Practice 2. Functions of a real variable: Graphing, limits and continuity

1. Vectors

One of the most remarkable aspects of MATLAB is the way it manipulates and operates on vectors and matrices.

1.1. Row vectors

- In general, vectors are introduced writing between brackets each of their coordinates separated by a space or a comma. For example:

```
>> v=[4 -6 5]
>> v=[4,-6,5]
```

- You can also enter them by specifying the value of each coordinate in the desired order:

```
>> w(2)=-6, w(1)=4, w(3)=5
>> w
```

- Other commands for particular cases :

`v=[a:h:b]`	Defines a row vector whose first coordinate is `a` and the other coordinates are obtained by adding `h` successively without exceeding `b`. If not specified `h` takes, by default, the value 1. It can also be written without the brackets.
`v=linspace(a,b,n)`	Defines a row vector of `n` coordinates, whose first coordinate is `a` and whose last coordinate is `b`, with a constant difference between consecutive coordinates. If not specified `n` takes, by default, the value 100.

```
>> u=linspace(-4,7,6)
>> u=linspace(1,3,5)
>> u=linspace(1,100)
>> v=[-4:2:7]
>> v=-4:2:7   % it can be written without the brackets
>>v=linspace(-4,7,6)
>> v=1:10
```

Notice the difference between $v = [-4 : 2 : 7]$ and linspace(-4,7,6), the former command uses a step of 2, produces six elements and finishes at 6, the latter finishes at 7 and also creates six elements, but the step is $(7 + 4)/5 = 2.2$

1.2. Column Vectors

In general, they are entered as row vectors, but separating the elements with a semicolon, we can enter them as column vectors:

```
>> u=[0;1;-5]
>> w=[1 2 3 4]'
```

Besides, the command `u'` calculates the transpose of a given vector `u` (as well as the transpose of any matrix).

1.3. Operations with vectors

Let v and u be vectors with n coordinates and α a scalar:

α*v	multiplies α by v
v/α	multiplies $1/\alpha$ by v
u+v	sums u and v
u-v	subtracts v from v

```
>> u=[1,5,6];
>> v=[2,6,-1];
>> u+v
>> 3*u
>> u/2
>> u-v
```

In addition to the mathematical operations with vectors the program MATLAB allows other *operations* that are performed coordinate by coordinate .

u.*v	multiplies u by v coordinate by coordinate
u./v	divides u by v coordinate by coordinate
u.^v	raises u to v coordinate by coordinate
α+v	sums α to each coordinate of v
v-α	subtracts α from each coordinate of v
v.^α	raises each coordinate of v to α
α.^v	raises α to each coordinate of v
α./v	divides α by each coordinate of v

It is important to remember that the "." that appears in some of the previous operations is part of the operator syntax. It indicates that we work with the vectors coordinate by coordinate.

```
>> u=[1,2,1];
>> v=[3,2,0];
>> u.*v
>> u.^v
>> v.^u
>> u.^2
>> 2.^u
>> u+3
>> u-5
>> 1./u
```

Many of the functions defined in MATLAB can be applied directly to vectors (coordinate by coordinate) and the result is again a vector.

```
>> u=[1,0,2];
>> exp(u)
>> cos(u)
>> sqrt(u)
```

2. Plotting Graphs

2.1. The plot command

The command `plot` is one of the basic commands of MATLAB to draw the graph of a function.

- `plot(x,y)`: draws the set of points (x_i, y_i), where x and y are row vectors with the same number of coordinates. To draw the graph of $y = f(x)$ we must specify the points $(x_i, f(x_i))$.

  For instance, to plot the function $y = x^2$ on the interval $[-3, 3]$ we can type:

  ```
  >> x=[-3:0.01:3];
  >> y=x.^2;
  >> plot(x,y)
  ```

  The graph features (color, line, axis, title, texts, etc.) can be changed using the Toolbar in the graph window.

- `plot(x,y,S)`: means the same as `plot(x,y)` but with the options specified in `S`. The types of line, color and marking are described in `S` in quotes. The codes of lines, colors and markings can be found using `doc plot`.

  To draw $y = x^3 - 1$ in the interval $[0, 5]$ with a dashed red line and asterisks on the points:

  ```
  >> x=[0:0.1:5];
  >> y=x.^3-1;
  >> plot(x,y,'--r*')
  ```

- `plot(x1,y1,S1,x2,y2,S2,...)`: presents on the same axes the graphs defined by the triples `xi, yi, Si`. This command allows you to represent several functions in the same graph.

  For instance, to plot the function cosine in green color and the function sine in red color on $[-\pi, \pi]$:

```
>> x=[-pi:0.01:pi];
>> y1=cos(x);
>> y2=sin(x);
>> plot(x,y1,'g',x,y2,'r')
>> axis([-pi pi -1 1])
```

We have redefined the axes with x in $[-\pi,\pi]$ and y in $[-1,1]$.

