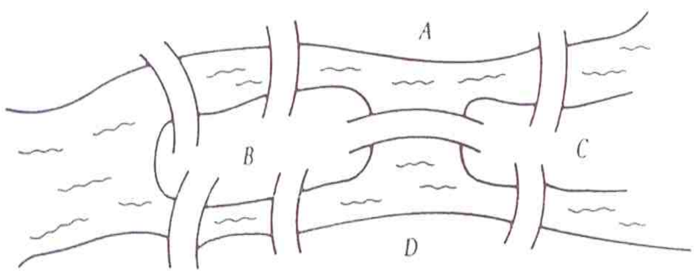
DISCRETE MATHEMATICS

Chapter 5: Graphs

5.1 Basic concepts

The Konigsberg Bridge Problem is considered to be the beginning of graph theory.

The Konigsberg Bridge Problem:
The Pregolya River passes through a city once known as Konigsberg (now Kaliningrad). In the 1700s seven bridges were situated across the river connecting various land areas:



The problem is to find a path which crosses each bridge exactly once.

In 1736 the Swiss mathematician Leonhard Euler solved this problem by showing that no such path could exist.

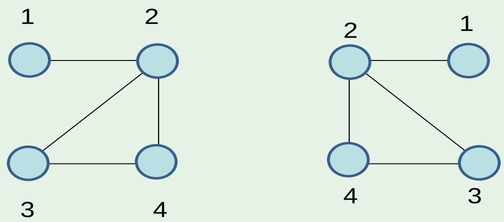
5.1 Basic concepts

Definition

A **graph** consists of two finite sets, V and E. Each element of V is called a **vertex**. The elements of E, called **edges**, are unordered pairs of vertices.

Example

$V = \{1, 2, 3, 4\}$, $E = \{\{1, 2\}, \{2, 3\}, \{2, 4\}, \{3, 4\}\}$.



These are two representations of the same graph.

5.1 Basic concepts

Definition

Two graphs are said to be **isomorphic** if there is a one-to-one correspondence between their vertices, which carries over to become a one-to-one correspondence between their edges.

Example

Let $V = \{1, 2, 3, 4, 5\}$, $E = \{\{1, 2\}, \{1, 5\}, \{2, 4\}, \{3, 4\}\}$ and $V' = \{a, b, c, d, e\}$, $E' = \{\{c, d\}, \{c, e\}, \{a, b\}, \{d, b\}\}$. These graphs are isomorphic by the one-to-one correspondence:

$1 \leftrightarrow c, \quad 2 \leftrightarrow d, \quad 3 \leftrightarrow a, \quad 4 \leftrightarrow b, \quad 5 \leftrightarrow e.$

5.1 Basic concepts

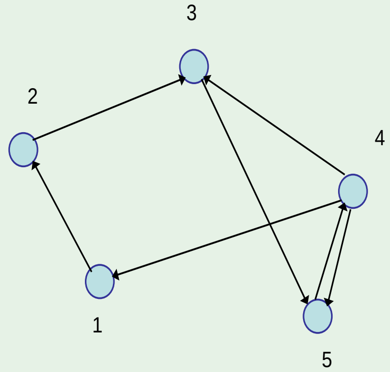
We can obtain similar structures by altering our definition in various ways:

Definition

By replacing our set E with a set of ordered pairs of vertices, we obtain a **directed graph or digraph**. Each edge of a digraph has a specific orientation.

Example

Let $V = \{1, 2, 3, 4, 5\}$, $E = \{(1, 2), (2, 3), (3, 5), (4, 1), (4, 3), (4, 5), (5, 4)\}$.



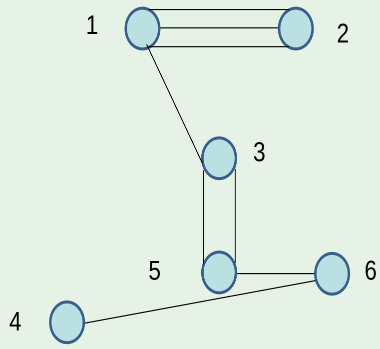
5.1 Basic concepts

Definition

If we allow repeated elements in our set of edges, technically replacing our set E with a multiset, we obtain a **multigraph**.

