DISCRETE MATHEMATICS

Chapter 3: Generating functions and recurrence relations

3.1 Generating functions

3.1.1 Power series

Theorem (Convergence)

Let $(a_n)_{n\geq 0}$ be a sequence of real or complex numbers, $\sum_{n=0}^{\infty} a_n x^n$ a power series and $\rho = \frac{1}{\lim_n \sup \sqrt[n]{|a_n|}}$.

Then,

- *If $\rho = 0$ the series converges only for $x = 0$.*
- *If $\rho > 0$:*
  - *For $|x| < \rho$ the series is absolutely convergent.*
  - *For $|x| > \rho$ the series diverges.*
  - *For $|x| = \rho$ nothing can be said (there are cases in which converges and cases in which it doesn't).*

ρ is called the **radius of convergence** *of the series.*

Let $r = (r_n)_{n\geq 0}$ be a sequence of real numbers,
$\mathcal{S}_r$ = collection of subsequences of r with limit (finite or infinite).
Let $\mathcal{L}$ = collection of the limits of the subsequences in $\mathcal{S}_r$.
It turns out that $\lim_n \sup r_n = \max \mathcal{L}$ and $\lim_n \inf r_n = \min \mathcal{L}$.

Example

$r = (1, 0, 1, 0, \ldots)$, $r_n = \left\{ \begin{array}{cc} 0 & \text{if } n \text{ odd} \\ 1 & \text{if } n \text{ even} \end{array} \right\} = \frac{1+(-1)^n}{2}$
$\mathcal{L} = \{0, 1\}$ and $\lim_n \sup r_n = 1$, $\lim_n \inf r_n = 0$.

3.1 Generating functions

3.1.1 Power series

Proposition

- *Assume that* $r = (r_n)_{n\geq 0}$ *has limit,* $l = \lim_n r_n$. *Then every subsequence has limit* l. *Therefore* $\mathcal{L} = \{l\}$ *and* $\lim_n \sup r_n = l$, $\lim_n \inf r_n = l$.
- $\lim_n \inf(-r_n) = -\lim_n \sup r_n$ *and* $\lim_n \sup(-r_n) = -\lim_n \inf r_n$.
- *If* $r_n \leq r'_n$ *for all* n *then* $\lim_n \inf r_n \leq \lim_n \inf r'_n$ *and* $\lim_n \sup r_n \leq \lim_n \sup r'_n$.
- *If* $\lim_n |\frac{r_n}{r_{n-1}}| = l$ *then* $\lim \sqrt[n]{|r_n|} = l$.

Example

Let $a_n = \alpha^n, n \in \mathbb{N}^*, \alpha \in \mathbb{C}$.
Let $\sum_{n=0}^{\infty} (\alpha x)^n$.
$\rho = \frac{1}{\lim_n \sup \sqrt[n]{|\alpha^n|}} = \frac{1}{\lim_n \sup |\alpha|} = \frac{1}{|\alpha|}$.

- If $\alpha = 1$ then $a_n = 1$ for all n and the series is $1 + x + x^2 + \cdots$, which is the geometric series of ratio x and converges for $|x| < 1$ $(\rho = 1)$.
- If $\alpha = 0$ (but $a_0 = 1$) then the series is $1 + 0x + 0x^2 + \cdots = 1$, which converges for all x $(\rho = \frac{1}{0} = \infty)$.

Example

Let $a_n = \binom{\alpha}{n}$, $n \in \mathbb{N}^*$, $\alpha \in \mathbb{R} \setminus \mathbb{N}^*$.
Notice that $\left|\frac{a_n}{a_{n-1}}\right| = \left|\frac{\binom{\alpha}{n}}{\binom{\alpha}{n-1}}\right| = \frac{|\alpha - n + 1|}{n}$.
$\lim_n \left|\frac{a_n}{a_{n-1}}\right| = \lim_n \frac{|\alpha-n+1|}{n} = 1$ and, therefore, $\lim_n \sqrt[n]{|a_n|} = 1$.
Thus, $\lim_n \sup \sqrt[n]{|a_n|} = 1$ and $\rho = 1$.
Hence,

$$\sum_{n=0}^{\infty} \binom{\alpha}{n} x^n \text{ converges for } |x| < 1.$$

Furthermore, if $\alpha \in \mathbb{N}^*$, for example $\alpha = m$:
$\binom{m}{0}, \binom{m}{1}, \ldots, \binom{m}{m}, 0, 0, \ldots$.
$\lim_n \sqrt[n]{|a_n|} = 0$ and $\rho = \infty$.

