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S3-Euclidean Spaces Least-Squares

Linear Algebra

Ingeniería del Software-Universidad de Oviedo

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Class-4 November 2020

Euclidean Geometry in \mathbb{R}^3



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Euclidean Geometry in \mathbb{R}^3

Planes $\Pi : \{ (x, y, z) : ax + by + cz = 0 \} \rightarrow \dim \Pi = 2 \Rightarrow x = -\frac{b}{a}y - \frac{c}{a}z$

Lines $r : \{ (x, y, z) : \frac{x}{a} = \frac{y}{b} = \frac{z}{c} = \lambda \} \rightarrow \dim r = 1$ $(x, y, z) = \left(-\frac{b}{a}y - \frac{c}{a}z, y, z \right) =$

$$\Downarrow$$
$$(x, y, z) = \lambda (a, b, c)$$

$$r = \langle (a, b, c) \rangle$$

$$= y \left(-\frac{b}{a}, 1, 0 \right) + z \left(-\frac{c}{a}, 0, 1 \right) =$$

$$= \langle (-b, a, 0), (-c, 0, a) \rangle$$

Euclidean Geometry in \mathbb{R}^3



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Euclidean Geometry in \mathbb{R}^3

Planes $\Pi : \{ (x, y, z) : ax + by + cz = 0 \} \rightarrow \dim \Pi = 2 \Rightarrow$

$\Pi = \langle (-b, a, 0), (-c, 0, a) \rangle$. or I can define Π^\perp

which is the orthogonal complement of Π

$$\dim \Pi^\perp = 3 - \dim \Pi = 1.$$

$$\mathbb{R}^3 = \Pi \oplus \Pi^\perp$$

$$\Pi^\perp = \langle (a, b, c) \rangle$$

$$ax + by + cz = 0$$

$$\downarrow$$
$$(x, y, z) \cdot (a, b, c) = 0$$

\uparrow
Euclidean Scalar Product
in \mathbb{R}^3 .

(a, b, c) is orthogonal to Π

\downarrow
normal vector to Π .

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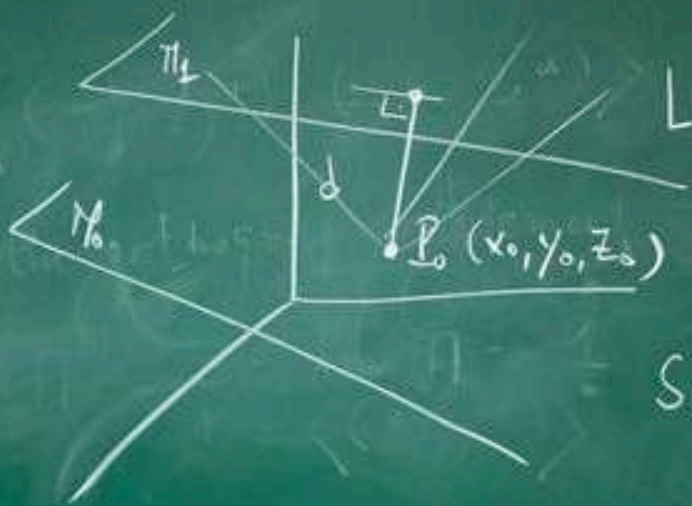
Distance of $P_0(x_0, y_0, z_0)$ to $\Pi_1 = \{ax + by + cz = d\} \Rightarrow$ linear variety

\downarrow passes through $(0, 0, \frac{d}{c}) \neq (0, 0, 0)$

if $d \neq 0$

$$L = \left\{ (x, y, z) = \left(0, 0, \frac{d}{c}\right) + \underbrace{\langle (-b, a, 0), (-c, 0, a) \rangle}_{\substack{\uparrow \\ \Pi_0 \\ \text{director plane}}} \right\}$$

$d = d_{P_0 \Pi_1}$
(minimum)



