THE OXFORD MURDERS

1

- Definition of the triangular numbers:

  $\{T(n)\}_{n \in \mathbb{N}}$ is a recurrence sequence of natural numbers such that $T(1) = 1$. Then, $\forall n \geq 2$, we define the n-th triangular number (denoted by $T(n)$) as a geometric pattern of triangles of sizes 1 through n, arranged in the unique way possible such that they overlap each other and share a common vertex.

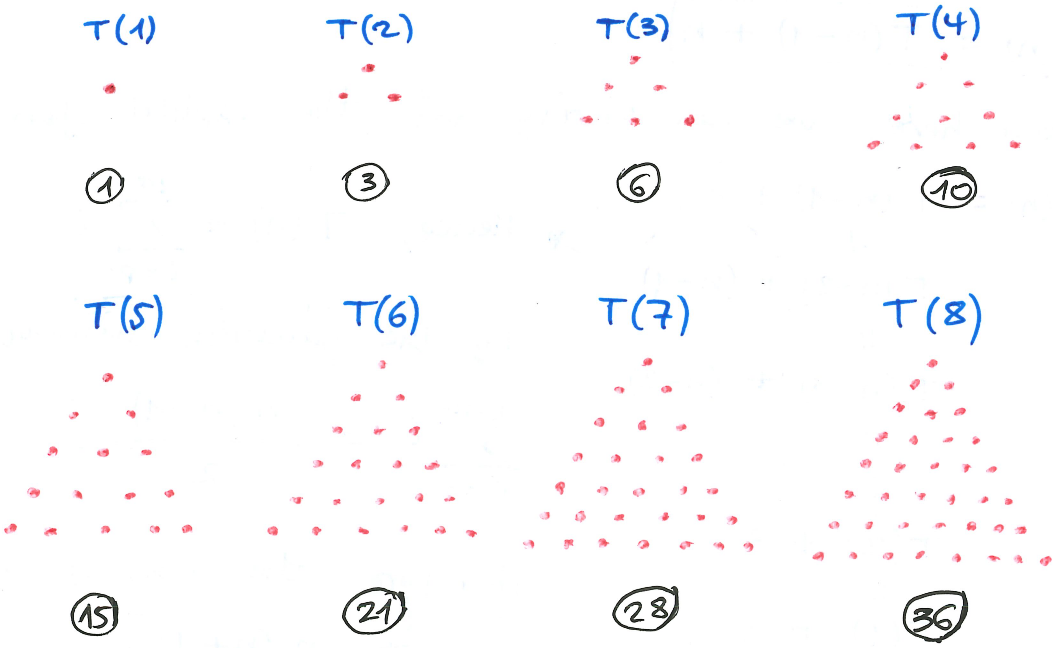
- The first numbers of the sequence:

  According to the OEIS, the first numbers of the triangular sequence (A000217) are:

  $T(1) = 1$, $T(2) = 3$, $T(3) = 6$, $T(4) = 10$,

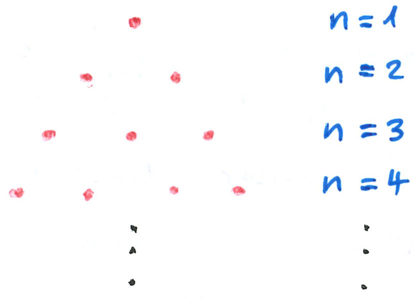
  $T(5) = 15$, $T(6) = 21$, $T(7) = 28$, $T(8) = 36$

  PICTORIAL REPRESENTATION OF THE NUMBERS:



2. Explanation of the formulas for $T(n)$.

From the definition, we observe the following relation among two consecutive triangular numbers:



We could think about the n-th number of the sequence as adding n dots in a straight line pararallel to any of the sides of the triangle associated to $T(n-1)$.

Note: If we choose the same side of the previous triangle for the n dots, the vertex opposite to this side will be the common vertex of all the triangles.

