# Practice 3. Functions of a real variable: Derivatives

## 1. Calculation of derivatives

To calculate the *derivative function* of a function, MATLAB has the command `diff`:

| | |
|---|---|
| `diff(f,x)` | Calculates the derivative of the symbolic expression $f$ with respect to the variable $x$. If the variable is not specified, MATLAB will choose one by default. |
| `diff(f,x,n)` | Calculates the $n$-order derivative of $f$ with respect to the variable $x$. |

For example: (1) For $f(x) = x^2$, calculate $f'(x)$ , that is, $\dfrac{df}{dx}$ .

(2) For $z = \cos(xy)$ , calculate $\dfrac{dz}{dx}$ and $\dfrac{dz}{dy}$.

(3) For $g(x) = x \sin x$ , calculate $g'''(x)$.

(4) For $h(x) = \ln x$ , calculate $h''(x)$.

(5) For $f(x) = \dfrac{1}{x}$ , calculate $f'(x)$, $f''(x)$ y $f'''(x)$.

```
>> syms x y
>> diff(x^2,x)
>> diff(cos(x*y),x)
>> diff(cos(x*y),y)
>> diff(x*sin(x),x,3)
>> diff(log(x),x,2)
>> f=1/x
>> diff(f,x), diff(f,x,2), diff(f,x,3)
```

### 1.1. Example

Plot the graphs of the functions

$$f(x) = \frac{x}{1+x^2},$$

of its derivative and of its second derivative in the interval $[-3, 3]$ (in the same graph). Notice the relation between the extrema and the inflection points in the graphs of the three functions.

*Solution.*- First, we calculate the first and the second derivative of $f$.

```
>> syms x
>> f=x/(1+x^2); pretty(f)
>> df=diff(f,x)
>> pretty(df)
>> simplify(df)
>> pretty(ans)
>> d2f=diff(f,x,2), pretty(d2f)
>> d2f=simplify(d2f), pretty(d2f)
```

We plot the function and its derivatives on the interval $[-3,3]$. After having graphed the three functions, we could change the color of the different lines by using the buttons on the graphic window. In order to distinguish the three functions, we can add a legend with the command `legend` .

```
>> ezplot(f,[-3,3])
>> hold  on
>> ezplot(df,[-3,3])
>> ezplot(d2f,[-3,3])
>> grid on
>> legend('f','df','d2f')
```

### 1.2. Example

Identify the intervals on which the following function is increasing and on which it is decreasing. Analyze its concavity

$$f(x) = \frac{x^2-4}{x^3}.$$

Calculate its asymptotes and the intercepts with the axes.

*Solution.-* We calculate the first derivative of $f$ and the points at which this derivative is zero (the critical points):

```
>> syms x
>> f=(x^2-4)/x^3; pretty(f)
>> df=diff(f)
>>crit=solve(df,x)
crit =
-2*3^(1/2)
 2*3^(1/2)
```

Thus, the monotonicity intervals of the function are $(-\infty,-2\sqrt{3})$, $(-2\sqrt{3},0)$, $(0,2\sqrt{3})$, $(2\sqrt{3},+\infty)$ (Note that the function is not defined at 0). We study the sign of the first derivative on the monotonicity intervals. To evaluate a symbolic expression at a point we use the command `subs`:

```
>> subs(df,x,-4)
>> subs(df,x,-1)
```

We obtain $f'(-4) = -0.0156 < 0$ and $f'(-1) = 11 > 0$ , therefore, the function is decreasing on $(-\infty,-2\sqrt{3})$ and increasing on $(-2\sqrt{3},0)$ , besides, since $f$ is odd, it is increasing on $(0,2\sqrt{3})$ and decreasing on $(2\sqrt{3},+\infty)$ . In this case, the function is odd and by studying its monotonicity on the negative semiaxis, we can deduce its behaviour on the positive semiaxis. If there were no symmetries, we should carry out similar calculations for the positive semiaxis. We see that the function has a local minimum at the first critical point and a local maximum at the second. Why ?

We obtain the points at which the second derivative is zero (possible inflection points):

```
>> d2f=diff(f,2)
>> infl=solve(d2f,x)
infl =
2*6^(1/2)
-2*6^(1/2)
```

Thus, The concavity intervals are $(-\infty,-2\sqrt{6})$, $(-2\sqrt{6},0)$, $(0,2\sqrt{6})$, $(2\sqrt{6},+\infty)$. We study the sign of the second derivative on these intervals:

```
>> subs(d2f,x,-5)
>> subs(d2f,x,-1)
```

We obtain that $f''(-5) = -6.4000 \cdot 10^{-4} < 0$ and $f''(-1) = 46 > 0$ , therefore, the function is concave down on $(-\infty, -2\sqrt{6})$ and $(0, 2\sqrt{6})$ and concave up on $(-2\sqrt{6}, 0)$ and $(2\sqrt{6}, +\infty)$ (using again the symmetry of the function). Both points where the second derivative is zero are inflection points in this case. Why ?