$$\sum_{n=0}^{m} \binom{m}{n} x^n \text{ converges for all } x.$$

3.1 Generating functions

3.1.2 Generating function of a sequence of numbers

Definition

Let $a = (a_k)_{k \geq 0} = (a_0, a_1, a_2, \ldots)$ be a sequence of real (complex) numbers. The **generating function of** a is

$$g_a(x) = \sum_{k=0}^{\infty} a_k x^k = a_0 + a_1 x + a_2 x^2 + \cdots$$

in the real (complex) x in which the series converges.

3.1 Generating functions

3.1.3 Examples (direct problems)

Example

$$a = (C_{n,k})_{k \geq 0} = (C_{n,0}, C_{n,1}, C_{n,2}, \ldots, C_{n,n}, C_{n,n+1}, C_{n,n+2}, \ldots) =$$

$$= \left(\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n}, \binom{n}{n+1}, \binom{n}{n+2}, \ldots \right)$$

The generating function is

$$g_a(x) = \sum_{k=0}^{\infty} C_{n,k} x^k = \sum_{k=0}^{n} C_{n,k} x^k = \sum_{k=0}^{n} \binom{n}{k} x^k = (1+x)^n, \quad x \in \mathbb{R}.$$

Example

$$a = (CR_{n,k})_{k \geq 0}$$

$$g_a(x) = \sum_{k=0}^{\infty} CR_{n,k} x^k = \sum_{k=0}^{\infty} \binom{n+k-1}{k} x^k = \sum_{k=0}^{\infty} (-1)^k \binom{-n}{k} x^k = (1-x)^{-n}, \quad |x| < 1.$$

Example

$\alpha \in \mathbb{R} \setminus \mathbb{N}^*$

$$a = \left(\binom{\alpha}{k} \right)_{k \geq 0}$$

$$g_a(x) = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k = (1+x)^\alpha, \quad |x| < 1.$$

Example

$$a = (1, 1, 1, \dots) \Rightarrow a_n = 1 \quad \forall n$$

$$g_a(x) = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \cdots = \frac{1}{1-x}, \quad |x| < 1.$$

3.1 Generating functions

3.1.3 Examples (direct problems)

Example

$$a = (1, 0, 1, 0, 1, 0, \dots) \Rightarrow a_n = \begin{cases} 1 & n \equiv 0 \pmod 2 \\ 0 & n \equiv 1 \pmod 2 \end{cases} \Rightarrow a_n = \frac{(-1)^n + 1}{2}$$

$$g_a(x) = 1 + x^2 + x^4 + \cdots = \frac{1}{1-x^2}, \quad |x^2| < 1 \quad (\Leftrightarrow |x| < 1).$$

Example

$$a = (1, 0, 0, 1, 0, 0, 1, 0, 0, \dots) \Rightarrow a_n = \begin{cases} 1 & n \equiv 0 \pmod 3 \\ 0 & \text{otherwise} \end{cases}$$

$$g_a(x) = 1 + x^3 + x^6 + \cdots = \frac{1}{1-x^3}, \quad |x^3| < 1 \quad (\Leftrightarrow |x| < 1).$$

3.1 Generating functions

3.1.3 Examples (direct problems)

Example

$$a = (1, 5, 5^2, 5^3, 5^4, \ldots) \Rightarrow a_n = 5^n = VR_{5,n}$$

$$g_a(x) = 1 + 5x + 5^2x^2 + 5^3x^3 + \cdots = 1 + 5x + (5x)^2 + (5x)^3 + \cdots = \frac{1}{1-5x}, \quad |x| < \frac{1}{5}.$$