Thus, we get the recurrence formula:

$$\boxed{T(n) = T(n-1) + n}$$

From here, we can easily get the explicit formula:

$$
\begin{aligned}
T(n) &= T(n-1) + n \\
&= T(n-2) + (n-1) + n \\
&= T(n-3) + (n-2) + \cdots \\
&\;\;\vdots \\
&= T(2) + 3 + \cdots \\
&= T(1) + 2 + \cdots \\
&= 1 + \cdots
\end{aligned}
$$

by definition → $T(1) = 1$

Hence, $T(n) = \sum_{i=0}^{n} i$

By the gaussian summation,

$$\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$$

Therefore, the explicit formula:

$$\boxed{T(n) = \frac{n(n+1)}{2}}$$

- Propierties of the triangular numbers:

① $T(n) = \binom{n+1}{2}$

* **Analytical proof:**

$$\binom{n+1}{2} = \frac{(n+1)!}{2!\,(n+1-2)!} = \frac{(n+1)!}{2!\,(n-1)!} = \frac{n(n+1)}{2} = T(n) \blacksquare$$

* **Combinatorial proof:**

By definition of combinatorial number:

$\binom{n+1}{2}$ = "number of different pairs we can form with the elements of a set of $n+1$ elements"

Let $\Omega = \{1, 2, \ldots, n, n+1\}$

we define:

$$P = \{B \subset \Omega : |B| = 2\} \Rightarrow |P| = \binom{n+1}{2}$$

Now, we define:

$$A(1) = \{B \subset P : 1 \in B\}$$

$$A(2) = \{B \subset P : 2 \in B, \ B \notin A(1)\}$$

$$\vdots$$

$$A(k) = \{B \subset P : k \in B, \ B \notin A(k-1)\}, \quad \forall k \in \{2, \ldots, n+1\}$$

Then, we observe that:

$$\left.\begin{aligned} |A(1)| &= (n+1)-1 = n \\ |A(2)| &= (n)-1 = n-1 \\ &\vdots \\ |A(n-1)| &= (n-(n-3))-1 = 2 \\ |A(n)| &= (n-(n-2))-1 = 1 \\ |A(n+1)| &= (n-(n-1))-1 = 0 \end{aligned}\right\} \Rightarrow |P| = \sum_{i=1}^{n} |A(n)| =$$

[since $|A(n+1)| = 0$]

$$= \sum_{i=1}^{n} i = \frac{n(n+1)}{2} = T(n) \blacksquare$$

② TRIANGULAR NUMBER OF A SUM:

$$T(n+m) = T(n) + T(m) + n\cdot m$$

Proof:

By the explicit formula:

$$T(n+m) = \frac{(n+m)(n+m+1)}{2} = \frac{n^2+nm+n+mn+m^2+m}{2} =$$

$$= \frac{n^2+n}{2} + \frac{m^2+m}{2} + \cancel{2}\frac{nm}{\cancel{2}} =$$

$$= \frac{n(n+1)}{2} + \frac{m(m+1)}{2} + n\cdot m = T(n) + T(m) + n\cdot m. \blacksquare$$

③ TRIANGULAR NUMBER OF A PRODUCT:

$$T(n\cdot m) = T(n)\cdot T(m) + T(n-1)\cdot T(m-1)$$

Proof:

Going from right to left is easier in this case:

$$T(n)\cdot T(m) + T(n-1)\cdot T(m-1) =$$

$$= \frac{n(n+1)}{2}\cdot\frac{m(m+1)}{2} + \frac{n(n-1)}{2}\cdot\frac{m(m-1)}{2} =$$

$$= \frac{nm}{4}\left[(n+1)(m+1)\right] + \frac{nm}{4}\left[(n-1)(m-1)\right] =$$

$$= \frac{nm}{4}\left[(n+1)(m+1)+(n-1)(m-1)\right] =$$

$$= \frac{nm}{4}\left[(nm+\cancel{n}+\cancel{m}+1) + (nm-\cancel{n}-\cancel{m}+1)\right] =$$

$$= \frac{nm}{4}\left[2nm+2\right] = \frac{nm}{4}\left[2(nm+1)\right] =$$

$$= \frac{nm}{2}(nm+1) = T(n\cdot m) \blacksquare$$

④ $T(n) + T(n-1) = n^2$

Proof:

By the explicit formula:

$$T(n) + T(n-1) = \frac{n(n+1)}{2} + \frac{n(n-1)}{2} =$$

$$= \frac{n}{2}\left[(n\cancel{+1}) + (n\cancel{-1})\right] = \frac{n}{\cancel{2}}\left[\cancel{2}n\right] = n^2 \blacksquare$$

