Chapter I

BASIC COMBINATORICS

List of exercises 1

Floor and ceiling functions

1. Prove that for any real numbers x and y:

   (a) $\lfloor x+y \rfloor \geq \lfloor x \rfloor + \lfloor y \rfloor$. (superadditive)

   (b) $\lceil x+y \rceil \leq \lceil x \rceil + \lceil y \rceil$. (subadditive)

   When are the equalities true? Generalize (a) and (b).

2. Let $b \geq 2$ be an integer. Taking a base b numeral system, each natural number n is expressed (uniquely) by digits in $\{0, 1, 2, \ldots, b-1\}$. Put $d_b(n)$ as the number of digits of n in base b. For example,
$$d_{10}(357) = 3 \quad \text{and} \quad d_2(8) = d_2((1000)_2) = 4.$$
   Express $d_b(n)$ by $\log_b(n)$.

3. Prove or refute the following statements:

   (a) $\left\lfloor \sqrt{\lfloor x \rfloor} \right\rfloor = \lfloor \sqrt{x} \rfloor, \ \forall x \geq 0.$

   (b) $\left\lceil \sqrt{\lceil x \rceil} \right\rceil = \lceil \sqrt{x} \rceil, \ \forall x \geq 0.$

4. Let D be the real line or the interval $[0, \infty)$ and let $f : D \to \mathbb{R}$ be a continuous increasing function with the following property:
$$f(x) \in \mathbb{Z} \Rightarrow x \in \mathbb{Z}.$$
   Prove:

   (a) $\lfloor f(x) \rfloor = \lfloor f(\lfloor x \rfloor) \rfloor$.

   (b) $\lceil f(x) \rceil = \lceil f(\lceil x \rceil) \rceil$.

5. Let $m \in \mathbb{Z}$, $n \in \mathbb{N}$. Prove for each real x:

   (a) $\left\lfloor \frac{x+m}{n} \right\rfloor = \left\lfloor \frac{\lfloor x \rfloor + m}{n} \right\rfloor$.

(b) $\left\lceil \frac{x+m}{n} \right\rceil = \left\lceil \frac{\lceil x \rceil + m}{n} \right\rceil$.

6. Prove or refute:

   (a) $\left\lceil \sqrt{\lfloor x \rfloor} \right\rceil = \lceil \sqrt{x} \rceil \,, \ \forall x \geq 0.$

   (b) $\left\lfloor \sqrt{\lceil x \rceil} \right\rfloor = \lfloor \sqrt{x} \rfloor \,, \ \forall x \geq 0.$

7. Find a necessary and sufficient condition for:

$$\left\lceil \sqrt{\lfloor x \rfloor} \right\rceil = \lceil \sqrt{x} \rceil \,, \ \forall x \geq 0.$$

8. Let $a, b \in \mathbb{Z},\ a \leq b$. Calculate the number of integers in the following intervals:

$$(a, b], \quad [a, b), \quad (a, b), \quad [a, b].$$

9. Solve the same problem but now with $a, b \in \mathbb{R}$.

10. Every positive integer n can be expressed (uniquely) as $n = 2^m + l$, where m and l are non-negative integers and $0 \leq l < 2^m$. For example:

$$17 = 2^4 + 1, \quad 83 = 2^6 + 19.$$

    Use the floor function $\lfloor \cdot \rfloor$ and/or the ceiling function $\lceil \cdot \rceil$, to express m and l as functions of n.

11. For every real x that is not the midpoint of an interval with integer endpoints, call $\epsilon(x)$ the closest integer. Write $\epsilon(x)$ using the functions $\lfloor \cdot \rfloor$ and/or $\lceil \cdot \rceil$.

12. (*The Dirichlet Pigeonhole Principle*) If n pigeons occupy m pigeonholes then one of the pigeonholes has at least $\left\lceil \frac{n}{m} \right\rceil$ pigeons and one of the pigeonholes has at most $\left\lfloor \frac{n}{m} \right\rfloor$ pigeons.

