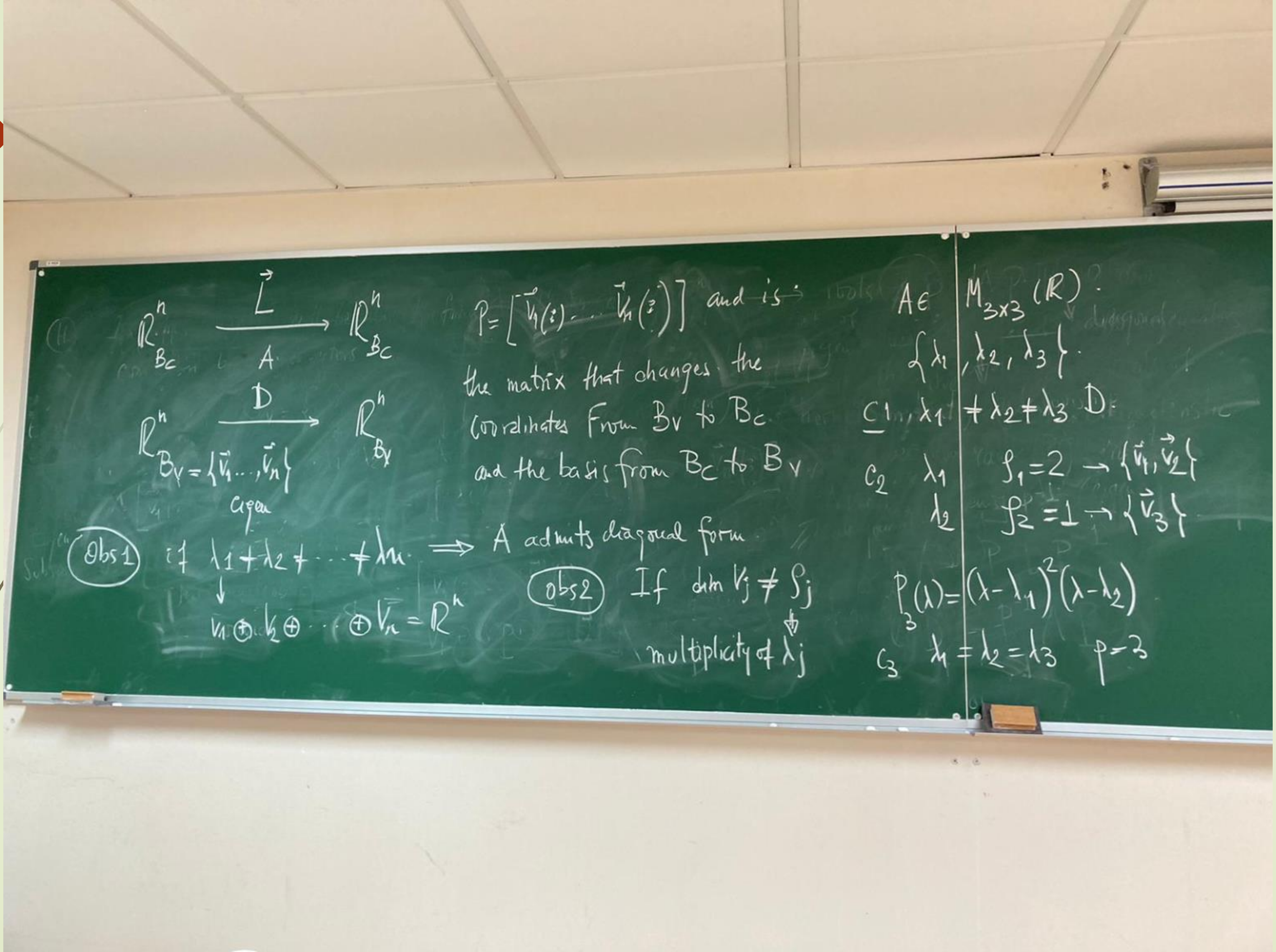

(H) $\vec{v}_1$: $A\vec{v}_1 = \lambda_1 \vec{v}_1$
$\vec{v}_2$: $A\vec{v}_2 = \lambda_2 \vec{v}_2$
$\lambda_1 \neq \lambda_2$
(Th) $\vec{v}_1$ is independent from $\vec{v}_2$
Let us supose $\alpha \vec{v}_1 + \beta \vec{v}_2 = \vec{0} \Rightarrow \vec{v}_1 = -\frac{\beta}{\alpha}\vec{v}_2$
they are dependent
$A\vec{v}_1 = -\frac{\beta}{\alpha} A\vec{v}_2 =$
$= -\frac{\beta}{\alpha}\lambda_2 \vec{v}_2 =$
$= \lambda_2 \vec{v}_1$
$= \lambda_1 \vec{v}_1$
$\Leftrightarrow \boxed{\lambda_1 = \lambda_2}$
impossible
$\vec{v}_1, \vec{v}_2$ independent.