Example

Let $V = \{1, 2, 3, 4, 5, 6\}$,
$E = \{\{1, 2\}, \{1, 2\}, \{1, 2\}, \{1, 3\}, \{3, 5\}, \{3, 5\}, \{4, 6\}, \{5, 6\}\}$.



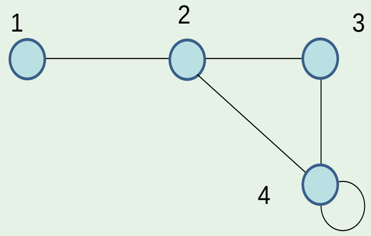
5.1 Basic concepts

Definition

By allowing edges to connect a vertex to itself ("loops"), we obtain a **pseudograph**.

Example

Let $V = \{1, 2, 3, 4\}$,
$E = \{\{1, 2\}, \{2, 3\}, \{2, 4\}, \{3, 4\}, \{4, 4\}\}$.



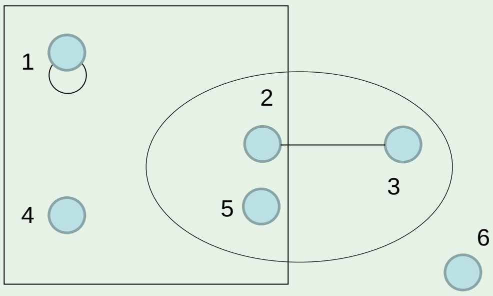
5.1 Basic concepts

Definition

Allowing our edges to be arbitrary subsets of vertices gives us **hypergraphs**.

Example

Let $V = \{1, 2, 3, 4, 5, 6\}$,
$E = \{\{1, 1\}, \{2, 3\}, \{3, 5, 2\}, \{1, 2, 4, 5\}\}$.



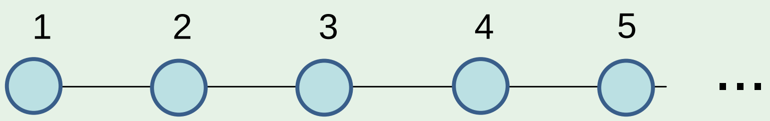
5.1 Basic concepts

Definition

By allowing V or E to be an infinite set, we obtain **infinite graphs**.

Example

Let $V = \mathbb{N}$, $E = \{\{i, i+1\} : i \in \mathbb{N}\}$.



5.1 Basic concepts

From now on let $G = \{V, E\}$ be a finite, simple graph: without loops or multiple edges.

Definition

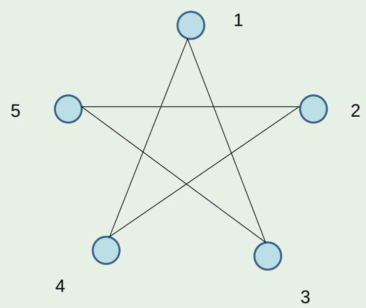
The **order of a graph** G is the cardinality of V: $|V|$.

Definition

The **size of a graph** G is the cardinality of E: $|E|$.

Example

Let $G = \{V, E\}$ with $V = \{1, 2, 3, 4, 5\}$ and $E = \{\{1, 3\}, \{1, 4\}, \{2, 4\}, \{2, 5\}, \{3, 5\}\}$.



The order of G is 5 and the size is 5.

5.1 Basic concepts

Definition

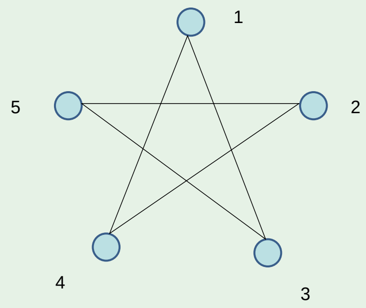
Given two vertices u and v, if $\{u, v\} \in E$, then u and v are said to be **adjacent**. If $\{u, v\} \notin E$, u and v are **nonadjacent**.

