Chapter II

COMBINATORIAL IDENTITIES

List of exercises 5

Combinatorial numbers

1. Show that:
$$\frac{1}{n+1}\binom{2n}{n} = \binom{2n-1}{n-1} - \binom{2n-1}{n+1} = \binom{2n}{n} - \binom{2n}{n-1}.$$
Draw an interesting conclusion about divisibility.

2. Prove that $\binom{2n}{n}$ is even.

3. Prove that the sequence of combinatorial numbers
$$\binom{n}{0},\ \binom{n}{1},\ \binom{n}{2},\ \ldots,\binom{n}{n}$$
first increases and then decreases. If n is even, the middle term is the biggest; if n is odd, the two middle terms are the biggest.

4. From the known formula
$$\binom{k}{0} + \binom{k+1}{1} + \ldots + \binom{k+r}{r} = \binom{k+r+1}{r},$$
deduce the following formulas:

   (a) $1 + 2 + \ldots + n = \dfrac{n(n+1)}{2}$

   (b) $1 \cdot 2 + 2 \cdot 3 + \ldots + n \cdot (n+1) = \dfrac{n(n+1)(n+2)}{3}$

   (c) $1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \ldots + n \cdot (n+1) \cdot (n+2) = \dfrac{n(n+1)(n+2)(n+3)}{4}$

   (d) $1^2 + 2^2 + \ldots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$

   (e) $1^3 + 2^3 + \ldots + n^3 = \dfrac{n^2(n+1)^2}{4}$

   (f) generalize formulas (a), (b) and (c).

5. Using the same formula in the beginning of the previous exercise,

   (a) Deduce that
   $$\binom{k}{k} + \binom{k+1}{k} + \binom{k+2}{k} + \ldots + \binom{n}{k} = \binom{n+1}{k+1}.$$

   (b) Interprete this equality combinatorially.

   (c) From (a) deduce once again that
   $$1 + 2 + \ldots + n = \frac{n(n+1)}{2}.$$

6. Prove combinatorially the formula
   $$k^2 = 2\binom{k}{2} + \binom{k}{1}.$$
   From this expression and the previous exercise deduce that
   $$1^2 + 2^2 + \ldots + n^2 = 2\binom{n+1}{3} + \binom{n+1}{2} = \frac{n(n+1)(2n+1)}{6}$$

7. We know that
   $$\binom{n}{r-1} = \binom{n+1}{r} - \binom{n}{r}.$$
   Applying the same idea in each term of the right hand side we obtain
   $$\binom{n}{r-1} = \binom{n+2}{r+1} - 2\binom{n+1}{r+1} + \binom{n}{r+1}.$$
   Show that if this process is repeated $k-2$ more times, the result is
   $$\binom{n}{r-1} = \binom{k}{0}\binom{n+k}{r+k-1} - \binom{k}{1}\binom{n+k-1}{r+k-1} + \ldots + (-1)^k\binom{k}{k}\binom{n}{r+k-1}.$$

8. Prove the formula
   $$\binom{n-1}{k} = \binom{n}{k} - \binom{n}{k-1} + \ldots + (-1)^k\binom{n}{0}.$$

9. Prove the identity
   $$\binom{m}{0}\binom{m}{n} + \binom{m}{1}\binom{m-1}{n-1} + \ldots + \binom{m}{n}\binom{m-n}{0} = 2^n\binom{m}{n}.$$

10. Decompose in simple fractions
    $$\frac{n!}{x(x+1)(x+2)\ldots(x+n)}.$$
    Use this result to show (once again) that
    $$\sum_{k=0}^{n} \frac{(-1)^k}{k+1}\binom{n}{k} = \frac{1}{n+1}.$$

11. Prove that four consecutive combinatorial numbers

$$\binom{n}{r}, \binom{n}{r+1}, \binom{n}{r+2}, \binom{n}{r+3},$$

can never be on arithmetic progression.