Example

$$a = (1, r, r^2, r^3, r^4, \ldots) \Rightarrow a_n = r^n = VR_{r,n}, r \neq 0$$

$$g_a(x) = 1 + rx + r^2x^2 + r^3x^3 + \cdots = 1 + rx + (rx)^2 + (rx)^3 + \cdots = \frac{1}{1-rx}, \quad |x| < \frac{1}{r}.$$

Example

$$a = (a_0, a_1, \ldots, a_n, 0, 0, \ldots)$$

$$g_a(x) = a_0 + a_1x + \cdots + a_nx^n + 0x^{n+1} + 0x^{n+2} + \cdots = a_0 + a_1x + \cdots + a_nx^n, \quad \forall x \in \mathbb{R}.$$

Example

$$a = \left(\frac{1}{0!}, \frac{1}{1!}, \frac{1}{2!}, \frac{1}{3!}, \ldots\right) \Rightarrow a_n = \frac{1}{n!}$$

$$g_a(x) = \frac{1}{0!} + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots = e^x, \quad \forall x \in \mathbb{R}.$$

3.1 Generating functions

3.1.4 Examples (inverse problems)

Example

$\alpha \in \mathbb{R} \setminus \mathbb{N}^*$

$$g_a(x) = (1+x^m)^\alpha, \quad |x| < 1.$$

$$g_a(x) = (1+x^m)^\alpha = \binom{\alpha}{0} + \binom{\alpha}{1} x^m + \binom{\alpha}{2} x^{2m} + \binom{\alpha}{3} x^{3m} + \cdots .$$

Thus,

$$a_n = \begin{cases} \binom{\alpha}{\frac{n}{m}} & n \equiv 0 \pmod m \\ 0 & \text{otherwise} \end{cases}$$

Example

$$g_a(x) = e^{-x^2} = \sum_{k=0}^{\infty} \frac{(-x^2)^k}{k!} = 1 - \frac{1}{1!}x^2 + \frac{1}{2!}x^4 - \frac{1}{3!}x^6 + \cdots .$$

Thus,

$$a_n = \begin{cases} (-1)^{\frac{n}{2}} \frac{1}{(\frac{n}{2})!} & n \equiv 0 \pmod 2 \\ 0 & \text{otherwise} \end{cases}$$

3.1 Generating functions

3.1.4 Examples (inverse problems)

Example

$$g_a(x) = \frac{\alpha}{\beta + \gamma x}, \quad \alpha, \beta, \gamma \neq 0, \quad |x| < \left|\frac{\beta}{\gamma}\right| \quad \text{(geometric series)}.$$

Thus,

$$a_n = (-1)^n \frac{\alpha}{\beta} \left(\frac{\gamma}{\beta}\right)^n = (-1)^n \frac{\alpha \gamma^n}{\beta^{n+1}}.$$

Example

$$g_a(x) = \frac{3}{x^2 - 2x + 1}, \quad |x| < 1.$$

Thus,

$$a = (3, 6, 9, \ldots).$$

3.1 Generating functions

3.1.4 Examples (inverse problems)

Example

$$g_a(x) = \frac{\alpha x + \beta}{x^2 + x - 6}, \quad |x| < 2.$$

$$g_a(x) = \frac{\alpha x + \beta}{(x-2)(x+3)} = \frac{A}{x-2} + \frac{B}{x+3}.$$

Thus,

$$a_k = -\frac{A}{2^{k+1}} + \frac{(-1)^k B}{3^{k+1}}.$$

3.1 Generating functions

3.1.5 Operations with generating functions

Proposition

1. $g_{ca}(x) = c g_a(x), \quad ca = (ca_0, ca_1, \dots).$
2. $g_{a+b}(x) = g_a(x) + g_b(x).$
3. $g_a(x) g_b(x) = g_c(x), \quad c_k = a_0 b_k + a_1 b_{k-1} + \cdots + a_k b_0, \quad k = 0, 1, 2, \dots.$

Definition

In the last item of the previous proposition c is the **convolution of a and b**, which is written $c = a * b$.

