CALCULUS
DEGREE IN SOFTWARE ENGINEERING
CHAPTER 15. INTEGRATION OF RATIONAL AND TRIGONOMETRIC FUNCTIONS.

INTEGRATION OF RATIONAL FUNCTIONS.

As you know, a function is called rational if it is the quotient of two polynomials

$$F(x) = \frac{P(x)}{Q(x)}$$

In order to integrate these functions, first we must analyze whether the fraction is improper or proper. If it is improper, that is, the degree of the numerator is greater than or equal to the degree of the denominator, we must divide $P(x)$ by $Q(x)$ and express the rational function as the quotient (a polynomial $G(x)$) plus the remainder $R(x)$ divided by the denominator, that is

$$F(x) = G(x) + \frac{R(x)}{Q(x)}$$

Then, we integrate the polynomial and are left with the integration of a proper fraction- the new numerator (remainder) divided by the denominator.
Once we have to deal with a proper fraction, we use the method of partial fractions. This technique consists of several steps.

1) We search for the roots of the denominator, solving $Q(x) = 0$. There can be two types of roots: real and complex. We factorize $Q(x)$ as the product of the factors corresponding to each real root r of multiplicity m and of each complex root $\alpha + i\beta$ of multiplicity n.

$$Q(x) = (x-r)^m((x-\alpha)^2+\beta^2)^n..................$$

where we should include the factors coming from other roots, real or complex, in a similar way.

2) We decompose the proper rational function into the sum of partial fractions (Method of Partial Fractions) according to the following scheme.

$$\frac{R(x)}{Q(x)} = \frac{A_1}{(x-r)} + \frac{A_2}{(x-r)^2} + ... + \frac{A_m}{(x-r)^m} + \frac{B_1 x + C_1}{(x-\alpha)^2+\beta^2} + ... + \frac{B_n x + C_n}{((x-\alpha)^2+\beta^2)^n}$$

The coefficients $A_1, ...A_m, B_1, C_1, ...B_n, C_n$ are the undetermined coefficients and our next goal will be to find their value.

3) After we express the rational function as a sum of partial fractions, we place these fractions over a common denominator, clear the equations of fractions and equate the numerators on both sides.

4) We equate the coefficients of polynomials of like powers of x and/or substitute the real roots in both sides. In this way, we are always capable of determining the unknown coefficients.

5) We integrate each partial fraction.
Now, the question is how to integrate the partial fractions. We will show how to do it with some examples

EXAMPLE 1. A SIMPLE REAL ROOT.

$$\int \frac{dx}{(x-11)}$$

Each term of this type is integrated in a staightforward way

$$\int \frac{dx}{(x-11)} = \ln|x-11| + C$$

EXAMPLE 2. A MULTIPLE REAL ROOT.

In this case, we have to integrate several terms, imagine that one of them is

$$\int \frac{dx}{(x-6)^3} = -\frac{1}{2(x-6)^2} + C$$

or in general

$$\int \frac{dx}{(x-6)^k} = \frac{(x-6)^{(-k+1)}}{(-k+1)} + C$$

EXAMPLE 3. A SIMPLE COMPLEX ROOT.

$$\int \frac{3x+5}{(x-1)^2+4}\,dx = \int \frac{3(x-1)}{(x-1)^2+4}\,dx + \int \frac{8}{(x-1)^2+4}\,dx$$

We have created a factor $(x-1)$ in the numerator by subtracting and adding 3. Now, we make the substitution $(x-1)^2+4=t$ in the first integral and write

$$\int \frac{3(x-1)}{(x-1)^2+4}\,dx = \int \frac{3\,dt}{2t}$$

since $dt = 2(x-1)\,dx$

Finally,

$$\int \frac{3(x-1)}{(x-1)^2+4}\,dx = 3/2\ln\left[(x-1)^2+4)\right] + C$$

In the second integral, we make the change $t = (x-1)$ and obtain

$$\int \frac{8}{(x-1)^2+4}\,dx = \int \frac{8}{t^2+4}\,dt = 8/2\arctan\left(t/2\right)$$

as you can check by taking the derivative. Reversing the change

$$\int \frac{8}{t^2+4}\,dt = 8/2\arctan\left(t/2\right) = 4\arctan\left[(x-1)/2\right]$$

If we add up the two integrals

$$\int \frac{3x+5}{(x-1)^2+4}\,dx = 3/2\ln\left[(x-1)^2+4)\right] + 4\arctan[(x-1)/2] + C$$

This method can be used with any general expression

$$\int \frac{Cx+D}{(x-\alpha)^2+\beta^2}\,dx$$

EXAMPLE 4. A MULTIPLE COMPLEX ROOT.

The process for this case is quite complicated. We can use a method similar to the previous one, calculating a first integral by making the change $(x-\alpha)^2+\beta^2 = t$, but we have to integrate a second one of the type

$$I_n = \int \frac{dt}{(t^2+\beta^2)^n}$$

with $n > 1$. This can be done by using the Reduction Formula, which can be proved by integration by parts

$$I_n = \frac{t}{2\,\beta^2\,(n-1)\,(t^2+\beta^2)^{n-1}} + \frac{(2n-3)}{2\beta^2\,(n-1)} I_{n-1}$$

where

$$I_{n-1} = \int \frac{dt}{(t^2+\beta^2)^{n-1}}$$

and for each value of n, we can use the formula to reduce the exponent till we obtain $n = 1$ and find the arctangent as our antiderivative. Of course, the process can be cumbersome and boring and today, it is incorporated in programs of mathematical computing written in languages such as Matlab.