(Th1) The eigenvectors associated to different eigen values are independent.

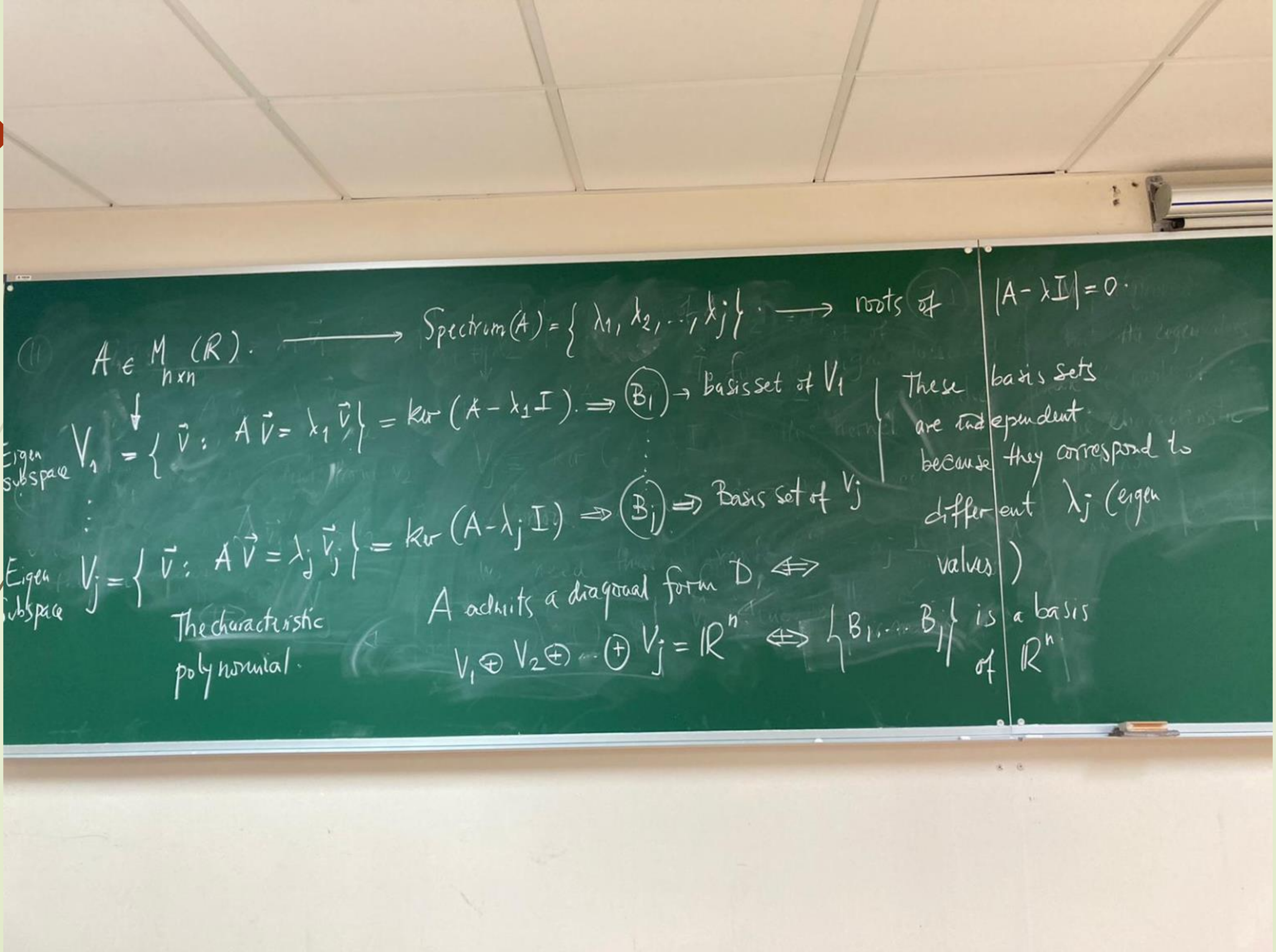

$A \in M_{n \times n}(\mathbb{R})$ → $\{\lambda_1, \dots, \lambda_j\}$ eigenvalues of A
spectrum of A ⇒ To find the Set of eigen values
$V_1 = \{\vec{v} : A\vec{v} = \lambda_1 \vec{v}\}$
$V_j = \{\vec{v} : A\vec{v} = \lambda_j \vec{v}_j\}$
$V_j = \ker(A - \lambda_j I)$ this Kernel to be nontrivial and containing eigen vectors.
we need that rank $(A - \lambda_j I) < n$.
Singular matrix
The characteristic polynomial ⇐ $\det(A - \lambda_j I) = 0$
We just proved that the eigenvalues are the roots of the characteristic polynomial.
$P_n(\lambda) = |A - \lambda I|$