Euclidean Geometry in \mathbb{R}^3

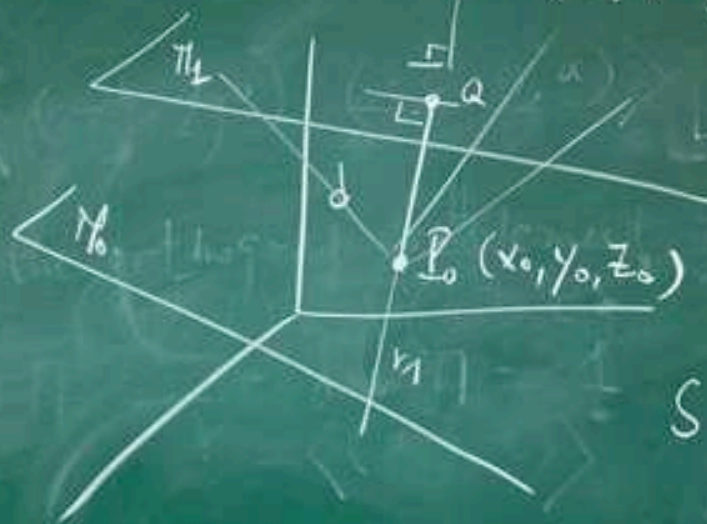


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Euclidean Geometry in \mathbb{R}^3

Distance of $P_0(x_0, y_0, z_0)$ to the plane π defined by (a, b, c) .

= distance $P_0 \Pi$
(minimum)



Method 1

P_0Q is in the direction of

Let us find Q .

$$\begin{cases} ax + by + cz = d \\ \frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c} \end{cases}$$

$$d = \|\vec{P_0Q}\|$$

$$\vec{n} = (a, b, c)$$

intersection
 \Downarrow

Q

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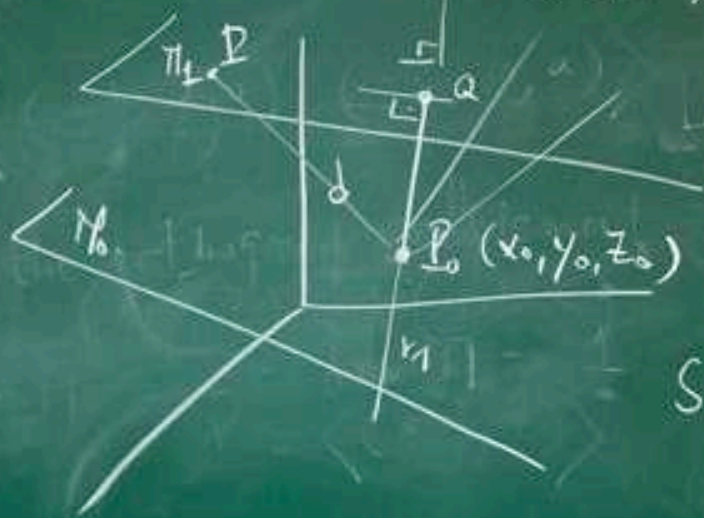


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Euclidean Geometry in \mathbb{R}^3

Distance of $P_0(x_0, y_0, z_0)$ to plane π defined by (a, b, c)

$d = d_{P_0/\pi}$
(minimum)



Method 2

$$\vec{P_0P} = (x - x_0, y - y_0, z - z_0)$$

$$(x, y, z) \in \pi$$

$\|\vec{P_0Q}\|$ is the length of the orthogonal projection of $\vec{P_0P}$ onto $\langle (a, b, c) \rangle$

$$\vec{P} = \vec{P_0Q} = \lambda \cdot (a, b, c) = \lambda \vec{n}$$

$$\lambda = \frac{\vec{P_0P} \cdot \vec{n}}{\vec{n} \cdot \vec{n}} = \frac{(x - x_0)a + (y - y_0)b + (z - z_0)c}{\|\vec{n}\|^2}$$