Example

Let $a = (a_k)_{k \geq 0}$ with generating function $g_a(x)$ defined for $|x| < \rho$.
Let $c_k = a_0 + a_1 + \cdots + a_k$, $k = 0, 1, 2, \dots$.
Then, $c = a * b$ with $b = (1, 1, 1, \dots)$.
Thus, $g_c(x) = g_a(x) g_b(x) = g_a(x)(1 + x + x^2 + \cdots) = \frac{g_a(x)}{1-x}$, $|x| < \min\{1, \rho\}$.

3.1 Generating functions

3.1.5 Operations with generating functions

Example

Let $d_k = 2^k a_0 + 2^{k-1} a_1 + \cdots + 2^0 a_k$, $k = 0, 1, 2, \ldots$.
Then $d = a * b'$ with $b' = (2^0, 2^1, 2^2, \ldots)$.
Thus, $g_{b'}(x) = 1 + 2x + (2x)^2 + \cdots = (1-2x)^{-1}$ and $g_d(x) = \frac{g_a(x)}{1-2x}$, $|x| < \min\{1/2, \rho\}$.

Example

Let $r_k = \binom{\alpha}{0} + \binom{\alpha}{1} + \cdots + \binom{\alpha}{k}$.

Then $r = a * b$ with $a = \left(\binom{\alpha}{n}\right)_{n \geq 0}$ and $b = (1, 1, \ldots)$.

Hence, $g_r(x) = \frac{g_a(x)}{1-x} = \frac{(1+x)^\alpha}{1-x}$, $|x| < 1$.

Proposition (Derivatives)

*Let $\rho > 0$ and $g_a(x) = \sum_{k=0}^{\infty} a_k x^k$, $|x| < \rho$.
Then g_a is indefinitely differentiable in the interval of convergence and $g_a^{(r)}(x) = \sum_{k=r}^{\infty} k^{\underline{r}} a_k x^{k-r}$, $r = 0, 1, 2, \ldots$.
Taking $x = 0$, $g_a^{(r)}(0) = r^{\underline{r}} a_r$ and $a_r = \frac{g_a^{(r)}(0)}{r!}$, $r = 0, 1, 2, \ldots$.*

Remark

If g_a is the generating function of $a = (a_0, a_1, a_2, \ldots)$ then

$$g_a^{(r)}(x) = r^{\underline{r}} a_r + (r+1)^{\underline{r}} a_{r+1} x + (r+2)^{\underline{r}} a_{r+2} x^2 + \cdots .$$

This is the generating function of $a^{(r)} = (a_0^{(r)}, a_1^{(r)}, \ldots)$ with $a_k^{(r)} = (r+k)^{\underline{r}} a_{r+k}$.

3.2 Generating functions and combinatorial problems

3.2.1 Number of solutions of an equation

Theorem

Let $B = \{b_1, b_2, \ldots\}, C = \{c_1, c_2, \ldots\}, \ldots, H = \{h_1, h_2, \ldots\}$ be k subsets of $\mathbb{N} \cup \{0\}$.
Let a_n be the number of solutions of $x_1 + x_2 + \cdots + x_k = n$ where
$x_1 \in B, x_2 \in C, \ldots, x_k \in H$.
Then the generating function of $(a_n)_{n\geq 0}$ is

$$g_a(x) = \sum_{n=0}^{\infty} a_n x^n = (x^{b_1} + x^{b_2} + \cdots)(x^{c_1} + x^{c_2} + \cdots)\cdots(x^{h_1} + x^{h_2} + \cdots).$$

Remark

If $|x| < 1$

$$|x|^{b_1} + |x|^{b_2} + \cdots \leq 1 + |x| + |x|^2 + \cdots + |x|^{b_1} + |x|^{b_1+1} + \cdots + |x|^{b_2} + |x|^{b_2+1} + \cdots = \frac{1}{1-|x|}.$$

Hence, $x^{b_1} + x^{b_2} + \cdots$ is absolutely convergent for $|x| < 1$. In the same way, $x^{c_1} + x^{c_2} + \cdots$ and $x^{h_1} + x^{h_2} + \cdots$ are absolutely convergent for $|x| < 1$. Thus, $g_a(x)$ is well defined for $|x| < 1$.