In this case I will be able to find a basis set of $\mathbb{R}^n$.
composed by eigenvectors.
$B_v = \{ \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \}$.
$\lambda_1 \quad \lambda_2 \quad \ldots \quad \lambda_n$
λ_i might be equal
(some of them)
$A \left[\vec{v}_1(:) \ \vec{v}_2(:) \ldots \vec{v}_n(:) \right] = \left[\lambda_1 \vec{v}_1(:) \ \lambda_2 \vec{v}_2(:) \ldots \lambda_n \vec{v}_n(:) \right] =$
P
$= \left[\vec{v}_1(:) \ldots \vec{v}_n(:) \right] \begin{pmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ 0 & \ldots & \ldots & \lambda_n \end{pmatrix}$
P
D
$A.P = P.D$
diagonal
Similarity relationship
$\mathrm{rank}(P) = n$
$\exists P^{-1}$
$A = P D P^{-1}$
$D = P^{-1} A P$

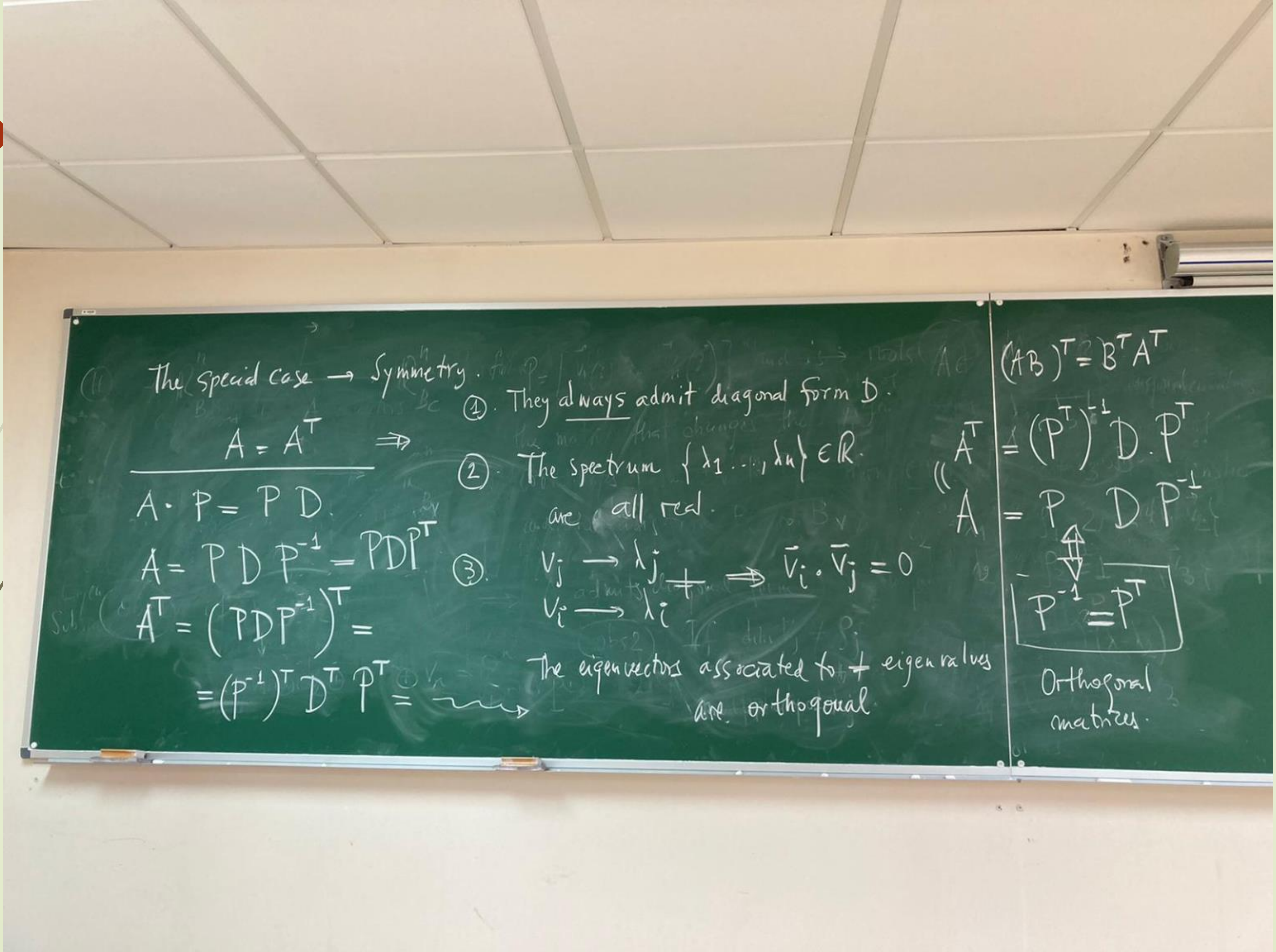


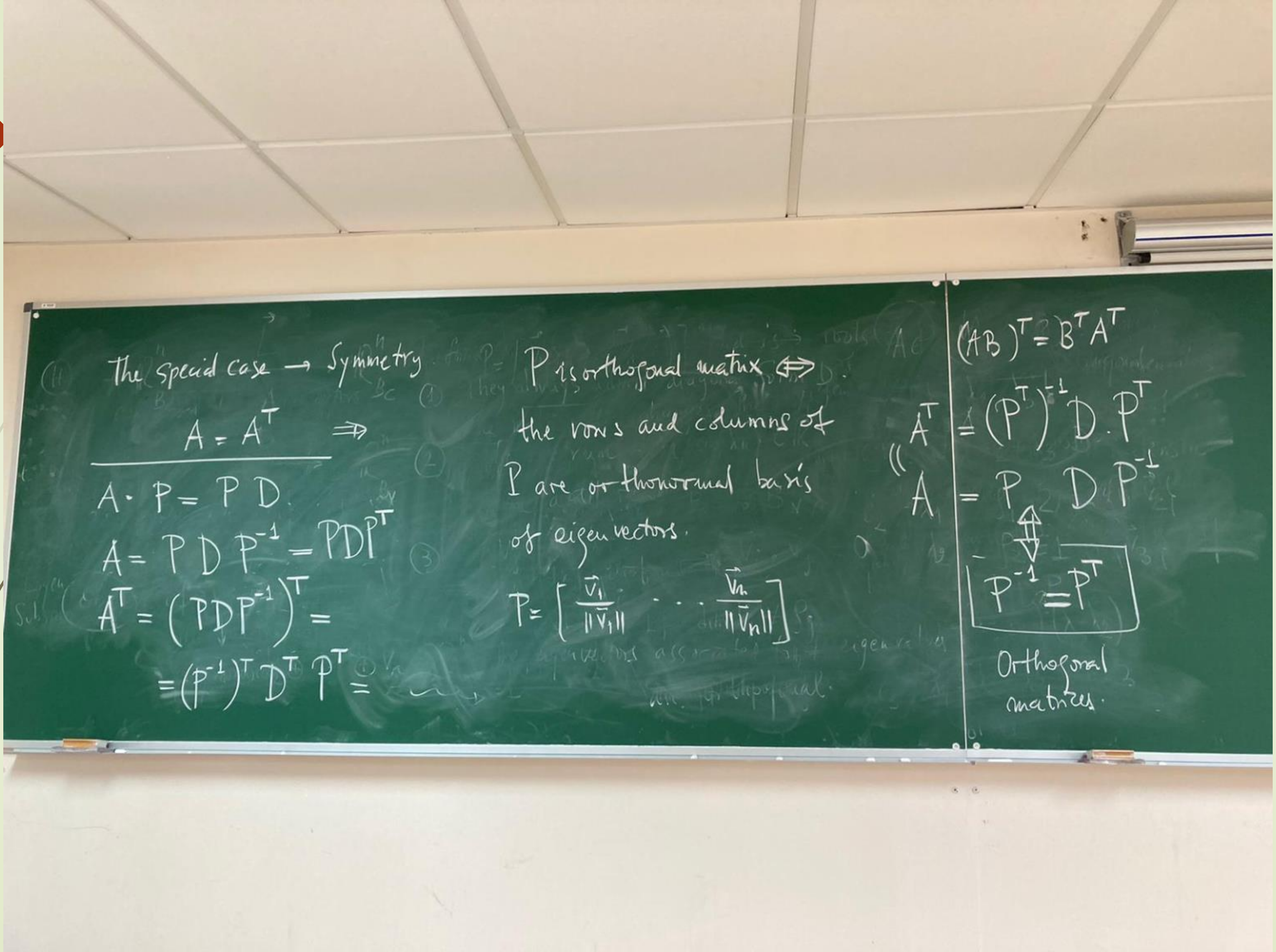
$\mathbb{R}^n_{B_c} \xrightarrow[A]{\vec{L}} \mathbb{R}^n_{B_c}$
$\mathbb{R}^n_{B_V} \xrightarrow{D} \mathbb{R}^n_{B_V}$
$B_V = \{\vec{v}_1, \dots, \vec{v}_n\}$
eigen
$P = [\vec{v}_1(\vdots) \dots \vec{v}_n(\vdots)]$ and is
the matrix that changes the
coordinates from B_V to B_c
and the basis from B_c to B_V
Obs 1
if $\lambda_1 \neq \lambda_2 \neq \dots \neq \lambda_n \Rightarrow$ A admits diagonal form
$V_1 \oplus V_2 \oplus \dots \oplus V_n = \mathbb{R}^n$
A is not diagonalizable
Obs 2
If $\dim V_j \neq \rho_j$
multiplicity of λ_j
$A \cdot P = P \cdot D$ diagonal
Similarity relationship
$\text{rank}(P) = n$
$\exists P^{-1}$
$A = P D P^{-1}$
$D = P^{-1} A P$


$\mathbb{R}^n_{B_c} \xrightarrow[A]{\vec{L}} \mathbb{R}^n_{B_c}$
$\mathbb{R}^n_{B_v} \xrightarrow{D} \mathbb{R}^n_{B_v}$
$B_v = \{\vec{v}_1 \dots, \vec{v}_n\}$
eigen
$P = [\vec{v}_1(:) \dots \vec{v}_n(:)]$ and is
the matrix that changes the
coordinates From B_v to B_c
and the basis from B_c to B_v
Obs1 if $\lambda_1 \neq \lambda_2 \neq \dots \neq \lambda_n \Longrightarrow$ A admits diagonal form
$V_1 \oplus V_2 \oplus \dots \oplus V_n = \mathbb{R}^n$
obs2 If $\dim V_j \neq \rho_j$
multiplicity of λ_j
$A \in M_{3\times 3}(\mathbb{R})$.
