PRACTICE 7: Numerical Integration

The need to integrate a function $f(x)$ arises often in mathematical modeling. This session is devoted to techniques for approximating this operation numerically. Such techniques are at the heart of approximation methods for solving differential and integral equations, which are important and active areas of research.

Given a function $f(x)$ on a short interval $[a, b]$, we can choose a set of nodes $x_0, x_1, \ldots, x_n \in [a, b]$ and construct a polynomial interpolant $p_n(x)$. From this it follows that

$$\int_a^b p_n(x)dx \text{ approximates } \int_a^b f(x)dx$$

Depending on the number of nodes we obtain different quadrature rules.Ã§

1 Midpoint Rule

The Midpoint Rule is a numerical method for approximating the definite integral of a function $f(x)$ over an interval $[a, b]$. The interval is divided into n subintervals of equal width:

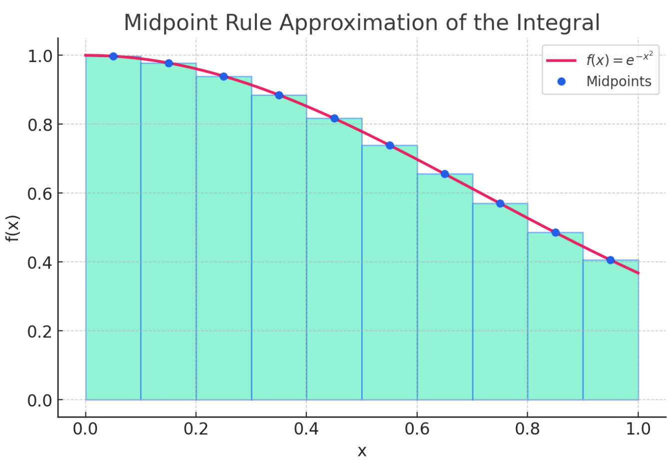
$$h = \frac{b-a}{n}$$

For each subinterval, the function is evaluated at the midpoint x_i^*:

$$x_i^* = a + \left(i - \frac{1}{2}\right)h, \quad \text{for } i = 1, 2, \ldots, n.$$

The integral is then approximated as the sum of the areas of n rectangles:

$$\int_a^b f(x)dx \approx h \sum_{i=1}^{n} f(x_i^*).$$

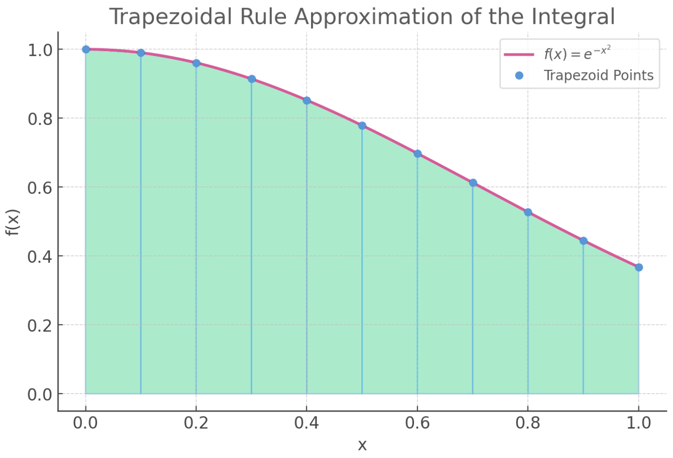


2 Trapezoidal rule

One of the simplest rules is

$$\int_a^b f(x)dx \approx \int_a^b p(x)dx$$

being $p(x)$ the polynomial which interpolates the function at the ends, a and b.



Graphically, this involves calculating the area of the trapezoid formed by the straight line joining the points $(a, f(a))$ and $(b, f(b))$, along with the x-axis and the straight $x = a$ and $x = b$, thus:

$$\int_a^b f(x)dx \approx \frac{b-a}{2}[f(a) + f(b)]$$

In addition, denoting E the difference between the exact value of the integral and approximate value, it is shown that there is $c \in (a, b)$, such that

$$E = -\frac{(b-a)^3}{12}f''(c) \implies |E| \leq \frac{(b-a)^3}{12}M, \text{ with } M = \max\{|f''(x)| : \ a < x < b\}$$

therefore, the trapezoidal rule has precision 1 (or degree of accuracy 1) that is $E = 0$ for all polynomials $p_n(x)$ with $n \leq 1$.

