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S3-Euclidean Spaces Design of scalar products

Linear Algebra

Ingeniería del Software-Universidad de Oviedo

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Euclidean Spaces V

↑ scalar product

Vector Spaces $V, +, \cdot$ $\Uparrow$ λ $\quad \lambda \cdot \vec{v}$

metric problems

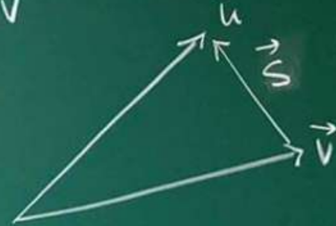
(1) Finding the size of a vector $\quad \|\vec{v}\| \overset{def}{=} \sqrt{\vec{v} \cdot \vec{v}}$

Norm

$\vec{u}_1 \cdot \vec{u}_2$

(2) Distances between two vectors. $\quad d(\vec{u}, \vec{v}) \overset{def}{=} \|\vec{u} - \vec{v}\|$

$d(\vec{u}, \vec{v}) \geq 0$ & $d(\vec{u}, \vec{v}) = 0 \Leftrightarrow \vec{u} = \vec{v}$



$\vec{v} + \vec{s} = \vec{u}$

$\vec{s} = \vec{u} - \vec{v}$

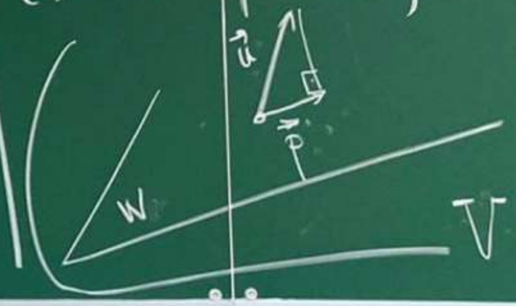
$d(\vec{u}, \vec{v}) = \|\vec{s}\|$

(3) Angle

$$\text{angle}(\vec{u}, \vec{v}) \overset{def}{=} \cos^{-1}\left(\frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \cdot \|\vec{v}\|}\right)$$

$\vec{u}$ & $\vec{v}$ are said to be orthogonal $\quad \text{angle}(\vec{u}, \vec{v}) = 90^\circ \ (\pi/2)$

(4) Orthogonal Projections



Design of scalar Products

V is Euclidean Space of dimension n

$B = \{\vec{b}_1, \vec{b}_2, \dots, \vec{b}_n\}$ a basis set in V $\quad \langle \vec{b}_1, \dots, \vec{b}_n \rangle \equiv V$

- dot product (scalar)

$$\vec{u} = (x_1, x_2, \dots, x_n)_{\{\vec{b}_1, \dots \vec{b}_n\}} = x_1 \vec{b}_1 + x_2 \vec{b}_2 + \dots + x_n \vec{b}_n = \sum_{k=1}^{n} x_k \vec{b}_k = B \cdot \vec{x}$$

$$B = \left[\vec{b}_1(:)\ \vec{b}_2(:) \dots \vec{b}_n(:) \right]$$

$$\vec{v} = (y_1, y_2, \dots y_n)_{\{\vec{b}_1, \dots \vec{b}_n\}} = y_1 \vec{b}_1 + y_2 \vec{b}_2 + \dots + y_n \vec{b}_n = \sum_{k=1}^{n} y_k \vec{b}_k = B \cdot \vec{y}$$

$$\vec{u} \cdot \vec{v} = (x_1 \vec{b}_1 + x_2 \vec{b}_2 + \dots + x_n \vec{b}_n) \cdot (y_1 \vec{b}_1 + y_2 \vec{b}_2 + \dots + y_n \vec{b}_n) = \sum_{i=1}^{n} \sum_{j=1}^{n} x_i\, y_j\, \vec{b}_i \cdot \vec{b}_j$$

i loop

bilinearity

scalar products of the basis terms.

$\in \mathbb{R}$ (scalar)

$\vec{x}^T$

NOR $= (x_1, x_2, \dots, x_n)^T$

row vector

$(1, n)$

j loop

$$\begin{pmatrix} \vec{b}_1 \cdot \vec{b}_1 & \vec{b}_1 \cdot \vec{b}_2 & \dots & \vec{b}_1 \cdot \vec{b}_n \\ \vdots & & & \vdots \\ \vec{b}_n \cdot \vec{b}_1 & \vec{b}_n \cdot \vec{b}_2 & \dots & \vec{b}_n \cdot \vec{b}_n \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \rightarrow \vec{y}$$

$(n, 1)$

$G \Rightarrow$ Gram Matrix of $\cdot$

(n, n)

$$G(i,j) = \vec{b}_i \cdot \vec{b}_j$$

$$\boxed{\vec{u} \cdot \vec{v} = \vec{x}^T G \cdot \vec{y}}$$

$$\vec{u}\cdot\vec{v} = (x_1\vec{b}_1 + x_2\vec{b}_2 + \dots + x_n\vec{b}_n)\cdot(y_1\vec{b}_1 + y_2\vec{b}_2 + \dots + y_n\vec{b}_n) = \sum_{i=1}^{n}\sum_{j=1}^{n} x_i\, y_j\, \vec{b}_i\cdot\vec{b}_j$$

i loop

bilinearity

scalar products of the basis terms.