3.2 Generating functions and combinatorial problems

3.2.2 Applications and examples

Example

Let a_n be the number of solutions of $x_1 + \cdots + x_k = n$ where
$x_1 \in \{0, 1, 2, \ldots\}, \ldots, x_k \in \{0, 1, 2, \ldots\}$.
The generating function of $a = (a_n)_{n\geq 0}$ is

$$g_a(x) = (x^0 + x^1 + x^2 + \cdots)(x^0 + x^1 + x^2 + \cdots)\cdots(x^0 + x^1 + x^2 + \cdots) =$$

$$= (x^0 + x^1 + x^2 + \cdots)^k =$$

$$= \left(\frac{1}{1-x}\right)^k = (1-x)^{-k} = \sum_{n=0}^{\infty}(-1)^n \binom{-k}{n} x^n, |x| < 1.$$

Thus,

$$a_n = (-1)^n \binom{-k}{n} = (-1)^n(-1)^n \binom{k+n-1}{n} = \binom{k+n-1}{n} = CR_{k,n}.$$

3.2 Generating functions and combinatorial problems

3.2.2 Applications and examples

Example

Let a_n be the number of ways of gathering n euros with coins or bills of 1 euro, 2 euros and 5 euros.
Put x_1 = number of coins of 1 euro, x_2 = number of coins of 2 euros and x_3 = number of bills of 5 euros.
Then a_n is the number of solutions of $x_1 + 2x_2 + 5x_3 = n$ where $x_1, x_2, x_3 \in \mathbb{N} \cup \{0\}$ or a_n is the number of solutions of $y_1 + y_2 + y_3 = n$ where $y_1 \in \{0, 1, 2, \ldots\}, y_2 \in \{0, 2, 4, \ldots\}, y_3 \in \{0, 5, 10, \ldots\}$.
The generating function of $(a_n)_{n\geq 0}$ is

$$g_a(x) = (x^0 + x^1 + x^2 + \cdots)(x^0 + x^2 + x^4 + \cdots)(x^0 + x^5 + x^{10} + \cdots) =$$

$$= \frac{1}{1-x}\frac{1}{1-x^2}\frac{1}{1-x^5} = \frac{1}{(1+x)(1-x)^2(1-x^5)} =$$

$$= \frac{A}{1+x} + \frac{B}{1-x} + \frac{C}{(1-x)^2} + \frac{Dx^4 + Ex^3 + Fx^2 + Gx + H}{1-x^5}, |x| < 1.$$

Now it can be worked out in order to obtain a_n.

Example

A die is rolled 10 times. Determine the probability that the outcome is 30 points.

Example

In how many ways can a student get a total of 18 points with 6 tasks where students are awarded 0, 2 or 4 points for each task?

3.3 Recurrence relations

3.3.1 Combinatorial problems and recurrence relations. Examples

Example

Let $a_n =$ number of subsets of $\{1, 2, \ldots, n\}$.
Its recurrence relation is:
$a_0 = 1, a_n = 2a_{n-1},\ \ n \geq 1.$

Example

Let $a_n =$ number of sequences of length n formed with 0s and 1s where there are not two consecutive 0s.
Its recurrence relation is:
$a_0 = 1, a_1 = 2, a_n = a_{n-1} + a_{n-2},\ \ n \geq 2.$

Example

Let $a_n =$ number of ways of tiling an $n \times 2$ floor with 1×2 and 2×2 tiles.
Its recurrence relation is:
$a_0 = 1, a_1 = 1, a_n = a_{n-1} + 2a_{n-2},\ \ n \geq 2.$

3.3 Recurrence relations

3.3.1 Combinatorial problems and recurrence relations. Examples

Example

Let $a_n =$ number of subsets of $\{1, 2, \ldots, n\}$ in which there are not two consecutive integers.
Its recurrecne relation is:
$a_0 = 1, a_1 = 2, a_n = a_{n-1} + a_{n-2},\ \ n \geq 2.$

Example

Let $a_n =$ number of ways of climbing a stair of n steps if in each move we climb 1 or 2 steps.
Its recurrence relation is:
$a_0 = 1, a_1 = 1, a_n = a_{n-1} + a_{n-2},\ \ n \geq 2.$