Definition

If an edge e has a vertex v as an endpoint, we say that v and e are **incident**.

Example

Let $G = \{V, E\}$ with $V = \{1, 2, 3, 4, 5\}$ and $E = \{\{1, 3\}, \{1, 4\}, \{2, 4\}, \{2, 5\}, \{3, 5\}\}$.



Vertices 1 and 2 are nonadjacent while 1 and 3 are adjacent. Moreover, vertex 1 and edge $\{1, 3\}$ are incident.

5.1 Basic concepts

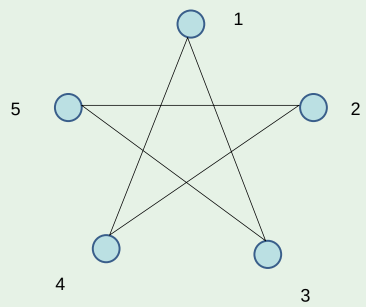
Definition

The **neighborhood of a vertex** v, denoted by $N(v)$, is the set of vertices adjacent to v:

$$N(v) = \{x \in V : \{v, x\} \in E\}.$$

Example

Let $G = \{V, E\}$ with $V = \{1, 2, 3, 4, 5\}$ and $E = \{\{1, 3\}, \{1, 4\}, \{2, 4\}, \{2, 5\}, \{3, 5\}\}$.



$N(1) = \{3, 4\}$, $N(2) = \{4, 5\}$, $N(3) = \{1, 5\}$, $N(4) = \{1, 2\}$, $N(5) = \{2, 3\}$.

5.1 Basic concepts

Definition

The **degree of a vertex** v, denoted by $\deg(v)$, is the number of edges incident with v. That is,

$$\deg(v) = |\{e \in E : \exists x \in V \text{ such that } e = \{v, x\}\}|.$$

In simple graphs, this is the same as the cardinality of the neighborhood of v.
We say that a vertex is even/odd if its degree is even/odd.

Definition

The **maximum degree of a graph** G, denoted by $\Delta(G)$, is defined to be

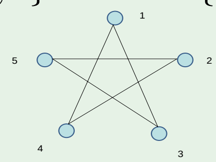
$$\Delta(G) = \max\{\deg(v) : v \in V\}.$$

The **minimum degree of a graph** G, denoted by $\delta(G)$, is defined to be

$$\delta(G) = \min\{\deg(v) : v \in V\}.$$

Example

Let $G = \{V, E\}$ with $V = \{1, 2, 3, 4, 5\}$ and $E = \{\{1, 3\}, \{1, 4\}, \{2, 4\}, \{2, 5\}, \{3, 5\}\}$.



$\deg(1) = 2, \deg(2) = 2, \deg(3) = 2, \deg(4) = 2, \deg(5) = 2, \Delta(G) = 2, \delta(G) = 2.$

5.1 Basic concepts

Theorem

Let G be a graph or a multigraph with $|E|$ edges and the vertices $v_1, v_2, \ldots$. Then

$$\deg(v_1) + \deg(v_2) + \cdots = 2|E|$$

Corollary

Every graph or multigraph has an even number of odd vertices.

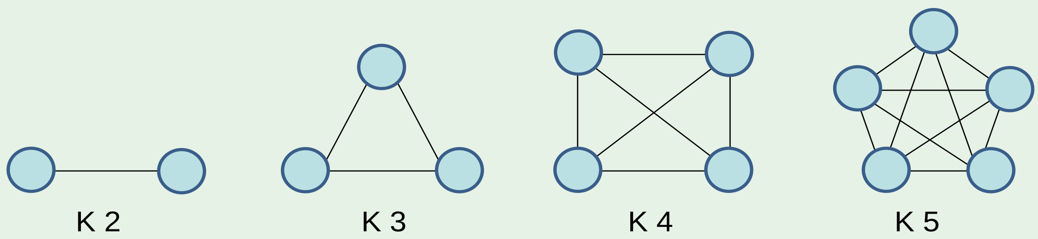
5.1 Basic concepts

Special types of graphs:

Definition

The **complete graph on n vertices**, denoted by K_n, is the graph of order n where $\{u, v\} \in E$ for all u and $v \in V$.