MATLAB Code

```
function r=trapezoidal_rule(f,a,b)
     % approximates the integral of the numerical
     %  function f, at the interval [a,b], using the trapezoidal rule
r=(b-a)*(f(a)+f(b))/2;
```

Example 2.1 *Calculate, using the trapezoidal rule,* $\int_0^2 (3x+4)dx$, $\int_0^\pi \sin\frac{x}{4}dx$ *and the absolute errors obtained.*

Example 2.2 *Consider the function:*

$$f(x) = e^{-x^2}$$

on the interval $[0, 1]$*. Approximate the definite integral* $I = \int_0^1 e^{-x^2}\,dx$ *using two numerical methods: the* ***Midpoint Rule****, and the* ***Trapezoidal Rule****, both with* $n = 10$ *subintervals.*

Compute the absolute errors of both methods. Which method provides a more accurate approximation for this function?

Plot the function $f(x)$ *along with the rectangles used in the Midpoint Rule and the trapezoidal segments used in the Trapezoidal Rule.*

3 Simpson rule

In this case, we approximate the integral with the polynomial which interpolates the function at the ends a, b and the middle point of the interval. Denoting $x_M = \dfrac{a+b}{2}$ the middle point of the interval, we interpolate the 2nd degree polynomial $p(x) = \alpha(x - x_M)^2 + \beta(x - x_M) + \gamma$, and it results

$$\int p(x)dx = \alpha\frac{(x - x_M)^3}{3} + \beta\frac{(x - x_M)^2}{2} + \gamma x$$

Denoting $h = \dfrac{b-a}{2}$, we have $b - x_M = h$ and $a - x_M = -h$, we find:

$$\int_a^b p(x)dx = \frac{\alpha}{3}[h^3 - (-h)^3] + \frac{\beta}{2}[h^2 - (-h)^2] + \gamma(b-a) = \frac{2\alpha h^3}{3} + 2\gamma h = \frac{h}{3}(2\alpha h^2 + 6\gamma)$$

On the other hand, as $p(a) = f(a),\ p(x_M) = f(x_M),\ p(b) = f(b)$, then

$$\alpha h^2 - \beta h + \gamma = f(a),\ \gamma = f(x_M),\ \alpha h^2 + \beta h + \gamma = f(b)$$

so, adding the first equality and the last one,

$$2\alpha h^2 + 2\gamma = f(a) + f(b) \implies 2\alpha h^2 = f(a) + f(b) - 2f(x_M)$$

thus:

$$\int_a^b p(x)dx = \frac{h}{3}\left(f(a) + f(b) - 2f(x_M) + 6f(x_M)\right) \implies$$

$$\implies \int_a^b f(x)dx \approx \frac{h}{3}\left(f(a) + 4f(x_M) + f(b)\right) = \frac{b-a}{6}\left(f(a) + 4f(x_M) + f(b)\right)$$

In addition, denoting E the difference between the exact value of the integral and approximate value, it is shown that there is $c \in (a, b)$, such that

$$E = -\frac{h^5}{90}f^{(iv)}(c) = -\frac{(b-a)^5}{2880}f^{(iv)}(c) \implies |E| \le \frac{(b-a)^5}{2880}M,\ \text{with } M = \max\{|f^{(iv)}(x)| :\ a < x < b\}$$

therefore, the Simpson rule has precision 3 (or degree of accuracy 3) that is $E = 0$ for all polynomials $p_n(x)$ with $n \le 3$.