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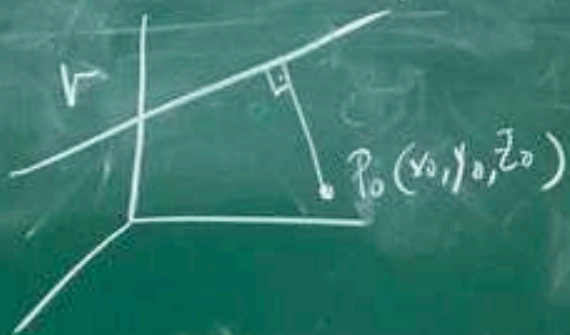
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$$r = \left\{ \frac{x - x_p}{a} = \frac{y - y_p}{b} = \frac{z - z_p}{c} \right\} \rightarrow \text{straight line passing through } (x_p, y_p, z_p).$$

$$r \equiv (x, y, z) = (x_p, y_p, z_p) + \langle (a, b, c) \rangle$$

$P_0(x_0, y_0, z_0) \Rightarrow$ finding the distance from P_0 to L



Euclidean Geometry in \mathbb{R}^3



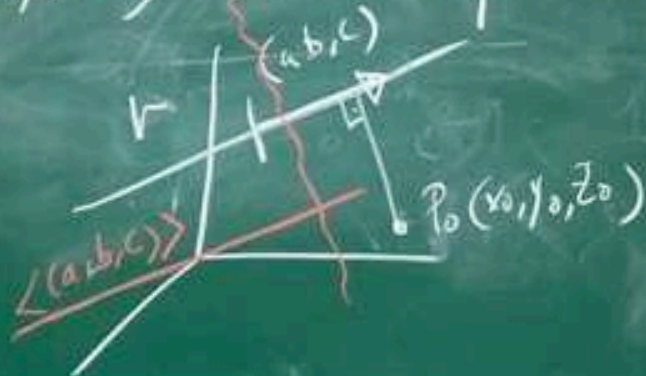
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Euclidean Geometry in \mathbb{R}^3

$$r = \left\{ \frac{x - x_p}{a} = \frac{y - y_p}{b} = \frac{z - z_p}{c} \right\} \rightarrow \text{straight line passing through } (x_p, y_p, z_p).$$

$$r \equiv (x, y, z) = (x_p, y_p, z_p) + \langle (a, b, c) \rangle$$

$P_0(x_0, y_0, z_0) \Rightarrow$ finding the distance from P_0 to L



What is the plane passing through P_0 and
being orthogonal to $\langle (a, b, c) \rangle$

$$\begin{aligned} ax + by + cz &= d \\ d &= ax_0 + by_0 + cz_0 \end{aligned}$$

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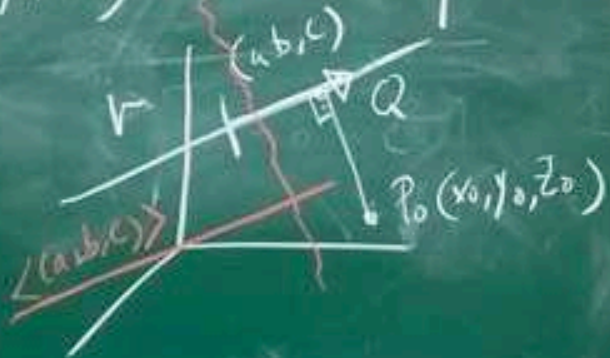
$$r = \left\{ \frac{x - x_p}{a} = \frac{y - y_p}{b} = \frac{z - z_p}{c} \right\}$$

→ straight line passing through (x_p, y_p, z_p) .

