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S4-Linear Applications matrices and linear systems

Linear Algebra

Ingeniería del Software-Universidad de Oviedo

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Linear Applications



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Linear Applications

$$U_{B_1} \xrightarrow{L} V_{B_2}$$

$$\dim U = n$$

$$\dim V = m$$

$$B_1 = \{\vec{e}_1, \dots, \vec{e}_n\} \quad \vec{x} = (x_1, \dots, x_n)_{B_1} \xrightarrow{L} \vec{y} = L(\vec{x}) = (y_1, \dots, y_m)_{B_2}$$

$$y_1 = a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n$$

$$\vdots$$

$$\vec{y}_m = a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n$$

$$\Rightarrow \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix} = \vec{y}_{B_2} = A \cdot \vec{x}_{B_1}$$

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} = [L(\vec{e}_1) \dots L(\vec{e}_n)]$$

$$L(\vec{e}_1) = A \cdot \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}$$

$$\vdots$$

$$L(\vec{e}_n) = A \cdot \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

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$$U_{B_1} \xrightarrow{\vec{L}} V_{B_2}$$

$$\dim U = n$$

$$\dim V = m$$

$$\vec{x} = (x_1, \dots, x_n)_{B_1}$$

$$A \rightarrow \begin{matrix} \vec{B}_2 \\ \vec{B}_1 \end{matrix}$$

$$A_{B_1 B_2}$$

$$A = \begin{bmatrix} \vec{c}_1(\cdot) & \vec{c}_2(\cdot) & \dots & \vec{c}_n(\cdot) \end{bmatrix}$$

$$\text{where } \vec{c}_i(\cdot) \text{ is } \vec{L}(\vec{e}_i)$$

that is the transform of the i -th vector of basis B_1 expressed in B_2

$$\vec{y} = \vec{L}(\vec{x}) = (y_1, \dots, y_m)_{B_2}$$

Characterization of \vec{L}

The linear application \vec{L} in basis sets B_1 and B_2

is represented by a matrix $A \in M(\mathbb{R})_{m \times n}$ that is built as follows:

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Ex 1 $\mathbb{R}^2_{B_1} \longrightarrow \mathbb{R}^3_{B_2}$ B_1 & B_2 the canonic

$$\{(1,0), (0,1)\} \quad \{(1,0,0), (0,1,0), (0,0,1)\}$$

$$\vec{x} = (x_1, x_2) \longmapsto \vec{y} = (y_1, y_2, y_3)$$

Ad-hoc $\vec{L} \begin{pmatrix} y_1 \stackrel{\text{def}}{=} 3x_1 - x_2 \\ y_2 \stackrel{\text{def}}{=} x_1 \\ y_3 \stackrel{\text{def}}{=} x_1 + x_2 \end{pmatrix}$

$$\vec{L}(\vec{e}_1) = \vec{L} \begin{pmatrix} 1,0 \\ x_1, x_2 \end{pmatrix} = (3, 1, 1)_{B_2}$$

$$\vec{L}(\vec{e}_2) = \vec{L} \begin{pmatrix} 0,1 \\ x_1, x_2 \end{pmatrix} = (-1, 0, 1)_{B_2}$$

$$A = \begin{bmatrix} 3 & -1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$A \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3x_1 - x_2 \\ x_1 \\ x_1 + x_2 \end{pmatrix}$$

C9d

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Ex 2

$$U = P_3(x)$$

$$P(x) = a_0 + a_1x + a_2x^2 + a_3x^3$$

$$B_1 = \{1, x, x^2, x^3\}$$

Finding $A_{B_1B_2}$

$$\vec{L}: \int \rightarrow HW \quad P_4(x)$$

$$V = P_2(x)$$

$$p'(x) = \frac{dp}{dx} = a_1 + 2a_2x + 3a_3x^2$$

$$B_2 = \{1, x, x^2\}$$

$$A \in M(\mathbb{R})_{3 \times 4} = \left[\frac{d\vec{1}}{dx}, \frac{d\vec{x}}{dx}, \frac{d\vec{x^2}}{dx}, \frac{d\vec{x^3}}{dx} \right]$$