Solutions 1

1. (a) $\{x\} + \{y\} \in [0, 1)$; (b) $< x > + < y > \in [0, 1)$. **2.** $d_b(n) = \lfloor \log_b(n) \rfloor + 1$. **3.** (a) true; (b) true. **6.** (a) false; (b) false. **7.** $\lfloor x \rfloor > (n-1)^2$ and $x \leq n^2$. **8.** $b - a$; $b - a$; $b - a - 1$; $b - a + 1$. **9.** $\lfloor b \rfloor - \lfloor a \rfloor$; $\lceil b \rceil - \lceil a \rceil$; $\lceil b \rceil - \lfloor a \rfloor - 1$; $\lfloor b \rfloor - \lceil a \rceil + 1$. **10.** $m = \lfloor \log_2(n) \rfloor$, $l = n - 2^{\lfloor \log_2(n) \rfloor}$. **11.** $\left\lfloor x + \frac{1}{2} \right\rfloor$ or $\left\lceil x - \frac{1}{2} \right\rceil$.

List of exercises 2

Elementary problems (I)

1. How many multiples of 3 are there between 52 and 311?
2. How many multiples of 6 with three digits and beginning by 4 are there?
3. How many different words made up of 7 different letters can be formed with A, I, R, S, T , U, V, W so that there is a consonant in the second position?
4. How many different words made up of 6 letters (different or not) can be formed with E, F, G, H, I, J, K, L so that they end with a vowel?
5. How many different words made up of 4 different letters can be formed with A, B, C, D, E , F so that they begin with a consonant and finish with a vowel?
6. How many different words made up of 6 letters (different or not) can be formed with A, B, C, D, E , F so that there are two different consonants in the two central positions?
7. How many different words are there made up of 4 letters (different or not) and formed with A, B, C, D, E beginning with a vowel and with the second letter different from the last one?
8. How many odd numbers with 5 different digits can be formed with digits 1,2,3,4,5 and 6? And if the digits can be repeated?
9. How many multiples of 5, greater than 2000 and made of 4 digits, can be formed with digits 0,1,2,3,4,5,6 and 7? And if the digits are all different?
10. How many palindromic numbers of 5 digits are there? How many of them are odd?
11. Mr. Casanova has 7 girlfriends and each day he takes a walk with one of them. One Sunday he decides not to date the same girl two consecutive days in the following week. In how many ways can he date the girls that week?
12. How many different words can be formed with all the letters of the word ABRACADABRA?

13. There are 7 athletes in a race. How many possible results are there? If there are 3 English, 2 French, 1 Swedish and 1 Russian, in how many ways can the national flags appear in the list of results?

14. How many different results can be obtained from shuffling a 40-card Spanish deck?

15. How many different results can be obtained from shuffling two identical 40-card Spanish decks together?

16. In 2000 a tourist decided to visit 4 of the 15 capitals of the European Union. How many different itineraries could he consider?

17. The combination of a safe is 5 digits. How many combinations must we try in order to be sure the door will be opened? And in the case that each digit should be different from the previous one?

18. A boat has a set of 12 different flags and can raise up to 3 in the mast. How many different messages can be shown? And if the boat has 3 sets of flags?

19. In how many ways can 40 cards be placed in two decks with the same number of cards so that there are two aces in each? And if one of the decks has 10 cards and the other 30?

20. If we toss 6 coins, in how many cases do we obtain 4 heads and 2 tails?

21. How many bets can be done in *bonoloto* (mark 6 out of 49 numbers)? And how many bets can be done marking only in the first 10 numbers?

22. In how many ways can a poker player get a full (three cards of one rank and two of a second rank)? And a poker (four cards of one rank)? And two pairs? And three cards of one rank?

23. How many different soccer pools are possible?

24. Mr.So-and-so has forgotten the last three digits of her girlfriend's phone number. How many calls must he make in order to be sure he will contact her? And if he remembers that the three digits are different and the last one is odd?

25. How many words of 5 different letters can be formed with A, B, C, D, E and F so that the first and second are consonants and the third is a vowel? And if the letters can be repeated?

26. The flight numbers of an airline are formed with 3 digits and 2 vowels, in any order but all different How many flight numbers are possible?

