CALCULUS
DEGREE IN SOFTWARE ENGINEERING
EXERCISES 10. MULTIVARIABLE FUNCTIONS. LIMITS AND CONTINUITY

DOMAINS AND RANGES

1. Find the domains and ranges of the following functions

   (a) $\sqrt{xy}$

   $$D = \{(x, y) \in \mathbb{R}^2 / x \geq 0 \textit{ and } y \geq 0 \textit{ or } x \leq 0 \textit{ and } y \leq 0\}$$

   that is, the first quadrant and the third quadrant.
   The range is clearly $f(D) = [0, \infty]$.

   (b) $\dfrac{x^2 + y^2}{x^2 - y^2}$

   $$D = \{(x, y) \in \mathbb{R}^2 / x \neq y \textit{ or } x \neq -y\}$$

   So, we exclude from the domain the lines $y = x$ and $y = -x$ where the denominator is zero.
   The calculation of the range is more complicated. We write

   $$\frac{x^2 + y^2}{x^2 - y^2} = K$$

   where K is a number in the range. Notice that $K = 1$ if $y = 0$ and $K = -1$ if $x = 0$. Multiplying by the denominator and putting x in one side and y in the other, we find

   $$x^2(K - 1) = y^2(K + 1)$$

   Then, $K + 1$ and $K - 1$ must be both positive or both negative, if x and y are different from zero. In the first case $K > 1$, and in the second $K < -1$. Including the cases seen before

   $$f(D) = (-\infty, -1] \cup [1, \infty)$$

   (c) $\dfrac{e^x + e^y}{e^x - e^y}$
   The domain consists of all the points for which $e^y \neq e^x$ and since the exponential function is one-to-one $y \neq x$. Only the line $y = x$ is excluded from the domain. The range can be obtained as before, by writing

$$\frac{e^x + e^y}{e^x - e^y} = K$$

Then,

$$e^x(K+1) = e^y(K-1)$$

It is simple to conclude that the range is

$$f(D) = (-\infty, -1) \cup (1, \infty)$$

Now, $K \neq 1$ and $K \neq -1$. Why?

LIMITS AND CONTINUITY

2. Analyze the existence of the limit at $(0,0)$ for the following functions and calculate the limit when it exists.

   (a) $f(x,y) = xy \sin \dfrac{x}{y}$

   The limit of xy at $(0,0)$ is zero and the sine function is bounded. This means that the limit of their product is 0.

   (b) $f(x,y) = \dfrac{x^2 - y^2}{x^2 + y^2}$

   First, we compute the directional limits. Along $x = 0$ the function becomes $f(0,y) = -y^2/y^2 = -1$ and the limit is -1. If $y = mx$, the function can be written as

   $$f(x, mx) = \frac{x^2 - m^2x^2}{x^2 + m^2x^2} = \frac{1-m^2}{1+m^2}$$

   Therefore, the directional limits are

   $$\frac{1-m^2}{1+m^2}$$

   and depend on the slope of each particular line. In conclusion, the double limit does not exist.

   (c) $f(x,y) = \dfrac{xy^2}{x^2 + y^4}$

   We take the limit along $x = 0$

   $$\lim_{y \to 0} \frac{0}{y^4} = 0$$

   and along $y = mx$

$$\lim_{x\to 0} \frac{mx^3}{x^2+m^4x^4} = \lim_{x\to 0} \frac{mx}{1+m^4x^2} = 0$$

and all the directional limits are zero. However, we have not proved the existence of the double limit. If we consider the parabola $x = y^2$ and calculate the limit along it

$$\lim_{y\to 0} \frac{y^4}{2y^4} = \frac{1}{2}$$

Therefore, the limit along this curve is different from the directional limits, what implies that the double limit does not exist, since we have found a path along which the limit is different from zero (the value of all directional limits).

(d) $f(x,y) = \dfrac{xy}{\sqrt{x^2+y^2}}$

For $x = 0$ the limit is

$$\lim_{y\to 0} \frac{0}{|x|} = 0$$

and for $y = mx$

$$\lim_{x\to 0} \frac{mx^2}{|x|\sqrt{1+m^2}} = 0$$

Can we prove that the limit is zero? A hint is that the degree of the numerator, 2, is greater than that of the denominator, 1 and then, we can bound the function in a suitable way - this will work if the denominator depends on x^2+y^2-.

So,

$$\sqrt{x^2+y^2} \geq |x|$$

and then, for $x \neq 0$

$$\frac{1}{\sqrt{x^2+y^2}} \leq \frac{1}{|x|}$$

This implies

$$\left|\frac{xy}{\sqrt{x^2+y^2}}\right| \leq \left|\frac{xy}{|x|}\right| = |y|$$

Therefore, the function is bounded by a function that approaches zero as (x,y) approaches $(0,0)$ and in consequence, its double limit is zero.

