CALCULUS.
DEGREE IN SOFTWARE ENGINEERING
EXERCISES AND SOLUTIONS 2

1. Calculate the limits of these functions :

I) $\lim_{x\to -1} \frac{x^4+x-2}{x^3+2x^2+1}$ II) $\lim_{x\to -2} \frac{x^4+2x^3}{x^2-4}$

III) $\lim_{x\to 2} \frac{x-2}{1-\sqrt{3-x}}$ IV) $\lim_{x\to 1} \frac{x^3-1}{x^2-x}$

V) $\lim_{x\to 5} \sqrt{x^2+x+6}$ VI) $\lim_{x\to 2} (\frac{x+1}{x-1})^{(2x-6)}$

VII) $\lim_{x\to\infty} x^4-x^3+1$ VIII) $\lim_{x\to +\infty} \frac{e^x+cosx}{e^x-cosx}$

IX) $\lim_{x\to 0} \frac{\sin x + x}{x}$ X) $\lim_{x\to -1} \frac{x}{x+1}$

XI) $\lim_{x\to 0} \sin(1/x)$ XII) $\lim_{x\to\infty} \sqrt{x+2}-\sqrt{x}$

XIII) $\lim_{x\to -\infty} \frac{5x+10}{2x}$ XIV) $\lim_{x\to 2} \left(\frac{2x+1}{2x+3}\right)^{\frac{x^2-1}{(x-2)^2}}$

Solutions: I) Mere substitution: -1 , II) Factorize numerator and denominator and simplify:

$$\frac{x^3(x+2)}{(x-2)(x+2)} = \frac{x^3}{(x-2)}$$

. It is equal to 2 at $x=-2$. III) Multiply numerator and denominator by the conjugate of the denominator and simplify:

$$\frac{(x-2)(1+\sqrt{3-x})}{1-(3-x)} = (1+\sqrt{3-x})$$

at $x=2$, it is 2. IV) It is again an indeterminate form, factorize numerator and denominator and simplify

$$\frac{(x-1)(x^2+x+1)}{x(x-1)} = \frac{(x^2+x+1)}{x}$$

The value of the limit is 3. V) Substitution: 6. VI) Substitution: 1/9. VII) Using the limit laws for limits at ∞, the solution is ∞ . VIII) Dividing numerator and denominator by e^x, the limit is 1, since the cosine function is bounded. IX) Using the fact that $\lim_{x\to 0} \frac{\sin x}{x} = 1$, the solution is 2. X) the limit is $-1/0$, but the sign depends on how we approach -1, so $\lim_{x\to -1^+} x/(x+1) = -1/0^+ = -\infty$ and $\lim_{x\to -1^-} x/(x+1) = -1/0^- = \infty$. XI) In this case, the limit does not exist, $1/x$ takes values for which the sine oscillates between -1 and 1 and does not approach a given number. For instance, if $x = 1/(k\pi)$ with k an integer the value is always zero and for $x = 1/((2k+1)\pi)$ the value is 1 or -1. XII) We multiply and divide by the conjugate of the expression and obtain

$$\lim_{x\to\infty} \frac{2}{\sqrt{x+2}+\sqrt{x}} = 0$$

XIII) Dividing by x, the limit is 5/2. XIV) We substitute and obtain : $(5/7)^\infty$, that is, 0.

2. Let $f(x)$ be

$$f(x) = \begin{cases} x^2|x+2| & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ x^2 \sin \frac{1}{x} & \text{if } x > 0 \end{cases}$$

Study the continuity of $f(x)$

Solution: For $x < 0$ the function is continuous, as the product of continuous functions. At $x = 0$, we calculate $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} x^2|x+2| = 0$ and $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x^2 \sin 1/x = 0$. In this last limit, we have the product of a function with zero limit and a bounded function: $\sin 1/x$. So, the function is continuous at $x = 0$. For $x > 0$, the function is also continuous, as the product of two continuous functions. In conclusion, $f(x)$ is continuous on its domain, $\mathbb{R}$.