27. How many words of 2 vowels and 2 consonants can be formed with A, B, C, D, E and I? And if there are not repetitions?

28. How many words with A, B, C and D are there in which A and B are together? In how many are they separated?

29. In how many ways can A, B, C, D and E be ordered such that there is one letter between A and B?

30. In how many ways can 8 rooks be placed on a chessboard without being under attack among themselves?

31. How many different poker hands can be given among 4 players?

32. A restaurant offers 5 first dishes, 7 second dishes and 4 desserts. From how many menus of 3 different dishes can a client choose if

    (a) he holds the usual conventionalism?

    (b) he ignores the conventionalism but consumes a dish after the other?

    (c) he ignores the conventionalism and consumes the three dishes at the same time?

33. A basketball team has 11 players. In how many ways can the coach form the starting lineup? If there are 4 *forwards*, 2 *guards* and 5*centers* and he must consider 1 *guard* and 1 or 2 *forwards*, how many choices does he have?

34. Find the number of choices of a group of 6 cards of a Spanish deck so that there is

    (a) At least one king.

    (b) No club.

    (c) At least one king but no club.

35. Find the number of choices of 7 cards of a Spanish deck so that 3 of them are spades and two of them are horses.

36. If a gang is formed by 7 girls and 11 boys, in how many ways can 5 couples be formed so that they can dance simultaneously?

37. Find the number of different 6 letter words that can be formed if the letters are not repeated and they have exactly 3 vowels and the consonants can be chosen only among B, C, D and F. And if the letters do not have to be different?

38. In the building where I live I have 23 neighbors. 11 of us are going to be selected to form a soccer team. How many different choices are there? In how many of them am I included? In how many am I not?

Solutions 2

1. 86. **2.** 17. **3.** 25200. **4.** 65536. **5.** 96. **6.** 15552. **7.** 200. **8.** 360; 3888. **9.** 767; 330. **10.** 900; 500. **11.** 326592. **12.** 83160. **13.** 5040; 420. **14.** 40!. **15.** $\frac{80!}{2^{40}}$. **16.** 32760. **17.** 100000; 65610. **18.** 1465; 1885. **19.** $\frac{1}{2}\binom{4}{2}\binom{36}{18}$; $\binom{4}{2}\binom{36}{8}$. **20.** 15. **21.** $\binom{49}{6}$; $\binom{10}{6}$. **22.** 3744; 624; 123552; 54912. **23.** 3^{15}. **24.** 1000; 360. **25.** 144; 1152. **26.** 144000. **27.** 486; 216. **28.** 12; 12. **29.** 36. **30.** 40320. **31.** $\binom{52}{5}\binom{47}{5}\binom{42}{5}\binom{37}{5}$. **32.** 140; 3360; 560. **33.** 462; 200. **34.** $\binom{40}{6}-\binom{36}{6}$; $\binom{30}{6}$; $\binom{30}{6}-\binom{27}{6}$. **35.** 404352. **36.** $\binom{7}{5}\binom{11}{5}5!$. **37.** 28800; 160000. **38.** $\binom{24}{11}$; $\binom{23}{10}$; $\binom{23}{11}$.

List of exercises 3

Elementary problems (II)

1. Find the number of ways to divide 9 persons in 3 groups, of 3 people each, if the divisions differ only in the structure (that is, the order of the groups and the order of the persons in the groups do not matter).

2. Calculate the number of ways to divide 3 boys and 6 girls in 3 groups of 3 people each so that there is a boy in each group. (The order of the groups and the order of the people in the groups do not matter).

3. Let A, B and C be subsets of a finite set Ω. Show that the following statements are inconsistent:
$$|A \cup B \cup C| = 1000, \quad |A| = 510, \quad |B| = 490, \quad |C| = 427,$$
$$|A \cap B| = 189, \quad |A \cap C| = 140, \quad |B \cap C| = 85.$$

4. We have a collection of 100 cubes with sides painted in red, blue or green. We know that 80 of them have at least one red side, 85 have at least one blue side and 75 have at least one green side. Knowing only this we cannot calculate the number of cubes that have sides of the three colors, but we can say something about it. What?

