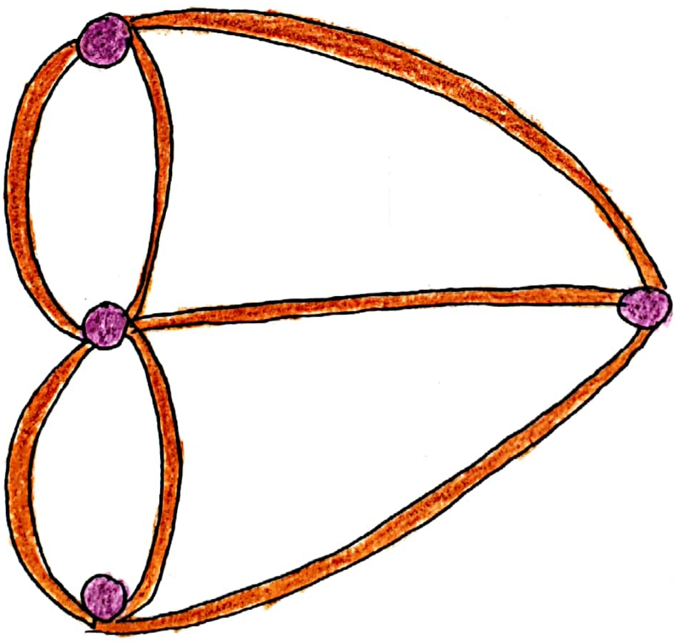
DISCRETAS



Puentes de könisberg

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Chapter 1

Basic Combinatorics

Combinatorial theory is the study of methods of counting:

- objects that fulfil a given description.
- how many ways can we do something.
- how many ways can a certain event occur.

We will work with sets:

$$A = \{x : x \text{ satisfies property } P\}$$

Our aim will be the number of elements of A, its cardinality.

$$|A| \quad \text{or} \quad \text{Card}(A) \quad \text{or} \quad \# A$$

Note:

$\mathbb{N} = \{1, 2, 3, \dots\}$, natural numbers.

$\mathbb{N}^* = \{0, 1, 2, \dots\}$, non-negative integers. $\mathbb{N}^* = \mathbb{N} \cup \{0\}$

A first idea is to "count by hand" and make lists:

$$A = \{(x,y) \in \mathbb{N}^* \times \mathbb{N}^* : x+y=5\}$$

List: $\{(0,5),(1,4),(2,3),(3,2),(4,1),(5,0)\}$ $(=A)$

$\Rightarrow |A| = 6$

A generalization to this problem:

$$A = \{(x,y) \in \mathbb{N}^* \times \mathbb{N}^* : x+y=n,\ n \in \mathbb{N}^*\}$$

List: $\{(0,n),(1,n-1),\ldots,(n-1,1),(n,0)\}$ $(=A)$

$\Rightarrow |A| = n+1$

Variation of the problem:

$$A = \{(x,y) \in \mathbb{N}^* \times \mathbb{N}^* : x+y=n,\ n \in \mathbb{N}^*,\ x,y \geq 2\}$$

If $n \leq 3$, $x+y \geq 4 \Rightarrow |A| = 0$

If $n \geq 4$, list: $\{(2,n-2),(3,n-3),\ldots,(n-2,2)\}$ $(=A)$

$\Rightarrow |A| = n-3$

Another variation, with triples instead of tuples.

$A = \{(x,y,z) \in \mathbb{N}^* \times \mathbb{N}^* \times \mathbb{N}^* : x+y+z = n,\ n \in \mathbb{N}^*\}$

List :

x	0	0		0	1	1		1	
y	0	1	$\dots$	n	0	1	$\dots$	$n-1$	$\dots$
z	n	$n-1$		0	$n-1$	$n-2$		0	
		$n+1$				n			

x		k	k		k		$n-1$	$n-1$	n
y	$\dots$	0	1	$\dots$	$n-k$	$\dots$	0	1	0
z		$n-k$	$n-k-1$		0		1	0	0
			$n-k+1$				2		1

(Each column is a possible triple. We create all of them by varing every possible x with every (y,z) that fulfils $x+y+z=n$ for that x (previous case more or less)).

$$|A| = (n+1) + n + \dots + (n-k+1) + \dots + 2 + 1$$

$$|A| = 1 + 2 + \dots + (k+1) + \dots + n + n+1$$

$$2|A| = (n+2) + (n+2) + \dots + (n+2) + \dots + (n+2) + (n+2)$$

$= (n+1)(n+2) \Rightarrow$

$$\Rightarrow |A| = \frac{(n+1)(n+2)}{2}$$

Suppose a general case:

$$A = \{(x_1, \dots, x_m) \in \mathbb{N}^* \times \dots \times \mathbb{N}^* : x_1 + \dots + x_m = n \in \mathbb{N}\}$$

Note a list cannot be used for this case; remark:

- Building the list may be easy, less easy or impracticable.
- A list cannot have repetitions or absences.
- Before trying to solve a problem it may be convenient to analyze a particular case.

Example:

$A = \{x : 1 \leq x \leq 1871,\ x \equiv 0 \bmod (4)\} \to ¿|A|?$

1 2 3 4 5 6 7 8 9 ... 1867 1868 1869 1870 1871

$$\begin{array}{l|l} 1871 & 4 \\ \hline 27 & 467 \\ \ \ 31 & \\ \ \ \ 3 & \end{array} \rightarrow 1871 = \underbrace{4 \cdot 467}_{1868} + 3$$

$\Rightarrow |A| = 467$

A more generalized problem:

$A_n = \{x : 1 \le x \le n,\ x \equiv 0 \bmod (4)\} \rightarrow ¿|A|?$

$n = \underbrace{c}_{\text{quotient}} \cdot 4 + \underbrace{r}_{\text{reminder}} \longrightarrow |A| = c = \lfloor n/4 \rfloor$ (floor function)

Even more:

$A(a,n) = \{x : 1 \le x \le n,\ x \equiv 0 \bmod (a)\} \rightarrow ¿|A|?$

$|A| = \lfloor n/a \rfloor$

<u>Definition:</u> (Floor function)

$\lfloor x \rfloor$ = largest integer not greater than x $\Rightarrow$

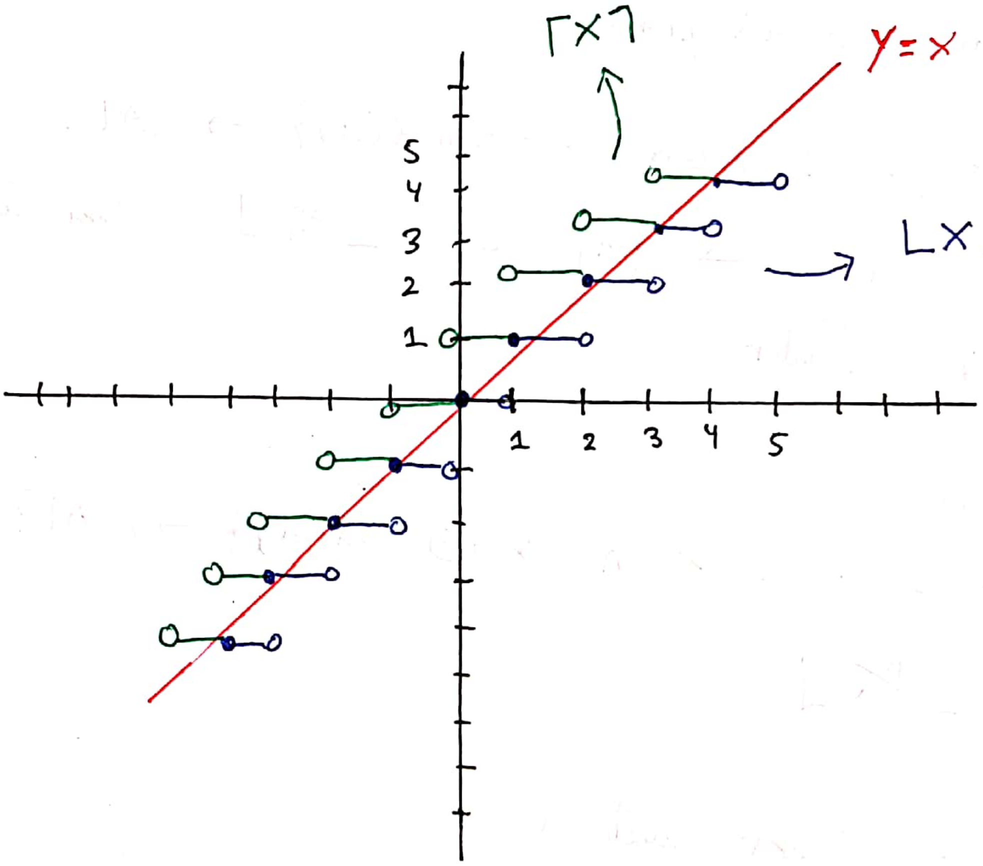
$$\begin{cases} \lfloor x \rfloor \in \mathbb{Z}, \quad x \in \mathbb{R} \\ \lfloor x \rfloor \le x \le \lfloor x \rfloor + 1 \end{cases}$$

Definition: (Ceiling function)

$\lceil x \rceil$ = smallest integer greater than x $\Rightarrow$

$\lceil x \rceil \in \mathbb{Z}, \quad x \in \mathbb{R}$

$\lceil x \rceil \ge x \ge \lceil x \rceil - 1$



Properties of these functions:

(i) $\lceil x \rceil = \lfloor x \rfloor \iff x \in \mathbb{Z}$ $(x = \lceil x \rceil = \lfloor x \rfloor \iff x \in \mathbb{Z})$

(ii) $\lceil x \rceil = \lfloor x \rfloor + 1 \iff x \notin \mathbb{Z}$

(iii) $\lfloor x \rfloor \leq x \leq \lceil x \rceil$

(iv) $\lfloor -x \rfloor = -\lceil x \rceil$

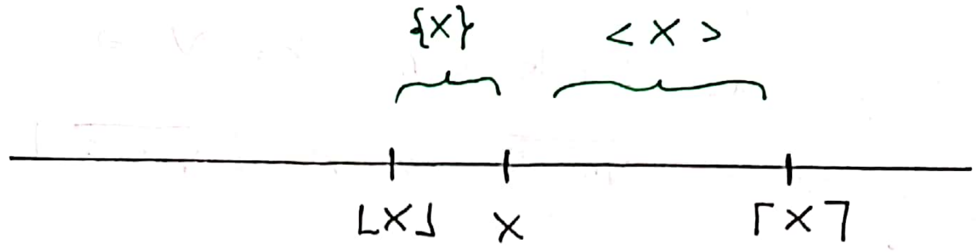
(v) $\lceil -x \rceil = -\lfloor x \rfloor$

Definition: (fractional part of x)

$\{x\} = x - \lfloor x \rfloor \in [0, 1)$

Definition: (pseudofractional part of x)

$\langle x \rangle = \lceil x \rceil - x \in [0, 1)$



(If $x \in \mathbb{Z}$, $\{x\} = \langle x \rangle = 0$)

(If $x \in \mathbb{Q}, \mathbb{R} \Rightarrow \{x\} + \langle x \rangle = 1$)

Example:

$A(a,n) = \{x : 1 \le x \le n \mid x \equiv a \bmod (0)\}$

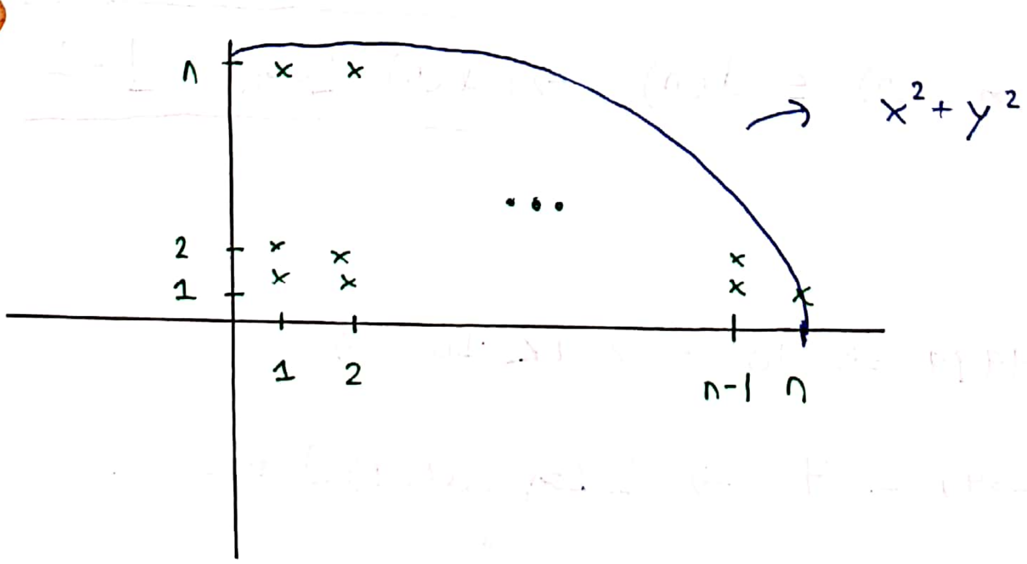
$|A(a,n)| = c = [\frac{n}{a}] = \lfloor n/a \rfloor$ $\qquad [0,1]$ ψ

$$\boxed{\lim_{n\to\infty} \frac{|A_{n,a}|}{n}} = \lim_{n\to\infty} \frac{\lfloor n/a \rfloor}{n} \overset{\lim}{=} \frac{n/a - \{n/a\}}{n} \overset{\lim}{=} \frac{1}{a} - \frac{\{n/a\}}{n} =$$

$$= \frac{1}{a} + 0 = \boxed{\frac{1}{a}} \qquad \left(\lim_{n\to\infty} \frac{x}{n} = 0, \quad x \in [0,1]\right)$$

Example:

$A_n = \{(x,y) \in \mathbb{N} \times \mathbb{N} \mid x^2 + y^2 \le n^2\} \to |A_n|$?



First column:

$(1,1), \dots, (1,k)$ where $1^2 + k^2 \le n^2$, $k \in \mathbb{N}$ $\Rightarrow$

$\Rightarrow k^2 \le n^2 - 1 \Rightarrow k \le \sqrt{n^2-1} \Rightarrow k = \lfloor \sqrt{n^2-1} \rfloor$

Second:

$(2,1), \dots, (2,k)$ where $2^2 + k^2 \le n^2 \Rightarrow \dots \Rightarrow k = \lfloor \sqrt{n^2 - 2^2} \rfloor$

$$\Rightarrow |A_n| = \sum_{j=1}^{n} \lfloor \sqrt{n^2 - j^2} \rfloor$$

Example:

$n \in \mathbb{N}$, number of digits radix - 10 ?

$n = 1234 \to d(n) = 4$
$n = 300 \to d(n) = 3$
$n = 78 \to d(n) = 2$
$n = x \in \{0, \dots, 9\} \to d(n) = 1$

$1000 \le abcd \le 9999 \Rightarrow 10^3 \le abcd < 10^4 \xrightarrow{\text{General}}$

$10^{d(n)-1} \le n < 10^{d(n)} \Rightarrow \log_{10}(10^{d(n)-1}) \le \log_{10}(n) <$

$< \log_{10}(10^{d(n)}) \Rightarrow$

$\Rightarrow d(n) - 1 \le \log_{10}(n) < d(n) \Rightarrow \boxed{d(n) = \lfloor \log_{10}(n) \rfloor + 1}$

Example:

$1000 \le 1234 \le 9999 \Rightarrow 10^3 \le \underset{\underset{10^{3'09\dots}}{\shortparallel}}{1234} < 10^4 \Rightarrow$

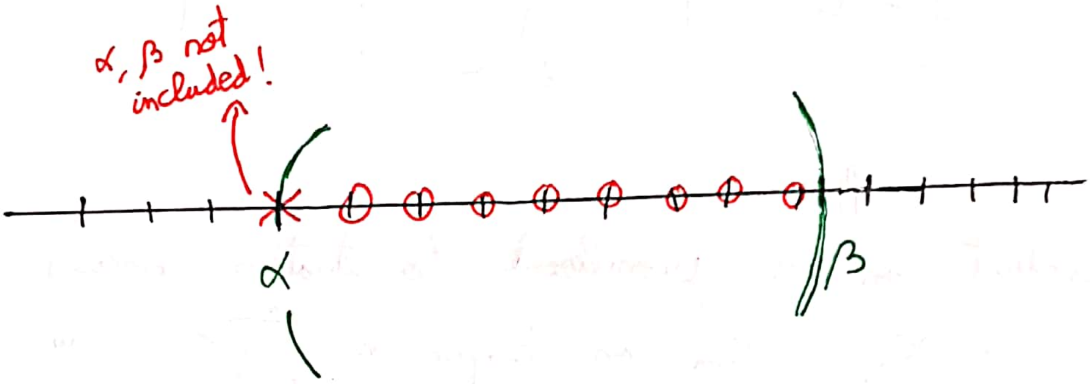
$\Rightarrow 3 \le \log_{10}(1234) < 4 \Rightarrow \underset{\underset{3}{\shortparallel}}{\lfloor \log_{10}(1234) \rfloor} + 1$

(We don't use $\lceil x \rceil$ because if $\log(x) \in \mathbb{Z} \Rightarrow \lfloor \log(x) \rfloor + 1 > d(n)$).

Example:

How many integers are there in the interval $(\alpha, \beta), \in \mathbb{R}^2$?

$$A(\alpha, \beta) = \{x \in \mathbb{Z} : x \in (\alpha, \beta)\}$$



$$A(\alpha, \beta) = \{\lfloor \alpha \rfloor + 1, \lfloor \alpha \rfloor + 2, \dots, \lceil \beta \rceil - 1\} =$$

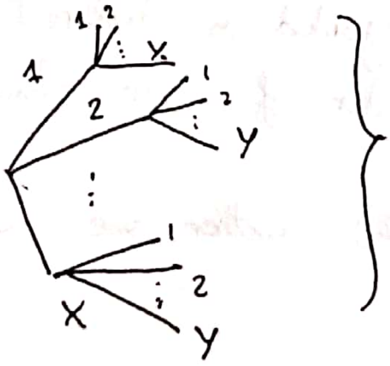
$$= \{\lfloor \alpha \rfloor + 1, \lfloor \alpha \rfloor + 2, \dots, \lfloor \alpha \rfloor + \lceil \beta \rceil - 1 - \lfloor \alpha \rfloor\} \Rightarrow$$

$$\Rightarrow \boxed{|A(\alpha, \beta)| = \lceil \beta \rceil - 1 - \lfloor \alpha \rfloor}$$

1.4 Tree diagram

Definition:

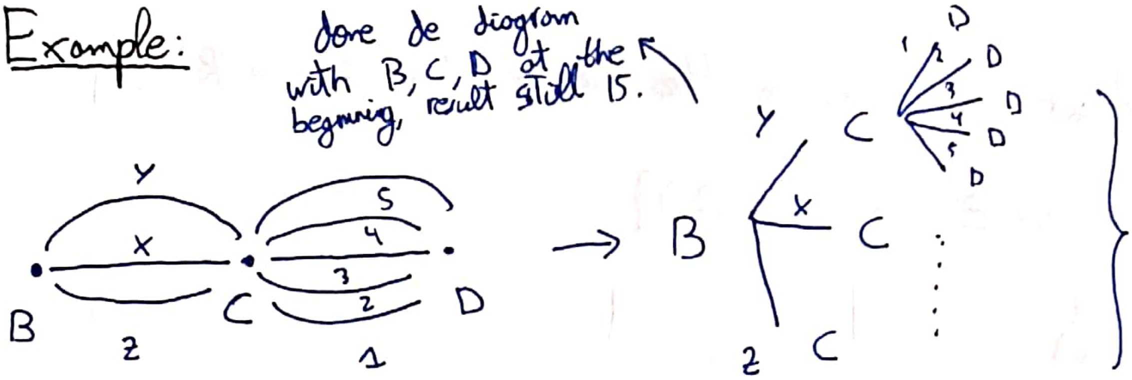
The rule of product states that if event X can happen in x ways and a distinct event Y can happen in y ways, X and Y can happen in $x \cdot y$ ways. Representation:



$x \cdot y$ branches

Example:

we could have done de diagram with B, C, D at the beginning, result still 15.



$3 \cdot 5 = 15$ ways

The rule product can be generalized to situations involving m events: $X_1, X_2, \dots, X_m$ can happen in $\prod_{i=1}^{m} x_i$ ways.

Definition:
A set is an unordered collection of distinct objects. The objects of a set are called elements. $\{1, 2, \dots, n\}$

Definition:
A sequence is an ordered collection of not necessarily distinct objects. $(1, 1, \dots, n)$

Definition:
Let $\Omega = \{a_1, \dots, a_n\}$ a set with $n \geq k \geq 1$.
A variation without repetition of Ω or k-permutation without repetition of n is an arrangement of the elements of Ω taken k at a time, where two arrangement are regarded as different if they differ in composition or in the order of their elements.

Ex. $\rightarrow \{1, 2, 3\} \rightarrow \Omega_2 = (1, 2)$ another one $(2, 1)$

(And no repetitions are allowed).

Let $V_{n,k}$ denote all possibles k-variations without repetition given n distinct objects, how much is $V_{n,k}$?

$V_{n,k} = |\{k\text{-var without rep. of } \Omega\}|$

$\underset{n}{\underset{\uparrow}{\_\_}} \; \underset{n-1}{\underset{\uparrow}{\_\_}} \; \underset{n-2}{\underset{\uparrow}{\_\_}} \; \dots \; \underset{n-k+1}{\underset{\uparrow}{\_\_}}$ $\Rightarrow$

$$\boxed{V_{n,k} = n \cdot (n-1) \cdot \dots \cdot (n-k+1) = \frac{n!}{(n-k)!}}$$

Example:

- 25 elements
- 4 positions
- No rep.
- Order matters

$\rightarrow 25 \cdot 24 \cdot 23 \cdot 22 = 303600$ ways

(If order not mattered, divide by permutations of 4, 4!)