12. Putting

$$a_n := \frac{1}{\binom{n}{0}} + \frac{1}{\binom{n}{1}} + \ldots + \frac{1}{\binom{n}{n}},$$

prove that

$$a_n = \frac{n+1}{2n} a_{n-1} + 1 \quad \text{and} \quad \lim_{n\to\infty} a_n = 2.$$

13. Prove

(a) $(n!)^2 > n^n, \ \forall n > 2.$

(b) $2^n < \binom{2n}{n} < 4^n, \ \forall n \geq 2.$

(c) $\binom{2n-1}{n} < 4^{n-1}, \ \forall n \geq 2.$

14. As $\binom{n}{r}$ represents the number of r-combinations of n elements, we use the symbol $\left\langle {n \atop r} \right\rangle$ in order to represent the number of r-combinations with repetition of n elements. We know that

$$\left\langle {n \atop r} \right\rangle = \binom{n+r-1}{r}.$$

Prove

(a) $\left\langle {n \atop r} \right\rangle = \left\langle {n \atop r-1} \right\rangle + \left\langle {n-1 \atop r} \right\rangle.$

(b) $\left\langle {n \atop r} \right\rangle = \frac{n}{r}\left\langle {n+1 \atop r-1} \right\rangle = \frac{n+r-1}{r}\left\langle {n \atop r-1} \right\rangle.$

(c) $\left\langle {n+1 \atop r} \right\rangle = \left\langle {n \atop 0} \right\rangle + \left\langle {n \atop 1} \right\rangle + \ldots + \left\langle {n \atop r} \right\rangle.$

(d) Show that in the Pascal's triangle, the diagonals give the numbers $\left\langle {n \atop r} \right\rangle$ with fix n

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \end{array}$$

15. The *Gaussian binomial coefficients* $\left[{n \atop r}\right]$ are defined in the following way:

$$\begin{bmatrix} n \\ 0 \end{bmatrix} = 1,$$

$$\begin{bmatrix} n \\ r \end{bmatrix} = \frac{q^n - 1}{q - 1} \frac{q^{n-1} - 1}{q^2 - 1} \cdots \frac{q^{n-r+1} - 1}{q^r - 1}, \ r = 1, 2, \ldots, n.$$

For example,
$\begin{bmatrix} 4 \\ 2 \end{bmatrix} = \frac{q^4 - 1}{q - 1} \frac{q^3 - 1}{q^2 - 1} = \frac{(q^2+1)(q+1)(q-1)(q^2+q+1)(q-1)}{(q-1)(q+1)(q-1)} = (q^2+1)(q^2+q+1) = q^4 + q^3 + 2q^2 + q + 1.$

(In consequence, the Gaussian binomial coefficients are not numbers but functions of q. By simplicity reasons, that is not in the notation.)

(a) Calculate $\begin{bmatrix} 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \end{bmatrix}.$

(b) Prove the *addition law*:

$$\begin{bmatrix} n \\ r \end{bmatrix} + \begin{bmatrix} n \\ r-1 \end{bmatrix} q^{n+1-r} = \begin{bmatrix} n+1 \\ r \end{bmatrix}.$$

(c) Prove that the Gaussian binomial coefficients are polynomials on q.

(d) Prove that

$$\lim_{q \to 1} \begin{bmatrix} n \\ r \end{bmatrix} = \binom{n}{r}.$$

(e) Prove that

$$\begin{bmatrix} n \\ r \end{bmatrix} = \begin{bmatrix} n \\ n-r \end{bmatrix}.$$

(f) Prove that

$$(1+x)(1+qx)(1+q^2x) = \begin{bmatrix} 3 \\ 0 \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \end{bmatrix} x + \begin{bmatrix} 3 \\ 2 \end{bmatrix} qx^2 + \begin{bmatrix} 3 \\ 3 \end{bmatrix} q^3x^3.$$

(g) Prove that, in general,

$$\prod_{r=0}^{n-1} (1 + q^r x) = \sum_{r=0}^{n} \begin{bmatrix} n \\ r \end{bmatrix} q^{1+2+\ldots+(r-1)} x^r.$$

(h) Use the previous in order to give a new proof for Newton binomial formula.