the simplest thing to do is finding the intersection

between $\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$ and the plane

$P_0(x_0, y_0, z_0) \Rightarrow$ finding the distance



$$ax + by + cz = ax_0 + by_0 + cz_0$$

this will provide Q

$$d = \|\vec{P_0Q}\|$$

Euclidean Geometry in \mathbb{R}^3

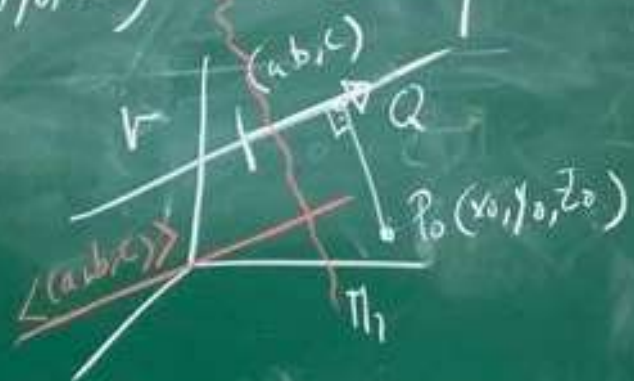


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$$r = \left\{ \frac{x - x_p}{a} = \frac{y - y_p}{b} = \frac{z - z_p}{c} \right\}$$

$P_0(x_0, y_0, z_0) \Rightarrow$ finding the distance



Ex $\frac{x-1}{2} = \frac{y-1}{1} = \frac{z-1}{1}$

$$P_0 = (0, 1, -1)$$

$$(a, b, c) = (2, 1, 1)$$

$$r \begin{cases} x-1 = 2(y-1) \rightarrow x-2y = -1 \\ x-1 = 2(z-1) \rightarrow x-2z = -1 \end{cases}$$

$$\pi_1: 2x + y + z = 0$$

$$2x + y + z = 2x_0 + y_0 + z_0$$

$$\Rightarrow Q \text{ is the projection of } P_0 \text{ onto } \pi_1$$

$$\| \vec{P_0 Q} \|$$

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Euclidean Geometry in \mathbb{R}^3

$$\gamma = \left\{ \frac{x-x_p}{a} = \frac{y-y_p}{b} = \frac{z-z_p}{c} \right\}$$

$P_0(x_0, y_0, z_0) \rightarrow$ finding the distance



Second method $P(x, y, z) \in \gamma$

$$\vec{PP}_0 = (x-x_0, y-y_0, z-z_0)$$

π_0 is the plane orthogonal to $\langle (a, b, c) \rangle$

$$\pi_0 = \left\{ (x, y, z) : ax + by + cz = 0 \right\} =$$

$$= \langle (-b, a, 0), (-c, 0, a) \rangle$$

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Now I have to compute the orthogonal
projection of $\vec{P}_0 P$ onto $\langle (-b, a, 0), (-c, 0, a) \rangle$

$$\vec{P} = \alpha \begin{pmatrix} -b \\ a \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -c \\ 0 \\ a \end{pmatrix}$$

non-orthogonal



$$\begin{pmatrix} \vec{u}_1 \cdot \vec{u}_1 & \vec{u}_1 \cdot \vec{u}_2 \\ \vec{u}_2 \cdot \vec{u}_1 & \vec{u}_2 \cdot \vec{u}_2 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} A \\ B \end{pmatrix} \rightarrow \begin{matrix} \alpha, \beta \\ \downarrow \\ \vec{P} \\ d = \|\vec{P}\| \end{matrix}$$

$$\vec{P}_0 P = \vec{P} + \vec{r} \quad \vec{r} \perp \vec{P}$$

$$A = \vec{P}_0 P \cdot \vec{u}_1 = \alpha \vec{u}_1 \cdot \vec{u}_1 + \beta \vec{u}_1 \cdot \vec{u}_2 + 0$$

$$B = \vec{P}_0 P \cdot \vec{u}_2 = \alpha \vec{u}_1 \cdot \vec{u}_2 + \beta \vec{u}_2 \cdot \vec{u}_2 + 0$$