$\{\lambda_1, \lambda_2, \lambda_3\}$.
C1, $\lambda_1 \neq \lambda_2 \neq \lambda_3$ D.
C2 λ_1 $\rho_1 = 2 \rightarrow \{\vec{v}_1, \vec{v}_2\}$
λ_2 $\rho_2 = 1 \rightarrow \{\vec{v}_3\}$
$P_3(\lambda) = (\lambda-\lambda_1)^2(\lambda-\lambda_2)$
C3 $\lambda_1 = \lambda_2 = \lambda_3$ $\rho = 3$

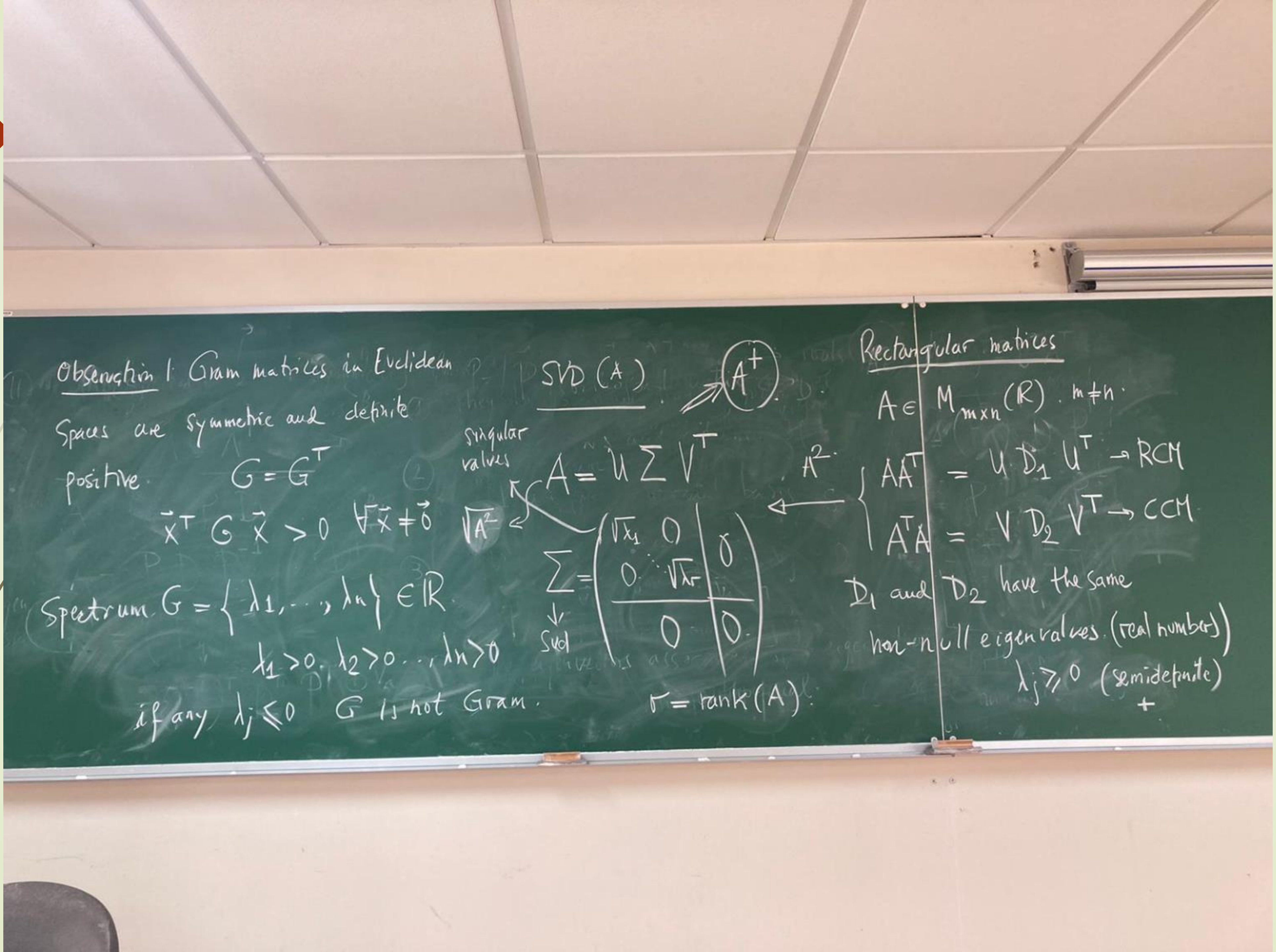


$Spectrum(A) = \{ \lambda_1, \lambda_2, \dots, \lambda_j \}$ → roots of $|A - \lambda I| = 0$.