⑤ $(T(n))^2 - (T(n-1))^2 = n^3$

Proof:

By the explicit formula:

$$(T(n))^2 - (T(n-1))^2 = \left(\frac{n(n+1)}{2}\right)^2 - \left(\frac{n(n-1)}{2}\right)^2 =$$

$$= \frac{n^2(n+1)^2}{4} - \frac{n^2(n-1)^2}{4} = \frac{n^2}{4}\left[(n+1)^2 - (n-1)^2\right] =$$

$$= \frac{n^2}{4}\left[n^2 + 2n + 1 - (n^2 - 2n + 1)\right] =$$

$$= \frac{n^2}{4}(\cancel{n^2} + 2n \cancel{+1} \cancel{-n^2} + 2n \cancel{-1}) = \frac{n^2}{\cancel{4}}\left[\cancel{4}n\right] = n^3 \blacksquare$$

⑥ In 1796, Gauss proved that every natural number can be expressed as a sum of three (or less, naturally) triangular numbers.

NOTE: We have thougth this is a remarkable propierty of the triangular numbers. However, its proof is really difficult and needs some results from other theorems.

$\boxed{3}$. Triangular numbers cannot finish by 2, 4, 7, 9.

PROOF:

We observe that, if $n \equiv k \mod (10) \Rightarrow$

$$\Rightarrow \underbrace{\frac{n(n+1)}{2}}_{\boxed{T(n)}} \equiv \frac{k(k+1)}{2} \mod (10)$$

Hence, we'll focus on $k \in [0, \dots, 9]$. That is, on the last digit of n.

Now, we see all the possible products:

$$\frac{0 \cdot \cancel{1}}{\cancel{2}} = 0 \equiv 0 \mod (10)$$

$$\frac{1 \cdot \cancel{2}}{\cancel{2}} = 1 \equiv 1 \mod (10)$$

$$\frac{\cancel{2} \cdot 3}{\cancel{2}} = 3 \equiv 3 \mod (10)$$

$$\frac{3 \cdot 4}{2} = 6 \equiv 6 \mod (10)$$

$$\frac{4 \cdot 5}{2} = 10 \equiv 0 \mod (10)$$

$$\frac{5 \cdot 6}{2} = 15 \equiv 5 \mod (10)$$

$$\frac{6 \cdot 7}{2} = 21 \equiv 1 \mod (10)$$

$$\frac{7 \cdot 8}{2} = 28 \equiv 8 \mod (10)$$

$$\frac{8 \cdot 9}{2} = 36 \equiv 6 \mod (10)$$

Therefore, we conclude that $T(n) \mod (10) \neq 2, 4, 7, 9$.

Because of that, $T(n)$ cannot finish by 2, 4, 7 or 9. ∎

4. "If you take 8 times a triangular number and afterwards you add one, it turns out a square".

By the explicit formula:

$$T(n) = \frac{n(n+1)}{2}$$

Now, if we apply the previous statement:

$$8T(n) + 1 = 8\,\frac{n(n+1)}{2} + 1 = 4n(n+1) + 1 =$$

$$= 4n^2 + 4n + 1 = (2n+1)^2$$

Then, we see that $8T(n)+1$ is always going to be a perfect square.

5. a) Explicit formulas for:

SQUARE NUMBERS $[S(n)]$

$S(1)$ $S(2)$ $S(3)$ $S(4)$

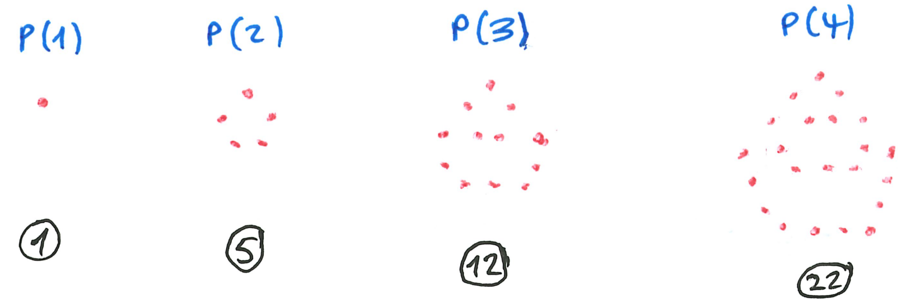
1 4 9 16

We easily see that: $\boxed{S(n) = n^2}$

We can also observe the recurrence relation:

$$\left.\begin{aligned} S(n+1) &= S(n) + 2n + 1 \\ S(n) &= S(n-1) + 2n - 1 \end{aligned}\right\} \Rightarrow \sum_{i=1}^{n} 2i - 1 = n^2 = S(n)$$