MATLAB Code: Simpson Rule

```
function r=simpson(f,a,b)
     % approximates the integral of the num.
     % function f, at the interval [a,b], using the Simpson rule
r=(b-a)*(f(a)+4*f((a+b)/2)+f(b))/6;
```

Example 3.1 *Calculate, using the Simpson rule,* $\int_0^2 (x^3 + x^2 + 2x - 1)dx$, $\int_0^\pi \sin\frac{x}{4}dx$ *and the absolute errors obtained.*

Solution:

MATLAB Code

```
function r=simpson(f,a,b)
%  approximates the integral of the numerical function f at [a,b]
r=(b-a)*(f(a)+4*f((a+b)/2)+f(b))/6;
```

Examples

```
 %% Example 1
f=@(x) x^3+x^2+2*x-1
% Approx. integral:
simpson_integral=simpson(f,0,2)
% Exact integral:
syms x, f_sym(x)=f(x)
exact_integral=double(int(f_sym,0,2))
error=abs(exact_integral-simpson_integral)
 %% Example 2
f=@(x) sin(x/4)
simpson_integral=simpson(f,0,pi)
syms x, f_sim(x)=f(x)
exact_integral=double(int(f_sim,0,pi))
error=abs(exact_integral-simpson_integral)
```

4 The composite trapezoidal method

The basic quadrature rules may be ineffective when the integration is performed over a long interval. More sampling of $f(x)$ is intuitively required in such circumstances. Increasing the order of the Newton-Cotes formulas is one way to deal with this problem, but high precision formulas of this sort go through the same problems that high degree polynomial interpolation experiences over long intervals.

Instead, we proceed with the approach that is equivalent to approximating $f(x)$ with piecewise polynomials. The resulting quadrature formulas, called composite rules, or composite quadrature methods, are the techniques most often used in practice.

Thus, in its simplest form we divide the interval $[a, b]$ into r equal subintervals of length $h = \frac{b-a}{r}$ each, that is

$$x_0 = a,\ x_1 = a + h,\ x_2 = a + 2h,\ \dots\ ,\ x_n = a + nh = b$$

and we obtain

$$\int_a^b f(x)dx = \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} f(x)dx$$

Now, we apply the trapezoid rule to each of the integrals on the right-hand:

$$\int_a^b f(x)dx \approx \frac{h}{2} \sum_{i=0}^{n-1} [f(x_i) + f(x_{i+1})]$$

In addition, denoting E the difference between the exact value of the integral and approximate value, it is shown that there is $c \in (a, b)$, such that

$$E = -\frac{(b-a)^3}{12n^2} f''(c) \implies |E| \leq \frac{(b-a)^3}{12n^2} M, \text{ with } M = \max\{|f''(x)| :\ a < x < b\}$$

A way of programming this method is to create the file composite_trapezoid.m as it follows

MATLAB Code

```
  function r=composite_trapezoid(f,a,b,n)
      Function r=composite_trapezoid(f,a,b,n) which apply the composite
      trapezoidal rule to approximate the integral of f on the interval [a,b]
  INPUT ARGUMENTS:
    f ........ Elementwise numerical function
    a,b ...... Ends of the integration interval
    n ........ Number of subintervals
  OUTPUT ARGUMENTS:
    r ........ Approximation of the integral
 x=linspace(a,b,n+1); h=(b-a)/n;
 r=(2*sum(f(x))-f(a)-f(b))*h/2;
```

Example 4.1 *Given* $\int_0^3 x^2 \cos 2x dx$,

1. *Approximate the value of the integral, using the composite trapezoid rule with* 20 *subintervals, and find the absolute error.*
2. *Calculate the number of subintervals which should be used to ensure that the absolute error is less than* 0.001 *and verify the result.*

Solution:

Item a): Approximation of the integral and the absolute error.

Item b): The error is bounded by $\frac{(b-a)^3}{12n^2} M$, with $M = \max\{|f''(x)| :\ a < x < b\}$, therefore, we need n such that

$$\frac{(b-a)^3}{12n^2} M = \varepsilon$$

thus, we guarantee that the error is less than ε, so that n must be:

$$\frac{(b-a)^3M}{12\varepsilon} = n^2 \implies n = \sqrt{\frac{(b-a)^3M}{12\varepsilon}}$$

Therefore, we denote $f_2(x) = f''(x)$, so we must calculate the maximum of the absolute value of the function $f_2(x)$ at the interval $[0,3]$.

We calculate the local maximum x_0 that is shown in the plot. As $f_2'(x)$ must be zero, we compute the zeros of f_2'.