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

$$\frac{d\vec{1}}{dx} = 0 = (0, 0, 0)_{B_2}$$

$$\frac{d\vec{x}}{dx} = 1 = (1, 0, 0)_{B_2}$$

$$\frac{d\vec{x^2}}{dx} = 2x = (0, 2, 0)_{B_2}$$

$$\frac{d\vec{x^3}}{dx} = 3x^2 = (0, 0, 3)_{B_2}$$

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Linear Applications $\int \rightarrow \mathbb{H}W$ $\mathcal{P}_4(x)$
 $\vec{L}: \frac{d}{dx} \rightarrow \vec{V} = \mathcal{P}_2(x)$ $A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$

Ex2. $\mathcal{U}_4 = \mathcal{P}_3(x)$

$P(x) = a_0 + a_1x + a_2x^2 + a_3x^3$

$B_1 = \{1, x, x^2, x^3\}$

$P'(x) = \frac{dP}{dx} = a_1 + 2a_2x + 3a_3x^2$

$B_2 = \{1, x, x^2\}$

Find $q(x) = 1 + 2x - x^2 + 3x^3 = (1, 2, -1, 3)_{B_1}$

$\frac{dq(x)}{dx} = A \cdot \begin{pmatrix} 1 \\ 2 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ 9 \end{pmatrix}_{B_2} = 2 - 2x + 9x^2$

$\vec{L} \rightarrow q'(x)$
 $\vec{q}_{B_1} \rightarrow \boxed{A} (A\vec{q})_{B_2}$

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Endomorphism ($U \equiv V$)

\vec{L}

$M_{2 \times 2}(\mathbb{R})$

E_{X3}

$M_{2 \times 2}(\mathbb{R})$

$\dim = 4$

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$\vec{L}(A) \stackrel{\text{def}}{=} A^T = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix}$$

$$B_1 = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

$$P \rightarrow \begin{matrix} \vec{L} \\ B_1 B_1 \end{matrix} \in M(\mathbb{R})_{4 \times 4}$$

$$C = \begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix}$$

$$\vec{C}^T = \begin{pmatrix} 3 \\ 1 \\ -2 \\ 0 \end{pmatrix}_{B_1} = P \cdot \begin{pmatrix} 3 \\ -2 \\ 1 \\ 0 \end{pmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Review: From continuous to discrete least squares



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$$f(x) = \lg(x) \quad \text{in } [0, 1].$$

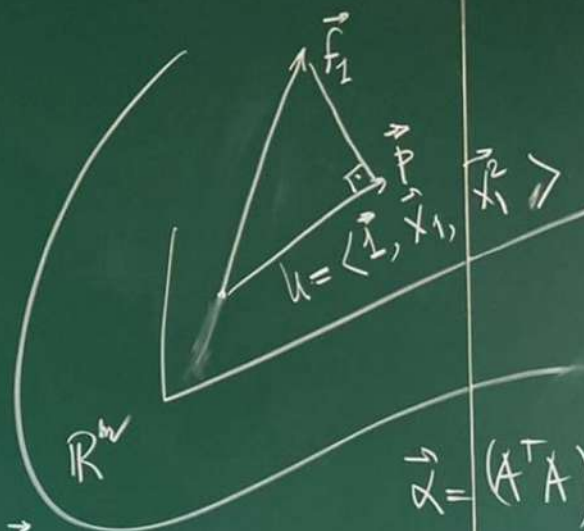
$$\vec{x}_1 = [0; 0.01; 1] \in \mathbb{R}^m \quad \text{Sampling in } x$$

$$\vec{f}_1 = f(x_1) \leadsto \text{Sampling of } f(x) \text{ in } x_1 \in \mathbb{R}^m$$

$$x_i \in \vec{x}_1 \quad f(x_i) \simeq a_0 + a_1 \cdot x_i + a_2 x_i^2$$

$$i = 1, \dots, m$$

$$\begin{pmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ \vdots & \vdots & \vdots \\ 1 & x_m & x_m^2 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \vec{f}_1$$



$$A \cdot \vec{x} = \vec{f}_1$$

↓ LS

$$A^T A \vec{x} = A^T \vec{f}_1$$

$$\vec{x} = (A^T A)^{-1} (A^T \vec{f}_1)$$

$$\vec{p} = A \cdot \vec{x}$$