Practice 3: Euclidean Spaces

1. Scalar Product

1.1. Definition

$$\cdot : \begin{array}{ccc} V\times V & \to & \mathbb{R} \\ (\vec{v},\vec{u}) & \to & \vec{v}\cdot\vec{u} \end{array}$$ is a **scalar product** if:

1. $\begin{cases} (\alpha\vec{v}_1+\beta\vec{v}_2)\cdot\vec{u} = \alpha\,(\vec{v}_1\cdot\vec{u})+\beta\,(\vec{v}_2\cdot\vec{u}) \\ \vec{u}\cdot(\alpha\vec{v}_1+\beta\vec{v}_2) = \alpha\,(\vec{u}\cdot\vec{v}_1)+\beta\,(\vec{u}\cdot\vec{v}_2) \end{cases} \quad \forall\vec{v}_1,\vec{v}_2,\vec{u}\in V,\ \forall\alpha,\beta\in\mathbb{R},$

2. Symmetric: $\vec{v}\cdot\vec{u}=\vec{u}\cdot\vec{v},\quad \forall\vec{v},\vec{u}\in V.$

3. Positive definite:$\vec{v}\cdot\vec{v}\geq 0,\quad \forall\vec{v}\in V$, besides $\vec{v}\cdot\vec{v}=0\Rightarrow\vec{v}=\vec{0}$

$(V,\cdot)$ is an **Euclidean Space**.

Example 1 *The following application is a scalar product in $\mathbb{R}^2$:*

$$(x_1,x_2)\cdot(y_1,y_2)=2x_1y_1+x_2y_1+x_1y_2+x_2y_2$$

1.2. Matrix Representation

Let $B=\{\vec{v}_1,\vec{v}_2,\ldots,\vec{v}_n\}$ be a basis set of V, and $\vec{u}$, $\vec{v}\in V$

$$\left.\begin{array}{l} \vec{u}=(x_1,\cdots,x_n)_B=x_1\vec{v}_1+x_2\vec{v}_2+\cdots+x_n\vec{v}_n \\ \vec{w}=(y_1,\cdots,y_n)_B=y_1\vec{v}_1+y_2\vec{v}_2+\cdots+y_n\vec{v}_n \end{array}\right\} \quad\Longrightarrow\quad \vec{u}\cdot\vec{w}=X^tGY$$

where

$$G=\begin{pmatrix} \vec{v}_1^2 & \vec{v}_1.\vec{v}_2 & \ldots & \vec{v}_1.\vec{v}_n \\ \vec{v}_1.\vec{v}_2 & \vec{v}_2^2 & \ldots & \vec{v}_2.\vec{v}_n \\ \vdots & \vdots & & \vdots \\ \vec{v}_1.\vec{v}_n & \vec{v}_2.\vec{v}_n & \ldots & \vec{v}_n^2 \end{pmatrix},\quad X=\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix},\quad Y=\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$$

G is the associated matrix, named **Gram Matrix**.

Example 2 *We know that the Gram matrix of a scalar product in $\mathbb{R}^4$, in the standard basis is*

$$A=\begin{pmatrix} 2 & 1 & 0 & -2 \\ 1 & 2 & -1 & -2 \\ 0 & -1 & 3 & -2 \\ -2 & -2 & -2 & 6 \end{pmatrix}$$

compute the scalar product of the vectors $\vec{u}=(3,5,2,-9)$ and $\vec{v}=(4,2,6,-1)$

Example 3 *Nevertheless, in this practice we will work with the dot product in* $\mathbb{R}^n$*:*

$$\left.\begin{array}{l}\vec{v}=(x_1,x_2,\ldots,x_n)\\ \vec{w}=(y_1,y_2,\cdots,y_n)\end{array}\right\} \Longrightarrow \vec{v}\cdot\vec{w}=\sum_{k=1}^{n}x_ky_k$$

Let us compute the dot product of two vectors in $\mathscr{M}_3(\mathbb{R})$*, for instance, matrices A and B:*

$$A=\begin{pmatrix}1&0&-3\\0&2&5\\1&0&1\end{pmatrix}, B=\begin{pmatrix}-2&2&-3\\0&1&1\\0&0&1\end{pmatrix}$$

```
» A=[1 0 -3;0 2 5;1 0 1]; % Matrix A
» B=[-2 2 -3;0 1 1; 0 0 1]; % Matrix B
» A=A(:) %First, we pass the matrices to column vectors
» B=B(:)
» prod=dot(A,B) % the dot product of A and B
```

1.3. Norm, Angle, Distance

The **norm** of a vector (length) $\vec{v}\in V$ is:

$$\|\vec{v}\|=\sqrt{\cdot\vec{v}}=\sqrt{\vec{v}^2}$$

Function	OutPut
dot(u,v)	the scalar product of vectors $\vec{u}$ and $\vec{v}$ when the Gram matrix is the identity. For column vectors it is the same as u' *v
abs(v)	the absolute value of the elements of the vector $\vec{v}$.
norm(v,p)	the p^{th} root of the sum of p powers of the coordinates of vector $\vec{v}$. That means, $\sqrt[p]{\sum_{k=1}^{n}v_k^p}$, in Matlab: `norm(v,p)=sum(abs(v).∧(p)∧(1/p)`
norm(v)	the Euclidean norm of $\vec{v}$, that mens, `norm(v)=norm(v,2)`

Example 4 *Compute the norm of* $\vec{u}=(2,5,-3,8)$ *de* $\mathbb{R}^4$.

We can compute the **angle** between two vectors $\vec{u}$ and $\vec{v}$ by calculating its cosine:

$$\cos(\alpha)=\frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|}$$

Example 5 *Use the cosine to compare vectors in* $\mathbb{R}^3$*:*

$$\vec{v}_1=(-1,0,1),\quad \vec{v}_2=(1,-5,0),\quad \vec{v}_3=(2,3,-1),\quad \vec{w}=(1,2,3)$$

Find out which of the vectors $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$ *is closer to* $\vec{w}$.

Example 6 *We will apply the cosine of the angle of two vectors to the* Face Recognition *problem:*

- *Compute the cosine of the angle between a new image (that we want to classify) and all the face images in the database.*

```
» cosk=dot(I,Ik)/(norm(I)*norm(Ik)); % cos angle between image I and im
```

- *Descending sort the cosines*
- *Select the image with the greatest cosine (and the smallest angle).*

Now apply the distance between two vectors to the face recognition problem.

The **distance** between two vectors: Let $\vec{u}, \vec{v} \in \mathbb{R}^n$. Then the **Distance** between $\vec{u}$ and $\vec{v}$ is:

$$d(\vec{u},\vec{v}) = \|\vec{u}-\vec{v}\| = \sqrt{(u_1-v_1)^2+(u_2-v_2)^2+\cdots+(u_n-v_n)^2}$$

```
» dist_uv=norm(u-v); % distance between vector u and v
```