Example

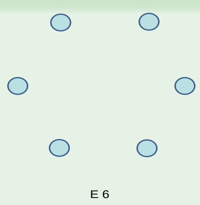


5.1 Basic concepts

Definition

The **empty graph on n vertices**, denoted by E_n, is the graph of order n where E is the empty set.

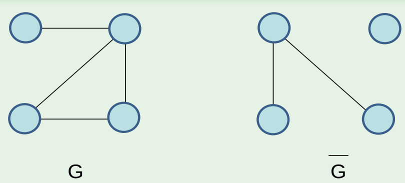
Example



Definition

Given a graph G, the **complement of** G, denoted by $\overline{G}$, is the graph whose vertex set is the same as G's and whose edge set consists of all the edges that are not present in G.

Example

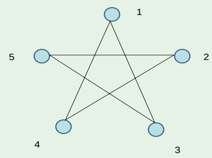


5.1 Basic concepts

Definition

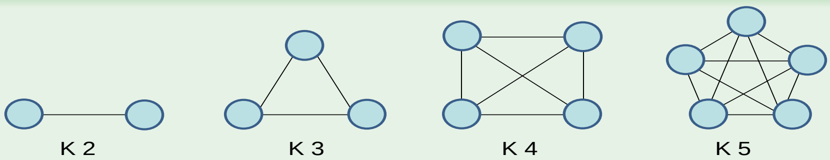
A graph G is **regular** if every vertex has the same degree. G is said to be **regular of degree** r (or r-**regular**) if $\deg(v) = r$ for all vertices v in G.

Example



Complete graphs of order n are regular of degree $n-1$, and empty graphs are regular of degree 0.

Example

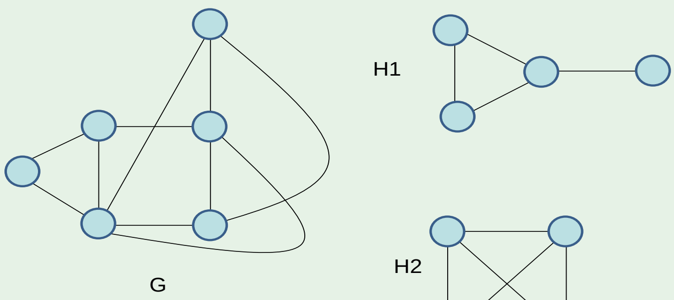


5.1 Basic concepts

Definition

A graph H is a **subgraph of a graph** G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. In this case we write $H \subseteq G$ and we say that G contains H.

Example

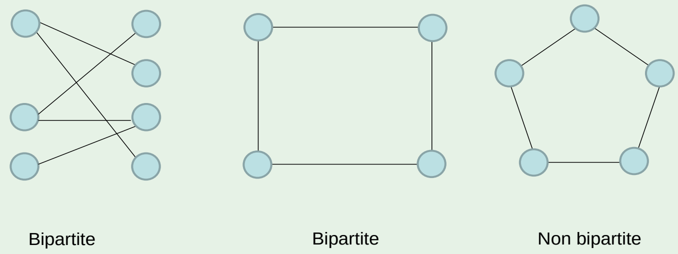


5.1 Basic concepts

Definition

A graph G is **bipartite** if its vertex set can be partitioned into two sets X and Y in such a way that every edge of G has one vertex in X and another in Y. In this case, X and Y are called the **partite sets**.

Example

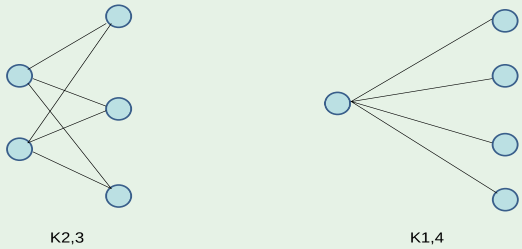


5.1 Basic concepts

Definition

A bipartite graph with partite sets X and Y is called a **complete bipartite graph** if its edge set is of the form $E = \{\{x, y\} : x \in X, y \in Y\}$. Such a graph is denoted by $K_{|X|,|Y|}$.

