CALCULUS
DEGREE IN SOFTWARE ENGINEERING
EXERCISES 12. EXTREME VALUES OF MULTIVARIABLE FUNCTIONS

1. Find the local extreme values of $f(x,y) = xy - x^2 - y^2 - 2x - 2y + 4$ and classify them.

   The function is at least $C^2(\mathbb{R}^2)$. First, we search for the critical points

$$\frac{\partial f}{\partial x} = y - 2x - 2 = 0$$

$$\frac{\partial f}{\partial y} = x - 2y - 2 = 0$$

   We solve a linear system, from the first equation $y = 2x + 2$, and substituting in the second $x - 2(2x+2) - 2 = 0$. The only solution is the point $x = -2$, $y = -2$.

   We calculate the Hessian matrix

$$Hf = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$$

   $\dfrac{\partial^2 f}{\partial x^2}(-2,-2) = -2$ and $|Hf(-2,-2)| = 3$. In consequence, the function has a local maximum at $(-2,-2)$ and the value of the maximum is $f(-2,-2) = 8$. Since the Hessian matrix is constant, it will be definite negative at all points and then, by applying Taylor's theorem, the maximum is absolute. There is no local minimum.

2. Find the local extrema of $f(x,y) = 3y^2 - 2y^3 - 3x^2 + 6xy$.

   We find the critical points

$$\frac{\partial f}{\partial x} = -6x + 6y = 0$$

$$\frac{\partial f}{\partial y} = 6y - 6y^2 + 6x = 0$$

   From the first equation, $y = x$ and the second becomes $6x - 6x^2 + 6x = 0$, that is, $x(2-x) = 0$. Then, there are two critical points: $P_1 = (0,0)$ and $P_2 = (2,2)$. The Hessian matrix is, in general

$$Hf = \begin{bmatrix} -6 & 6 \\ 6 & 6-12y \end{bmatrix}$$

and at $(0,0)$, $\dfrac{\partial^2 f}{\partial x^2}(0,0) = -6$, $|Hf(0,0)| = -72$. $(0,0)$ is a saddle point.

At $(2,2)$, $\dfrac{\partial^2 f}{\partial x^2}(2,2) = -6$, $|Hf(2,2)| = 72$ and the function has a local maximum at $(2,2)$. However, it is not an absolute maximum, for instance $f(2,2) = 8$ and $f(0,-10) = 300 + 2000 = 2300$. There are neither absolute maximum nor absolute minimum for this function on $\mathbb{R}^2$.

3. Has the function $f(x,y) = y^2 + 3y - x^2y - 2xy$ any local extreme values ?

   The critical points are

$$\frac{\partial f}{\partial x} = -2xy - 2y = 0$$

$$\frac{\partial f}{\partial y} = 2y + 3 - x^2 - 2x = 0$$

   From the first equation, either $y = 0$ or $x = -1$. In the first case, the second equation is $x^2 + 2x - 3 = 0$ and $x = -3$ or $x = 1$. In the second case, $x = 2$, the second equation can be written $2y + 4 = 0$ and $y = -2$. Thus, we have the following critical points

$$P_1 = (1,0)$$
$$P_2 = (-3,0)$$

   and

$$P_3 = (-1,-2)$$

   The Hessian matrix is

$$Hf = \begin{bmatrix} -2y & -2x-2 \\ -2x-2 & 2 \end{bmatrix}$$

   At $P_1 = (1,0)$, $\dfrac{\partial^2 f}{\partial x^2} = 0$ and $|Hf| = -16$. There is a saddle point.

   At $P_1 = (-3,0)$, $\dfrac{\partial^2 f}{\partial x^2} = 0$ and $|Hf| = -16$. There is another saddle point.

   Finally, at $P_3 = (-1,-2)$, $\dfrac{\partial^2 f}{\partial x^2} = 4$ and $|Hf| = 8$. There is a local minimum. It is not an absolute minimum, since $f(-1,-2) = -4$ and $f(10,1) = 4 - 100 - 20 = -116$.

4. What are the critical points of $f(x,y) = x^4 + y^4 - 4xy$? Classify them

$$\frac{\partial f}{\partial x} = 4x^3 - 4y = 0$$

$$\frac{\partial f}{\partial y} = 4y^3 - 4x = 0$$

Then, $y = x^3$ and, substituting this in the second equation, $4x^9 - 4x = 0$. The critical points are $P_1 = (0,0)$, $P_2 = (1,1)$ and $P_3 = (-1,-1)$. The Hessian matrix is

$$Hf = \begin{bmatrix} 12x^2 & -4 \\ -4 & 12y^2 \end{bmatrix}$$

At $P_1 = (0,0)$, $\dfrac{\partial^2 f}{\partial x^2} = 0$ and $|Hf| = -16$. There is a saddle point.

