Linear Systems 3

Rank Deficient

Rank deficient.

$A\vec{x} = \vec{b}$ $\quad A \in M_{m\times n}(\mathbb{R}) \longrightarrow \boxed{A^T A\, \vec{x}_{LS} = A^T \vec{b}}$

$\text{rank}(A) = r < \min(m,n)$

$\Downarrow$

has ∞ solutions

In this case $\vec{b} \notin \text{Col}(A)$ generally

$\nexists\ \vec{x}$ $\longrightarrow$ has ∞ solutions

$$\boxed{\begin{aligned} \vec{x}_{LS} &= \vec{x}_p + \ker(A^T A) = \\ &= \vec{x}_p + \ker(A) \end{aligned}}$$

$\dim \ker(A) = n - r \neq 0$

not injective

$\Downarrow$

$\exists!\ x^+ = A^+ \cdot \vec{b}$ (the pseudoinverse solution)

x^+ is the minimum norm solution of $\vec{x}_{LS}$

$\Downarrow$

x^+ is orthogonal to $\ker(A)$

$\boxed{\text{pin}(A) * b(:)}$

Rank deficient

$A\vec{x} = \vec{b} \longrightarrow A^TA\,\vec{x}_{LS} = A^T\vec{b}$

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$\exists!\ x^+ = A^+ \cdot \vec{b}$ (the pseudoinverse solution)

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$\Downarrow$

x^+ is orthogonal to $\ker(A)$

$$\mathbb{R}^3 \xrightarrow{\vec{L}} \mathbb{R}^3$$

$$\vec{x} = (x, y, z) \longrightarrow \vec{L}(\vec{x}) = (x+y,\ 2x+2y,\ z).$$

matrix representing $\vec{L}$ in $B_C - B_C$

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\text{rank}(A) = 2 < 3.$

rank-deficient $\underline{L\ A}$

Traveller Pb $\vec{b} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$ What is $\vec{L}^{-1}(\vec{b}) \Longleftrightarrow \vec{x} : A\vec{x} = \vec{b}$

$$\begin{bmatrix} 1 & 1 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}.$$

$$Ab = \begin{bmatrix} 1 & 1 & 0 & 2 \\ 2 & 2 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$$\text{rank}(Ab) = 3.$$

$$\Updownarrow$$

$$\vec{b} \notin Col(A).$$

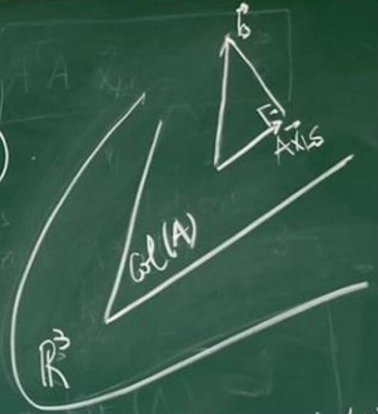
$$\mathbb{R}^3 \xrightarrow{\ \vec{L}\ } \mathbb{R}^3$$

$$\vec{x} = (x, y, z) \longrightarrow \vec{L}(\vec{x}) = (x+y,\ 2x+2y,\ z)$$

Let us find the least-squares solution (pseudo solution)

Finding $\vec{x}_{LS}$: $A\vec{x}_{LS} = \vec{p} = P^{\perp}_{\underset{ColA}{\text{proj}}}(\vec{b})$

$$A^TA\,\vec{x}_{LS} = A^T\vec{b} \qquad \begin{pmatrix} 5 & 5 & 0 \\ \cancel{5} & \cancel{5} & \cancel{0} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}_{LS} = \begin{pmatrix} 4 \\ \cancel{4} \\ -1 \end{pmatrix}$$



$\text{rank}(A^TA) = \text{rank}(A) = 2$

$$\begin{cases} 5x + 5y = 4 \\ z = -1 \end{cases} \rightarrow$$

$$\begin{cases} x + y = 4/5 \\ z = -1 \end{cases}$$

$$(x, y, z) = \left(x, \tfrac{4}{5} - x, -1\right)$$

∞ least-squares sol.

$\Uparrow$

$m = 2$ Purely

$n = 3$ Underdetermined LS.

∞ solutions

$$\mathbb{R}^3 \xrightarrow{\ \vec{L}\ } \mathbb{R}^3$$

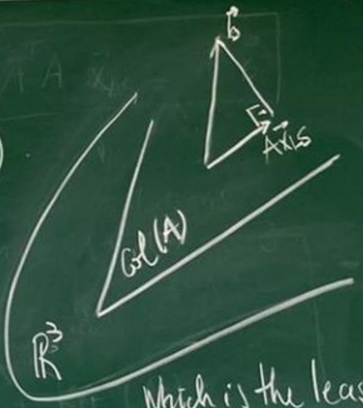
$$\vec{x} = (x,y,z) \longrightarrow \vec{L}(\vec{x}) = (x+y,\ 2x+2y,\ z)$$

Linear variety of least-squares solutions

$$(x,y,z) = \left(0, \frac{4}{5}, -1\right) + x\underbrace{(1,-1,0)}_{} \equiv \langle \ker A \rangle$$

$\vec{x}_p^{LS}$ → particular least-squares solution

$\ker A = \langle (1,-1,0) \rangle$



Which is the least-squares solution of minimum norm?

$$\Downarrow$$

$$(x,y,z)\cdot(1,-1,0) = 0$$

$$\begin{cases} 5x + 5y = 4 \\ z = -1 \\ x - y = 0 \end{cases}$$

$$\Downarrow$$

pseudo inverse solution

$$x^{+} = \begin{pmatrix} 0.4 \\ 0.4 \\ -1 \end{pmatrix}$$

$$\mathbb{R}^3 \xrightarrow{\vec{L}} \mathbb{R}^3$$

$$\vec{x} = (x, y, z) \longrightarrow \vec{L}(\vec{x}) = (x+y,\ 2x+2y,\ z)$$

Linear variety of least-squares solutions

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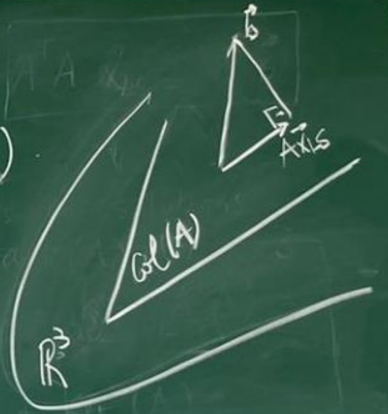

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