Example:

a)

$\underline{B} \; \underline{T} \; \underline{C}$

$n = |\Omega| = 100$

$k = 3$

$\rightarrow V_{100,3} = 100 \cdot 99 \cdot 98 = 970200$

(One can only have one prize).

b) Possibilities of winning the car?

$$\underset{\underset{99}{\uparrow}}{\underline{B}} \quad \underset{\underset{98}{\uparrow}}{\underline{T}} \quad \underset{\underset{5}{\uparrow}}{\underline{C}} \longrightarrow 99 \cdot 98 \cdot 5 = 48510$$

c) Winning the three prizes?

$$\underset{5}{\underline{B}} \quad \underset{4}{\underline{T}} \quad \underset{3}{\underline{C}} \longrightarrow 5 \cdot 4 \cdot 3 = 60$$

Definition:

Let $\Omega = \{a_1, \dots, a_n\}$ a set and $k \geq 1$

A variation with repetition of Ω or k-permutation with repetition is a sequence of length k that can be formed with the elements in Ω being able to repeat them.

$$VR_{n,k} = |\{k\text{-var with rep. of } \Omega\}|$$

$$\overbrace{\underset{n}{\underline{\quad}} \; \underset{n}{\underline{\quad}} \; \underset{n}{\underline{\quad}} \; \cdots \; \underset{n}{\underline{\quad}}}^{k} \Rightarrow VR_{n,k} = n^k$$

Example:

(a) 3 digits numbers? ($\underset{9}{\cancel{0}}\underset{10}{\underline{1}}\underset{10}{\underline{9}}$ is valid)

$\rightarrow 9 \cdot 10 \cdot 10 = 900$ (1000 if zero is allowed).

(b) Palindromic? (Capicúa)

$$\underset{10}{\underline{\quad}} \; \underset{10}{\underline{\quad}} \; \underset{1}{\underline{\quad}} \longrightarrow 10 \cdot 10 \cdot 1 = 100 = VR_{10,2}$$

(c) How many are even?

$$\underset{10}{\underline{\quad}} \; \underset{10}{\underline{\quad}} \; \underset{5}{\underline{\quad}} \longrightarrow 5 \cdot VR_{10,2} = 5 \cdot 10 \cdot 10 = 500$$

Example:

- $\Omega = \{0, 1, 2\} \Rightarrow n = |\Omega| = 3$
- Repetitions ✓
- Order matters
- $K = 15$

$\longrightarrow V_{3,15} = 3^{15}$

Example:

ⓐ

$\Omega = \{a, e, o, u, b, c, d\}$

$A = \{$ words of 4 letters of Ω, first cons, last vowel $\}$

$|A| \longrightarrow \underline{\ \ }\ \underline{\ \ }\ \underline{\ \ }\ \underline{\ \ }$
$\quad 3 \quad 7 \quad 7 \quad 4$ $\rightarrow |A| = 3 \cdot 7 \cdot 7 \cdot 4 = 588$ //

ⓑ

$\underline{\ \ }\ \underline{\ \ }\ \underline{\ \ }\ \underline{\ \ }\ \underline{\ \ }$
$\quad 7 \quad 6 \quad 6 \quad 6 \quad 6$ $\rightarrow |B| = 7 \cdot 6 \cdot 6 \cdot 6 \cdot 6 = 9072$ //

Example:

$|A| = ? \longrightarrow A = \{$ 5 digits numbers, starting and finishing in different digits $\}$

$\underline{\ \ }\ \underline{\ \ }\ \underline{\ \ }\ \underline{\ \ }\ \underline{\ \ }$
$\quad 4 \quad 8 \quad 6 \quad 7 \quad 4$ $\longrightarrow 4 \cdot 8 \cdot 6 \cdot 7 \cdot 4 = 5376$

Example:

We need to be exhaustive and count every single possibilities.
But we don't know how to do this yet.

For the record: (and to prepare the classical scheme).

Variations (of n elements form groups with k elements, order <u>does</u> matter).

- repetitions allowed $\rightarrow VR_{n,k} = n^k$
- repetitions not allowed $\rightarrow V_{n,k} = \dfrac{n!}{(n-k)!}$

$$\left(V_{n,k} = \binom{n}{k} \cdot (k)! \right)$$

<u>Definition:</u>

Let $\Omega = \{a_1, \dots, a_n\}$ be a set of elements, $n \geq k \geq 1$. A k-combination without repetition of Ω is an arrangement of the elements of Ω taken k at a time, where two arrangements are regarded as different if and only if they differ in composition.

In oder words, it is a subset of k elements of Ω.

The number of k-combinations of n elements without repetition is denoted by $C_{n,k}$. (Note: order doesn't matter, only elements chosen and its number).

<u>Example:</u>

$\Omega = \{a_1, a_2, a_3\}$

2-combinations without repetition of Ω are $\{a_1, a_2\} = \{a_2, a_1\}$

2-var. wo. rep. of Ω : (a_1, a_2), (a_2, a_1) with $(a_1, a_2) \neq (a_2, a_1)$.

Example:

$\Omega = \{a, b, c, d, e\}$. Compute all possible subsets of 3 elements (no rep.) (order matters)

$$\underset{5}{\_}\ \underset{4}{\_}\ \underset{3}{\_} \rightarrow V_{5,3} = 5 \cdot 4 \cdot 3 = 60$$

Another way: Choose all possible distinct sets and then order them:

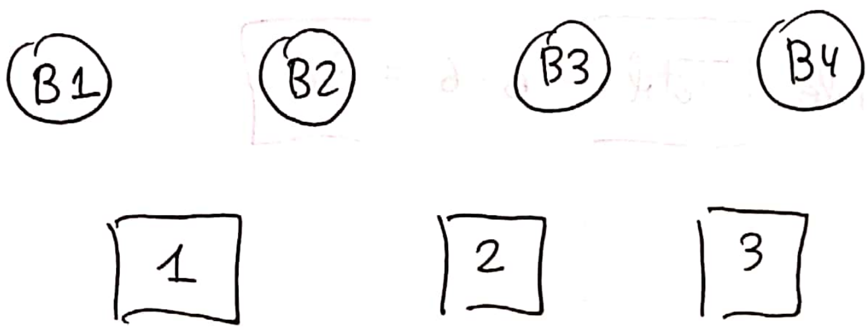
$$V_{5,3} = C_{5,3} \cdot V_{3,3} = C_{5,3} \cdot 3! \Rightarrow$$

$$\Rightarrow C_{5,3} = \frac{V_{5,3}}{3!} = \frac{5 \cdot 4 \cdot 3}{3!} = \frac{\frac{5!}{2!}}{3!} = \frac{5!}{2! \cdot 3!} = \binom{5}{3}$$

We generalize (it can be done).

$$C_{n,k} = \frac{V_{n,k}}{k!} = \binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

Example: (of the balls). How many ways can they be arranged? (1 box with two balls, no box empty, every box with elements.



?

1) Choose the box with 2 balls: $V_{3,1} = C_{3,1} = 3$

2) Choose two balls for that box: $C_{4,2} = \binom{4}{2}$

3) Order the remaining balls in the remaining boxes:

$V_{2,2} = 2! = 2$

→ Total: their product (the product rule) → $T = 3 \cdot \binom{4}{2} \cdot 2$

⇒ $\boxed{T = 36 \text{ number of ways to do it.}}$

More ways:

1) Choose 3 balls: $C_{4,3} = \binom{4}{3} = \frac{4!}{3! \cdot 1!} = 4$

2) Place one in each box: $V_{3,3} = \frac{3!}{0!} = 3! = 6$ ways to do it.

3) Place the 4^{th} ball: $C_{3,1} = V_{3,1} = 3$

4) Divide by two because we are counting each option twice.

→ By product rule: $\boxed{\text{Total} = 4 \cdot 6! \cdot 3 \cdot \frac{1}{2} = 72 \cdot \frac{1}{2} = 36}$

1) Choose two balls that go together: $C_{4,2} = \binom{4}{2} = \frac{4!}{2! \cdot 2!} = 6$

2) Order the remaining balls + the pack: $V_{3,3} = \frac{3!}{0!} = 3! = 6$

→ By product rule: $\boxed{\text{Total} = 6 \cdot 6 = 36}$

General case: $n+1$ balls and n boxes:

(Method 1).

1) Choose the box with 2 balls: $V_{n,1} = C_{n,1} = n$

2) Choose two balls for that box: $C_{n+1,2} = \binom{n+1}{2}$

3) Order the remaining balls in the remaining order:

$V_{n-1,n-1} = (n-1)!$

By product rule:

Total $\rightarrow$ $\boxed{T_1 = n \cdot \binom{n+1}{2} \cdot (n-1)!} = n \cdot \frac{(n+1)!}{2!}$

(Ex: $n=3 \rightarrow \frac{3 \cdot 4!}{2} = \frac{72}{2} = 36$ checks out).

Example: generalization ($n+2$ balls, n boxes)

1) 1 box, 3 balls:

1.1) Choose the box: $V_{n,1} = C_{n,1} = n$

1.2) Choose 3 balls: $C_{n+2,3} = \binom{n+2}{3}$

1.3) Place the remaining $n-1$ balls: $V_{n-1,n-1} = (n-1)!$

($n+2-3 = n-1$)

Product rule $\rightarrow$ $\boxed{T_1 = n \cdot (n-1)! \cdot \binom{n+2}{3} = n \cdot \frac{(n+2)!}{3}}$

2) 2 boxes with 2 balls:

2.1) Choose the double boxes: $C_{n,2} = \binom{n}{2}$

(Next page)

2.2) Select the balls in first double box: $C_{n+2,2} = \binom{n+2}{2}$

2.3) Select the balls in second box: $C_{n,2} = \binom{n}{2}$

2.4) Place the remaining $n-2$ balls: $V_{n-2,n-2} = (n-2)!$

$(n+2-4) \longrightarrow$ $(n-2$ boxes

Product rule $\longrightarrow$ $\boxed{\binom{n}{2}^2 \cdot \binom{n+2}{2} \cdot (n-2)! = T_2}$

So the total possible ways is all the ways to do method 1, and all the ways in 2 $\Rightarrow$

$$\boxed{T = T_1 + T_2 = n \cdot (n-1) \cdot \binom{n+2}{3} + (n-2)! \cdot \binom{n}{2}^2 \cdot \binom{n+2}{2}}$$

Properties of $C_{n,k}$

1.- $C_{n,k} = \binom{n}{k} = \frac{V_{n,k}}{k!} = \frac{n!}{(n-k)! \cdot k!}$, $k = 0, \dots, n$

2.- $\binom{n}{k} = \binom{n}{n-k}$, $k = 0, \dots, n$

This property means that choosing k elements among n is the same as rejecting n-k elements among n.

3.- $\binom{n}{n} = 1 = \frac{n!}{1! \cdot n!} = \frac{n!}{(n-n)! \cdot n!}$

4.- $\binom{n}{0} = 1 = \frac{n!}{1! \cdot n!} = \frac{n!}{0! \cdot (n-0)!}$

5.- $\binom{n}{k} = 0$, $k > n$

6.- $\binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}$, $n \geq 1$, $k = 0, \dots, n$

Pascal's identity.