3.3 Recurrence relations

3.3.1 Combinatorial problems and recurrence relations. Examples

Example (Tower of Brahma or Tower of Hanoi)

The Tower of Hanoi (also called the Tower of Brahma or Lucas' Tower) is a mathematical game or puzzle. It consists of three rods, and a number of disks of different sizes which can slide onto any rod. The puzzle starts with the disks in a neat stack in ascending order of size on one rod, the smallest at the top, thus making a conical shape.
The objective of the puzzle is to move the entire stack to another rod, obeying the following rules:

- Only one disk may be moved at a time.
- Each move consists of taking the upper disk from one of the rods and sliding it onto another rod, on top of the other disks that may already be present on that rod.
- No disk may be placed on top of a smaller disk.

Let $a_n =$ the minimum number of moves to solve the problem when there are n disks.
$a_1 = 1, a_n = 2a_{n-1} + 1,\ \ n \geq 2.$

3.3 Recurrence relations

3.3.1 Combinatorial problems and recurrence relations. Examples

Example

Let $a_n =$ number of regions in the plane that are determined by n lines that are concurrent in pairs but there are not three that intersect at a single point.
Its recurrence relation is:
$a_0 = 1, a_1 = 2, a_2 = 4, a_n = a_{n-1} + n,\ \ n \geq 3.$

Example

Let $a_n =$ number of maps $f : \{1, \ldots, n\} \to \{1, \ldots, n\}$ such that $f \circ f = identity$.
Notice that:

- $f(1) = 1$, there are a_{n-1}
- $f(1) = 2 \Rightarrow f(2) = 1$, there are a_{n-2}
- $\vdots$
- $f(1) = n \Rightarrow f(n) = 1$, there are a_{n-2}

Therefore, its recurrence relation is:

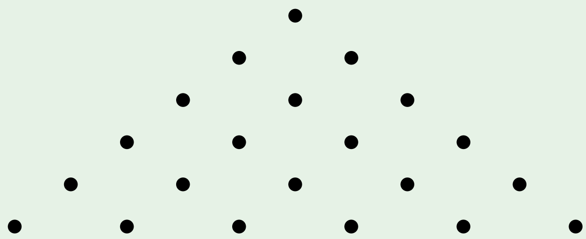
$$a_1 = 1, a_2 = 2, a_n = a_{n-1} + (n-1)a_{n-2}, \quad n \geq 3.$$

3.3 Recurrence relations

3.3.1 Combinatorial problems and recurrence relations. Examples

Example

Let $T_n =$ number of points that are necessary to build a triangle of n rows as the following:



$$\begin{aligned} T_1 &= 1 \\ T_2 &= 3 \\ T_3 &= 6 \\ T_4 &= 10 \\ T_5 &= 15 \\ T_6 &= 21 \end{aligned}$$

Its recurrence relation is:

$$T_0 = 0,\, T_n = T_{n-1} + n, \;\; n \geq 1.$$

3.3 Recurrence relations

3.3.2 Solving recurrences

When we have a recurrence relation, can we give an explicit formula for the general term?

Example

We saw that $T_0 = 0,\, T_n = T_{n-1} + n, \;\; n \geq 1.$
Therefore,

$$\begin{aligned} T_0 &= 0 \\ T_1 &= 0 + 1 = 1 \\ T_2 &= 0 + 1 + 2 = 3 \\ T_3 &= 0 + 1 + 2 + 3 = 6 \\ T_4 &= 0 + 1 + 2 + 3 + 4 = 10 \\ &\vdots \end{aligned}$$

Thus, $T_n = 0 + 1 + 2 + \cdots + (n-1) + n$.
Since this can be written also as $T_n = n + (n-1) + (n-2) + \cdots + 1 + 0$, $2T_n = n(n+1)$ and

$$T_n = \frac{n(n+1)}{2}, \;\; n \geq 0.$$

Example

Suppose we are given the following recurrence relation $a_0 = a$, $a_n = \alpha a_{n-1} + \beta, n \geq 1$.

- If $\alpha = 1$ then $a_n = a_{n-1} + \beta$ is an arithmetic progression with common difference β.
- If $\beta = 0$ then $a_n = \alpha a_{n-1}$ is a geometric progression with common ratio α.