PENTAGONAL NUMBERS [P(n)]



As we reasoned on the triangular numbers, in this case we could think about the n-th pentagonal number as adding an n-dot straight line parallel to each of the sides opposite (not touching) to the common vertex of the pentagon associated to P(n-1). These will always be $5-2=3$ n-dot lines, $\forall n \geq 2$. Since the line in the middle shares a vertex with the line at his left and another one with the line at his right, we have to substract 2 dots to $3 \cdot n$.

Hence, we get the recurrence formula:

$$P(n) = P(n-1) + 3n - 2.$$

From here, we can deduce the explicit formula:

$$P(n) = \sum_{i=1}^{n} 3i - 2 = 3\sum_{i=1}^{n} i - 2\sum_{i=1}^{n} 1 =$$

$$= 3\,\frac{n(n+1)}{2} - 2n = \frac{n}{2}\left(3(n+1) - 4\right) =$$

$$= \frac{n}{2}(3n-1) \Rightarrow \boxed{P(n) = \frac{n(3n-1)}{2}}$$

HEXAGONAL NUMBERS [H(n)]

H(1) H(2) H(3) H(4)

1 6 15 28

As we reasoned on the pentagonal numbers, we could think about the n-th hexagonal number as adding n-dot straight lines to the hexagon associated to H(n−1) aswell. But, in this case, we will need to add 6 − 2 = 4 lines instead of 5. since the lines in the middle share a vertex (each one shares two vertizes, but note that "two" of those vertices is actually a single one touching both lines simultaneously) with the lines at their sides, we have to substract 3 dots to 4·n.

Hence, we get the recurrence formula:

$$H(n) = H(n-1) + 4n - 3$$

From here, we can deduce the explicit formula:

$$H(n) = \sum_{i=1}^{n} 4i - 3 = 4\sum_{i=1}^{n} i - 3\sum_{i=1}^{n} 1 =$$

$$= 4\,\frac{n(n+1)}{2} - 3n = 2n(n+1) - 3n =$$

$$= 2n^2 + 2n - 3n = 2n^2 - n = n(2n-1)$$

$$\Rightarrow \boxed{H(n) = n(2n-1)}$$

GENERALIZATION : POLYGONAL NUMBERS [$K(n)$]

Once observed the recurrence relation of the pentagonal and hexagonal numbers, we can now generalize these results to get the explicit formula for a K-gon (a K-sided polygon).

The explanation of arriving to the recurrence formula will be done in part b). Therefore, we are going to base on it:

$$K(n) = K(n-1) + n(K-2) - (K-3)$$

Moreover, we are using that every sequence of polygonal numbers starts at 1. that is, $K(1) = 1$ ($\forall K \geq 3$, obviously).

$$K(n) = \sum_{i=1}^{n} \left[i(K-2) - (K-3) \right] =$$

$$= (K-2) \sum_{i=1}^{n} i - (K-3) \sum_{i=1}^{n} 1 =$$

$$= (K-2) \frac{n(n+1)}{2} - (K-3) \cdot n =$$

$$= \frac{n}{2} \left[(n+1)(K-2) - 2(K-3) \right] =$$

$$= \frac{n}{2} \left[nK - 2n + K - 2 - 2K + 6 \right] =$$

$$= \frac{n}{2} \left[nK - 2n - K + 4 \right] = \frac{n}{2} \left[n(K-2) - (K-4) \right] =$$

$$= \frac{n^2(K-2) - n(K-4)}{2}$$

Alternative formula

$$\Rightarrow \boxed{K(n) = \frac{n^2(K-2) - n(K-4)}{2}} = \boxed{K(n) = (K-2)\frac{n(n-1)}{2} + n}$$

• b) Recurrence relationship for the polygonal numbers:

As we have seen in the previous part, the recurrence relations of the first polygonal sequences of numbers are:

SQUARE $\Rightarrow$ $S(n) = S(n-1) + 2n - 1$

PENTAGONAL $\Rightarrow$ $P(n) = P(n-1) + 3n - 2$

HEXAGONAL $\Rightarrow$ $H(n) = H(n-1) + 4n - 3$

By construction, in each increasement of n we are overlaping the previous polygon by fixing a vertex (the "common vertex"). This way, each new side of the new polygon will be formed by n dots.