MATLAB Code

```
% a)  We approximate the integral and the absolute error:
f=@(x) x.^2.*cos(2*x)
composite_trapezoid_integral=composite_trapezoid(f,0,3,20)
syms x, f_sim(x)=f(x)
exact_integral=double(int(f_sim,0,3))
error=abs(exact_integral-composite_trapezoid_integral)

% b) Error Bound
f2=diff(f_sim,2)
fplot(matlabFunction(f2),[0 3])

% c) Zeros of f2' to find M
x0=fzero(matlabFunction(diff(f2)),2)
max=double(abs(f2([x0,3])));% so, the maximum is reached at 3
M=double(abs(f2(3)))
a=0; b=3; e=1e-3;
n=ceil(sqrt((b-a)^3*M/12/e));% This means that we need 242 subintervals
% to approximate the integral

composite_trapezoid_integral=composite_trapezoid(f,0,3,n); %  the approx

error=abs(exact_integral-composite_trapezoid_integral)
```

5 Composite Simpson rule

In this method, we divide the interval $[a,b]$ into n equal subintervals of length $h = \frac{b-a}{n}$ each, that is

$$x_0 = a,\ x_1 = a+h,\ x_2 = a+2h,\ \dots\ ,\ x_n = a+nh = b$$

and we obtain

$$\int_a^b f(x)dx = \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} f(x)dx$$

now, we apply the Simpson rule to each of the integrals on the right-hand, so if we denote $y_i = x_i + \dfrac{h}{2}$ the middle point of x_i and x_{i+1}:

$$\int_a^b f(x)dx \approx \frac{h}{6}\sum_{i=0}^{n-1}[f(x_i) + 4f(y_i) + f(x_{i+1})]$$

In addition, denoting E the difference between the exact value of the integral and approximate value, it is shown that there is $c \in (a,b)$, such that

$$E = -\frac{(b-a)}{180} f^{(iv)}(c)h^4 = -\frac{(b-a)^5}{2880n^4} f^{(iv)}(c) \implies |E| \leq \frac{(b-a)^5}{2880n^4} M,$$

with $M = \max\{|f^{(iv)}(x)| : \ a < x < b\}$

MATLAB Code: Composite Simpson Rule Function

```
function r=composite_simpson(f,a,b,n)
%     Apply the composite Simpson rule to approximate
%     the integral of f on the interval [a,b]
% INPUT ARGUMENTS:
%   f ........ Elementwise numerical function
%  a,b ...... Ends of the integration interval
%   n ........ Number of subintervals
% OUTPUT ARGUMENTS:
%   r ........ Approximation of the integral
x=linspace(a,b,n+1); h=(b-a)/n; y=x+h/2; y(n+1)=[];
r=(2*sum(f(x))-f(a)-f(b)+4*sum(f(y)))*h/6;
```

Example 5.1 *Approximate the value of* $\int_0^{\pi} x^2 \sin 5x dx$, *using the composite Simpson rule with* 10 *subintervals and find the absolute error.*

Solution: we approximate the value of the integral and compute the error.

5.1 MATLAB commands

MATLAB has its own functions for approximating the value of the integrals. We highlight **quad** and **trapz**. Both functions are used to approximate

$$\int_a^b f(x)dx$$

and they work as follows:

`quad(f,a,b)`: approximates the integral of a function `f` from `a` to `b` to within an error of $1e-6$ using recursive adaptive Simpson quadrature. `f` must be an elementwise numerical function.

`trapz(x,y)`: apply the composite trapezoidal rule. `x` is a vector with the ends of the subintervals and `y` is a vector whose elements are $y_i = f(x_i)$.

Example 5.2 *Approximate* $\int_0^3 x^2 \cos 2x dx$, *using the command* **quad** *and the command* **trapz** *with* 20 *subintervals. Find the absolute errors.*

Solution:

MATLAB Code

```
f=@(x) x.^2.*cos(2*x)

% quad:
quad_integral=quad(f,0,3)

%trapez:
xx=linspace(0,3,21);
yy=f(xx);
trapz_integral=trapz(xx,yy)

% Exact integral:
syms x, f_sim(x)=f(x)
exact_integral=double(int(f_sim,0,3))
quad_error=abs(exact_integral-quad_integral)
trapz_error=abs(exact_integral-trapz_integral)
```