(e) $f(x,y) = e^y \dfrac{\sin x}{x}$
The limit will be the product of the limits and we know that

$$\lim_{(x,y)\to(0,0)} e^y = 1$$

and

$$\lim_{(x,y)\to(0,0)} \frac{\sin x}{x} = 1$$

So, the double limit is 1.

(f) $f(x,y) = \dfrac{2xy^2}{x^2+y^2}$
We omit the calculation of the directional limits. You can check that their value is 0 and as in Item d, we can bound the function. We use for x and y different from zero- for x or y equal to zero the limit is 0-

$$\left|\frac{2xy^2}{x^2+y^2}\right| \leq \left|\frac{2xy^2}{y^2}\right| = |2x|$$

The function is bounded by a function that goes to zero, so the limit of the original function is 0.

3. Study the continuity of the functions:

(a) $f(x,y) = \begin{cases} \dfrac{x^2y^2}{x^2y^2+(x-y)^2} & \text{if} \quad (x,y) \neq (0,0) \\ \\ 1 & \text{if} \quad (x,y) = (0,0) \end{cases}$

The function is continuous on its natural domain

$$D = \{(x,y) \in \mathbb{R}^2/(x,y) \neq (0,0)\}$$

since it is the quotient of two continuous functions with non-zero denominator. We have to study its behaviour at $(0,0)$.
First, we calculate the limit at $(0,0)$ along $x = 0$.

$$\lim_{y\to 0} \frac{0}{x^2} = 0$$

and the directional limits along $y = mx$

$$\lim_{x\to 0} \frac{m^2x^4}{m^2x^4 + x^2(1-m)^2} = \lim_{x\to 0} \frac{m^2x^2}{m^2x^2+(1-m)^2}$$

which is zero, provided $m \neq 1$. However, for $m = 1$ the limit is 1. This means that the function has no double limit at the origin and thus, it is not continuous at $(0,0)$. The discontinuity is not removable, due to the non-existence of the limit. Notice that just the existence of a line along which the limit is different has meant the non-existence of the double limit.

(b) $$f(x,y) = \begin{cases} (x^2+y^2)\sin\left(\dfrac{1}{x^2+y^2}\right)^{\frac{1}{2}} & \text{if} \quad (x,y) \neq (0,0) \\ \\ 0 & \text{if} \quad (x,y) = (0,0) \end{cases}$$

This exercise is very simple. At $(x,y) \neq (0,0)$, the function is continuous as the product of two continuous functions. At $(0,0)$, the function is the product of a function tending to zero times a bounded function. Therefore, the limit is 0 and the function is continuous on $\mathbb{R}^2$.

(c) $$f(x,y) = \begin{cases} \dfrac{2x^5+4x^2y^3-2y^5}{(x^2+y^2)^2} & \text{if} \quad (x,y) \neq (0,0) \\ \\ 0 & \text{if} \quad (x,y) = (0,0) \end{cases}$$

If we calculate all the directional limits at the origin, we can easily check that they are 0. We try to bound our function by a function that tends to zero. For that, we use the triangle inequality

$$\left|\frac{2x^5+4x^2y^3-2y^5}{(x^2+y^2)^2}\right| \leq \left|\frac{2x^5}{(x^2+y^2)^2}\right| + \left|\frac{4x^2y^3}{(x^2+y^2)^2}\right| + \left|\frac{2y^5}{(x^2+y^2)^2}\right|$$

Now we apply

$$\frac{1}{(x^2+y^2)^2} \leq \frac{1}{x^4}$$

$$\frac{1}{(x^2+y^2)^2} \leq \frac{1}{2x^2y^2}$$

and

$$\frac{1}{(x^2+y^2)^2} \leq \frac{1}{y^4}$$

These inequalities, conveniently applied, lead to

$$\left|\frac{2x^5}{(x^2+y^2)^2}\right| + \left|\frac{4x^2y^3}{(x^2+y^2)^2}\right| + \left|\frac{2y^5}{(x^2+y^2)^2}\right| \leq \left|2x\right| + \left|2y\right| + \left|2y\right|$$

We have bounded the function by a function that tends to zero and proved that the limit at the origin is zero. Since the function is continuous at all other points, it is continuous on $\mathbb{R}^2$

(d) $f(x,y) = \begin{cases} \dfrac{x^4 - y^4}{(x^2+y^2)^2} & \text{if} \quad (x,y) \neq (0,0) \\ \\ 0 & \text{if} \quad (x,y) = (0,0) \end{cases}$

The directional limits are

$$\lim_{x \to 0} \frac{(1-m^4)x^4}{(1+m^2)^2 x^4} = \lim_{x \to 0} \frac{(1-m^4)}{(1+m^2)^2}$$

Then, they depend on m and the double limit does not exist. The function is continuous on $\mathbb{R}^2 \setminus \{(0,0)\}$.