Example



5.2 Paths

Definition

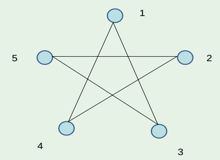
A **path** in a graph is a sequence of distinct edges of the graph such that each pair of consecutive edges shares a common vertex. It is denoted by $v_1, v_2, \dots, v_k$ or $v_1v_2\ v_2v_3\ \cdots\ v_{k-1}v_k$ where $\{v_i, v_{i+1}\} \in E$ for $i = 1, 2, \dots, k-1$.

Definition

The **length of a path** is the number of edges on the path.

Example

Let $G = \{V, E\}$ with $V = \{1, 2, 3, 4, 5\}$ and $E = \{\{1, 3\}, \{1, 4\}, \{2, 4\}, \{2, 5\}, \{3, 5\}\}$.



For example, $1, 3, 5, 2, 4$ is a path of length 4.

5.2 Paths

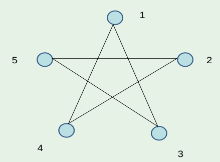
Definition

A **cycle** or **circuit** in a graph is a sequence of vertices $w_1, w_2, \dots, w_{r-1}, w_r$ such that $w_1, w_2, \dots, w_{r-1}$ is a path, $w_1 = w_r$, and $\{w_{r-1}, w_r\} \in E$.

Essentially, a cycle is a closed path. The length of a cycle is defined to be the number of edges on the cycle. When we speak of an odd/even cycle, we mean that the cycle has an odd/even length.

Example

Let $G = \{V, E\}$ with $V = \{1, 2, 3, 4, 5\}$ and $E = \{\{1, 3\}, \{1, 4\}, \{2, 4\}, \{2, 5\}, \{3, 5\}\}$.



For example, $1, 3, 5, 2, 4, 1$ is a cycle of length 5.

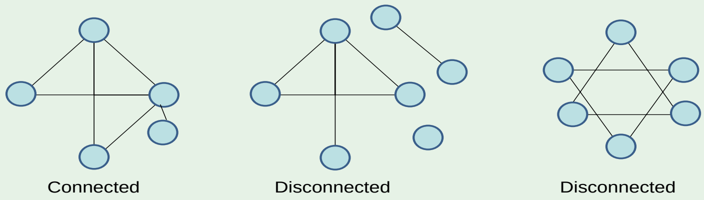
5.2 Paths

Definition

A graph is **connected** if every pair of vertices can be joined by a path.

Informally, if one can pick up an entire graph by grabbing just one vertex, then the graph is connected.

Example



Definition

Each maximal connected piece of a graph is called a **connected component**.

5.2 Paths

Theorem

A graph of order at least two is bipartite if and only if it contains no odd cycles.

5.2 Paths

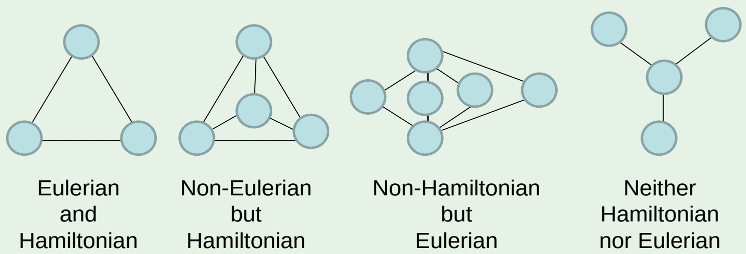
Definition

- A path that goes through each vertex of the graph exactly once is called **Hamiltonian path**.
- If the path ends up back at the same vertex it started at it is called a **Hamiltonian cycle**.
- A graph is **Hamiltonian** if it contains a Hamiltonian cycle.