7.- If $k \geq 1$, $\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}$

dem.:

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!} = \frac{n\cdot(n-1)!}{k\cdot(k-1)!\,(n-k)!} = \frac{n}{k}\cdot\underbrace{\cancel{\frac{(n-1)!}{(k-1)!\cdot((n-1)-(k-1))!}}}_{\binom{n-1}{k-1}} =$$

$$= \frac{n}{k}\cdot\binom{n-1}{k-1} \quad \square$$

8.- If $0 \leq k \leq n-1$, $\binom{n}{k} = \frac{n}{n-k}\binom{n-1}{k}$

dem.:

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!} = \frac{n}{n-k}\cdot\frac{(n-1)!}{k!\,(n-k-1)!} = \frac{n}{n-k}\cdot\underbrace{\cancel{\frac{(n-1)!}{k!\,((n-1)-k)!}}}_{\binom{n-1}{k}} =$$

$$= \frac{n}{n-k}\cdot\binom{n-1}{k}$$ (if $k=n$ it doesn't work, $\binom{n-1}{n}=0$)

Formula valid if $0 \leq k \leq n-1$ $\square$

Definition:

A permutation of n elements or n-permutation, is an n-variation without repetition of n elements. Permutations of n elements are all the possible n-arrangements each of which contains every element at once, with two such arrangements differing only in the order of their elements.

Order matters. No repetition.

Number of n-permutations is denoted by P_n.

$$\boxed{P_n = V_{n,n} = \frac{n!}{(n-n)!} = n\cdot(n-1)\cdot\ldots\cdot 2\cdot 1 = n!}$$

$(n-n)! = 0! = 1$

Definition:

A multiset is a generalization of the notion of set in which elements are allowed to appear more than once.

Definition:

Let M be a multiset with n_i elements of the i-type:

$$M = \{\underbrace{a_1, \dots, a_1}_{n_1}, \underbrace{a_2, \dots, a_2}_{n_2}, \dots, \underbrace{a_k, \dots, a_k}_{n_k}\}$$

$$|M| = n_1 + n_2 + \dots + n_k = n$$

The different arrangements of M are the permutations with repetition.

The number of such permutations (given n_i, $i \in \{1, k\}$ elements of each type and $|M|$ total elements) it's denoted by:

$$P(n_1, n_2, \dots, n_k)$$

How much is $P(n_1, \dots, n_k)$?

Briefing:

Variation: sequence of length k of n elements (with/without rep.)
Combination: subset of k of n elements (with/without rep.)
Permutation: rearrangement of n elements (with/without rep.).

Briefing up to now:

$\Omega = \{a_1, \dots, a_n\}$

- k-Variation wo. rep. = sequences of length k wo. rep.

$V_{n,k} = \frac{n!}{(n-k)!} = n(n-1)\cdot \ldots \cdot (n-k+1)$

- k-Variation with. rep. = sequences of length k with rep.

$VR_{n,k} = n^k$

- k-Combination wo. rep. = subsets of k elements wo. rep.

$C_{n,k} = \binom{n}{k}$

- Permutations wo. rep. = n-Variations wo. rep. = sequences of length k without repetition (wo. rep.).

$P_n = V_{n,n} = n!$

- Permutations with repetition (with rep.). = sequences of length k with rep.

$P(n_1, n_2, \dots, n_k)$?

Let's find out little by little.

We will start with n_1, n_2

__Example:__

Calculate number of sequences that can be formed with p zeros and q ones.

$$M = \{\underbrace{0, \dots, 0}_{p}, \underbrace{1, \dots, 1}_{q}\}$$ p 0s and q 1s.

$$\overbrace{\underline{\ \ }_{1}\ \underline{\ \ }_{2}\ \dots\ \underline{\ \ }_{p+q}}^{p+q \text{ positions}}$$

We choose from the total p positions for 0 or q positions for ones, which is the same:

$$\boxed{P(p,q) = C_{p+q,p} = C_{p+q,q} = \binom{p+q}{p} = \binom{p+q}{q} = \frac{(p+q)!}{p! \cdot q!}}$$

__Example:__

Same but with r 2s.

p 0s, q 1s and r 2s

$$M = \{\overbrace{0, \dots, 0}^{p}, \overbrace{1, \dots, 1}^{q}, \overbrace{2, \dots, 2}^{r}\}$$

$$\overbrace{\underline{\ \ }_{1}\ \underline{\ \ }_{2}\ \dots\ \underline{\ \ }_{p+q+r}}^{p+q+r \text{ positions}}$$

We choose from p+q+r p for the 0s. Then, from q+r we choose q for the 1s and the rest are for the 2s.

$$\Rightarrow \boxed{P(p,q,r)} = C_{p+q+r,p} \cdot C_{q+r,q} \cdot \cancel{C_{r,r}}^{1} =$$

$$= \binom{p+q+r}{p}\binom{q+r}{q} = \frac{(p+q+r)!}{p! \cdot \cancel{(q+r)!}} \cdot \frac{\cancel{(q+r)}!}{q! \cdot r!} =$$

$$= \boxed{\frac{(p+q+r)!}{p! \cdot q! \cdot r!}}$$

Now we generalize:

Let $n = n_1 + \ldots + n_k$

$$\boxed{P(n_1, n_2, \ldots, n_k)} = \binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3} \cdot \ldots \cdot \binom{n-\ldots-n_{k-2}}{n_{k-1}}$$

$$= \frac{n!}{n_1! \cdot \cancel{(n-n_1)!}} \cdot \frac{\cancel{(n-n_1)}!}{n_2 \cdot (n-n_1-n_2)!} \cdot \ldots \cdot \frac{(n-n_1-\ldots-n_{k-2})!}{n_{k-1}!\,\underbrace{(n-n_1-\ldots-n_{k-1})!}_{(n_k)!}} =$$

$$= \boxed{\frac{n!}{n_1! \cdot n_2! \cdot \ldots \cdot n_k!}}$$

<u>Definition:</u>

$\Omega = \{a_1, \ldots, a_n\}$. Let n and $k \geq 1$ be positive integers. Then, a k - combination with repetition of Ω is a multiset of k elements of Ω. If $CR_{n,k}$ denotes the number of k-combination with repetition given n distinct objects, how much is $CR_{n,k}$?

Example:

$\Omega = \{a_1, a_2\}$ $(n=2,\ k=3)$

$CR_{2,3}$?

$$\left.\begin{array}{l}\{a_1, a_1, a_1\} \\ \{a_1, a_1, a_2\} \\ \{a_1, a_2, a_2\} \\ \{a_2, a_2, a_2\}\end{array}\right\} \quad \boxed{CR_{2,3} = 4}$$

Example:

$\Omega = \{a, b, c, d\} \longrightarrow CR_{4,3}$?

		a	b	c	d		
$\{a,a,a\}$	$\leftrightarrow$	000				$\leftrightarrow$	000111
$\{a,a,b\}$	$\leftrightarrow$	00	0			$\leftrightarrow$	001011
$\{a,b,b\}$	$\leftrightarrow$	00		0		$\leftrightarrow$	010011
$\{a,b,c\}$	$\leftrightarrow$	0	0	0		$\leftrightarrow$	010101
$\{c,c,d\}$	$\leftrightarrow$			00	0	$\leftrightarrow$	110010

(Write zeros and if we find a line write 1).

Observe that every multiset has associated a sequence of 1s and 0s (3 ones and 3 ones), we can find it as shown above (two sets have a distinct representation).

Similarly, given a sequence we can find a multiset:

example:

$$100101 \rightarrow \overset{a}{\,}\big|\overset{b}{00}\big|\overset{c}{0}\big|\overset{d}{\,} \rightarrow \{b, b, c\}$$

There is a bijection!

$\{3\text{-combs with rep. of } 4\} \longrightarrow \{\text{sequences formed with } 3\ 0\text{s and } 3\ 1\text{s}\}$

$\qquad\quad \shortparallel \qquad\qquad\qquad\qquad\qquad\qquad\qquad \shortparallel$

$\qquad\quad A \qquad\qquad\qquad\qquad\qquad\qquad\qquad B$

$\Rightarrow |A| = |B|$ and we know $|B| \Rightarrow$

$\Rightarrow |A| = |B| \Rightarrow CR_{4,3} = \underbrace{C_{6,3}} = \binom{6}{3}$

(choose 3 places for the 0s, the rest are 1s)

In general:

$\{k\text{-combs with rep of } n \text{ elements}\} \xrightarrow{\text{bijective}} \{\text{sequences formed with } k\ 0\text{s and } (n-1)\ 1\text{s}\}$

(We have n letters $\Rightarrow$ $n-1$ bars, and k elements, with one 0 for each element)

$$\Rightarrow |A| = |B| \Rightarrow \boxed{CR_{n,k}} = P(k, n-1) = \underline{\underline{C_{n+k-1,\,k}}} = \binom{n+k-1}{k}$$

Choose from $n+k-1$ places, k places for the 0s and the rest must be 1s.

Example:

Number of sequences with p zeros and q ones:

$$CR_{q+1,\,p} = \binom{q+1+p-1}{p} = \binom{p+q}{q} \quad //$$

Example:

Number of sequences with 2 zeros and 3 ones such that there are no consecutive zeros.

2 0s and 3 1s. → ~~00111~~ (0s consecutive)

~ 1 ~ 1 ~ 1 ~ → We have to choose from these 4 possibilities two for the zeros:

$$\rightarrow \boxed{T = C_{4,2} = \binom{4}{2} = 6} \quad (T = \text{total})$$

Example:

Generalization of the previous:

p 0s, q 1s

~ 1 ~ 1 ~ .... ~ 1 ~ ⟶ From q+1 possible positions, we choose p.

$$\longrightarrow \boxed{\text{Total} = C_{q+1,p} = \binom{q+1}{p}}$$

Example:

2 0s, 3 1s and 4 2s:

~ x ~ x ~ x ... ~ x ~ → We choose the 1s or 2s ⟶ From 7 choose 3 for 1s $\binom{7}{3} = \binom{7}{4}$. Then choose the zeros

↓ possible 0 ↓ possible 3 or 2

$$\longrightarrow \boxed{\text{Total} = \binom{7}{3} \cdot \binom{8}{2} = 980}$$

Example:

Generalization.

p zeros, q 1s, r 2s :

$\sim x \sim x \sim x \dots \sim x \sim x \sim$ with $x = 1$ or 2

and $\sim$ its a position for a possible zero.

We choose the 1s and 2s, from $r+q$, choose q 1s (same to choose r 2s) $\rightarrow$ $\binom{r+q}{r} = C_{r+q,r}$.

Then choose the zeros, from $r+q+1$ positions, choose p.

(Note that $p \leq r+q+1$) $\rightarrow C_{r+q+1,p}$ $\rightarrow$ By product rule:

$\Rightarrow$ $\boxed{\text{Total} = C_{r+q,r} \cdot C_{r+q+1,p} = \binom{r+q}{r}\binom{r+q+1}{p}}$

Scheme:

Combinations without rep: $C_{n,k} = \binom{n}{k}$
(subset of k elements).

Combinations with rep: $CR_{n,k} = C_{n+k-1,k} = \binom{n+k-1}{k}$
(multiset of k elements).

Variations without rep: $V_{n,k} = \dfrac{n!}{(n-k)!}$
(sequence of k elements without rep).

Variations with rep: $V_{n,k} = n^k$
(sequence of k elements with rep).

Permutations without rep: $P_n = n!$
(rearrangement of the elements of a set)

Permutations with rep: $P(n_1,\dots,n_k) = \dfrac{(n_1+\dots+n_k)!}{n_1!\cdot n_2!\cdot\dots\cdot n_k!}$
(rearrangement of the elements of a multiset).

