CALCULUS
DEGREE IN SOFTWARE ENGINEERING
EXERCISES AND SOLUTIONS 6.

INTEGRATION OF RATIONAL FUNCTIONS.

Calculate the following integrals

1.
$$\int \frac{x^2+4x+1}{x^3+3x^2-x-3}\,dx$$

Solution:

Since the integrand is a proper rational function, we find the roots of the denominator and factorize it. We can see that the roots are real and simple

$$x^3+3x^2-x-3 = (x-1)(x+1)(x+3)$$

Now, we write the integrand as the sum of partial fractions

$$\frac{x^2+4x+1}{x^3+3x^2-x-3} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x+3}$$

and try to find the undetermined coefficients by putting the expression on the rigth-hand side over a common denominator and equating the numerators of both sides

$$x^2+4x+1 = A(x+1)(x+3) + B(x-1)(x+3) + C(x-1)(x+1)$$

This must be true for any x. Then, we substitute $x = 1, -1, -3$, the roots, and solve for the coefficients

$$A = 3/4$$
$$B = 1/2$$
$$C = -1/4$$

Therefore, the integral can be written

$$\int \frac{x^2+4x+1}{x^3+3x^2-x-3}\,dx = \int \frac{3}{4(x-1)}\,dx + \int \frac{1}{2(x+1)}\,dx + \int \frac{-1}{4(x+3)}\,dx$$

now, we can integrate in a straightforward way

$$\int \frac{x^2+4x+1}{x^3+3x^2-x-3}\,dx = \frac{3}{4}\ln|x-1| + \frac{1}{2}\ln|x+1| + \frac{-1}{4}\ln|x+3| + C$$

and, as is the case with simple real roots, the solution is a sum of natural logarithms.

2.
$$\int \frac{2x^2+3}{x^3-2x^2+x}\,dx$$

Solution:

First, we find the roots of the denominator and factorize it

$$x^3-2x^2+x = x(x-1)^2$$

the roots are real, but one of them is double. Hence, the corresponding sum of partial fractions is

$$\frac{2x^2+3}{x^3-2x^2+x} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$

Note that we need two terms for the double real root, one with denominator $x-1$ and other with denominator $(x-1)^2$. Equating the numerators, we obtain

$$2x^2+3 = A(x-1)^2 + Bx(x-1) + Cx$$

If we substitute the roots, we find

$$A = 3$$
$$C = 5$$

we have run out of roots and we must equate the coefficients of x^2 to determine B

$$2 = A + B$$
$$B = -1$$

Our final integral is

$$\int \frac{2x^2+3}{x^3-2x^2+x}\,dx = \int \frac{3}{x}\,dx + \int \frac{-1}{x-1}\,dx + \int \frac{5}{(x-1)^2}\,dx$$

and the final solution I is

$$I = 3\ln|x| - \ln|x-1| - \frac{5}{(x-1)} + C$$

3.

$$\int \frac{-2x}{x^3+x^2+x+1}\,dx$$

Solution:

The roots of the denominator are $x=-1$ and $x=i,-i$ and its factorization is

$$x^3+x^2+x+1 = (x+1)(x^2+1)$$

Since there is a complex root, the integrand can be expressed as a sum of partial fractions in the following way

$$\frac{-2x}{x^3+x^2+x+1} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}$$

In this case, we must include a term with $Bx+C$ in the numerator. Equating the numerators

$$-2x = A(x^2+1) + (Bx+C)(x+1)$$

Substituting $x=-1$, we obtain

$$A = 1$$

and equating like powers of x on both sides

$$A + B = 0$$
$$B + C = -2$$

The solution of this simple system is

$$B = -1$$
$$C = -1$$

Therefore, the integral can be written

$$\int \frac{-2x}{x^3+x^2+x+1}\,dx = \int \frac{1}{x+1}\,dx - \int \frac{x}{x^2+1}\,dx - \int \frac{1}{x^2+1}\,dx$$

and the integration is immediate

$$\int \frac{-2x}{x^3+x^2+x+1}\,dx = \ln|x+1| - \frac{1}{2}\ln\left(x^2+1\right) - \arctan x + C$$

4.

$$\int \frac{dx}{x^3+x^2+x}$$

Solution: the denominator can be factorized as

$$x^3+x^2+x = x(x^2+x+1)$$

with the second factor corresponding to the complex root $-1/2+\sqrt{3}i/2$ and its conjugate $-1/2-\sqrt{3}i/2$. Of course, the second factor can also be expressed as

$$x^2+x+1 = (x+1/2)^2+3/4$$

In order to decompose the integral into partial fractions, we write

$$\frac{1}{x^3+x^2+x} = \frac{A}{x} + \frac{Bx+C}{x^2+x+1}$$

and we find the undetermined coefficients by solving

$$1 = A(x^2+x+1)+(Bx+C)x$$

with $x=0$, we find $A=1$, and the equations for the powers x^2 and x are

$$A+B=0$$

$$A+C=0$$

Hence, $B=-1$, $C=-1$.