2.1.1. Example

We have the following data of the decay of a radioactive substance:

Time (hours)	Mass (mg)
0	102.9
1	75.8
2	56.1
3	42.2
4	31.1
5	23.6

Verify graphically that the exponential curve $y = 102.04e^{-0.29x}$ fits roughly the previous data. *Solution.-* We represent simultaneously the curve and the points described in the table.(Use `*`, red for points in the table, and light blue for the exponential curve):

```
>> x1=[0:5];
>> y1=[102.9, 75.8, 56.1, 42.2, 31.1, 23.6];
>> x2=[0:0.01:5];
>> y2=102.04*exp(-0.29*x2);
>> plot(x1,y1,'r*',x2,y2,'c')
```

2.2. The command ezplot

The command `ezplot(function,[xmin xmax])` plots the function defined symbolically on the interval `[xmin,xmax]`.

For instance, to plot the function $y = x^2$ on the interval $[-3,3]$

```
>> syms x
>> ezplot(x^2,[-3,3])
```

You can also use `fplot`. There is a slight difference, can you see it?

2.3. Other important commands

Each time you run the command `plot` and `ezplot`, MATLAB creates a graphical window and removes the previous window. Sometimes it is interesting to draw two or more functions on the same window or have multiple graphical windows:

`hold on`	The graph remains fixed so that subsequent graphing commands are applied on it.
`hold off`	Resets all properties of a graph to their default values
`figure(n)`	Selects the graphic window `Figure n` as the active window; if it does not exist, it is created.
`close all`	Close all the graphic windows.
`grid`	To use a grid on the plot, `grid on` to create it, `grid off` to remove it.

Besides, we can modify the appearance and scaling of the axes of a graph with the command `axis` and its options . For example (the list is not exhaustive)

`axis([x1 x2 y1 y2])`	Determines the visible intervals for the axis OX and OY.
`axis equal`	Sets the same scale for each axis.

2.4. Example

Represent graphically on the interval $[-5,5]$ the function

$$f(x)=\begin{cases} \dfrac{2x^2+3}{5} & \text{si} \quad x\leq 1,\\ 6-5x & \text{si} \quad 1<x<3,\\ x-3 & \text{si} \quad x\geq 3. \end{cases}$$

Solution.- We will solve it with `plot` on the left and with `ezplot` on the right. We plot simultaneously with `plot` the three sections of the function on the intervals $[-5,1]$, $[1,3]$ and $[3,5]$

```
>> x1=[-5:0.01:1];
>> x2=[1:0.01:3];
>> x3=[3:0.01:5];
>> y1=(2*x1.^2+3)/5;
>> y2=6-5*x2;
>> y3=x3-3;
>> plot(x1,y1,x2,y2,x3,y3)
```

```
>> syms x
>> f1=(2*x^2+3)/5;
>> f2=6-5*x;
>> f3=x-3;
>> figure(2)
>> ezplot(f1,[-5,1])
>> hold on
>> ezplot(f2,[1,3])
>> ezplot(f3,[3,5])
>> axis([-5,5,-10 12])
```

We mus redefine the axes-the x-axis from -5 to 5 and the y-axis approximately from the minimum to the maximum of the function. Of course, we can edit the graph by inserting labels and legends. You can avoid this redefinition by using fplot from the beginning. Close the figures con `close all`.

2.5. Example

Plot the function

$$f(x)=\frac{1}{1+2^{1/x}},\quad x\in[-1,1],\ x\neq 0.$$

- We can use the command `ezplot` taking the interval $[-1,1]$:

```
>> syms x
>> f=1/(1+2^(1/x))
>> pretty(f)
>> ezplot(f,[-1 1])
```

Is the graph correct? What happens at $x = 0$?

- We plot it on $[-1, 0)$ y en $(0, 1]$, entering the following commands:

```
>> figure(2)
>> ezplot(f,[-1 0])
>> hold on
>> ezplot(f,[0 1])
```

Do you get the expected result? Compare the intervals of the axes of the graph with those of the previous section. Do they agree? If they do not, modify them to obtain a similar graph by adding the command

```
>> axis([-1,1,-0.2,1.2])
```

2.6. Example

Let $f(x) = e^{-x}$ and $g(x) = x^2$. Calculate and show in different windows and on the interval $[-3, 3]$ the functions $f \circ g$ and $g \circ f$.

```
>> syms x
>> f=exp(-x)
>> g=x^2
>> h1=compose(g,f)
>> h2=compose(f,g)
>> ezplot(h1,[-3,3])
>> figure(2)
>> ezplot(h2,[-3,3])
```

3. Limits and continuity

As outlined below, the symbolic computation module Symbolic Math Toolbox provides Matlab with the ability to perform some of the fundamental operations of mathematical analysis of functions of one variable. We begin with the calculation of limits and the analysis of the continuity of a function.