These are very useful tools. To compute number of possible subsets of a set: $C_{n,k}$. To compute number of rearrangements of a set P_n. To compute the number of rearrangements of a multiset $P_{n_1,\dots,n_k}$. Notice this last one is the same as computing all the possible arrangements $(n!)$ and then for each solution divide by the permutation of equal elements (we've counted $1_1 1_2$ as different from $1_2 1_1$, but they are the same, in this case divide by $2!$), which means $(n)!\cdot\left(\frac{1}{n_1!}\right)\left(\frac{1}{(n_2)!}\right)\cdot\dots\cdot\left(\frac{1}{n_k!}\right)$.

To compute the number of possible multisets that can be formed from a set: $CR_{n,k}$. The intuitive way to see it is with the bijection from multisets of length k from n elements with sequences formed with k zeros and $(n-1)$ ones:

(n-1 bars, because n letters and one zero for each element in the multiset, k)

$$0110\dots \rightarrow \overset{A}{0}\,|\,\overset{B}{}\,|\,\overset{C}{0}\,|\,\dots \rightarrow \{a, c, \dots\}$$

$$\Rightarrow CR_{n,k} = C_{n+k-1,k} = C_{n+k-1,n-1} = \binom{n+k-1}{k} = \binom{n+k-1}{n-1}$$

(From $(n-1+k)$ positions, choose either k zeros or $n-1$ ones, which is the same).

1.5 Factorial Powers

Definition:

Let $a \in \mathbb{C}$. Its descending or lower or falling factorial (power) of order k, $k \in \mathbb{N}^*$ is:

$$a^{\underline{k}} = \begin{cases} a(a-1)\dots(a-k+1) & k \geq 1 \\ 1 & k = 0 \end{cases}$$

(It has $k-1+1 = k$ terms).

Analogous: ascending / upper / rising factorial power of order k,

$$a^{\overline{k}} = \begin{cases} a(a+1)\dots(a+k-1) & k \geq 1 \\ 1 & k = 0 \end{cases}$$

(It has k terms).

Properties:

1.1

$$(-a)^{\underline{k}} = (-1)^k \cdot a^{\overline{k}}$$

1.2

$$(-a)^{\overline{k}} = (-1)^k \cdot a^{\underline{k}}$$

dem.: (Proof)
(next page)

$(-a)^{\underline{k}} = (-a)(-a-1)\cdot\ldots\cdot(-a-k+1) = (-1)^k \cdot \underbrace{a(a+1)\cdot\ldots\cdot(a+k-1)}_{a^{\overline{k}}} =$

$= (-1)^k \cdot a^{\overline{k}}$ $\square$

1.3

$(-a)^{\overline{k}} = (-1)^k \cdot a^{\underline{k}}$ (because of 1.2)

2.1

$a^{k+\ell} = a^k \cdot a^\ell$

2.3

$a^{\underline{k+\ell}} = a^{\underline{k}} \cdot (a-k)^{\underline{\ell}}$

proof:

$a^{\underline{k+\ell}} = \underbrace{a(a-1)\cdot\ldots\cdot(a-k+1)}_{a^{\underline{k}}} \cdot \underbrace{(a-k)\cdot(a-k-1)\cdot\ldots\cdot(\underbrace{a-(k+\ell+1)}_{a-k-\ell+1)})}_{(a-k)^{\underline{\ell}}}$

$= a^{\underline{k}} \cdot (a-k)^{\underline{\ell}}$ $\square$

2.4

$a^{\overline{k+\ell}} = a^{\overline{k}} (a+k)^{\overline{\ell}}$ (similar to 2.3)

3.1

$a = n \in \mathbb{N}^*$

if $0 \leq k \leq n$:

$n^{\underline{k}} = n(n-1)\ldots(n-k+1) = \frac{n!}{(n-k)!} = V_{n,k}$

3.2

If $k > n$:

$$n^{\underline{k}} = 0$$

proof:

If $k > n \Rightarrow n^{\underline{k}} = \underset{\vee\atop 0}{n}(n-1)\cdot\ldots\cdot\underset{\shortparallel\atop 0}{(n-n)}\cdot\ldots\cdot\underset{\wedge\atop 0}{(n-k+1)} = 0$ □

1.6 Classifications

Definition:

Rule of sum: Let Ω be a set. If $\{A_1, \ldots, A_k\}$ is a partition of Ω, that is:

$$\bigcup_{i=1}^{k} A_i = \Omega \quad \text{and} \quad A_i \cap A_j = \emptyset \quad \forall i \neq j \quad \Rightarrow$$

$$\Rightarrow |\Omega| = |A_1| + \ldots + |A_k|$$

Classification is very useful. Sometimes there no other solution. We can find some identities. And it is the main idea to get recurrence relations.

Example:

A = ways to place 5 distinguishable balls in 3 boxes.

A_1 = ways in which two boxes have 2 balls.

A_2 = ways " " one box has 3 balls.

Since $\{A_i\}_{i=1,2}$ is a partition of A $(A = A_1 \cup A_2$ and $A_1 \cap A_2 = \emptyset)$

$\Rightarrow |A| = |A_1| + |A_2|$ which is how we solved it before.

Example:

For $n = 1$:

$\Omega_1 = \{a_1\}$

$\mathcal{P}(\Omega_1) = \{\emptyset, \{a_1\}\} \Rightarrow |\mathcal{P}(\Omega_1)| = 2 = 2^1$

$n = 2$:

$\Omega_2 = \{a_1, a_2\}$

$P(\Omega_2) = \{\emptyset, \{a_1\}, \{a_2\}, \{a_1, a_2\}\} \Rightarrow$

$\Rightarrow |P(\Omega_2)| = 4 = 2^2$

If $\Omega = \{a_1, \dots, a_n\}$, $P(\Omega) = ¿?$

All possible subsets of Ω:

- with 0 elements: $\emptyset$
- with 1 : $\{a_1\}, \{a_2\}, \dots, \{a_n\}$

$\vdots$

- with n: Ω

So; we define: $\emptyset = A_i = \{\text{subsets of } \Omega \text{ with } i \text{ elements}\} \subseteq \mathcal{P}(\Omega)$ ($i \in \{0, \dots, n\}$). We observe:

$$\underbrace{\bigcup_{i=0}^{n} A_i = \mathcal{P}(\Omega)}_{\text{We do count every possible element}} \quad \text{and} \quad \underbrace{A_i \cap A_j \neq \emptyset \text{ if } i \neq j}$$

for $i \neq j$ $a \in A_i$ and $b \in A_j$ have $|a| = i \neq j = |b| \Rightarrow a \neq b \Rightarrow$

$\Rightarrow a \notin A_j$ and $a \notin A_i \Rightarrow A_i \cap A_j = \emptyset$

$\Rightarrow \{A_i\}_{i \in \{0, \dots, n\}}$ is a partition of Ω.

We have:

$A_0 = \{\emptyset\} \to |A_0| = 1$

$A_1 = \{\{a_1\}, \dots, \{a_n\}\} \to |A_1| = n$

$\vdots$

$A_k = \{X \subset \Omega : |X| = k\} \to |A_k| = C_{n,k} = \binom{n}{k}$

$\vdots$

$A_n = \{\Omega\} \to |A_n| = |\Omega| = n$

Thus:

$$|P(\Omega)| = \sum_{i=0}^{n} |A_i| = \sum_{i=0}^{n} C_{n,i} = \sum_{i=0}^{n} \binom{n}{i}$$

Brief:

$\Omega = \{a_1, \dots, a_n\}$

$$|P(\Omega)| = \sum_{i=0}^{n} (C_{n,i}) = \sum_{i=0}^{n} \binom{n}{i}$$

We define now:

$\emptyset \neq B_1 = \{$ subsets of Ω that contain $a_n\} \subseteq P(\Omega)$

$\emptyset \neq B_2 = \{$ subsets of Ω that do not contain $a_n\} \subseteq P(\Omega)$

Trivial to prove that $\{B_i\}_{i \in \{1,2\}}$ is a partition of Ω.

So:

$$|P(\Omega)| = |B_1| + |B_2|$$

$x_n = |\mathcal{P}(\Omega)|$ when $|\Omega| = n$

$x_i = |\mathcal{P}(\Omega_i)|$ when $|\Omega_i| = i$

In this notation:

~~But not that there is a bijection from $B_1 \to B_2$, $B \to B - \{a_n\}$~~

$\Rightarrow |B_1| = |\Omega_{n-1}| = x_{n-1}$

$$\left\{\begin{array}{ccc} \mathcal{P}(\Omega_{n-1}) & \longrightarrow & B_1 \\ A & \longrightarrow & A \cup \{a_n\} \end{array}\right. \quad \text{bijective} \Rightarrow |B_1| = |\Omega_{n-1}| = x_{n-1}$$

$$\left\{\begin{array}{ccc} \mathcal{P}(\Omega_{n-1}) & \longrightarrow & B_2 \\ A & \longrightarrow & A \end{array}\right. \quad \text{bijective} \Rightarrow |B_2| = |\Omega_{n-1}| = x_{n-1}$$

$$\Rightarrow |\mathcal{P}(\Omega)| = |B_1| + |B_2| = x_{n-1} + x_{n-1} = 2x_{n-1}, \quad \forall n \geq 1$$

(We've proved $x_1 = |\mathcal{P}(\Omega_1)| = |\mathcal{P}(\{a_1\})| = 2$ and
$x_0 = |\mathcal{P}(\Omega_0)| = |\mathcal{P}(\emptyset)| = 1$).

$$\Rightarrow 2x_{n-1} = 2 \cdot (2 \cdot x_{n-2}) = 2^2 \cdot x_{n-2} = 2^2 \cdot (2x_{n-3}) =$$

$$= 2^3 \cdot x_{n-3} = \ldots = 2^{n-1} x_{n-(n-1)} = 2^{n-1} \cdot x_1 = 2^n \cdot \cancel{x_0}^{1} = 2^n$$

$$\Rightarrow \boxed{\begin{array}{l} |\mathcal{P}(\Omega)| = 2^n \quad \text{and also:} \\ |\mathcal{P}(\Omega)| = \sum_{i=0}^{n} \binom{n}{i} = 2^n \end{array}}$$

Note: this was thanks to the classification rule!

Example:

How many k-combinations without rep? $C_{n,k}$

Let's see it another way: $(|\Omega| = k$

$A_1 = \{\text{subsets of } \Omega \text{ contain } a_n\} \xrightarrow{(1)} \{a_n\} \cup \{\text{subsets of } k-1 \text{ elements of } \Omega - a_n\}$

$A_2 = \{\text{subsets of } \Omega \text{ don't contain } a_n\} \xrightarrow{(2)} \{\text{subsets of } k \text{ elements of } \Omega - a_n\}$

$\{A_1, A_2\}$ is a partition of $A = \{\text{subsets of } \Omega \text{ with } k \text{ elements}\}$.

$|A| = |A_1| + |A_2|$

Put it in terms of $C_{n,k}$ we obtain:

$$C_{n,k} = \underbrace{C_{n-1,k-1}}_{(1)} + \underbrace{C_{n-1,k}}_{(2)} \Rightarrow \binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

we already know and A_1

We've proved Pascal's identity. This is Pascal's triangle:

$$\binom{0}{0}$$
$$\binom{1}{0} \quad \binom{1}{1}$$
$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$
$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$
$$\vdots$$

$$1$$
$$1 \quad 1$$
$$1 \quad 2 \quad 1$$
$$1 \quad 3 \quad 3 \quad 1$$
$$\vdots$$

Example:

a_n = number of sequences of n elements, 1s and 0s, in which there are no consecutive zeros.