At $P_2 = (1,1)$, $\dfrac{\partial^2 f}{\partial x^2} = 12$ and $|Hf| = 144 - 16 = 128$. There is a local minimum.
At $P_3 = (-1,-1)$, $\dfrac{\partial^2 f}{\partial x^2} = 12$ and $|Hf| = 144 - 16 = 128$. There is another local minimum.

$f(1,1) = f(-1,-1) = -2$. If we take points far away from the origin, for instance on the sides of the rectangle $D = [-R,R] \times [-R,R]$, you can check that the function takes values dominated by R^4. Since the function must have an absolute minimum on a closed and bounded set, the rectangular region, and the function is greater on the boundary than at $(-1,-2)$, it has an absolute minimum at that point.

5. Find the minimum distance from the surface $z = \dfrac{1}{xy}$ to the origin.

The distance to the origin from any point on the surface is

$$d(x,y) = \sqrt{x^2 + y^2 + \frac{1}{x^2y^2}}$$

If the square root must be minimum, the radicand must be minimum too. Then, we determine the critical points of the radicand $f(x,y)$

$$\frac{\partial f}{\partial x} = 2x - \frac{2}{x^3y^2} = 0$$

$$\frac{\partial f}{\partial y} = 2y - \frac{2}{x^2y^3} = 0$$

Therefore, $x^4y^2 = 1$ and $x^2y^4 = 1$. From these equations we obtain the following critical points

$P_1 = (1, 1)$, $P_2 = (1, -1)$, $P_3 = (-1, -1)$, $P_4 = (-1, 1)$

The Hessian matrix at a general point of the domain, $x \neq 0$ and $y \neq 0$, is

$$Hf = \begin{bmatrix} 2 + 6/x^4y^2 & 4/x^3y^3 \\ 4/x^3y^3 & 2 + 6/x^2y^4 \end{bmatrix}$$

You can check that the first element is always positive and the Hessian determinant is $|Hf| = 4+20/x^6y^6+12/x^4y^2+12/x^2y^4$. Then, the Hessian matrix is always definite positive and we have the absolute minimum in all the selected points $(f(x, y) > f(1, 1)$, but this only works inside each quadrant). This minimum distance is

$$d = \sqrt{3}$$

There is no maximum distance.

6. Find and classify the local extrema of $f(x, y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$.

The critical points must satisfy

$$\frac{\partial f}{\partial x} = 4x^3 - 4x - 4y = 0$$

$$\frac{\partial f}{\partial y} = 4y^3 + 4x - 4y = 0$$

Adding both equations

$$4(x^3 + y^3) = 0$$

and $y = -x$, $4x^3 - 8x = 0$. The critical points are $P_1 = (0, 0)$, $P_2 = (-\sqrt{2}, \sqrt{2})$ and $P_3 = (\sqrt{2}, -\sqrt{2})$.

$$Hf = \begin{bmatrix} 12x^2 - 4 & 4 \\ 4 & 12y^2 - 4 \end{bmatrix}$$

At $(0, 0)$, $|Hf| = 0$ and the criterion is inconclusive. However, if we take $f(x, 0) = x^4 - 2x^2$, for very small x the function is negative and for $y = x$, $f(x, x) = 2x^4$, positive; then there is a saddle point at $(0, 0)$.

At the other points, the Hessian matrix is

$$Hf = \begin{bmatrix} 20 & 4 \\ 4 & 20 \end{bmatrix}$$

and the function has a local minimum with the same value -8 at both points. Is it an absolute minimum ? the answer is yes. Give reasons for it.

7. Analyze if $f(x,y) = xy\sin x$ has a local extremum at the following points: $P_1 = (0, \pi/2)$, $P_2 = (\pi/4, \pi/2)$, $P_3 = (0, \pi/2)$

   These points have to be critical points, let us check it

$$\frac{\partial f}{\partial x} = y\sin x + xy\cos x = 0$$

$$\frac{\partial f}{\partial y} = x\sin x = 0$$

   At $(\pi/2, 0)$ and $(\pi/4, \pi/2)$, the system does not hold. At $(0, \pi/2)$, it holds, and the Hessian matrix is

$$Hf = \begin{bmatrix} 2y\cos x - xy\sin x & \sin x + x\cos x \\ \sin x + x\cos x & 0 \end{bmatrix}$$

$$Hf(0, \pi/2) = \begin{bmatrix} \pi & 0 \\ 0 & 0 \end{bmatrix}$$

   The criterion does not solve the problem. At any rate, we can see that, in some neighbourhood of $(0, \pi/2)$, $y > 0$, this means that $f(x,y)$ is always positive or zero in that nearby region. The function has a local minimum at that point.