Find the explicit formula for a_n:
$a_0 = a$,
$a_1 = \alpha a + \beta$,
$a_2 = \alpha(\alpha a + \beta) + \beta = \alpha^2 a + \beta(1 + \alpha)$,
$a_3 = \alpha(\alpha^2 a + \beta(1 + \alpha)) + \beta = \alpha^3 a + \beta(1 + \alpha + \alpha^2)$.
By induction on n, $a_n = \alpha^n a + \beta(1 + \alpha + \alpha^2 + \cdots + \alpha^{n-1}) = \begin{cases} \alpha^n a + \beta\frac{1-\alpha^n}{1-\alpha} & \alpha \neq 1 \\ a + n\beta & \alpha = 1 \end{cases}$.

Example

Let $a_0 = a, a_1 = b, a_n = \alpha a_{n-1} + \beta a_{n-2}, n \geq 2$, α, β constants and $\beta \neq 0$.
Find the explicit formula for a_n:
$a_2 = \alpha a_1 + \beta a_0 = \alpha b + \beta a$,
$a_3 = \alpha a_2 + \beta a_1 = \alpha(\alpha b + \beta a) + \beta b = b(\alpha^2 + \beta) + \alpha\beta a$,
$a_4 = \alpha a_3 + \beta a_2 = \alpha(b(\alpha^2 + \beta) + \alpha\beta a) + \beta(\alpha b + \beta a) = b(\alpha^3 + 2\beta\alpha) + a(\alpha^2\beta + \beta^2)$.
It is not as simple as the previous example. We have to find another method.

3.3 Recurrence relations

3.3.2 Generating functions and recurrence relations

Method: given the recurrence relation find the generating function in order to obtain the explicit formula for a_n.

Example

Let $a_0 = a$, $a_n = \alpha a_{n-1} + \beta, n \geq 1$.
Find the explicit formula for a_n:
Let $g(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$ be the generating function of $(a_n)_{n\geq 0}$.
Suppose $|x| < 1$.

$$g(x) = a_0 + (\alpha a_0 + \beta)x + (\alpha a_1 + \beta)x^2 + (\alpha a_2 + \beta)x^3 + \cdots =$$

$$= a_0 + \alpha x(a_0 + a_1 x + a_2 x^2 + \cdots) + \beta x(1 + x + x^2 + \cdots) = a_0 + \alpha x g(x) + \beta x \frac{1}{1-x}.$$

Thus,

$$g(x) = \frac{a_0 + \beta \frac{x}{1-x}}{1 - \alpha x} = \frac{a}{1-\alpha x} + \frac{\beta x}{(1-x)(1-\alpha x)}.$$

Solving for a_n:
If $\alpha = 1$, $a_n = a + n\beta$, $n \geq 0$.

If $\alpha \neq 1$, $a_n = a\alpha^n + \frac{\beta(1-\alpha^n)}{1-\alpha}$, $n \geq 0$.

3.3 Recurrence relations

3.3.2 Generating functions and recurrence relations

Example

Let $a_0 = a, a_1 = b, a_n = \alpha a_{n-1} + \beta a_{n-2}, n \geq 2$, α, β constants and $\beta \neq 0$.
Find the explicit formula for a_n:
Let $g(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$ be the generating function of $(a_n)_{n\geq 0}$.

$$g(x) = a_0 + a_1 x + (\alpha a_1 + \beta a_0)x^2 + (\alpha a_2 + \beta a_1)x^3 + \cdots =$$

$$= a_0 + a_1 x + \alpha x(a_1 x + a_2 x^2 + \cdots) + \beta x^2(a_0 + a_1 x + \cdots) = a_0 + a_1 x + \alpha x(g(x) - a_0) + \beta x^2 g(x).$$

Thus,

$$g(x) = \frac{a_0 + (a_1 - \alpha a_0)x}{1 - \alpha x - \beta x^2}.$$

Solving for a_n when $\alpha = 1, \beta = 2$:
$a_0 = 1, a_1 = 1$ then $a_n = \frac{(-1)^n + 2^{n+1}}{3}$, $n \geq 0$.