$a_1 = 2 \rightarrow (0), (1)$

$a_2 = 3 \rightarrow (0,1), (1,0), (1,1)$

$a_3 = 5 \rightarrow (0,1,0), (0,1,1), (1,1,0), (1,1,1), (1,0,1)$

We classify by the first number.

$$\begin{cases} A_1 = \{\text{sequences that start with a } 0\} \\ A_2 = \{\text{ " " " " } 1\} \end{cases}$$

$\{A_1, A_2\}$ is a partition set of A.

$\Rightarrow |A| = |A_1| + |A_2| \Rightarrow a_n = |A_1| + |A_2|$

For A_1. If they start with a $0 \rightarrow$

$$\underset{\substack{\downarrow \\ \text{start} \\ \text{with } 0}}{\underline{0}} \quad \underset{\substack{\downarrow \\ \text{next} \\ \text{must} \\ \text{be } 1}}{\underline{1}} \quad \underbrace{\_\ \_\ \dots}_{\text{¡} a_{n-2}\text{!}} \qquad \Rightarrow |A_1| = a_{n-2}$$

$$\underset{\substack{\downarrow \\ \text{starts} \\ \text{with} \\ 1}}{\underline{1}} \quad \underbrace{\_\ \dots\ \_}_{\text{¡} a_{n-1}\text{!}} \qquad \Rightarrow |A_2| = a_{n-1}$$

$\Rightarrow$ By the rule of sum: $a_n = a_{n-1} + a_{n-2}$, $n \geq 3$

(Recurrence relation of the Fibonacci sequence).

Example:

Similar, but 0s, 1s, 2s and no consecutive 1s.

$a_1 = 3 \rightarrow (0), (1), (2)$

$a_2 = 8 \rightarrow (0,0), (0,1), (0,2), (1,0), (1,2), (2,0), (2,1), (2,2)$

For $n \geq 3$

- Starts with 0: $\underline{0}\ \underbrace{\_\ \cdots\ \_}_{a_{n-1}}$ (we can put there any sequence of a_{n-1})

- Starts with 1, must be followed by 0 or 2, because there are no consecutive 1s:

  - starts with 10: $\underline{1}\ \underline{0}\ \underbrace{\cdots\ \_}_{a_{n-2}}$

  - Starts with 12: $\underline{1}\ \underline{2}\ \underbrace{\cdots\ \_}_{a_{n-2}}$

- Starts with 2: $\underline{2}\ \underbrace{\_\ \cdots\ \_}_{a_{n-1}}$ (similar to 0).

$\Rightarrow$ Rule of sum: $\boxed{a_n = 2 \cdot a_{n-1} + 2 \cdot a_{n-2}}$

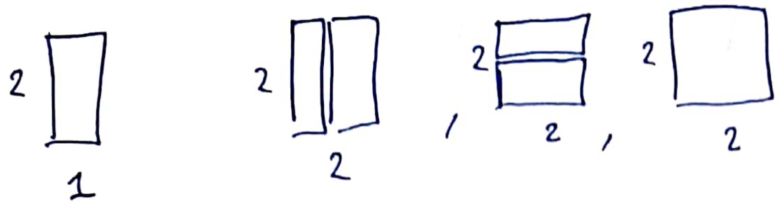
Example:

Two types of floor tiles: 2×1  2×2 

How many ways to tile a rectangular floor of $2\times n$?

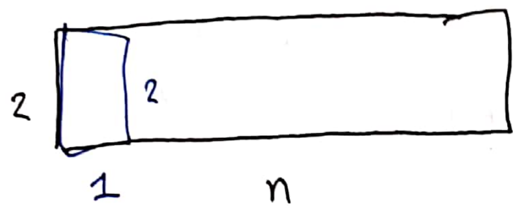
We classify:

$n=1$ $\quad$ $n=2$

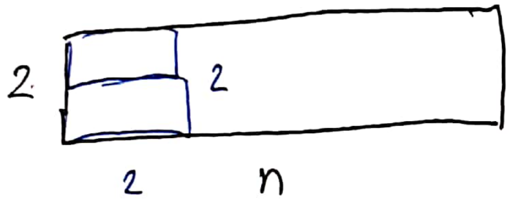


$$\begin{pmatrix} a_1 = 1 \\ a_2 = 3 \end{pmatrix}$$

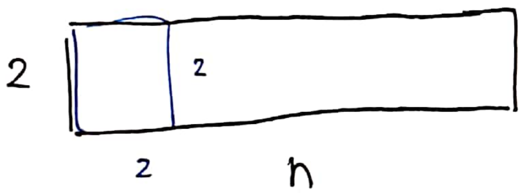
n:



$\rightarrow \quad a_{n-1}$



$\rightarrow \quad a_{n-2}$



$\rightarrow \quad a_{n-2}$

 is included in the first one! It is a partition $\Rightarrow$

By the rule of sum:

$$a_n = a_{n-1} + 2a_{n-2}$$

Example: (where classification it's not possible).

$$A_n = \{k : 1 \le k \le n : k \bmod (2) = 0 \vee k \bmod (3) = 0\} =$$

$$= \{k : 1 \le k \le n : k \bmod (2) = 0\} \cup \{k : 1 \le k \le n : k \bmod (3) = 0\}$$

$|A_n| = ?$

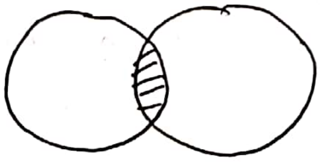
We define:
$$\begin{cases} B_n = \{k : 1 \le k \le n : k \bmod (2) = 0\} \\ C_n = \{k : 1 \le k \le n : k \bmod (3) = 0\} \end{cases}$$

Note it is not a partition, despite $A_n = B_n \cup C_n$, $B_n \cap C_n \neq \emptyset$!

Example: $6 \in B_n \cap C_n$

B_n C_n

$\Rightarrow |A_n| = |B_n| + |C_n| - |B_n \cap C_n|$



Venn diagram

A_n

From past exercises:

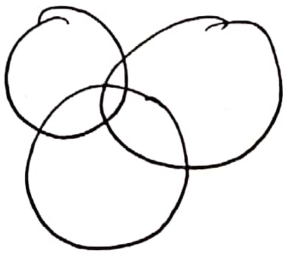
$$\left.\begin{aligned} |B_n| &= \lfloor n/2 \rfloor \\ |C_n| &= \lfloor n/3 \rfloor \\ |B_n \cap C_n| &= \lfloor n/6 \rfloor \end{aligned}\right\} \Rightarrow \boxed{|A_n| = \lfloor n/2 \rfloor + \lfloor n/3 \rfloor - \lfloor n/6 \rfloor}$$

Example:

Same but:

A_n

$$\begin{cases} A_n = \{k : 1 \le k \le n,\ k \equiv 0 \bmod (2)\} \\ B_n = \{k : 1 \le k \le n,\ k \equiv 0 \bmod (3)\} \\ C_n = \{k : 1 \le k \le n,\ k \equiv 0 \bmod (5)\} \end{cases}$$



C_n

B_n

$$|D_n| = |A_n \cup B_n \cup C_n| = |(A_n \cup B_n) \cup C_n| \overset{\text{(principle of inclusion-exclusion)}}{=}$$

$$= |A_n \cup B_n| + |C_n| - |(A_n \cup B_n) \cap C_n| =$$

$$= |A_n| + |B_n| - |A_n \cap B_n| + |C_n| - |(A_n \cap C_n) \cup (B_n \cap C_n)| =$$

$$= |A_n| + |B_n| + |C_n| - |A_n \cap B_n| - \left[|A_n \cap C_n| + |B_n \cap C_n| - \underbrace{|A_n \cap C_n \cap B_n \cap C_n|}_{|A_n \cap B_n \cap C_n|} \right]$$

$$= |A_n| + |B_n| + |C_n| - |A_n \cap B_n| - |A_n \cap C_n| - |B_n \cap C_n| + |A_n \cap B_n \cap C_n| =$$

in this case

$$\overset{\text{in this case}}{=} \boxed{\lfloor n/2 \rfloor + \lfloor n/3 \rfloor + \lfloor n/5 \rfloor - \lfloor n/6 \rfloor - \lfloor n/10 \rfloor - \lfloor n/15 \rfloor + \lfloor n/30 \rfloor = |D_n|}$$

We generalize (the following formula can be proven by induction but we won't do it (Básicos del año pasado)).

1.7 Principle of Inclusion - Exclusion

Let $A_1, \dots, A_n$ be subsets of a finite set Ω. Then:

$$\left| \bigcup_{i=1}^{n} A_i \right| = \sum_{i=1}^{n} |A_i| - \sum_{1 \le i < j \le n} |A_i \cap A_j| + \sum_{1 \le i < j < k \le n} |A_i \cap A_j \cap A_k| - \dots$$

$$\dots + (-1)^{r-1} \sum_{1 \le i_1 < \dots < i_r \le n} |A_{i_1} \cap \dots \cap A_{i_r}| + \dots + (-1)^{n-1} |A_1 \cap \dots \cap A_n|$$

The terms in the sums are the following by order:

$$\binom{n}{1}, \binom{n}{2}, \binom{n}{3}, \dots\dots, \binom{n}{r}, \dots, \binom{n}{n}$$

(We choose from n elements r distinct ones and they must have an order because they are distinct. All possible r-subsets: $\binom{n}{r}$).

Example:

$(M_1 W_1) \ldots (M_n W_n)$

How many ways can we pair MW so that no man dances with his wife?

$\Omega = \{$all the possible dancing couples$\}$ (no constrains)

$\frac{n}{M_1} \; \frac{n-1}{M_2} \; \ldots \; \frac{1}{M_n} \Rightarrow |\Omega| = (n!) = P_n = V_{n,n}$

$\Omega^* = \{$ways in which no man dances with his wife$\}$

$(\Omega^*)^c = \{$ways in which at least one man dances with his wife$\}$

By the rule of sum: $|\underbrace{\Omega}_{n!}| = |\Omega^*| + |(\Omega^*)^c|$

It easier to compute $|(\Omega^*)^c|$. Principle of inclusion-exclusion.

$A_k = \{$ways in which married couple k dances together$\}$, $k = 1, \ldots, n$

$(\Omega^*)^c = \bigcup_{k=1}^{n} A_k$ ($\supset$ and $\subset$: yes)

Since $\{A_k\}_{k \in I}$ is not a partition, we use the principle of inclusion-exclusion.

$|(\Omega^*)^c| = \left|\bigcup_{k=1}^{n} A_k\right| \rightarrow$ we need $|A_k|$, $|A_k \cap A_j|$ ...