8. Has the function $f(x,y) = x^2y + y^2x$ any local maximum or minimum ?

   The critical points of f must satisfy

$$\frac{\partial f}{\partial x} = 2xy + y^2 = y(2x + y) = 0$$

$$\frac{\partial f}{\partial y} = x^2 + 2xy = x(x + 2y) = 0$$

   You can check that the only solution of this system is $(0,0)$. The Hessian matrix is

$$Hf = \begin{bmatrix} 2y & 2x + 2y \\ 2x + 2y & 2x \end{bmatrix}$$

$$Hf(0,\pi/2) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

It is clear that the Hessian matrix criterion is not conclusive. We see that

$$f(x,x) = 2x^3$$

and then the function has a saddle point at $(0,0)$. Why ?

9. Find the absolute maximum and the absolute minimum of $f(x,y) = 2 + 2x + 2y - x^2 - y^2$ on the triangular region R defined by: $x \geq 0$, $y \geq 0$ and $y \leq 9 - x$.

   The region is closed and bounded. Therefore, this continuous function must have an absolute maximum and an absolute minimum- according to THE EXTREME VALUE THEOREM. First, we search for the critical point in the interior of the region

$$\frac{\partial f}{\partial x} = 2 - 2x = 0$$

$$\frac{\partial f}{\partial y} = 2 - 2y = 0$$

   $(1,1)$ is the only critical point and in fact, it is in the interior of R. Second, we explore the boundary of R

   1. $x = 0$ with $0 \leq y \leq 9$, the function is $g(y) = f(0,y) = 2 + 2y - y^2$. The critical points of this function must satisfy

$$g'(y) = 2 - 2y = 0$$

   then $y = 1$. We have to choose the point $(0,1)$ and the endpoints of the interval $[0,9]$, $(0,0)$ and $(0,9)$

   2. We do the same for $y = 0$ with $x \in [0,9]$. The situation is symmetric, we obtain $(1,0)$,$(9,0)$ and $(0,0)$ again.

   3. Finally, we consider $y = 9 - x$, with $x \in [0,9]$.

$$h(x) = f(x, 9-x) = -61 - 2x^2 + 18x$$

   and

$$h'(x) = -4x + 18 = 0$$

The only critical point is $(9/2, 9/2)$ and the endpoints are $(0, 9)$ and $(9, 0)$

We have the following candidates to extremum points:

$(1, 1)$, $(0, 1)$, $(1, 0)$, $(0, 0)$, $(0, 9)$, $(9, 0)$ and $(9/2, 9/2)$.
We calculate the function at all these points

$$f(1, 1) = 4$$

$$f(0, 0) = 2$$

$$f(0, 1) = f(1, 0) = 3$$

$$f(0, 9) = f(9, 0) = -61$$

and

$$f(9/2, 9/2) = -41/2$$

Thus, the absolute maximum value is 4 attained at $(1, 1)$ and the absolute minimum value is -61 reached at $(0, 9)$ and $(9, 0)$.

Imagine that f stands for the temperature on that triangular region. We have found the maximum and minimum temperature. The method is very powerful.