3.3 Recurrence relations

3.3.2 Generating functions and recurrence relations

Example

$a_0 = a, a_1 = b, a_2 = c, a_n = \alpha a_{n-1} + \beta a_{n-2} + \gamma a_{n-3}, n \geq 3, \alpha, \beta, \gamma \in \mathbb{Z}, \gamma \neq 0.$
Thus,

$$g(x) = \frac{a_0 + a_1 x + a_2 x^2 + \alpha x(-a_0 - a_1 x) + \beta x^2(-a_0)}{1 - \alpha x - \beta x^2 - \gamma x^3} =$$

$$= \frac{a + (b - \alpha a)x + (c - b\alpha - a\beta)x^2}{1 - \alpha x - \beta x^2 - \gamma x^3}.$$

Example

$a_0 = a, a_1 = b, a_n = \alpha a_{n-1} + \beta a_{n-2} + \gamma, n \geq 2.$ Thus,

$$g(x) = \frac{a_0 + (a_1 - \alpha a_0)x + \gamma \frac{x^2}{1-x}}{1 - \alpha x - \beta x^2}.$$

3.3 Recurrence relations

3.3.3 Other methods (algebraic method)

Let $a_0 = a, a_1 = b, a_n = \alpha a_{n-1} + \beta a_{n-2}, n \geq 2.$

- $S = \{(x_n)_{n \geq 0} : x_n = \alpha x_{n-1} + \beta x_{n-2} \forall n \geq 2\}$. S is a vector space over $\mathbb{C}$.
- $s_1 = (1, 0, \beta, \dots) \in S$, $s_2 = (0, 1, \alpha, \dots) \in S$. It turns out that $\{s_1, s_2\}$ is a basis for S. Therefore, $\dim S = 2$. Now we look for bases of S with known general term.
- **Recurrence characteristic equation**: $x^2 = \alpha x + \beta$ or $x^2 - \alpha x - \beta = 0$. If r is a root of this equation then the progression $(r^n)_{n \geq 0} \in S$, because $r^n - \alpha r^{n-1} - \beta r^{n-2} = r^{n-2}(r^2 - \alpha r - \beta) = r^{n-2} \cdot 0 = 0$.
-
  - If the characteristic equation has two different roots r_1 and r_2, that is, $\alpha^2 + 4\beta \neq 0$, then $(r_1^n)_{n \geq 0}, (r_2^n)_{n \geq 0} \in S$ and it is easy to see that they are linearly independent. Thus, they form a basis of S. Therefore, any sequence of S can be uniquely formed as a linear combination of them. We get $a_n = \lambda r_1^n + \mu r_2^n$ such that $a = \lambda + \mu$ and $b = \lambda r_1 + \mu r_2$.
  - If the characteristic equation has a double root $r = \frac{\alpha}{2}$ then it is easy to see that $(nr^n)_{n \geq 0} \in S$ and that it is linearly independent of $(r^n)_{n \geq 0}$. Therefore they form a basis for S. Thus, $a_n = \lambda r^n + \mu n r^n$ such that $a = \lambda$ and $b = \lambda r + \mu r = (\lambda + \mu)r$.

3.3 Recurrence relations

3.3.3 Other methods (algebraic method)

Example

$a_0 = 0, a_1 = 1, a_n = a_{n-1} + a_{n-2}, n \geq 2.$
The recurrence characteristic equation is $x^2 - x - 1 = 0$.
The roots are $x_1 = \frac{1+\sqrt{5}}{2}$ and $x_2 = \frac{1-\sqrt{5}}{2}$.
Thus, $a_n = \lambda \left(\frac{1+\sqrt{5}}{2}\right)^n + \mu \left(\frac{1-\sqrt{5}}{2}\right)^n$ with $0 = \lambda + \mu$ and
$1 = \lambda \frac{1+\sqrt{5}}{2} + \mu \frac{1-\sqrt{5}}{2}$.
Working it out, $\lambda = \frac{1}{\sqrt{5}}$ and $\mu = -\frac{1}{\sqrt{5}}$.
Hence, $a_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2}\right)^n$.