$\forall k$, $|A_k| = ?$

We have one pair: $(M_k, W_k) \rightarrow \frac{n-1}{M_1} \; \frac{n-2}{M_2} \ldots \frac{\overset{1}{\underset{\shortparallel}{W_k}}}{M_k} \ldots \frac{1}{M_n}$

$\Rightarrow |A_k| = (n-1)!$ (True for any k)

$\forall i,j \to |\underbrace{A_k \cap A_j}_{}|$

$(M_k W_k)$ and $(M_j W_j)$

$$\begin{array}{ccccccccc} M_1 & M_2 & M_3 & \dots & M_j & M_k & \dots & M_n \\ \hline n-2 & n-3 & n-4 & & 1 & W_k & & 1 \\ & & & & \shortparallel & \shortparallel & & \\ & & & & W_j & 1 & & \end{array} \Bigg\} \; |A_k \cap A_j| = (n-2)!$$

By the same reasoning:

$|A_{k_1} \cap A_{k_2} \cap .. \cap A_{k_r}| = (n-r)!$

$\big(|A_1 \cap \dots \cap A_n| = 1 = (n-n)!\big)$

We apply the principle:

$$n! = |\Omega| = |(\Omega^*)^c| + |\underbrace{\Omega^*}_{?}| \Rightarrow |\Omega^*| = n! - |(\Omega^*)^c|$$

$$|(\Omega^*)^c| = \left|\bigcup_{k=1}^{n} A_k\right| = n \cdot (n-1)! - C_{n,2} \cdot (n-2)! + C_{n,3} \cdot (n-3)! - \dots$$

$$\dots (-1)^{r-1} \cdot C_{n,r} \cdot (n-r)! \pm \dots (-1)^{n-1} \cdot 1 =$$

$$= \sum_{i=1}^{n} (-1)^{i-1} \cdot C_{n,i} \cdot (n-i)!$$

$$\Rightarrow |\Omega^*| = n! + \sum_{i=1}^{n} (-1)^i \cdot \binom{n}{i} (n-i)! \Rightarrow$$

$$\Rightarrow |\Omega^*| = \sum_{i=0}^{n} (-1)^i \cdot \binom{n}{i} (n-i)! \Rightarrow$$

$$\Rightarrow \boxed{|\Omega^*| = \sum_{i=0}^{n} (-1)^i \cdot \frac{n!}{i!}}$$

Simpler expression for the principle of exclusion-inclusion:

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{\emptyset \neq I \subseteq \{1,\dots,n\}} (-1)^{|I|-1} |A_I|$$

$(A_I = \bigcap_{i\in I} A_i)$

1.7.1 Euler's totien or ϕ function

Definition:

$n \in \mathbb{N}$.

$A_n = \{k \in \{1,\dots,n\} : k$ is prime with $n\} = \{k \in \{1,\dots,n\} : \gcd(k,n) = 1\}$

$\underbrace{\text{prime with}}_{\text{they are coprimes}}$

(gcd = greater common divisor).

Euler's totien or phi function is defined as:

$$\phi(n) = |A_n|$$

If $n = 1$, $\phi(1) = 1$ $(A_1 = \{1\})$

Example:

$A_{12} = \{1, 5, 7, 11\} \Rightarrow \phi(12) = 4 = |A_{12}|$

Theorem:

$$\phi(n) = n \prod_{i=1}^{\ell} \left(1 - \frac{1}{p_i}\right) = n\left(1 - \frac{1}{p_1}\right) \dots \left(1 - \frac{1}{p_\ell}\right)$$

(if $p_1^{\alpha_1} p_2^{\alpha_2} \dots p_\ell^{\alpha_\ell} = n$)

Proof: $(n = p_1^{\alpha_1} \cdot \ldots \cdot p_e^{\alpha_e})$

Let $A_n = \{k \in \{1, \ldots, n\} : k$ is not a multiple of $p_i \ \forall i \in \{1, \ldots, e\}\}$

and $A_n^c = \{k \in \{1, \ldots, n\} : k$ is a multiple of at least one $p_i \ \forall i \in \{1, \ldots, e\}\}$

Let: $B_i = \{k \in \{1, \ldots, n\} : k$ is a multiple of $p_i\} \ \forall i \in \{1, \ldots, e\}$

Note that: $A_n^c = \bigcup_{i \in I} B_i$

(E.g. $n = 12 = 2^2 \cdot 3 \Rightarrow 4 \in A_n^c,\ 4 \in B_2,\ 4 \in B_4,\ 6 \in B_2,\ 6 \in B_3,\ 6 \in B_6,$

Let's find out it's cardinality.

$|B_i| = \lfloor n/p_i \rfloor$ (we have already compute it).

Since p_i divides $n \Rightarrow |B_i| = n/p_i \quad \forall i \in \{1, \ldots, e\}$

This way:

$|B_i \cap B_j| = \lfloor n/p_i \cdot p_j \rfloor = \dfrac{n}{p_i \cdot p_j}$ (Multiples of p_i and p_j are multiples of their minimum common multiple: $p_i \cdot p_j = \underbrace{mcm(p_i, p_j)}_{\text{they are primes!}}$)

We generalize:

$|B_{i_1} \cap B_{i_2} \cap \ldots \cap B_{i_r}| = n \cdot (p_{i_1} \cdot \ldots \cdot p_{i_r})^{-1}$

And:

$|B_1 \cap \ldots \cap B_e| = n \cdot (p_1 \cdot \ldots \cdot p_e)^{-1}$

We apply the formula:

$$|A_n^c| = \left|\bigcup_{i=1}^{e} B_i\right| = \sum_{i=1}^{e} \frac{n}{p_i} - \sum_{1 \le i < j \le e} \frac{n}{p_i \cdot p_j} + \ldots + (-1)^{r-1} \cdot \sum_{1 \le i_1 < \ldots < i_r \le e} \frac{n}{p_{i_1} \cdots p_{i_r}} + \ldots$$

$$+ (-1)^{e-1} \cdot \frac{n}{p_1 \cdot \ldots \cdot p_e}$$

Note: if $k \in \{1, \dots, n\}$, $k \in A_n$ or A_n^c $\Rightarrow$ $|A_n| + |A_n^c| = n$

$$\Rightarrow \phi(n) = |A_n| = n - |A_n^c| = n\left[1 - \sum_{i=1}^{l} \frac{1}{p_i} + \dots \right.$$

$$\left. \dots (-1)^{r+1} \sum_{1 \le i_1 < \dots < i_r \le l} \frac{1}{p_{i_1} \cdot \dots \cdot p_{i_r}} + \dots + (-1)^l \cdot \frac{1}{p_1 \cdot \dots \cdot p_l}\right] =$$

$$= n \cdot (1 - 1/p_1)(1 - 1/p_2) \cdot \dots \cdot (1 - 1/p_l)$$

$\downarrow$

To be proven by induction:

- $l = 1$:

$(1 - 1/p_1)$

- $l = 2$:

$$(1 - 1/p_1)(1 - 1/p_2) = 1 - 1/p_1 - 1/p_2 + \frac{1}{p_1 \cdot p_2}$$

- $l = 3$:

$$(1 - 1/p_1)(1 - 1/p_2)(1 - 1/p_3) = \left[1 - 1/p_1 - 1/p_2 + \frac{1}{p_1 p_2}\right]\left(-1/p_3 + 1\right) =$$

$$= 1 - 1/p_1 - 1/p_2 + \frac{1}{p_1 p_2} - 1/p_3 + \frac{1}{p_1 p_3} + \frac{1}{p_2 p_3} - \frac{1}{p_1 p_2 p_3} =$$

$$= 1 - \frac{1}{p_1} - \frac{1}{p_2} - \frac{1}{p_3} + \frac{1}{p_1 p_2} + \frac{1}{p_1 p_3} + \frac{1}{p_2 p_3} - \frac{1}{p_1 p_2 p_3}$$

... (can be proven by induction).

$\Longrightarrow$

$\ell = n$:

$$(1-\tfrac{1}{p_1})\cdot\ldots\cdot(1-\tfrac{1}{p_{\ell-1}})(1-\tfrac{1}{p_\ell}) \overset{\substack{\text{Hip.}\\ \uparrow}}{=}$$

$$= (1-\tfrac{1}{p_\ell})\cdot\left(1-\sum_{i=1}^{\ell-1}\frac{1}{p_i}+\ldots+(-1)^{r-1}\cdot\sum_{1\le i_1<\ldots<i_r\le \ell-1}\left(\frac{1}{p_{i_1}\cdot\ldots\cdot p_{i_r}}\right)+\frac{(-1)^{\ell-2}\cdot 1}{(p_1\cdot\ldots\cdot p_{\ell-1})}\right)$$

$$= 1-\underbrace{\sum_{i=1}^{\ell-1}\frac{1}{p_i}-\frac{1}{p_\ell}}+\ldots\underbrace{(-1)^{r-1}\cdot\sum_{1\le i_1<\ldots<i_r\le \ell-1}\left(\frac{1}{p_{i_1}\cdot\ldots\cdot p_{i_r}}\right)+(-1)^{\cancel{r-2+1}\, r-1}\cdot\sum_{1\le i_1<\ldots<i_{r-1}\le \ell-1}\frac{1}{p_\ell}\cdot\frac{1}{p_{i_1}\cdot\ldots\cdot p_{i_{r-1}}}}$$

$$+\ldots+\frac{(-1)^{\ell-1}\cdot 1}{p_1\cdot\ldots\cdot p_{\ell-1}\cdot p_\ell} =$$

$$= 1-\sum_{i=1}^{\ell}\frac{1}{p_i}+\ldots+(-1)^{r-1}\sum_{1\le i_1<\ldots<i_r\le \ell-1}\frac{1}{p_{i_1}\cdot\ldots\cdot p_{i_r}}+\ldots+\frac{(-1)^{\ell-1}\cdot 1}{p_1\cdot\ldots\cdot p_\ell} \quad //$$

$$\Rightarrow \quad \phi(n) = n\cdot\prod_{i=1}^{\ell}(1-\tfrac{1}{p_i}) \quad \square$$

1.8 Translations

Definition:

To translate a problem is to state an equivalent problem in other terms.

Example:

Point in $\mathbb{R}^2 \Leftrightarrow (x,y)$ pairs of $\mathbb{R}\times\mathbb{R}$

geometric $\Leftrightarrow$ algebraic

Example: (Problem $\leftrightarrow$ Mapping) (Mapping)

(B1) ... (Bm)

$\boxed{1}$ ... $\boxed{n}$

$$f : \{1, \dots, m\} \to \{1, \dots, n\}$$
$$i \longmapsto f(i) = j$$
$$(B_i \in \boxed{j})$$

But if we don't more than 1 ball in a box:

$f : \{1, \dots, m\} \to \{1, \dots, n\}$ injective ($\Rightarrow m \le n$)

Without exclusion: no box empty:

$f : \{1, \dots, n\} \to \{1, \dots, n\}$ surjective ($\Rightarrow n \le m$)

(Bijective mappings: $m = n$)

Example: (Mapping $\leftrightarrow$ Sequences)

$$f : \{1, \dots, m\} \longrightarrow \{1, \dots, n\}$$
$$i \longmapsto f(i) = j$$

$$\underset{1}{\underline{f(1)}} \quad \underset{2}{\underline{f(2)}} \quad \underset{3}{\underline{f(3)}} \quad \dots \quad \underset{m}{\underline{f(m)}}$$

- For a general sequence with repetition:
  $n \cdot n \cdot \dots \cdot n = n^m$ possible mappings. ($VR_{n,m}$)
- If we want them to be injective, similar but without repetition:
  $n(n-1) \cdot \dots \cdot n-(m-1) = V_{n,m} = \frac{n!}{(n-m)!}$ (possible mappings!)

In the case of the surjective:
We don't know how the express it yet.

In the case of bijective:
$n \cdot (n-1) \cdot \ldots \cdot 1 = n! = P_n$ (possible sequences and mappings!)