5. In how many ways can n boys and n girls be on a row if two persons of the same gender cannot be together?

6. A, B and C are three persons of a group of n. In how many ways can n persons be placed on a row so that A, B and C are together and in that order?

7. How many sequences (of $m + n$ terms) can be formed with m zeros and n ones so that there are not two consecutive ones?

8. Calculate the number of ways of placing n distinguishable balls in n numbered boxes so that:

   (a) None of the boxes is empty.

   (b) Exactly one box is empty.

   (c) Exactly two boxes are empty.

9. Find the number of bijections from $\{1, 2, \ldots, n\}$ to itself that take exactly $k \leq n$ elements in themselves. State equivalent problems (translations) in terms of placing balls in boxes and in terms of dancing couples.

10. How many subsets of $\{a_1, \ldots, a_m, b_1, \ldots, b_n\}$ have at least one element a and one element b?

11. How many palindromic numbers are there with $n \geq 2$ significative digits in a base b numeral system?

12. Among the guests of a party let n be the number of those who greeted an odd number of guests. Show that n is even.

13. Let n be a natural number. Prove that n is a perfect square if and only if it has an odd number of divisors (including 1 and n).

14. How many divisors has the number whose prime decomposition is $p_1^{m_1} p_2^{m_2} \cdots p_k^{m_k}$? How much is the sum of all its divisors?

15. (ϕ Euler function) For each natural number n, $\phi(n) := |A_n|$, with

$$A_n := \{k \in \{1, 2, \ldots, n\} \ : \ k \text{ prime with } n\}.$$

Prove that if $p_1^{m_1} p_2^{m_2} \cdots p_l^{m_l}$ is the decomposition of n in prime factors then

$$\phi(n) = n \left(1 - \frac{1}{p_1}\right) \left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_l}\right).$$

16. (Legendre) Let p be a prime number and n be a natural number. Find the number of times that p divides $n!$. (For example, p divides n three times if n is divisible by p^3 but is not by p^4; i. e., if p^3 is the greatest power of p that divides n; in other words, if p^3 is the power of p in the prime decomposition of n.)

17. $10! = 3628800$ finishes with two zeros. With how many zeros does $1000!$ finish?

18. Let p and q be natural numbers, $p \geq q$. Find the number of U-D trajectories from $(0, 0)$ to (p, q) that cross OX, (that is, that reach the line $y = -1$) (Help: By symmetry this problem can be translated into another of U-D trajectories from $(0, -2)$ to (p, q) without constraints.)

19. In the ticket office of a movie theater there is not money at the beginning. A ticket is 5 euros. There are m persons with a 5 euro bill each and there are n ($n \leq m$) with a 10 euro bill each. In how many ways can they wait on a row in front of the ticket office so that the ticket office has change for all the clients? (Help: Is there any relationship between this problem and the previous one?)

20. In the three-dimensional affine space $\mathbb{E}_3$ over the field $\mathbb{Z}_2 := \{0, 1\}$ find:

    (a) the total number of points,

    (b) the total number of lines,

    (c) the total number of planes,

    (d) the number of points contained in each line,

(e) the number of lines passing through each point,

(f) the number of points on each plane,

(g) the number of planes through each point,

(h) the number of lines contained in each plane,

(i) the number of planes through each given line,

(j) the number of planes parallel to a given plane,

(k) the number of lines parallel to a given line,

(l) the number of lines parallel to a given plane,

(m) the number of planes parallel to a given line,

(n) the number of lines that intersect a given line.