(1) $A \in M_{n \times n}(\mathbb{R})$.
Eigen subspace $V_1 = \{ \vec{v} : A\vec{v} = \lambda_1 \vec{v} \} = \ker(A - \lambda_1 I) \Rightarrow B_1$ → Basis set of V_1
Eigen subspace $V_j = \{ \vec{v} : A\vec{v} = \lambda_j \vec{v}_j \} = \ker(A - \lambda_j I) \Rightarrow B_j \Rightarrow$ Basis set of V_j
These basis sets are independent because they correspond to different λ_j (eigen values)
A admits a diagonal form D $\Leftrightarrow$
$V_1 \oplus V_2 \oplus \dots \oplus V_j = \mathbb{R}^n \Leftrightarrow \{B_1, \dots, B_j\}$ is a basis of $\mathbb{R}^n$
The characteristic polynomial.


The special case → Symmetry.
A = A^T ⇒
A · P = P D.
A = P D P^-1 = P D P^T
A^T = (P D P^-1)^T =
= (P^-1)^T D^T P^T =
① They always admit diagonal form D.
② The spectrum {λ1 ..., λn} ∈ R.
are all real.
③ vj → λj
vi → λi
⇒ vi · vj = 0
The eigenvectors associated to ≠ eigenvalues
are orthogonal
(AB)^T = B^T A^T
A^T = (P^T)^-1 D . P^T
A = P D P^-1
P^-1 = P^T
Orthogonal matrices.



The special case → Symmetry
A = A^T ⇒
A · P = P D
A = P D P^-1 = PDP^T
A^T = (PDP^-1)^T =
= (P^-1)^T D^T P^T =
P is orthogonal matrix ⇔
the rows and columns of
P are orthonormal basis
of eigenvectors.
P = [v1/||v1|| . . . vn/||vn||]
(AB)^T = B^T A^T
A^T = (P^T)^-1 D . P^T
A = P D P^-1
P^-1 = P^T
Orthogonal matrices.


Observation I Gram matrices in Euclidean
Spaces are symmetric and definite
positive
$G = G^T$
$\vec{x}^T G \vec{x} > 0 \quad \forall \vec{x} \neq \vec{0}$
Spectrum $G = \{\lambda_1, \ldots, \lambda_n\} \in \mathbb{R}$
$\lambda_1 > 0, \lambda_2 > 0, \ldots, \lambda_n > 0$
if any $\lambda_j \leq 0$ G is not Gram.
SVD (A)
A^+
singular values
$\sqrt{A^2}$
$A = U \Sigma V^T$
A^2
$\Sigma = \begin{pmatrix} \sqrt{\lambda_1} & 0 & 0 \\ 0 & \sqrt{\lambda_r} & \\ 0 & & 0 \end{pmatrix}$
Svd
$r = \text{rank}(A)$
Rectangular matrices
$A \in M_{m \times n}(\mathbb{R})$. $m \neq n$.
$AA^T = U D_1 U^T \to$ RCM
$A^T A = V D_2 V^T \to$ CCM
D_1 and D_2 have the same
non-null eigenvalues (real numbers)
$\lambda_j \geqslant 0$ (semidefinite)
+