Solutions 5

1. $n+1$ divides $\binom{2n}{n}$. **2.** (Help: Join each subset with its complementary). **3.** (Help: Determine when the quotient of $\binom{n}{k}$ by $\binom{n}{k-1}$ is ≥ 1). **4.** (a) $k = 1, \ r = n-1$; (b) $k = 2, \ r = n-1$; (c) $k = 3, \ r = n-1$; (d) $\sum_{k=1}^{n} k^2 = \sum_{k=1}^{n} k(k+1) - \sum_{k=1}^{n} k$. **5.** (a) $r = n-k$. **6.** (Help: Classify ordered pairs according to the equality of their components). **7.** (Help: Induction on sobre k). **8.** (Help: Induction on k). **9.** (Help: Use that $\binom{m}{k}\binom{m-k}{n-k} = \binom{m}{n}\binom{n}{k}$. Combinatorially: number of pairs (A, B), where A is a subset of $\{a_1, a_2, \ldots, a_m\}$ with n elements and $B \subset A$). **10.** (a) $\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{1}{x+k}$. **12.** (Help: In the first place $\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}$ and then $\binom{n-1}{l} = \binom{n-1}{n-1-l}$.) **15.** a) $\begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} = 1, \ \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} = q^2 + q + 1.$

List of exercises 6

Newton binomial

1. Let $a, b, c, d \in \mathbb{N}$. Calculate:

   (a) The coefficient of x^6 in the series $(x^2 - 2x^{-1})^8$.

   (b) The coefficient of x^a in the series $(x^b + x^c)^d$.

   (c) The constant term in the series $(x^a + 1 + x^{-a})^b$.

2. (*Vandermonde*) Prove that

$$(x+y)^{\underline{n}} = \sum_{k=0}^{n} \binom{n}{k} x^{\underline{k}} y^{\underline{n-k}}.$$

3. (*Leibniz*) Let f and g be functions n times differentiable on the interval I. Prove that

$$(fg)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n-k)}.$$

4. Prove that, for each x,

$$(1+x)^n - \binom{n}{1} x(1+x)^{n-1} + \binom{n}{2} x^2 (1+x)^{n-2} - \ldots + (-1)^n x^n = 1.$$

5. (*Lucas*) In this exercise *operate* means *differentiate and then multiply by* x. Initially

$$-(1-x)^n = \sum_{r=0}^{n} (-1)^{r+1} \binom{n}{r} x^r.$$

   (a) *Operating* k times ($k < n$), obtain a formula that generalizes the following equality:

$$\binom{4}{1} 1^3 - \binom{4}{2} 2^3 + \binom{4}{3} 3^3 - \binom{4}{4} 4^3 = 0.$$

   (b) What do you get if you *operate* $k = n$ times?

6. Prove the identity
$$\sum_{l=0}^{k}(-1)^l x^l (1+x)^n = (1+x)^{n-1}(1-(-x)^{k+1}).$$
Comparing the coefficients of x^k on both sides, establish an identity with combinatorial numbers.

7. Use the identity
$$(1+x)^{-n}(1-x)^{-n} = (1-x^2)^{-n},$$
to find
$$\sum_{k=0}^{m}(-1)^k \binom{n+k-1}{k}\binom{n+m-k-1}{m-k}.$$

8. (*American Mathematical Monthly*) Given two non-negative integers m and n, find the value of the expression:
$$\sum_{k=0}^{m}\frac{(-1)^k}{n+k+1}\binom{m}{k}(1-y)^{n+k+1} + \sum_{k=0}^{n}\frac{(-1)^k}{m+k+1}\binom{n}{k}y^{m+k+1}.$$

Solutions 6

1. (a) 0; (b) $\binom{d}{k}$ if $k = \frac{a-bd}{c-b}$ is a non-negative integer; otherwise, the coefficient is 0; (c) $\sum \binom{b}{k}\binom{k}{l}$, where the sum is extended over the pairs (k,l) of non-negative integers such that $k \geq l$ and $k+l = b$ (in particular, if b is not an integer ≥ 0, the coefficient is 0). **2.** Since $\binom{x+y}{n} = \frac{(x+y)_n}{n!}$, Vandermonde's formula for binomial coefficients is deduced. **3.** Induction on n. **4.** The term on the left of the series $[(1+x)-x]^n$. **5.** (a) $\sum_{l=1}^{n}(-1)^{l+1}\binom{n}{l}l^k = 0$; (b) $\sum_{l=1}^{n}(-1)^{l+1}\binom{n}{l}l^n = (-1)^{n+1}n!$. **6.** The identity of exercise 8 of the List of exercises 5. **7.** If m is odd then 0; otherwise, if m is even then $\binom{n+m/2-1}{m/2}$. **8.** $\frac{m!n!}{(m+n+1)!}$ (Help: Write both sums as appropriate integrals).