Solutions 3

1. 280. **2.** 90. **3.** $c(ABC) = -13$. **4.** $c(ABC) \geq 40$. **5.** $2(n!)^2$. **6.** $(n-2)!$. **7.** $\binom{m+1}{n}$. **8.** (a) $n!$; (b) $n!\binom{n}{2}$; (c) $\binom{n}{2}^2\binom{n-2}{2}^2(n-4)! + \binom{n}{2}\binom{n}{3}(n-2)!$. **9.** $\frac{n!}{k!}\sum_{j=0}^{n-k}\frac{(-1)^j}{j!}$. **10.** $(2^m-1)(2^n-1)$. **11.** $(b-1)b^{\lceil \frac{n}{2} \rceil -1}$. **12.** (Help: think on a greeting as a line that connects two points). **13.** (Help: Associate with each divisor d the number $\frac{n}{d}$; or use the results of the next exercise). **14.** $\prod_{i=1}^{k}(m_i+1)$; $\prod_{i=1}^{k}\frac{1-p_i^{m_i+1}}{1-p_i}$. **15.** (Help: The Principle of Inclusion-Exclusion). **16.** $\sum_{i=1}^{\lfloor \log_p n \rfloor}\left\lfloor \frac{n}{p^i} \right\rfloor$. **17.** 249. **18.** $\binom{p}{\frac{p+q}{2}+1}$. **19.** $\binom{m+n}{n}\left(1-\frac{n}{m+1}\right)m!n!$. **20.** (a) 8; (b) 28; (c) 14; (d) 2; (e) 7; (f) 4; (g) 7; (h) 6; (i) 3; (j) 1; (k) 3; (l) 6; (m) 3; (n) 12.

List of exercises 4

Problems from Olimpiadas Matemáticas

1. 2000 members of a political party attended their Annual Congress. A journalist noticed that, among the participants in a session, the $12,\overset{\frown}{12}$ % were women and the $23,4\overset{\frown}{23}$ % belonged to the critical sector. Find the number of members that did not participate in that session.

2. Let Ω be a set of 14 different natural 3-digit numbers. Prove that there exist two nonempty disjoint sets $A \subset \Omega$ and $B \subset \Omega$ so that:

$$\text{the sum of the elements of } A = \text{the sum of the elements of } B$$

3. In a ball there are 8 boys and 8 girls, who are sat alternatively on a row (boy-girl-boy-...). In how many ways can a group of 5 dancing couples be formed if each couple must be made of a boy and a girl initially contiguous?

4. In a meeting there are 201 persons of 5 different nationalities. It is known that in each 6-person group at least 2 are of the same age. Prove that in the meeting there are at least 5 people of the same nationality, the same age and the same gender.

5. Let $A := \{1, 2, 3, 4, 5, 6\}$. Find the number of maps $f : A \to A$ that satisfy:
   (a) $(f \circ f)(x) = x,\ \forall x \in A$
   (b) $(f \circ f \circ f)(x) = x,\ \forall x \in A$

6. A subset $A \subset M := \{1, 2, \ldots, 10, 11\}$ is said to be *nice* if it has the following property:

$$2k \in A \Rightarrow 2k - 1 \in A \text{ and } 2k + 1 \in A.$$

   The empty set and M are nice. How many nice subsets does M have?

7. Put

$$q(n) := \left\lfloor \frac{n}{\lfloor \sqrt{n} \rfloor} \right\rfloor,\ n = 1, 2, \ldots.$$

(a) Find the table of values of $q(n)$ for $1 \leq n \leq 25$. Considering the table, conjecture the values of n for which $q(n) > q(n+1)$.

(b) Prove the conjecture, that is, find all the natural numbers for which $q(n) > q(n+1)$.

8. Can we draw 2003 segments on the plane so that each of them intersects exactly three other segments?

9. On a checkers (draughts) board 8×8 we set the 24 pieces on the upper three rows. We can move a piece according to the following: a piece can jump over another to an empty square horizontally (left or right), vertically (up or down) or diagonally. Can we place all the pieces on the lower three rows?

10. Find the minimum number of bets on a soccer pool to be sure that we get at least 5 good results in one of them. (A bet on a soccer pool consists of a prediction for 14 games in which there are 3 possible results).

Solutions 4

1. 779. **2.** (Help: apply the Pigeonhole Principle in order to show that the map that takes each nonempty subset of Ω to the sum of its elements cannot be injective.) **3.** 462. **4.** (Help: Pigeonhole Principle). **5.** (a) 76; (b) 81. **6.** 233. **7.** n such that $n+1$ is a perfect square. **8.** No (see exercise 12 of the List of exercises 3). **9.** No (the pieces placed on even (odd) rows only can be moved to even (odd) rows. **10.** 3.