CALCULUS
DEGREE IN SOFTWARE ENGINEERING
CHAPTER 3. COMPOSITION OF FUNCTIONS. INVERSE FUNCTIONS.

If $f : \mathbb{R} \longrightarrow \mathbb{R}$ and $g : \mathbb{R} \longrightarrow \mathbb{R}$ are two real functions of a single real variable, we can define their addition $(f+g)(x) = f(x) + g(x)$, their subtraction $(f-g)(x) = f(x) - g(x)$, their multiplication $(f.g)(x) = f(x).g(x)$ and their division $(f/g)(x) = f(x)/g(x)$, the results of these operations are called sum, difference, product and quotient of both functions, respectively. In this way, we have constructed, for example, polynomials and rational functions by just combining functions of the type x^n. In order to obtain general elementary functions, we must define the composition of functions

$$(f \circ g)(x) = f(g(x))$$

This means " f composed with g" and $f \circ g$ is the composite function. Of course, the composition operation can be applied to general functions with general domains and ranges, provided that the domain of f and the range of g have a non-empty intersection. It is very easy to compose functions. For example, if $f(x) = e^x$ and $g(x) = \cos x$, $(f \circ g)(x) = e^{\cos x}$ and $(g \circ f)(x) = \cos(e^x)$, we see that the composition of functions is not commutative. It can be easily proved that it is associative. We can write

$$(f \circ g \circ h)(x) = ((f \circ g) \circ h)(x) = (f \circ (g \circ h))(x)$$

The identity function $I(x) = x$ is a neutral element, as $(f \circ I)(x) = f(x) = (I \circ f)(x)$. It is natural to try to define inverse functions now. However, imagine that for a function f two or more input elements have the same image. The inverse function would be ill-defined since we could not associate a unique element to that image. For instance, if we associate one house to two people, the inverse function would assign two people to that house and it would not be a function. We avoid this problem by defining one-to-one functions, for which each output element comes from a unique input element.

DEFINITION 1

A function is one-to-one on a domain D if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$ in D.

Different input values imply different output values or, in other words, the same output value must correspond to the same input value.
If $f(x_1) = f(x_2)$, then $x_1 = x_2$. As we will normally deal with real functions of a single real variable, the following test is very useful to check if a function is one-to-one

THE HORIZONTAL LINE TEST

A real function of a single real variable is one-to-one on its domain if and only if its graph intersects each horizontal line at most once.

We can see that $y = x^3$ is one-to-one but $y = x^2$ is not. However, we can make this function one-to-one if we restrict its domain to $[0, \infty)$. The mention of the domain in the definition of one-to-one functions is essential. For a one-to-one function, we can always define its inverse. If $y = f(x) = x^3$, its inverse f^{-1} is obtained by isolating x in $y = x^3$, then $y = f^{-1}(x) = x^{1/3}$. Thus, $(f \circ f^{-1})(x) = x = (f^{-1} \circ f)(x)$, as we require from an inverse function.
Now, we give a precise definition of inverse function

DEFINITION 2

Suppose that f is a one-to-one function on a domain D with range Y. The inverse function f^{-1} is defined by

$$f^{-1}(b) = a \;\; if \;\; f(a) = b$$

The domain of f^{-1} is Y and the range of f^{-1} is D.

Given the exponential function $y = f(x) = e^x$, a one-to-one function with domain $D = \mathbb{R}$, we can define its inverse function, the natural logarithm $y = f^{-1}(x) = \ln x$, its domain is $(0, \infty)$. If we compose both functions, we obtain $(f \circ f^{-1})(x) = x$ and $(f^{-1} \circ f)(x) = x$. However, in the first composition the domain is $(0, \infty)$ and in the second $\mathbb{R}$. In "Figures", we can see both graphs, an interesting property is that a function and its inverse are symmetric with respect to the bisectrix of the first quadrant. Now, we will do a short exercise to see the importance of the domain in the definition of inverse: consider $y = f(x) = x^2$ and define it on the domain $D = (-\infty, 0]$ to make it one-to-one. Then, the inverse function is $y = f^{-1}(x) = -\sqrt{x}$, so that we return to the negative reals. In this case, the domain of f is $D = (-\infty, 0]$, its range $Y = [0, \infty)$. The domain of f^{-1} is Y and its range is D. When we compose both functions, we obtain

$$(f \circ f^{-1})(x) = (-\sqrt{x})^2 = x$$

and

$$(f^{-1} \circ f)(x) = (-\sqrt{x^2}) = x$$

In this last expression, we have taken into account that x is **negative**.
Finally, we will define the inverse trigonometric functions, by using the previous definitions.
The sine function is periodic with period 2π, it is a very good example of a non one-to-one function. However, we can make it one-to-one by defining it on a suitable domain so that it covers its whole range $[-1, 1]$. For the sake of simplicity, this interval is $[-\pi/2, \pi/2]$. The inverse of the sine function is the arcsine, $y = \arcsin(x)$,

defined as the angle (in radians) between $-\pi/2$ and $\pi/2$ whose sine is x. The domain of the arcsine is $[-1,1]$ and its range is $[-\pi/2,\pi/2]$. The restriction of the domain of the sine is essential for the definition of the arcsine. We can see the graph of the arcsine in "Figures". The graphs of sine and arcsine are symmetric with respect to the bisectrix of the first quadrant. If you write

$$\arcsin(\sin(x)) = x$$

this is true, provided that x is in $[-\pi/2,\pi/2]$. If you carry out the composition for $x = 3\pi/2$, you obtain $-\pi/2$, that is, the composition is not the identity function.
To make the cosine a one-to-one function, we restrict the domain to $[0,\pi]$, check that $[-\pi/2,\pi/2]$ will not work in this case. Therefore, we define $y = \arccos(x)$ as the angle (in radians) in that interval for which $x = \cos y$. The domain of arccos is $[-1,1]$ and the range is $[0,\pi]$.
In a similar way, we can define the arctangent, arctan, with domain $D = (-\infty,\infty)$ and range $Y = (-\pi/2,\pi/2)$, the arccotangent, arccot, with $D = (-\infty,\infty)$ and $Y = (0,\pi)$, the arcsecant, arcsec, with $D = (-\infty,-1]\cup[1,\infty)$ and $Y = [0,\pi/2)\cup(\pi/2,\pi]$ and the arccosecant, arccsc, with $D = (-\infty,-1]\cup[1,\infty)$ and $Y = [-\pi/2,0)\cup(0,\pi/2]$. You can have a look at all the graphs of the inverse trigonometric functions in Figures. We will also work out a few exercises about the domain of these functions in Exercises 1.