In Exercises 6, we will present problems in which we integrate rational functions, using the method of partial fractions: decomposing the integrand into simple fractions, finding the undetermined coefficients and integrating each term according to the examples shown.

INTEGRATION OF TRIGONOMETRIC FUNCTIONS.

In this section, we will integrate rational functions whose arguments are the basic trigonometric functions: sine and cosine. A general way of solving these integrals implies using the so-called Universal Change

$$t = \tan(x/2)$$

Since the application of this technique would be too lenghty and complicated, we just mention some cases in which other simpler substitutions can be carried out.

ODD FUNCTION IN SIN(X).

Here, we make the substitution $t = \cos x$. We will see how it works with an easy example.

EXAMPLE

$$\int \sin^5 x \, \cos^2 x \, dx$$

The integrand is a rational function, in particular a polynomial, whose variables are $\sin x$ and $\cos x$. As required, it is odd in $\sin x$. If we make the standard change $t = \cos x$, then $dt = -\sin x \, dx$ and we are left with a factor $\sin^4 x$ in the integrand. That is, the power of $\sin x$ is even and can be expressed in terms of $\cos x$ in a straightforward way .

$$\sin^4 x = (1 - t^2)^2$$

Finally, the integral is

$$\int -(1 - t^2)^2 \, t^2 \, dt$$

If we expand the binomial and integrate

$$\int -(1 - 2t^2 + t^4) \, t^2 \, dt = -t^3/3 + 2t^5/5 - t^7/7 + C = -(\cos^3 x)/3 + 2(\cos^5 x)/5 - (\cos^7 x)/7 + C$$

and the solution is a polynomial in $\cos x$.

ODD FUNCTION IN COS(X).

Now, the natural substitution is $t = \sin x$. We will see how to work out this type of integrals with an example. Instead of choosing a polynomial, whose solution would be very similar to that of the previous case, we are going to integrate

$$\int \sec x \, dx$$

EXAMPLE

If $t = \sin x$, $dt = \cos x\, dx$ and

$$\int \sec x\, dx = \int \frac{dt}{1-t^2}$$

This is one of our basic integrals, in terms of t instead of x. We know the solution

$$\int \frac{dt}{1-t^2} = 1/2 \ln\left|\frac{1+t}{1-t}\right| + C = 1/2 \ln\left|\frac{1+\sin x}{1-\sin x}\right| + C =$$

If we multiply and divide the argument of ln by $(1+\sin x)$,we obtain, after an easy simplification

$$1/2 \ln\left|\frac{1+\sin x}{1-\sin x}\right| + C = \ln|\sec x + \tan x| + C$$

In a similar way, now the integrand is odd in $\sin x$, we could find the integral of $\csc x$

$$\int \csc x\, dx = -\ln|\csc x + \cot x| + C$$

as we wrote in our Table of basic integrals.

EVEN FUNCTION IN COS(X) AND SIN(X).

Now, the recommended change is $t = \tan x$, $dt = \sec^2 x\, dx$, because, as in the previous cases, we always transform the integrand into a rational function in t. We will do a straightforward example

EXAMPLE

$$\int \sec^4 x\, dx = \int (1+\tan^2 x)\sec^2 x\, dx = \int (1+t^2)\, dt = t + t^3/3 + C = (\tan x) + (\tan^3 x)/3 + C$$

We will do different exercises applying these changes in Exercises 6. However, in some cases, it is better to use some basic trigonometric formulas. For instance, if we use $\cos 2x = 2\cos^2 x - 1$

$$\int \cos^2 x\, dx = \int \frac{(1+\cos 2x)}{2}\, dx = \frac{2x + \sin 2x}{4} + C$$

In a similar way,

$$\int \sin^2 x\, dx = \int \frac{(1-\cos 2x)}{2}\, dx = \frac{2x - \sin 2x}{4} + C$$

It is also very easy to integrate

$$\int \tan^2 x\, dx = \int (1+\tan^2 x - 1)\, dx = \tan x - x + C$$

or

$$\int \cot^2 x\,dx = \int (1+\cot^2 x - 1)\,dx = -\cot x - x + C$$

Our last example will be a little trickier since it involves integration by parts

$$\int \sec^3 x\,dx = \int \sec x\ \sec^2 x\,dx$$

We choose $u = \sec x$ and $dv = \sec^2 x\,dx$. Hence, $du = \sec x\ \tan x\,dx$ and $v = \tan x$. Applying integration by parts

$$\int \sec x\ \sec^2 x\,dx = \sec x\ \tan x - \int \sec x\ \tan^2 x\,dx = \sec x\ \tan x - \int \sec x\,(\sec^2 x - 1)\,dx$$

Moving the term with $\sec^3 x$ to the left-hand side, we solve

$$\int \sec^3 x\,dx = \frac{\sec x\ \tan x + \int \sec x\,dx}{2} = \frac{\sec x\ \tan x + \ln|\sec x + \tan x|}{2} + C$$

In the same way, we could integrate

$$\int \csc^3 x\,dx = -\frac{\csc x\ \cot x + \ln|\csc x + \cot x|}{2} + C$$

Try to prove it.