Let's take the same value n−1 fixed for any generic polygon and let's observe which is the increasement for n for each K-gon.

	sides to overlap	common vertices
SQUARE (4 sides) $\Rightarrow$	$4-2=2$	$4-3=1$
PENTAGON (5 sides) $\Rightarrow$	$5-2=3$	$5-3=2$
HEXAGON (6 sides) $\Rightarrow$	$6-2=4$	$6-3=3$
⋮	⋮	⋮
K-GON (K sides) $\Rightarrow$	$K-2$	$K-3$

That is, for each increasement of k we must sum n to the total of dots, but substract one more vertex aswell.

Therefore, $\boxed{K(n) = K(n-1) + n(K-2) - (K-3)}$

• c) Propierties of the polygonal numbers:

① RELATION BETWEEN POLYGON (FOR ANY K) AND TRIANGULAR NUMBERS:

$$K(n) = (K-2)\cdot T(n-1) + n$$

Proof:

We will use the alternative formula in order to get it proved directly:

$$k(n) = (k-2)\underbrace{\frac{n(n-1)}{2}}_{T(n-1)} + n = (k-2)\cdot T(n-1) + n.$$

② THE SUBSTRACTION OF POLYGONAL NUMBERS (OF CONSECUTIVE K) IS A TRIANGULAR NUMBER:

$$[k+1](n) - k(n) = T(n-1)$$

Proof:

$$[k+1](n) - k(n) = \left[(k-1)\underbrace{\frac{n(n-1)}{2}}_{T(n-1)} + n\right] - \left[(k-2)\underbrace{\frac{n(n-1)}{2}}_{T(n-1)} + n\right] =$$

$$= T(n-1)\left[\cancel{(k}-1) + \cancel{n} - \cancel{(k}-2) - \cancel{n}\right] = T(n-1)$$

③ n-TH POLYGONAL NUMBER OF A SUM OF DEGREES:

Let $k_1, k_2 \in \mathbb{N}$ such that $\underbrace{\frac{k_1+k_2}{2} \in \mathbb{N}}_{\text{(that is, } k_1 \text{ and } k_2 \text{ have same parity)}}$, $\frac{k_1+k_2}{2} \geq 3$

$$\left[\frac{k_1+k_2}{2}\right](n) = \frac{k_1(n) + k_2(n)}{2}$$

Proof:

$$\left[\frac{k_1+k_2}{2}\right](n) = \left(\frac{k_1+k_2}{2} - 2\right)\frac{n(n-1)}{2} + n =$$

$$= \frac{1}{2}\left[(k_1+k_2-4)\cdot\frac{n(n-1)}{2} + 2n\right] =$$

$$= \frac{1}{2}\left[(k_1+k_2-2-2)\frac{n(n-1)}{2} + n + n\right] =$$

$$= \frac{1}{2}\left[((k_1-2)+(k_2-2))\frac{n(n-1)}{2} + n + n\right] =$$

$$= \frac{1}{2}\left[\underbrace{(k_1-2)\frac{n(n-1)}{2} + n}_{= k_1(n)} + \underbrace{(k_2-2)\frac{n(n-1)}{2} + n}_{= k_2(n)}\right] =$$

$$= \frac{k_1(n) + k_2(n)}{2} \quad \blacksquare$$

④ POLYGONAL NUMBER OF A SUM:

$$K(n+m+1) = K(n+1) + K(m+1) + (K-2)\cdot n\cdot m + 1$$

Proof:

$$K(n+m+1) = (K-2)\cdot T((n+m+\cancel{1})-\cancel{1}) + (n+m+1)$$

Now, we apply the identity for the triangular number of a sum:

$$= (K-2)\cdot\left[T(n) + T(m) + n\cdot m\right] + (n+m+1) =$$

$$= (K-2)\cdot T(n) + (K-2)\cdot T(m) + (K-2)\cdot nm + (n+m+1) =$$

$$= \underbrace{\left[(K-2)\cdot T(n) + n\right]}_{= K(n+1)} + \underbrace{\left[(K-2)\cdot T(m) + m\right]}_{= K(m+1)} + (K-2)nm + 1 =$$