Now we graph the function and verify that the results previously obtained agree with the graphic representation.

```
>> ezplot(f,[-8,8])
>> grid on
```

To calculate the asymptotes:

```
>> limit(f,x,0,'right')
ans =
-Inf
>> limit(f,x,0,'left')
ans =
Inf
```

Then, the line $x = 0$ is a vertical asymptote.

To calculate the oblique asymptotes, first we calculate the slope $m$ and, if $m \in \mathbb{R}$, then we calculate $n$:

```
>> m=limit(f/x,x,inf)
m =
0
>> n=limit(f-m*x,x,inf)
n =
0
```

Therefore, the line $y = 0$ is a horizontal asymptote as $x$ approaches $+\infty$, and by symmetry , it is also a horizontal asymptote as $x$ approaches $-\infty$.

To calculate the intercepts of the graph of $f$ with the axis $x$, we solve the equation $f(x) = 0$:

```
>> solve(f,x)
ans =
 -2
  2
```

Therefore, it intersects the axis $x$ at points $(2, 0)$ and $(-2, 0)$. Note that there is no intersection with the $y$ axis, because the function is not defined at the point $x = 0$.

## 1.3. Example

Calculate the point of the line $2x + y = 1$ closest to the point $(2, 1)$.

*Solution.-* We must calculate the minimum of the euclidean distance on the plane

$$d((x_1, y_1), (x_2, y_2)) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

We can ignore the square root, because if the distance (as a function of $x$ and $y$) has a minimum at a point, its square (squared distance function) also attains a minimum at the same point. Thus, the problem is equivalent to finding a minimum of the function

$$F(x, y) = (x - 2)^2 + (y - 1)^2,$$

where $x$ e $y$ must satisfy the equation

$$2x + y = 1.$$

Substituting this expression in $F$ we obtain a function of one variable:

$$f(x) = (x - 2)^2 + (1 - 2x - 1)^2 = 5x^2 - 4x + 4.$$

We search for the critical points of $f$:

```
>> syms x y
>> f=(x-2)^2+(y-1)^2
>>f=subs(f,y,1-2*x)
f=(x - 2)^2 + 4*x^2
>> df=diff(f)
df =
10*x-4
>> solve(df)
ans =
2/5
```

We verify that the function has a minimum value at 2/5:

```
>> d2f=diff(f,2)
d2f =
10
```

The second derivative of $f$ is the constant function $f''(x) = 10$. Thus the value of $f''$ at $2/5$ is positive and the function has an absolute minimum at $2/5$.

The calculation of the second derivative can be complicated in some cases. To see if a function has a maximum or a minimum value at a point, we can use the criterion of the first derivative. The calculations — that are very simple in this case — can be carried out by using the command `subs`.

```
>> subs(df,x,0)
ans =
-4
>> subs(df,x,3)
ans =
26
```

The function is decreasing on $(-\infty, 2/5)$ and increasing on $(2/5, +\infty)$ , then it has an absolute minimum at $2/5$.

Now we compute the corresponding value of $y$:

```
>> y=1-2*2/5
y =
1/5
```

The requested point is $(2/5, 1/5)$.

We check the result graphically. To do this, we calculate the line perpendicular to $2x + y = 1$ that passes through the point $(2, 1)$ and obtain

$$y = x/2.$$

Finally we represent in the same graph the line $2x + y = 1$, its perpendicular $y = x/2$, and the points $(2, 1)$ and $(2/5, 1/5)$.

```
>> ezplot(1-2*x,[-1,4])
>> hold on
>> ezplot(x/2,[-1,4])
>> axis equal
>> axis([-1,4,-1,2])
>> plot(2,1,'r*')     % draws the point (2,1) in red
>> plot(2/5,1/5,'m*') %  draws (2/5,1/5) in magenta
```

We have fixed the same scale for the axes (axes equal) so that both lines look perpendicular.

## 2. Taylor polynomials

To find the Taylor polynomial of a given function we use the command:

| | |
|---|---|
| `taylor(f,x,a,'order',n)` | Calculates the Taylor polynomial of $f(x)$ of order $n-1$ at the point $a$. |
| `taylortool` | Is an interactive calculator of Taylor polynomials |

For example, for the functions $f(x) = e^x$ and $g(x) = \cos x$ , we are going to calculate:

(1) the Taylor polynomial of $f$ of order 4 at the point 0, (or the MacLaurin polynomial of $f$ of order 4)

(2) the Taylor polynomial of $f$ of order 6 at the point 0, (or the MacLaurin polynomial of $f$ of order 6)

(3) the Taylor polynomial of $f$ of order 3 at the point 2,

(4) the Taylor polynomial of $g$ of order 4 at the point $\pi/4$,

```
>> syms x
>> f=exp(x), g=cos(x)
>> taylor(f,x,0,'order',5)
>> taylor(f,x,0,'order',7)
>> taylor(f,x,2,'order',4)
>> taylor(g,x,pi/4,'order',5)
```

Note that, in this last case, we obtain the polynomial:

$$T(x) = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right) - \frac{\sqrt{2}}{4}\left(x - \frac{\pi}{4}\right)^2 + \frac{\sqrt{2}}{12}\left(x - \frac{\pi}{4}\right)^3 + \frac{\sqrt{2}}{48}\left(x - \frac{\pi}{4}\right)^4.$$

### 2.1. Example

Represent the function $f(x) = \sin x$ and its MacLaurin polynomials of order 1, 3 and 5 on the interval $[-\pi, \pi]$.