Definition

- A path that traverses each edge of a graph is called an **Euler path**.
- If a cycle includes each edge of a graph it is called an **Euler cycle**.
- A graph is **Eulerian** if it contains an Euler cycle.

Example



5.2 Paths

Proposition

If a graph or a multigraph has an Euler cycle then all its vertices are even.

Proposition

If a graph or a multigraph has an Euler path then at most two of its vertices are odd.

Proposition

Any connected graph or multigraph with no odd vertices possesses an Euler cycle.

Proposition

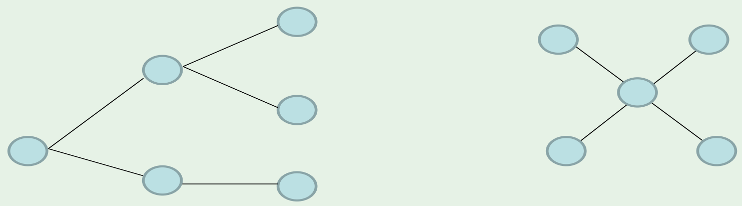
Any connected graph or multigraph with exactly 2 odd vertices contains an Euler path.

5.3 Trees

Definition

A **tree** is a connected graph that contains no cycles.

Example



Proposition

In every tree there is at least one vertex that has degree equal to 1.

Remark

We are excluding, of course, the trivial tree with only one vertex; it cannot have any edges at all.

5.3 Trees

Proposition

Every tree with n vertices has exactly $n-1$ edges.

Proposition

Between any two vertices in a tree there is one and only one path.

Proposition

If two nonadjacent vertices of a tree are connected by an edge the resulting graph will contain a cycle.

Proposition

If any edge is deleted from a tree the resulting graph is not connected.

Proposition

If a graph is connected and has n vertices and $n-1$ edges then it is a tree.

Proposition

If a graph has no cycles, has n vertices, and $n-1$ edges then it is a tree.

5.3 Trees

Definition

Given any two vertices in a tree A and B we define the **distance from A to B** to be the number of edges in the unique path from A to B.

- Every tree can be drawn with a tree-shaped picture. Moreover, we can choose a tree with a one-edge trunk.

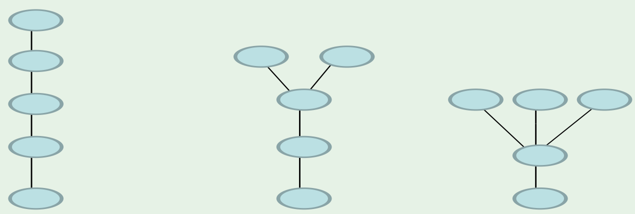
Question: How many trees are there with n vertices?

Definition

A **labeled tree** is one in which each vertex has a name.

Example

There are three different trees with 5 vertices:



The total number of labeled trees with 5 vertices is: $60 + 60 + 5 = 125$.

5.3 Trees

Theorem

The number of labeled trees in which v_1 has degree $d_1 + 1$, v_2 has degree $d_2 + 1$, $\dots$, v_n has degree $d_n + 1$ is exactly the multinomial coefficient

$$\binom{n-2}{d_1, d_2, \dots, d_n}$$

Lemma

The numbers $a_1, a_2, \dots, a_n$ (each an integer greater than 0) are the degrees of the vertices of some tree if and only if they add up to $2(n-1)$.

Theorem (Cayley's formula)

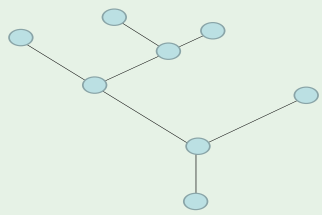
The total number of labeled trees with n vertices is n^{n-2}.

5.3 Trees

Definition
A **rooted tree** is a tree whose first vertex (**root**) has degree 1.

Definition
A rooted tree is **trivalent** if every vertex has degree 1 or 3.