$$= K(n+1) + K(m+1) + (K-2)n\cdot m + 1 \quad \blacksquare$$

⑤ POLYGONAL NUMBER OF DEGREE K AS A FUNCTION OF (K+r)-GON AND (K-r)-GON NUMBERS:

$$K(n) = \frac{[K+r](n) + [K-r](n)}{2}$$

Proof:

$$K(n) = (K-2)\,\frac{n(n-1)}{2} + n = \frac{2}{2}\left[(K-2)\,\frac{n(n-1)}{2} + n\right] =$$

$$= \frac{n(n-1)}{4}\left[2K-4\right] + n = \frac{n(n-1)}{4}\left[K+K-2-2+r-r\right] + n =$$

$$= \frac{1}{2}\left[\frac{n(n-1)}{2}\left[(K+r-2) + (K-r-2)\right] + 2n\right] =$$

$$= \frac{1}{2}\left[\underbrace{(K+r-2)\,\frac{n(n-1)}{2} + n}_{= [K+r](n)} + \underbrace{(K-r-2)\,\frac{n(n-1)}{2} + n}_{= [K-r](n)}\right] = \frac{[K+r](n) + [K-r](n)}{2} \quad \blacksquare$$

⑥ POLYGONAL NUMBER OF A PRODUCT:

$$(k-2)\,k(n\cdot m+1) = \big[k(n+1)-(n+1)\big]\big[k(m+1)-(m+1)\big] +$$
$$+\big[k(n)-n\big]\big[k(m)-m\big] + (k-2)\,n\cdot m$$

<u>Proof</u>:

$$(k-2)\,k(n\cdot m+1) = (k-2)\left[(k-2)\frac{nm(n\cdot m+1)}{2} + nm\right] =$$

$$= (k-2)^2\underbrace{\frac{nm(nm+1)}{2}}_{\displaystyle \overset{\shortparallel}{T(n\cdot m)}} + (k-2)\,n\cdot m$$

Now, we apply the identity for the triangular number of a product:

$$= (k-2)^2\big[T(n)\cdot T(m) + T(n-1)\cdot T(m-1)\big] + (k-2)\,n\cdot m =$$

By the alternative formula:

$$k(n) = (k-2)\underbrace{\frac{n(n-1)}{2}}_{\displaystyle \overset{\shortparallel}{T(n-1)}} + n \iff k(n)-n = (k-2)\cdot T(n-1)$$

Using this, we get:

$$= \underbrace{\big[(k-2)\cdot T(n)\big]}_{\displaystyle \overset{\shortparallel}{k(n+1)-(n+1)}}\cdot\underbrace{\big[(k-2)\cdot T(m)\big]}_{\displaystyle \overset{\shortparallel}{k(m+1)-(m+1)}} + \underbrace{\big[(k-2)\cdot T(n-1)\big]}_{\displaystyle \overset{\shortparallel}{k(n)-n}}\cdot\underbrace{\big[(k-2)\cdot T(m-1)\big]}_{\displaystyle \overset{\shortparallel}{k(m)-m}}$$

$$+ (k-2)\,n\cdot m =$$

$$= \big[k(n+1)-(n+1)\big]\cdot\big[k(m+1)-(m+1)\big] +$$

$$+ \big[(k(n)-n\big]\cdot\big[k(m)-m\big] + (k-2)\,n\cdot m \quad \blacksquare$$

EXTRA $\Rightarrow$ Proof of the alternative formula for the polygon numbers.

$$k(n) = (k-2)\frac{n(n-1)}{2} + n$$

Proof:

We had already proven the main formula:

$$k(n) = \frac{n^2(k-2) - n(k-4)}{2} = \frac{n^2(k-2) - n(k-2-2)}{2} =$$

$$= \frac{n^2(k-2) - n(k-2) + 2n}{2} =$$

$$= \frac{n^2(k-2) - n(k-2)}{2} + n =$$

$$= (k-2)\frac{n^2 - n}{2} + n =$$

$$= (k-2)\frac{n(n-1)}{2} + n \quad \blacksquare$$