3. Calculate a and b such that $f(x)$ is a continuous function at $x = 0$ and $x = 1$

$$f(x) = \begin{cases} e^x + a & \text{if } x < 0 \\ ax^2 + 2 & \text{if } 0 \le x \le 1 \\ \dfrac{b}{2x} & \text{if } x > 1 \end{cases}$$

Solution: The function is continuous except, maybe, at the points where there is a change of definition. First, we calculate

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} e^x + a = 1 + a$$

and

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} ax^2 + 2 = 2$$

For the function to be continuous at $x = 0$, a must be equal to 1. At $x = 1$,

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} x^2 + 2 = 3$$

and

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} b/(2x) = b/2$$

For the limits to be equal, b must be 6. So, with $a = 1$ and $b = 6$, the function is continuous on $\mathbb{R}$.

4. The function $f(x) = \dfrac{x}{1 + e^{\frac{1}{x+1}}}$ has a discontinuity at $x = -1$. Is the discontinuity removable? Analyze also the behaviour of the function at ∞ and $-\infty$

   Solution: First

   $$\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-} \frac{x}{1 + e^{\frac{1}{x+1}}} = -1$$

   In this case, $x + 1$ goes to zero with negative sign, so that $1/(x + 1)$ tends to $-\infty$ and the exponential approaches 0. Therefore, the limit is -1.

   Second

   $$\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} \frac{x}{1 + e^{\frac{1}{x+1}}} = 0$$

   Now, $x + 1$ goes to zero with positive sign, so that $1/(x + 1)$ tends to ∞ and the exponential approaches ∞. Therefore, the limit is 0. In conclusion, the discontinuity is not removable, it is a jump discontinuity.

   As regards the behaviour at ∞ and $-\infty$, we can check that in both cases $e^{\frac{1}{x+1}}$ tends to 1. Therefore, the limit of $f(x)$ is ∞ at $x = \infty$ and $-\infty$ at $x = -\infty$.

5. Prove that $f(x) = \sin x + 2x - 1$ has at least a real zero.

   Solution: By inspection, we see that $f(0) = -1$ and $f(\pi/2) = \pi$. Since the function is continuous, we apply Bolzano´s theorem and prove that there is a function's zero in $(0, \pi/2)$. We could use bisection for better estimates of that zero.

6. Prove that the graphs of the functions $h(x) = \ln x$ ans $g(x) = e^{-x}$ intersect at least at one point.

   Solution: We define $f(x) = g(x) - h(x)$ and calculate $f(1) = 1/e > 0$ and $f(2) = e^{-2} - \ln 2 < 0$. Applying Bolzano's theorem, we prove that there is a number c in $(1, 2)$ such that $e^{-c} = \ln c$. This means that both graph intersect at $x = c$.

7. Given the equation $f(x) = x^3 + \lambda x^2 - 2x - 1 = 0$. Prove that

   (a) if $\lambda > 2$ the equation has at least a solution less than 1.

   (b) If $\lambda < 2$ there is a solution of the equation that is greater than 1.

   Solution (a): It is clear that $f(0) = -1$ and $f(1) = \lambda - 2$. If $\lambda > 2$, there is a sign change of a continuous function in $[0, 1]$, then it has a zero in $(0, 1)$. We apply Bolzano´s theorem.

   Solution (b): Now, $f(1) = \lambda - 2 < 0$, we cannot use $f(0) = -1$. However, $\lim_{x\to\infty} f(x) = \infty$. So, $f(b) > 0$, for a certain $b > 1$. There must be a zero in $(1, b)$.

8. Has the equation $g(x) = ax^5 + bx^3 + cx + d = 0$ a real solution ? Use Bolzano´s theorem.

   Solution: $\lim_{x\to\infty} g(x) = \infty$ and $\lim_{x\to-\infty} g(x) = -\infty$, if $a > 0$. Then there must be a solution because there is a sign change. A similar argument holds if $a < 0$. If $a = 0$, we apply the same idea to the cubic polynomial. Can you see that there is always a real solution provided the polynomial is of odd degree ?