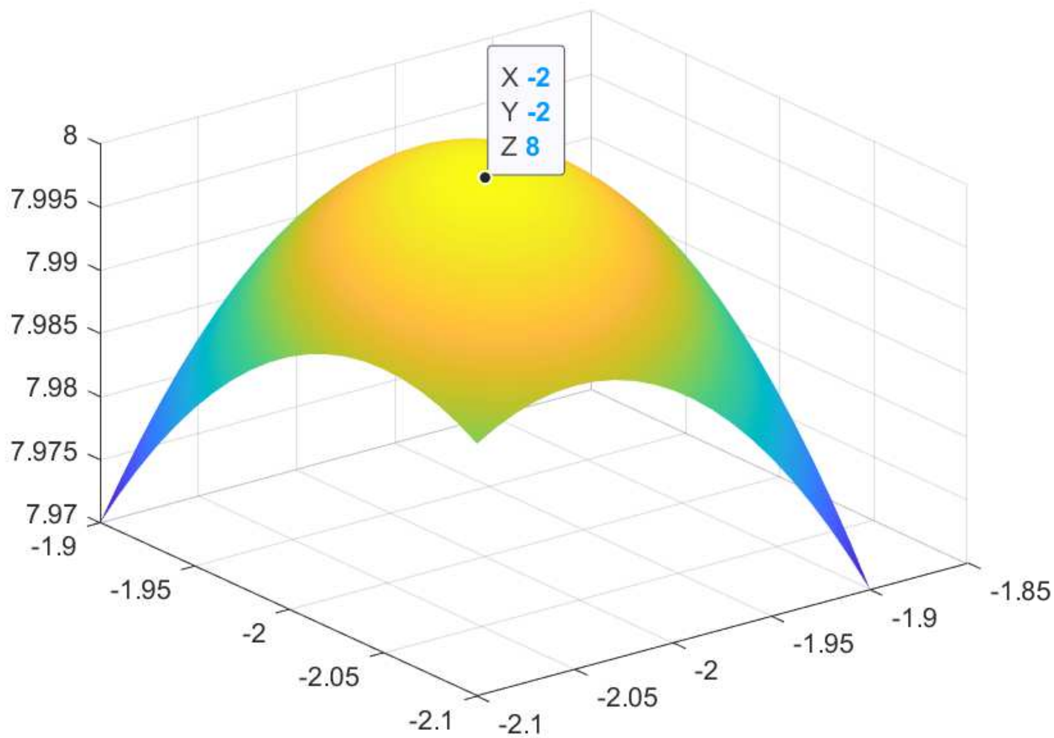

Figure 1: Exercise 1. $z = xy - x^2 - y^2 - 2x - 2y + 4$

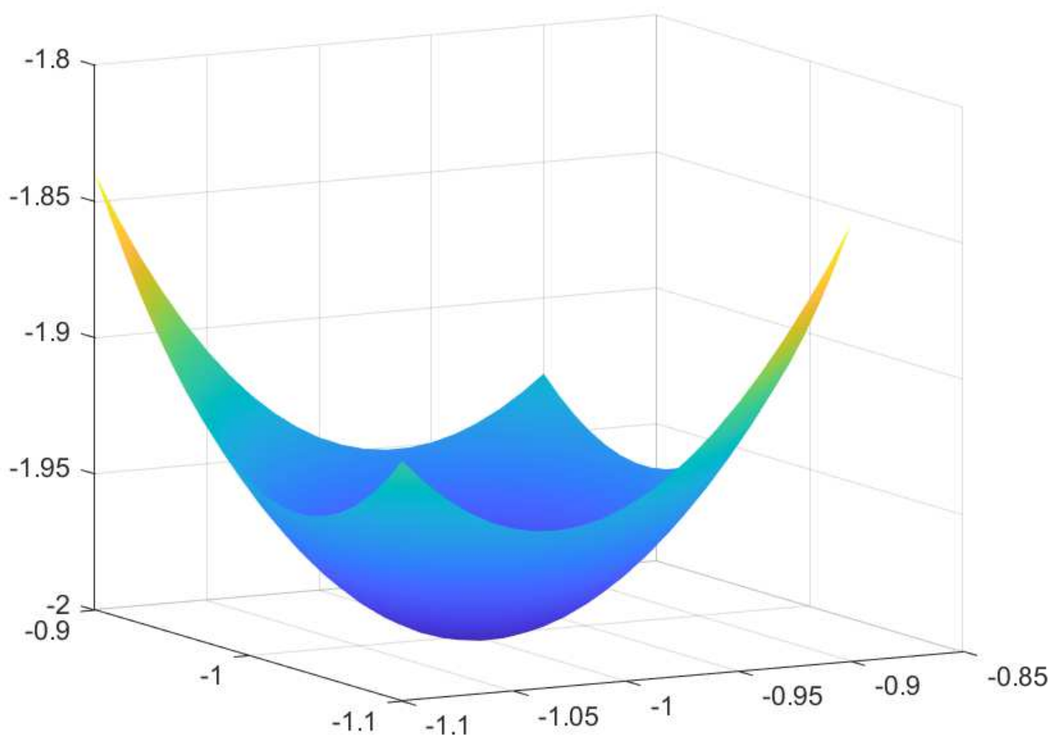

Figure 2: Exercise 4. $z = x^4 + y^4 - 4xy$