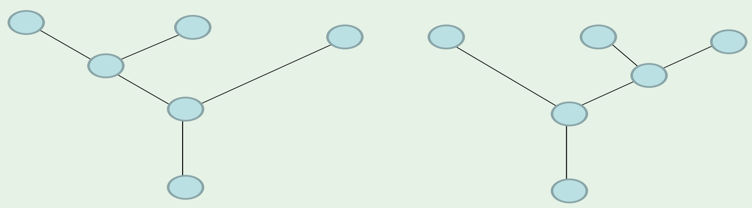
Definition
The **order** of a rooted tree is the number of the vertices of degree 1 except the root.

Example


It is a rooted trivalent tree of order 4.

5.3 Trees

Definition
An **ordered rooted tree** is a rooted tree in which the order of the branches matters.

Example


These are considered as two different ordered trees.

5.3 Trees

Theorem

The number of ordered rooted trivalent trees of order n is the $(n-1)$st Catalan number

$$C_{n-1} = \frac{1}{n}\binom{2n-2}{n-1}$$

Theorem

The number of ordered rooted trees with n vertices is the $(n-2)$nd Catalan number

$$C_{n-2} = \frac{1}{n-1}\binom{2n-4}{n-2}$$

5.4 Planarity

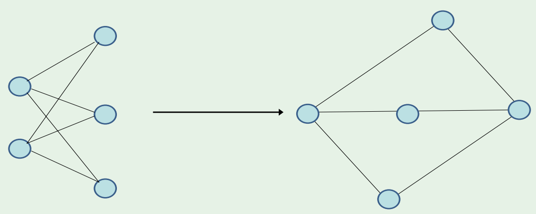
Definition

A graph G is said to be **planar** if it can be drawn in the plane in such a way that pairs of edges intersect only at vertices, if at all. If G has no such representation, G is called **nonplanar**.

Definition

A drawing of a planar graph G in the plane in which edges intersect only at vertices is called **planar representation** or a **planar embedding** of G.

Example

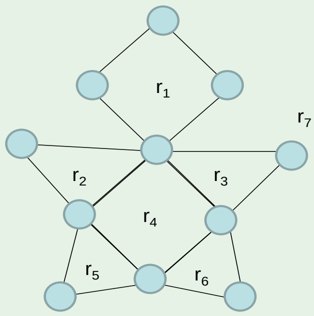


5.4 Planarity

Definition

Given a planar representation of a graph G, a **region** is a maximal section of the plane in which any two points can be joined by a curve that does not intersect any part of G.

Example



The number of regions in a planar representation of a graph does not depend on the representation itself.

Definition

The **bound degree** of a region r, denoted by $b(r)$, is the number of edges that bound region r.

5.4 Planarity

Theorem (Euler's formula)

In a planar connected graph let $|E|$ be the number of edges, $|V|$ the number of vertices, and $|R|$ the number of regions. Then

$$|V| - |E| + |R| = 2.$$

Corollary

$K_{3,3}$ is nonplanar. Moreover, any graph containing $K_{3,3}$ is nonplanar.

Theorem

If G is a planar graph with $|V| \geq 3$ vertices and $|E|$ edges, then $|E| \leq 3|V| - 6$. Furthermore, if equality holds, then every region is bounded by 3 edges.

Corollary

K_5 is nonplanar. Moreover, any graph containing K_5 is nonplanar.

Proposition

If G is a planar graph then G contains a vertex of degree at most 5. That is, $\delta(G) \leq 5$.

5.5 Colorings

Definition

Given a graph G, a k-**coloring** of the vertices of G is a partition of the vertex set V into k sets $C_1, C_2, \ldots, C_k$ such that for all i, no pair of vertices from C_i are adjacent. If such a partition exists, G is said to be k-**colorable**.

Another way of viewing k-coloring is: assign each vertex one of k colors in such a way that no pair of adjacent vertices gets the same color. Such coloring determines a specific partition of the vertex set.

Definition

Given a graph G, the **chromatic number** of G, denoted by $\chi(G)$, is the smallest integer k such that G is k-colorable.

Theorem (Four Color Theorem, 1976)

Every planar graph is 4-colorable.

Theorem (Five Color Theorem, 1890)

Every planar graph is 5-colorable.