`limit(f,x,a)`	Calculates the limit of f as x approaches a: $\lim\limits_{x\to a} f(x)$. If the variable is not specified, its default value is x. If a is not specified, its default value is 0
`limit(f,x,a,'right')`	Calculates the limit from the right : $\lim\limits_{x\to a^+} f(x)$
`limit(f,x,a,'left')`	Calculates the limit from the left : $\lim\limits_{x\to a^-} f(x)$
`limit(f,x,inf)`	Calculates the limit of f at *plus infinity*: $\lim\limits_{x\to +\infty} f(x)$
`limit(f,x,-inf)`	Calculates the limit of f at *minus infinity*: $\lim\limits_{x\to -\infty} f(x)$

For instance, calculate: $\lim\limits_{x\to 0}\dfrac{\sin x}{x}$, $\lim\limits_{x\to 2}\dfrac{x-2}{x^2-4}$, $\lim\limits_{x\to 0}\dfrac{1}{x}$, $\lim\limits_{x\to 0^+}\dfrac{1}{x}$, $\lim\limits_{x\to 0^-}\dfrac{1}{x}$, $\lim\limits_{x\to +\infty}\dfrac{1}{x}$, $\lim\limits_{x\to -\infty}\dfrac{1}{x}$

```
>> syms x
>> limit(sin(x)/x,x,0)
>> limit((x-2)/(x^2-4),x,2)
>> limit(1/x,x,0)
>> limit(1/x,x,0,'right')
>> limit(1/x,x,0,'left')
>> limit(1/x,x,+inf)
>> limit(1/x,x,-inf)
```

3.1. Example

Calculate the following limits

$$\lim_{x\to 0}\frac{\sin 5x}{\ln(1+4x)} \qquad \lim_{x\to 0}\frac{\ln(1+\sin 4x)}{e^{\sin 5x}-1} \qquad \lim_{x\to 0}(\cos x)^{\frac{1}{\sin x}}$$

```
>> syms x
>> f=sin(5*x)/log(1+4*x);pretty(f)
>> limit(f,x,0)
>> g=log(1+sin(4*x))/(exp(sin(5*x))-1);pretty(g)
>> limit(g,x,0)
>> h=cos(x)^(1/sin(x));pretty(h)
>> limit(h,x,0)
```

3.2. Example

Analyze the continuity of the function

$$f(x)=\begin{cases}\dfrac{2x^2+3}{5} & \text{si} \quad x\leq 1,\\ 6-5x & \text{si} \quad 1<x<3,\\ x-3 & \text{si} \quad x\geq 3.\end{cases}$$

Solution.- On the open intervals: $(-\infty,1)$, $(1,3)$ y $(3,+\infty)$ it is continuous because it is the result of basic operations with continuous functions. Now we have to study what happens at the points $x=1$ and $x=3$.

Analysis at the point $x=1$:

```
>> syms x
>> f1=(2*x^2+3)/5;
>> limit(f1,x,1,'left')
ans =
1
>> f2=6-5*x;
>> limit(f2,x,1,'right')
ans =
1
>> subs(f1,x,1)
ans =
     1
```

Since $\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^+} f(x) = f(1)$, f is continuous at $x = 1$.

Study at the point $x = 3$:

```
>> limit(f2,x,3,'left')
ans =
-9
>> f3=x-3;
>> limit(f3,x,3,'right')
ans =
0
```

Since $\lim\limits_{x\to 3^-} f(x) \neq \lim\limits_{x\to 3^+} f(x)$, $\lim\limits_{x\to 3} f(x)$ does not exist, therefore f is not continuous at $x = 3$.

Thus, the function f is continuous on $\mathbb{R} - \{3\}$.

4. Exercises

1. Plot the functions:

   a) $f(x) = \dfrac{x^2+2}{x-3}$ *b)* $f(x) = \sqrt{x^2+2}$

   c) $f(x) = x^2 e^{-x}$ *d)* $f(x) = \dfrac{\ln x^2}{x}$

2. Study the continuity of the following function and graph it on the interval $[-5, 5]$.

$$f(x) = \begin{cases} -2x+1 & \text{si} \quad x \leq -1 \\ x^2 & \text{si} \quad -1 < x < 0 \\ \sin x & \text{si} \quad x \geq 0 \end{cases}$$

Sol: f is continuous on $\mathbb{R} - \{-1\}$.

3. Calculate the following limits:

   a) $\lim\limits_{x\to 0} \dfrac{e^{-x}-1}{x}$ *b)* $\lim\limits_{x\to 0} (\cos x)^{\frac{1}{x}}$ *c)* $\lim\limits_{x\to +\infty} \left(\dfrac{2x+3}{2x+1}\right)^{x+1}$ *d)* $\lim\limits_{x\to \pi/2} e^{\tan x}$

   Sol.: *a)* -1, *b)* 1, *c)* e, *d)* it does not exist, infinity from the left, zero from the right.