$\vec{u}\cdot\vec{v} \in \mathbb{R}$ (scalar)

$\vec{x}^T$

j loop

$$\vec{u}\cdot\vec{v} = (x_1, x_2, \dots, x_n)^{T} \begin{pmatrix} \vec{b}_1\cdot\vec{b}_1 & \vec{b}_1\cdot\vec{b}_2 & \cdots & \vec{b}_1\cdot\vec{b}_n \\ \vdots & & & \vdots \\ \vec{b}_n\cdot\vec{b}_1 & \vec{b}_n\cdot\vec{b}_2 & \cdots & \vec{b}_n\cdot\vec{b}_n \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \to \vec{y}$$

row vector

$$G(i,j) = \vec{b}_i\cdot\vec{b}_j$$

G has special properties

$$\boxed{\vec{u}\cdot\vec{v} = \vec{x}^T G\, \vec{y}}$$

(2): $\vec{u}\cdot\vec{u} = \vec{x}^T G\, \vec{x} = \|\vec{u}\|_G^2 > 0$ if $\vec{u} \neq \vec{0}$ $\longrightarrow$ $G(i,j) = G(j,i)$

Gram matrix Definite positive Matrix. G is symmetric

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$$\vec{u}\cdot\vec{v} = (x_1\vec{b}_1 + x_2\vec{b}_2 + \dots + x_n\vec{b}_n)\cdot(y_1\vec{b}_1 + \vec{y}_2\vec{b}_2 + \dots + y_n\vec{b}_n) \underset{\text{bilinearity}}{=} \sum_{i=1}^{n}\sum_{j=1}^{n} x_i\, y_j\, \vec{b}_i\cdot\vec{b}_j$$

i loop

j loop

scalar products of the basis terms.

$\vec{u}\cdot\vec{v} \in \mathbb{R}$ (scalar)

$$\vec{u}\cdot\vec{v} = \underbrace{(x_1, x_2, \dots, x_n)}_{\vec{x}^T}{}^{T} \begin{pmatrix} \vec{b}_1\cdot\vec{b}_1 & \vec{b}_1\cdot\vec{b}_2 & \cdots & \vec{b}_1\cdot\vec{b}_n \\ \vdots & & & \vdots \\ \vec{b}_n\cdot\vec{b}_1 & \vec{b}_n\cdot\vec{b}_2 & \cdots & \vec{b}_n\cdot\vec{b}_n \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \rightsquigarrow \vec{y}$$

row vector

$$G(i,j) = \vec{b}_i\cdot\vec{b}_j$$

G has special properties

$$\boxed{\vec{u}\cdot\vec{v} = \vec{x}^T G\cdot\vec{y}}$$

(2): $\vec{u}\cdot\vec{u} = \vec{x}^T G\,\vec{x} = \|\vec{u}\|_G^2 > 0$ if $\vec{u} \neq \vec{0}$ $\longrightarrow$ $G(i,j) = G(j,i)$

Gram matrix ⇒ Definite positive Matrix. G is symmetric

$$\|\vec{u}\|^2 = \vec{u}\cdot\vec{u} = (1,2)\cdot(1,2) = 5 \Rightarrow \|\vec{u}\|_2 = \sqrt{5}$$

norm L2 or Euclidean norm.

$$\|\vec{u}\|_0 = \#_0 = 0$$

↓

Sparsity → number of 0s that a signal has.

Note.– There exist norms in $\mathbb{R}^n$ that are not induced by scalar products. → $\|\vec{u}\|_\infty = \max\{|1|, |2|\} = 2$

→ $\|\vec{u}\|_1 = |1| + |2| = 3$. L_1 norm.

→ $\|\vec{u}\|_p = \sqrt[3]{|1|^3 + |2|^3} = \sqrt[3]{9}$, $p = 3$

How to define scalar products in $\mathbb{R}^n$, $C^0[a,b]$, $M_{m\times n}(\mathbb{R})$?

(1) In $\mathbb{R}^n$ ⟶ the easiest scalar product is the Euclidean

$$\vec{x}\cdot\vec{y} = \sum_{k=1}^{n} x_k y_k$$

(High School Physics).