CALCULUS
DEGREE IN SOFTWARE ENGINEERING
CHAPTER 6. CONTINUOUS FUNCTIONS.

Let us assume that our real-valued function is defined on a closed interval $[a, b]$. Then, we define the concept of continuity of the function at a point.

DEFINITION OF CONTINUITY AT A POINT

Interior point: A function $y = f(x)$ is continuous at an interior point c of its domain if

$$\lim_{x \to c} f(x) = f(c)$$

Endpoint: A function $y = f(x)$ is continuous at a left endpoint a or is continuous at a right endpoint b if

$$\lim_{x \to a^+} f(x) = f(a)$$

or

$$\lim_{x \to b^-} f(x) = f(b)$$

The idea is quite simple: the limit at the point must exist and be equal to the value of the function. If the point is interior, the limit is a two-sided limit, if the point is an endpoint, the limit is one-sided. We summarize this information:

CONTINUITY TEST

A function is continuous at an interior point of its domain if and only if it meets the following three conditions:

1. $f(c)$ exists   (c lies in the domain of f)
2. $\lim_{x \to c} f(x)$ exists (f has a finite limit at $x = c$)
3. $\lim_{x \to c} f(x) = f(c)$ (the limit equals the function value)

At an endpoint, we just substitute the left-hand or right-hand limit for the limit in the definition. At an endpoint, we can say that the function is left-continuous or right-continuous.
If a function is not continuous at a point, we say that it is discontinuous at that point. Now, we will present the main types of discontinuities with the corresponding examples

1. Removable discontinuity. In this case, the limit at the point exists, but the function either is not defined at the point or its value does not equal the limit. It is clear that we can make the function continuous at the point just by assigning the value of the limit to the value of the function. For instance

$$f(x) = \frac{\sin x}{x}$$

is not defined at $x = 0$, but the limit is 1, by defining $f(0) = 1$ we have removed the discontinuity.

2. Jump discontinuity. In this second case, we have

$$\lim_{x \to c^-} f(x) = L_1$$

and

$$\lim_{x \to c^+} f(x) = L_2$$

Both are finite, but $L_1 \neq L_2$ and there is a jump in the graph of the function at $x = c$. For instance, the function defined as: $f(x) = x$ if $x < 0$, $f(x) = x + 2$ if $x \geq 0$ has a jump discontinuity at $x = 0$. The step function $f(x) = 1$ if $x > 0$, $f(x) = 0$ if $x \leq 0$ is a typical discontinuous function with a jump discontinuity at $x = 0$.

3. Infinite discontinuity. If the limit is infinite at the point, we have an infinite discontinuity. For instance $f(x) = 1/x$ is not defined at the origin. the limit from the left is $-\infty$ and the right-hand limit is ∞. There is clearly a discontinuity that is not removable and is not a jump discontinuity. As the limits are infinite, it is called an infinite discontinuity. Take into account that even if both limits have the same sign, there is an infinite discontinuity. As you can see from the example, this type of discontinuity is associated to vertical asymptotes.

4. Oscillating discontinuity. A function can have a different type of discontinuity. For instance $g(x) = \sin(1/x)$ at $x = 0$, the function oscillates and does not approach any value. We have the same behaviour on the left and the right of the point. The limit is not infinite, since the function is bounded. The function oscillates between -1 and 1 and for this reason, the discontinuity is oscillating.

Then, we present the basic properties of continuous functions

PROPERTIES OF CONTINUOUS FUNCTIONS

If the functions f and g are continuous at $x = c$, then the following combinations are continuous at $x = c$
Sum: $f + g$
Difference $f - g$
Constant multiple: $k.f$, for any number k
Products: $f.g$
Quotients: f/g, provided $g(c) \neq 0$
Powers: f^n, n a positive integer
Roots $f^{1/n}$, provided it is defined on an open interval containing c

These results follow from the limit laws. To combine all kind of continuous functions, we need the following property:

COMPOSITE OF CONTINUOUS FUNCTIONS

If f is continuous at c and g is continuous at $f(c)$, then the composite function $g \circ f$ is continuous at c.

A proof of this statement can be found in Thomas' chapter 2, pag. 78.
A function is continuous on an interval if it is continuous at all the points of the interval. At the endpoints, in the sense of left-continuous and right-continuous. We say that a function is continuous if it is continuous at every point of its domain. Using the properties of continuous functions, we can see that polynomials and rational functions are continuous on their domains. The elementary functions: absolute value, roots, exponentials, logarithms, trigonometric, inverse trigonometric.... are also continuous on their domains. Combining and composing, we can create as many continuous functions as we want. However, we have to be a little cautious with respect to the domain. For instance, is $1/x$ a continuous function ? The answer is yes, because it is continuous on its domain that excludes the point $x = 0$. However, is it continuous on the interval $[-1, 1]$? No, because it is not continuous at 0. This will be important when we study the theorems about continuous functions on closed intervals $[a, b]$. Another example, is $y = \ln x$ continuous ? Yes, it is continuous on its domain $(0, \infty)$. Is it continuous on $[0, \infty]$? No, it is not defined at the origin. The important properties of continuous functions on closed, bounded intervals will be treated in the next chapter.