CALCULUS
DEGREE IN SOFTWARE ENGINEERING
EXERCISES AND SOLUTIONS 3

1. Calculate the derivative of the following fuctions

(a) $y = x^5 + 6x^4 - 3x^3 + 2x^2 + 10x + 1$

$y' = 5x^4 + 24x^3 - 9x^2 + 4x$

(b) $y = \dfrac{x^2 + 3x}{x + 5}$

$y' = \dfrac{(2x + 3)(x + 5) - (x^2 + 3x)}{(x + 5)^2} = \dfrac{x^2 + 10x + 15}{(x + 5)^2}$

(c) $y = \sqrt{x^3 + 3x^2 + 5x}$

$y' = \dfrac{3x^2 + 6x + 5}{2\sqrt{x^3 + 3x^2 + 5}}$

(d) $y = x^x$

$y' = x^x(1 + \ln x)$

(e) $y = cscx$

$y' = -\csc x \cot x$

(f) $y = \sin(\cos(x^2 + 2))$

$y' = \cos(\cos(x^2 + 2)).(-\sin(x^2 + 2)).2x$

(g) $y = \sqrt{\arcsin(x - 1)}$

$y' = \dfrac{1}{2\sqrt{\arcsin(x - 1)}}.\dfrac{1}{\sqrt{2x - x^2}}$

(h) $y = \arccos \dfrac{2x}{5 + x}$

$y' = -\dfrac{10}{(5 + x)\sqrt{-3x^2 + 10x + 25}}$

(i) $y = x \, \tan x$

$y' = x(1 + \tan^2 x) + \tan x$

(j) $y = \dfrac{x + \sin x}{x - \sin x}$

$y' = \dfrac{2x \cos x - 2 \sin x}{(x - \sin x)^2}$

2. Given the function $f(x) = \dfrac{|x^2 - 4x| + 16}{x^2}$

   (a) Study the differentiability of $f(x)$.

   It is not differentiable at x=0 (the function is not even continuous there). It is not differentiable either at x=4 (the right and left derivatives are not equal at that point). We calculate the derivatives below.

   (b) Determine the derivative function where it exists.

   (c) $f'(x) = \dfrac{(2x-4)x^2 - (x^2 - 4x + 16).2x}{x^4} = \dfrac{4x - 32}{x^3}$ if $x < 0$ or $x > 4$.

   $f'(x) = \dfrac{(-2x+4)x^2 - (-x^2 + 4x + 16).2x}{x^4} = \dfrac{-4x - 32}{x^3}$ if $0 < x < 4$.

   You have to apply that $|x^2 - 4x| = x^2 - 4x$ in $(-\infty, 0) \bigcup [4, \infty)$ whereas $|x^2 - 4x| = -x^2 + 4x$ in $(0, 4)$. Then, $f'_+(4) = -1/4$ and $f'_-(4) = -3/4$

3. Find the points at which $f(x) = |x^2 + 6x + 8|$ has no derivative. Give reasons for the answer.
   $x = -2$ and $x = -4$. Right and left derivatives do not have the same value. We write the function as $f(x) = x^2 + 6x + 8$ in $(-\infty, -4] \bigcup [-2, \infty)$ and $f(x) = -x^2 - 6x - 8$ in $(-4, -2)$. Then, $f'_-(-4) = -2$, $f'_+(-4) = 2$, $f'_-(-2) = -2$ and $f'_+(-2) = 2$.

4. We know that $f : [0,\ 5] \to \mathbf{R}$

$$f(x) = \begin{cases} bx^2 + ax & 0 \le x \le 2 \\ c + \sqrt{x - 1} & 2 < x \le 5 \end{cases}$$

   is differentiable in $(0, 5)$ and $f(0) = f(5)$. What are the values of a, b and c ?

   For $f(x)$ to be differentiable, it has to be continuous. Therefore, $(bx^2 + ax)(2) = (c + \sqrt{x-1})(2)$, that is, $4b + 2a = c + 1$. Besides, $f(0) = 0 = f(5) = 2 + c$, $c = -2$ and $4b + 2a = -1$. The derivatives at 2 are $f'_-(2) = (2bx + a)(2) = 4b + a$, $f'_+(2) = 1/2$. They must be equal, so that $4b + a = 1/2$. Solving the system:

   $a = -3/2$, $b = 1/2$. We also found $c = -2$ before.