We have:

$[m] = \{1, \ldots, m\}$

m dist. balls n numb. boxes	Mappings, from $[m]$ to $[n]$	Sequences, of length m formed by $[n]$	Quantity,
Placements without exclusion	All	reps. allowed	$VR_{n,m}$
Placements with exclusion	Injective	without reps	$V_{n,m}$
Plac. without exc. nonempty boxes	Surjective	reps allowed. each element at least once	$n! \cdot \left\{ {m \atop n} \right\}$

$\left\{ {m \atop n} \right\}$: Stirling number of the second kind.

Example: (Subsets $\Leftrightarrow$ sequences of 0s and 1s)

$\Omega = \{1,2,3,4,5\}$
$\emptyset \Leftrightarrow 00000$
$\{2\} \Leftrightarrow 01000$
$\{2,5\} \Leftrightarrow 01001$

$\Rightarrow$ Bijection subsets of Ω and sequences of length $|\Omega|$ formed with 0s and 1s. Lets prove it. ($VR_{2,n} = 2^n$ for both cardinalities). $|A| = |B| \Rightarrow$ there is a bijection.

Example: (Subsets $\leftrightarrow$ sequences of 0s and 1s)

Subset of m elements of Ω $\Leftrightarrow$ of length n formed by m ones and $n-m$ zeros.

For both cases $|A| = |B| = C_{n,m} = \binom{n}{m}$ (n choose m)

$\Rightarrow$ There is a bijection.

Example: (Multisets $\leftrightarrow$ sequences of 0s and 1s)

(Example page 24)

Multisets of elements of Ω $\Leftrightarrow$ sequence of length $m+n-1$ formed by m zeros and $n-1$ ones.

For both cases $|A| = |B| = C_{n+m-1,\, m} = \binom{n+m-1}{m}$

$\Rightarrow$ There is a bijection.

Example: (Placement of indistinguishable balls $\Leftrightarrow$ sequences of 0s and 1s).

$$\boxed{0 \,\big|\, 00 \,\big|\, 0} \leftrightarrow 0110010$$

- Placement without inclusion $\Leftrightarrow$ sequences formed by m zeros and $n-1$ ones.

$$\underbrace{\_\ \_\ \_\ \_\ \_\ \_\ \_\ \_}_{m+n-1} \longrightarrow C_{m+n-1,\, m} = \binom{n+m-1}{m} = CR_{n,m}$$

- Placement with exclusion $\Leftrightarrow$ sequences formed by m zeros and $n-1$ ones with no consecutive zero

$\sim 1 \sim 1 \sim \ldots\ldots \sim 1 \sim \quad \to \quad \binom{n}{m}$ options

$(n-1\,(+1) = n$ places for $0)$

- Without exclusion but no box empty $\Leftrightarrow$ sequence formed by m zeros and $n-1$ ones which no consecutive ones and starting with 0

~~[illegible]~~ $\to 0 \sim 0 \sim 0 \sim \ldots \sim 0 \sim 0$

Whe have $m-1$ possible places for 1s $\Rightarrow$

$$\Rightarrow \binom{m-1}{n-1} = C_{m-1,\,n-1}$$

<u>Example:</u> (Placements of indistinguishable balls $\leftrightarrow$ solutions of an equation)

- Without exclusion $\Leftrightarrow$ solutions of $x_1 + \ldots + x_n = m$; $\underset{\smile}{x_i} \in \mathbb{N}^*$

they can be 0

$$\Rightarrow \binom{m+n-1}{m} \qquad (m \geq n)$$

- Whith exclusion $\Leftrightarrow$ each ball has at most one ball $\Leftrightarrow$

$\Leftrightarrow x_1 + \ldots + x_n = m$, $x_i \in \{0,1\}$ $\quad (n \geq m) \Rightarrow \binom{n}{m}$

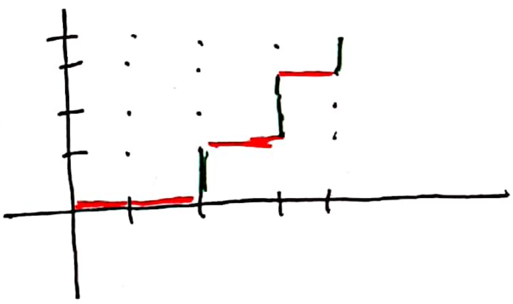
- Withot exclusion in which none box is empty $\Leftrightarrow$ solitions of $x_1 + \ldots + x_n = m$, $x_i \in \mathbb{N}$ $\Rightarrow \binom{m-1}{n-1}$

("How much we put inside x_i" $\rightsquigarrow$ how many balls inside a box $\to$ $\underbrace{1}_{x_1}\underbrace{0000}_{x_2}1 \ldots$ back to sequences).

$[n] = \{1, \dots, n\}$

m indist. balls n indist. boxes	Sequences of m 0s, $(n-1)$ 1s	Solutions of $x_1+\dots+x_n = m$	m elements of $[n]$	Quantity
Placements without exc.	All	$x_i \in \mathbb{N}^*$	Multisets	$CR_{n,m}$
Placements with exclusion	no consec. 0s	$x_i \in \{0,1\}$	Subsets	$C_{n,m}$
Plc. without exc. nonempty boxes	no consec. 1s, start and finish by 0	$x_i \in \mathbb{N}$	Multisets with all $[n]$	$CR_{n,m-n}$

Example: (H - V trajectories $\leftrightarrow$ sequences of 0s and 1s)



V step: $(x, y) \to (x, y+1)$

H step: $(x, y) \to (x+1, y)$

Let $T^{(p,q)}_{(0,0)}$ = set of H-V trajectories from $(0,0)$ to (p,q), with $p, q \in \mathbb{N}$. Put 0 for H and 1 for V, we get a sequence of p zeros and q 1s. E.g. (0010101).

Bijection:

$$T^{(p,q)}_{(0,0)} \to \{\text{sequence formed with } p \text{ 0s and } q \text{ 1s}\}$$

$$\left|T^{(p,q)}_{(0,0)}\right| = | \quad '' \quad | = \binom{p+q}{p} = \binom{p+q}{q}$$

Example: (U-D trajectorie $\leftrightarrow$ sequences formed by 0s and 1s)

U-D trajectory: continuous line formed by steps of the following types:

U-step: $(x,y) \to (x+1, y+1)$
D-step: $(x,y) \to (x+1, y-1)$

Let $\Theta_{(0,0)}^{(p,q)}$ = set of U-D trajectories from $(0,0)$ to (p,q) with $p,q \in \mathbb{N}$.

Let x = number of U and y = number of D then:

$$\begin{cases} x+y = p \\ x-y = q \end{cases} \Rightarrow x = \frac{p+q}{2} \text{ and } y = \frac{p-q}{2}$$

We observe:

$y \in \mathbb{N} \Rightarrow p \geq q$

$$\begin{cases} y \in \mathbb{N} \\ x \in \mathbb{N} \end{cases} \Rightarrow \begin{matrix} p+q & \text{even} & \Rightarrow p,q \text{ even or } p,q \text{ odd.} \\ p-q & \text{even} & \Rightarrow p,q \text{ ever or } p,q \text{ odd.} \end{matrix}$$

If the conditions are not met:

$\left|\Theta_{(0,0)}^{(p,q)}\right| = 0$

If they are met:
We translate this problem into sequences of 1s and 0s (0 are the Us and 1 the Ds). Bijective.

As we have stated:

$$\begin{cases} n^{\underline{o}} \text{ of } 1s \rightarrow y = \frac{p-q}{2} \\ n^{\underline{o}} \text{ of } 0s \rightarrow x = \frac{p+q}{2} \end{cases} \rightarrow C_{p,\frac{p+q}{2}} = \binom{p}{\frac{p+q}{2}}$$

$$\frac{p+q}{2} = x$$

$$p = \frac{p-q}{2} + \frac{p+q}{2} = x + y$$

$$\Rightarrow \left| \Theta_{(0,0)}^{(p,q)} \right| = \binom{p}{\frac{p+q}{2}} \longleftarrow \left(C_{x+y,y} = \binom{x+y}{y} = \binom{x+y}{x} \right)$$

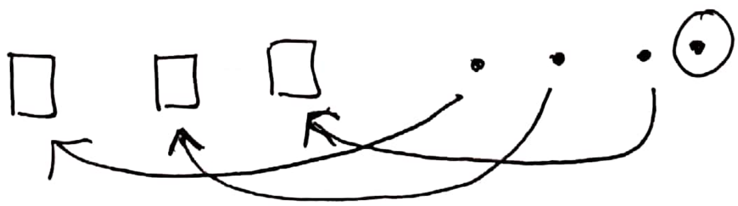
(put it in terms of p and q)

1.9 Dirichlet pigeonhole principle and the handshake lemma

Definition:

Dirichlet pigeonhole principle: if $m > n$, there is not any injective mapping from $\{a_1, \dots, a_m\}$ to $\{b_1, \dots, b_n\}$

(If m pigeons occupy n pigeonholes and $m > n$ then at least one will house at least two pigeons).



Must repeat house!

Example:

If a person can have at most $6 \cdot 10^6$ hairs on the head and in New York live more than $6 \cdot 10^6$ persons, applying the pigeonhole principle, at least two persons must have the same number of hairs on their head.

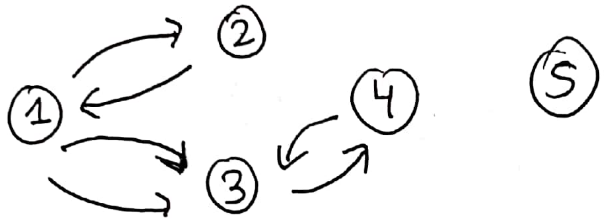
Lemma:

Handshake lemma: Let n be the number of guests in a party. The number of persons that shake hands with an $\underbrace{\text{odd}}$ number of persons is $\underbrace{\text{even}}$.

$\underbrace{3 \text{ letters} \to 3 = 1 \bmod (2)}_{\text{impar}}$

$\underbrace{4 \text{ letters} \to 4 \equiv 0 \bmod (2)}_{\text{par}}$

Proof:



$A = \{$ persons that shake hands with an odd number of persons $\}$

$|A|$ even?

Define $l_i = \underbrace{\#}_{\text{number}}$ of persons that guest i shakes with, $i = 1, \dots, n$.

$A = \{ i \in \{1, \dots, n\}$ such that l_i is odd $\}$

Then:

$\sum_{i=1}^{n} l_i$ must be even (it's the same as the sum of arrows, which come in pairs $\Rightarrow$ even)

$$\text{even} \to \sum_{i=1}^{n} l_i = \sum_{i \in A} l_i + \underbrace{\sum_{i \in A^c} \underbrace{l_i}_{\text{even}}}_{\text{even}} \Rightarrow$$

$$\Rightarrow \underbrace{\sum_{i \in A} \underbrace{l_i}_{\text{odd}}}_{\text{even}} = \underbrace{\sum_{i=1}^{n} l_i}_{\text{even}} - \underbrace{\sum_{i \in A^c} l_i}_{\text{even}} \Rightarrow$$

A sum of odd numbers that is even $\Rightarrow$ we are summing an even number of elements

(can also be proven this last part by contradiction)

$\Rightarrow |A|$ is even $\square$