4. We define $f(x,y) = \begin{cases} \dfrac{xy^3}{(x^2+y^6)} & \text{if} \quad (x,y) \neq (0,0) \\ \\ k & \text{if} \quad (x,y) = (0,0) \end{cases}$

(a) Calculate the limit of $f(x,y)$ as (x,y) approaches $(0,0)$ along the curve $x = y^3$.

We substitute $x = y^3$ in the definition of the function

$$f(y^3, y) = \frac{y^6}{2y^6} = \frac{1}{2}$$

The limit is $1/2$

(b) Prove that, regardless of the value of k, f is not continuous at $(0,0)$.

We find the limit along $x = 0$

$$f(0,y) = \frac{0}{y^6} = 0$$

For all $y \neq 0$ and since the limit along this line, 0, is not the same as the limit along $x = y^3$, $1/2$, the double limit at the origin does not exist and the function is not continuous at $(0,0)$, regardless of the value of k. If the limit is different along different paths (lines or curves), then the double limit does not exist.

We show the graphs (surfaces) of the functions which appear in Exercises 2b, 2c and 2f. The graphs are plotted on $[-1,1] \times [-1,1]$. Notice the discontinuity at the origin in the first two graphs.

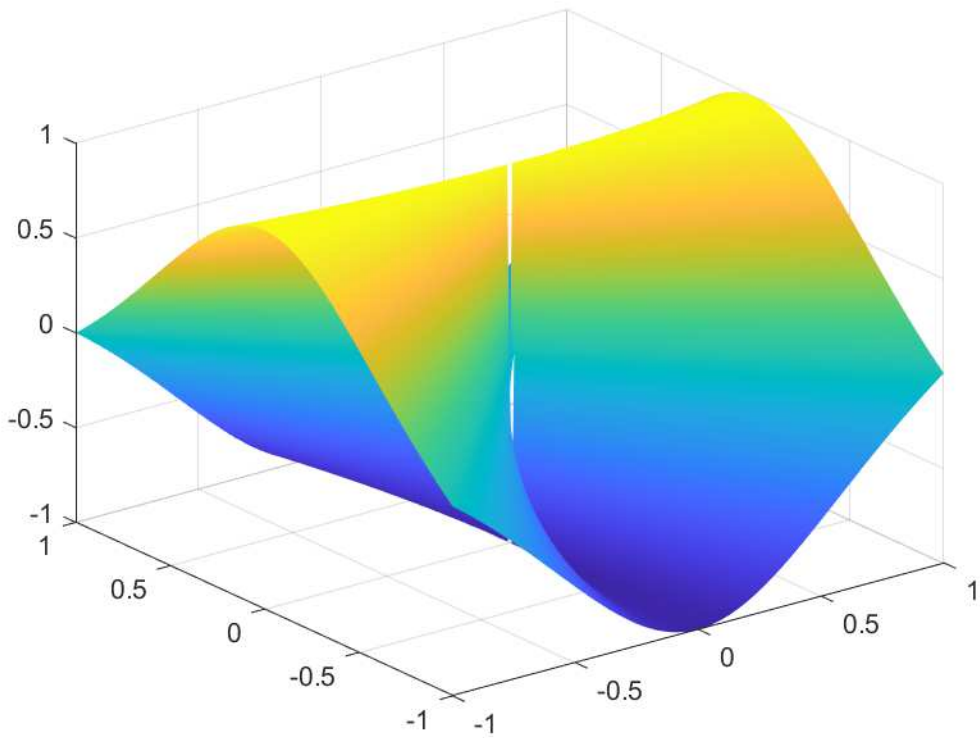

Figure 1: Exercise 2b. $f(x, y) = (x^2 - y^2)/(x^2 + y^2)$

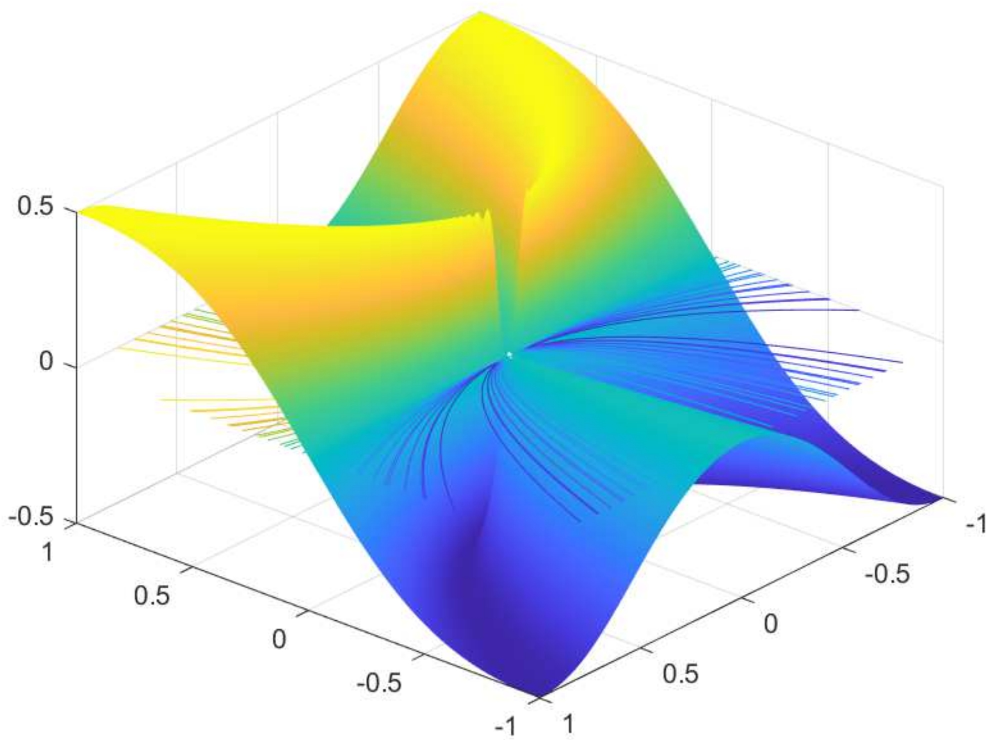

Figure 2: Exercise 2c. $f(x, y) = (xy^2)/(x^2 + y^4)$

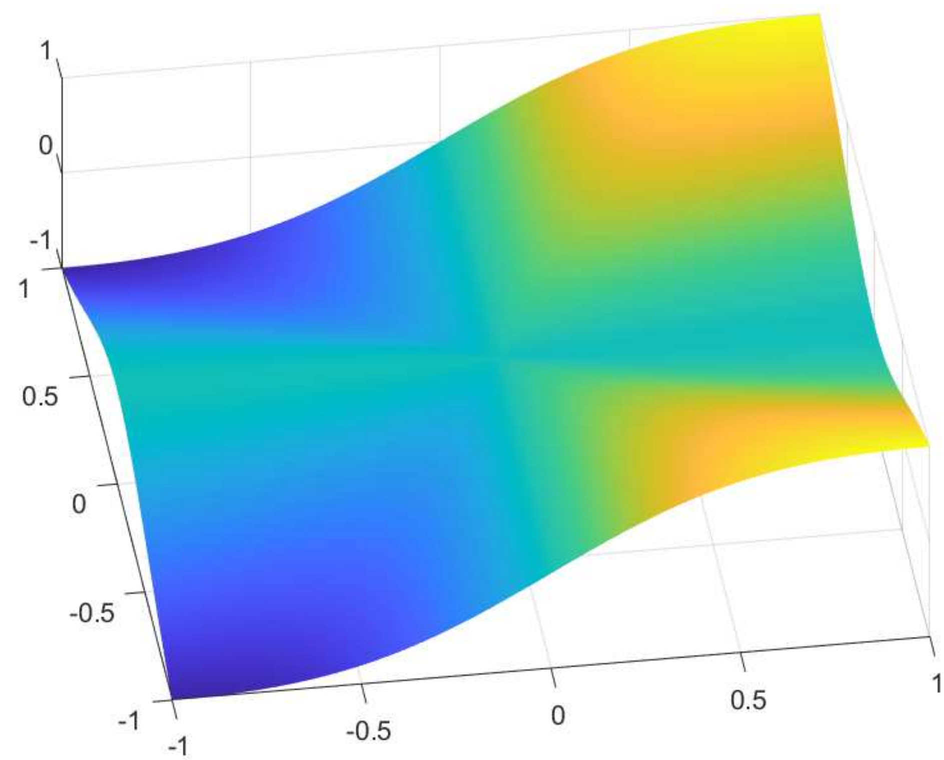

Figure 3: Exercise 2f. $f(x, y) = (2xy^2)/(x^2 + y^2)$