List of exercises 7

Multinomial coefficients

1. How many different permutations can be formed with the letters in the word MISSISSIPI? In how many aren't there two consecutive Is?

2. (Distinguishable balls in boxes)

   (a) In how many ways can $3n$ distinguishable balls be placed in 3 numbered boxes so that in each box there are n balls?

   (b) And if the boxes are not numbered?

   (c) In how many ways can kn distinguishable balls be placed in k numbered boxes so that in each box there are n balls?

   (d) And if the boxes are not numbered?

3. In how many ways can 6 apples, one pear, one orange, one peach, one banana, one strawberry and one grape be distributed among 3 persons?

4. Prove that the number 2^n divides $(2n)!$, and if $n \geq 2$ this division is an even number.

5. Prove that the number $[(n!)!]^{n+1}$ divides $(n^2!)!$.

6. Let $r_1, r_2, \ldots, r_k \geq 1$. Prove the following formula:

$$\binom{n}{r_1, r_2, \ldots, r_k} = \binom{n-1}{r_1 - 1, r_2, \ldots, r_k} + \binom{n-1}{r_1, r_2 - 1, \ldots, r_k} + \ldots + \binom{n-1}{r_1, r_2, \ldots, r_k - 1}$$

   in two ways: (a) using Combinatorics, (b) using Algebra by the multinomial theorem.

7. Prove the following formula:

$$\binom{n+s}{r_1, r_2, \ldots, r_m} = \sum \binom{n}{k_1, k_2, \ldots, k_m} \binom{s}{l_1, l_2, \ldots, l_m},$$

   where the sum is over all the pairs of m-tuples of non-negative integers $(k_1, k_2, \ldots, k_m)$, $(l_1, l_2, \ldots, l_m)$ such that

$$k_1 + k_2 + \ldots + k_m = n, \quad l_1 + l_2 + \ldots + l_m = s, \quad k_i + l_i = r_i, \quad \forall i = 1, 2, \ldots, m.$$

What known identity is generalized by this formula?

8. Calculate
$$\sum(-1)^{a+b}\binom{n}{a,b,c,d},$$
over all quadruples of non-negative integers (a,b,c,d) such that $a+b+c+d=n$. Generalize this result.

9. Find the coefficient of x^4yw^6 in the series $(x+y+z+w)^{11}$.

10. Let $m\in\mathbb{N}$. Show that the coefficient of x^m in the series $(1+x+x^2)^m$ is
$$1+\frac{m(m-1)}{(1!)^2}+\frac{m(m-1)(m-2)(m-3)}{(2!)^2}+\cdots.$$
What is the last term of the previous sum?

Solutions 7

1. $10!/(4!)^2$; $(6!/4!)\binom{7}{4}$. **2.** (a) $\binom{3n}{n,n,n}$ or $(1/3!)\binom{3n}{n,n,n}$ depending if the boxes are numbered or not; (b) $(kn)!/(n!)^k$ or $(1/k!)((kn)!/(n!)^k)$ depending if the boxes are numbered or not. **3.** $\binom{8}{6}3^6$. **4.** $(2n)!/2^n=\binom{2n}{2,2,\cdots,2}=(2n-1)!!n!$. **5.** $[(n!)!]^{n+1}$ divides $((n+1)!)!$ (multinomial coefficients), and, when $n\geq 2$, $((n+1)!)!$ divides $(n^2!)!$. **8.** 0; for all $n,k\geq 1$, $\sum(-1)^{r_1+r_2+\cdots+r_k}\binom{n}{r_1,\cdots,r_{2k}}=0$, where the sum is over all $2k$-tuples of non-negative integers $(r_1,\cdots,r_{2k})$ such that $r_1+\cdots+r_{2k}=n$. **9.** $11!/4!6!$.