*Solution.-* First we calculate the polynomials:

```
>> syms x
>> f=sin(x);
>> tf1=taylor(f,x,0,'order',2), tf3=taylor(f,x,0,'order',4), tf5=taylor(f,x,0,'order',6)
tf1 =
x
tf3 =
x-1/6*x^3
tf5 =
x-1/6*x^3+1/120*x^5
```

Now we make the graphical representation of $f$ and its Taylor polynomials. Once the functions are plotted, we could change the color, types, etc. of the individual curves. We add a legend to distinguish them.

```
>> ezplot(f,[-pi,pi])
>> hold on
>> ezplot(tf1,[-pi,pi])
>> ezplot(tf3,[-pi,pi])
>> ezplot(tf5,[-pi,pi])
>> grid on
>> legend('f','tf1','tf3','tf5')
```

## 3. Exercises

1. Find the intervals of monotonicity, extreme points and concavity of the function:

$$f(x) = x + \frac{1}{x-2}$$

Check the obtained results graphically. Calculate the asymptotes and the intercepts.

*Sol.:* $f$ is increasing on $(-\infty, 1)$ and $(3, +\infty)$, and decreasing on $(1, 2)$ and $(2, 3)$, local maximum at $x = 1$, local minimum at $x = 3$. $f$ is concave down on $(-\infty, 2)$ and concave up on $(2, +\infty)$. Asymptotes: $x = 2$ , $y = x$ . Intercepts: $(1, 0)$, $(0, -1/2)$

2. Let $f$ be a function defined as:

$$f(t) = \frac{e^{\left(\frac{1}{2}-\frac{1}{t}\right)}}{t}, \quad t > 0,$$

that describes the evolution with time of the concentration of a chemical compound in a certain chemical reaction. Calculate the time at which the concentration is maximum and this maximum value. Study the intervals of monotonicity and concavity of the function. What is the limit of $f(t)$ as $t$ approaches 0 from the right ?. Sketch the graph.

*Sol:* The maximum value is attained at $t = 1$ and the maximum concentration is $e^{-\frac{1}{2}}$. The function is increasing on $(0, 1)$ and decreasing on $(1, \infty)$. It is concave up on $(0, 1 - \sqrt{2}/2)$, concave down on $(1 - \sqrt{2}/2, 1 + \sqrt{2}/2)$ and concave up on $(1 + \sqrt{2}/2, \infty)$. The limit is zero.

3. Find the dimensions of the rectangle whose perimeter is 12 meters and has the shortest diagonal.

*Sol:* Square whose side is 3 meters.

4. Identify the intervals on which the function $f(x) = \dfrac{1}{x^2+1}$ is increasing and on which it is decreasing. Analyze its concavity. Approximate the function by a parabola at $x = 0$. Plot the function and the parabola in the same graph.

*Sol:* The function is increasing on $(-\infty, 0)$ and decreasing on $(0, \infty)$. It is concave up on $(-\infty, -1/\sqrt{3})$, concave down on $(-1/\sqrt{3}, 1/\sqrt{3})$ and concave up again on $(1/\sqrt{3}, \infty)$. The approximating parabola is $y = 1 - x^2$.

5. We consider the function $f$ defined as:

$$f(x) = \begin{cases} x^5 \ln x & \text{if} \quad x \in (0, +\infty) \\ a & \text{if} \quad x = 0 \end{cases}$$

*a*) Calculate $a$ so that the function is continuous on $[0, +\infty)$.

*b*) Find the extrema of $f$ in $[0, +\infty)$.

*c*) Study the monotonicity and concavity of the function. Plot the function.

*Sol:* *a*) $a = 0$. *b*) $f$ has its absolute minimum at $e^{-\frac{1}{5}}$ and its relative maximum at 0. *c*) f is decreasing on $(0, e^{-\frac{1}{5}})$ and increasing on $(e^{-\frac{1}{5}}, \infty)$, f is concave down on $(0, e^{-\frac{9}{20}})$ and concave up on $(e^{-\frac{9}{20}}, \infty)$

6. Calculate the McLaurin polynomials of order 2, 4, 6 y 8 of the function $f(x) = x \sin x$ and plot them together with the function in the same graph

*Sol:* $P_2(x) = x^2$

$P_4(x) = x^2 - 1/6x^4$

$P_6(x) = x^2 - 1/6x^4 + 1/120x^6$

$P_8(x) = x^2 - 1/6x^4 + 1/120x^6 - 1/5040x^8$