The original integral becomes

$$\int \frac{dx}{x^3+x^2+x} = \int \frac{dx}{x} - \int \frac{x+1}{(x+1/2)^2+3/4}\,dx$$

Now we write the numerator of the second integral in a suitable way

$$\int \frac{dx}{x^3+x^2+x} = \int \frac{dx}{x} - \int \frac{x+1/2}{(x+1/2)^2+3/4}\,dx - \int \frac{1/2}{(x+1/2)^2+3/4}\,dx$$

and the solution is

$$\ln|x| - \frac{1}{2}\ln\left[(x+1/2)^2+3/4\right] - \frac{1}{\sqrt{3}}\arctan{(2x+1)/\sqrt{3}} + C$$

The last integral has been carried out by using the change $x+1/2=t$ and remembering that

$$\int \frac{dt}{t^2+a^2} = \frac{\arctan{(t/a)}}{a}$$

5.

$$\int \frac{x^4-3x^2-3x-2}{x^3-x^2-2x}\,dx$$

Solution: This is not a proper rational function, then we must divide the numerator by the denominator. The result is

$$x+1-\frac{(x+2)}{x^3-x^2-2x}$$

The last fraction- remainder divided by divisor- is a proper rational function. If we factorize this divisor

$$x^3-x^2-2x = x(x-2)(x+1)$$

we can write

$$\frac{(x+2)}{x^3-x^2-2x} = \frac{A}{x}+\frac{B}{x-2}+\frac{C}{x+1}$$

Finding the undetermined coefficients is equivalent to solving

$$x+2 = A(x-2)(x+1)+Bx(x+1)+Cx(x-2)$$

we easilily find that the solution is

$$A=-1$$
$$B=2/3$$
$$C=1/3$$

Finally,

$$\int \frac{x^4-3x^2-3x-2}{x^3-x^2-2x}\,dx = \int (x+1)\,dx + \int \frac{1}{x}\,dx - \int \frac{2dx}{3(x-2)} - \int \frac{dx}{3(x+1)}$$

and integrating each term

$$\int \frac{x^4 - 3x^2 - 3x - 2}{x^3 - x^2 - 2x}\, dx = x^2/2 + x + \ln|x| - 2/3 \ln|x-2| - 1/3 \ln|x+1| + C$$

INTEGRATION BY PARTS.

6.

$$\int x \arctan x \, dx$$

Solution: We choose $u = \arctan x$, $dv = x dx$. Then, $du = \dfrac{dx}{1+x^2}$, $v = x^2/2$ and

$$\int x \arctan x \, dx = x^2/2 \, \arctan x - \int \frac{x^2 dx}{2(1+x^2)}$$

Dividing in the last integral, the solution is

$$\int x \arctan x \, dx = x^2/2 \, \arctan x - x/2 + 1/2 \arctan x + C$$

7.

$$\int e^x \sin x \, dx$$

Now, $u = e^x$, $dv = \sin x \, dx$, $du = e^x \, dx$ and $v = -\cos x$

$$\int e^x \sin x \, dx = -e^x \, \cos x + \int e^x \, \cos x \, dx$$

We integrate by parts again, but now
$u = e^x$, $dv = \cos x \, dx$, $du = e^x \, dx$ and $v = \sin x$

$$\int e^x \sin x \, dx = -e^x \, \cos x + e^x \, \sin x - \int e^x \, \sin x \, dx$$

Isolating the unknown integral

$$\int e^x \sin x \, dx = (-e^x \, \cos x + e^x \, \sin x)/2 + C$$

where we have added an arbitrary constant at the end of the process. You could try to solve

$$\int e^x \cos x \, dx$$

by using the same method

8.

$$\int x \ln(1 + 1/x) \, dx$$

Now, we choose $u = \ln(1 + 1/x)$, $dv = x\,dx$. Then,
$v = x^2/2$ and $du = -\dfrac{x}{2(x+1)x^2}$
Integrating by parts

$$\int x \ln(1 + 1/x) \, dx = x^2/2 \ln(1 + 1/x) + \int \frac{x}{2(x+1)} \, dx$$

The last integral is very simple and the final solution is

$$\int x \ln(1 + 1/x) \, dx = x^2/2 \ln(1 + 1/x) + (x - \ln|x + 1|)/2 + C$$

9.

$$\int x^2 \cos x \, dx$$

Solution:
if $u = x^2$ and $dv = \cos x \, dx$, then $du = 2x \, dx$ and $v = \sin x$. Integrating by parts,

$$\int x^2 \cos x \, dx = x^2 \sin x - \int 2x \sin x \, dx$$

this second integral can be calculated by parts again. Now $u = 2x$, $dv = \sin x \, dx$, $du = 2 \, dx$, $v = -\cos x$. Then

$$\int 2x \sin x \, dx = -2x \cos x + 2 \int \cos x \, dx = -2x \cos x + 2 \sin x$$

Putting together all the terms

$$\int x^2 \cos x \, dx = x^2 \sin x + 2x \cos x - 2 \sin x + C$$