5. Calculate the following limits, applying a suitable method:

I) $\lim\limits_{x\to 0}\left(\dfrac{1}{ln(1+x)}-\dfrac{1}{x}\right)$ II) $\lim\limits_{x\to+\infty}\dfrac{e^x+cosx}{e^x}$

III) $\lim\limits_{x\to 0}\dfrac{e^{3x}-e^x}{\sin 2x}$ IV) $\lim\limits_{x\to 0^+}\dfrac{ln(\sin\ 2x)}{ln(\sin\ x)}$

V) $\lim\limits_{x\to 0}\dfrac{\sin\ 5x}{\sin\ 2x}$ VI) $\lim\limits_{x\to 1}\dfrac{\tan(x^2-1)}{x-1}$

VII) $\lim\limits_{x\to 1}\dfrac{2^{x-1}-1}{x-1}$ VIII) $\lim\limits_{x\to\frac{1}{2}}\dfrac{ln(4x-1)}{2x-1}$

IX) $\lim\limits_{x\to 0^+} x\ln x$ X) $\lim\limits_{x\to 0^+} x^x$

We use L'Hôpital's Rule in most cases

I)

$$\lim_{x\to 0}\left(\frac{1}{ln(1+x)}-\frac{1}{x}\right)=\lim_{x\to 0}\frac{x-\ln(1+x)}{x\ \ln(1+x)}$$

Taking the derivatives and simplifying, we can write

$$\lim_{x\to 0}\frac{x}{x+(1+x)\ \ln(1+x)}$$

and differentiating again

$$\lim_{x\to 0}\frac{1}{2+\ln(1+x)}=1/2$$

II)

Here, we cannot solve the limit by using L'Hôpital's Rule. We divide numerator and denominator by e^x and see that $\cos x/e^x$ tends to zero. Why ? Then, the limit is 1.

III) The indeterminate form is of the 0/0 type. Taking derivatives, the limit can be written as

$$\lim_{x\to 0}\frac{3e^{3x}-e^x}{2\cos 2x}=1$$

IV) Now, we have ∞/∞. Differentiating

$$\lim_{x\to 0^+}\frac{2\sin x\cos 2x}{\sin 2x\cos x}=\lim_{x\to 0^+}\frac{2\sin x}{\sin 2x}=\lim_{x\to 0^+}\frac{2\cos x}{2\cos 2x}=1$$

We differentiated a second time after removing the cosines from the first quotient. Why ?

V) Now, the indeterminate form is 0/0. Taking derivatives

$$\lim_{x \to 0} \frac{\sin 5x}{\sin 2x} = \lim_{x \to 0} \frac{5 \cos 5x}{2 \cos 2x} = 5/2$$

VI) 0/0 again. Applying L'Hôpital's Rule

$$\lim_{x \to 1} \frac{\tan(x^2 - 1)}{x - 1} = \lim_{x \to 1} \frac{2x\,(1 + \tan(x^2 - 1))}{1} = 2$$

VII) 0/0

$$\lim_{x \to 1} \frac{2^{x-1} - 1}{x - 1} = \lim_{x \to 1} \frac{2^{x-1}\ \ln 2}{1} = \ln 2 = 0.6931....$$

VIII)

$$\lim_{x \to 1/2} \frac{\ln(4x - 1)}{2x - 1} = \lim_{x \to 1/2} \frac{2}{4x - 1} = 2$$

IX) An indeterminate form $0. - \infty$. We write it as a quotient and apply L'Hôpital's Rule

$$\lim_{x \to 0^+} \frac{\ln x}{1/x} = \lim_{x \to 0^+} \frac{1/x}{-1/x^2} = 0$$

The term x is dominant with respect to $\ln x$ as x approaches 0.

X) We have to write $x^x = e^{x \ln x}$ first. Then, we take the limit of the exponent. According to our last exercise, it is zero and since the exponential function is continuous, the final limit is

$$e^0 = 1$$