Problems 1.1 (My attempts)

$\boxed{4}$

$f:[0,\infty)\to\mathbb{R}$, conticuous and increasing and fulfils that:

- $f(x)\in\mathbb{Z}\Rightarrow x\in\mathbb{Z}$

(a) Prove that $\lfloor f(x)\rfloor=\lfloor f(\lfloor x\rfloor)\rfloor$

√ (i) Suppose $x\in\mathbb{Z}$:

$x=\lfloor x\rfloor \Rightarrow f(x)=f(\lfloor x\rfloor)\Rightarrow \lfloor f(x)\rfloor=\lfloor f(\lfloor x\rfloor)\rfloor$

√ (ii) Suppose $x\in\mathbb{R}$, $x\notin\mathbb{Z}$.

$\lfloor x\rfloor<x \Rightarrow f(\lfloor x\rfloor)<f(x)\Rightarrow \lfloor f(\lfloor x\rfloor)\rfloor\leq f(\lfloor x\rfloor)<f(x)$, since the function is continous and $\lfloor x\rfloor\in\mathbb{Z}$ might imply that $\lfloor f(\lfloor x\rfloor)\rfloor=f(\lfloor x\rfloor)$, but we don't know for sure. Let's suppose that $\exists n\in\mathbb{Z}:\ \underbrace{\lfloor f(\lfloor x\rfloor)\rfloor}_{\in\mathbb{Z}}\leq f(\lfloor x\rfloor)<\underbrace{f(n)}_{\in\mathbb{Z}}=\underbrace{\lfloor f(\lfloor x\rfloor)\rfloor+1}_{\in\mathbb{Z}}$

- If $f(x)<f(n)\Rightarrow \lfloor f(x)\rfloor=\lfloor f(\lfloor x\rfloor)\rfloor$.
- If $\lfloor f(\lfloor x\rfloor)\rfloor\leq f(\lfloor x\rfloor)<f(n)<f(x)$, then,

$\exists m\mid f(m)=\lfloor f(\lfloor x\rfloor)\rfloor \Rightarrow \underbrace{m}_{\in\mathbb{Z}}\leq\lfloor x\rfloor<\underbrace{n}_{\in\mathbb{Z}}<x$ and this is a contradiction #.

Hence, $f(x)<f(n)\Rightarrow \lfloor f(x)\rfloor=\lfloor f(\lfloor x\rfloor)\rfloor$ □

(It must be $\underbrace{f(m)}_{\in\mathbb{Z}}\leq f(\lfloor x\rfloor)<f(x)<\underbrace{f(m)+1}_{\in\mathbb{Z}}=f(n)$)

(b) Prove that $\lceil f(x) \rceil = \lceil f(\lceil x \rceil) \rceil$

(i)
If $x \in \mathbb{Z} \Rightarrow x = \lceil x \rceil \Rightarrow f(x) = f(\lceil x \rceil) \Rightarrow$

$\Rightarrow \lceil f(x) \rceil = \lceil f(\lceil x \rceil) \rceil$

(ii)
If $x \in \mathbb{R},\ x \notin \mathbb{Z} \Rightarrow \lceil x \rceil - 1 < x < \lceil x \rceil$

We focus that:

$\lceil f(\lceil x \rceil) \rceil - 1 < f(\lceil x \rceil) \leq \lceil f(\lceil x \rceil) \rceil$

$\exists m \in \mathbb{Z} \mid f(m) = \lceil f(\lceil x \rceil) \rceil - 1$ we have two possibilites:

- $f(m) < f(x) \Rightarrow \lceil f(\lceil x \rceil) \rceil - 1 < f(x) < f(\lceil x \rceil) \leq \lceil f(\lceil x \rceil) \rceil \Rightarrow$

  $\Rightarrow \lceil f(x) \rceil = \lceil f(\lceil x \rceil) \rceil$

- $f(m) > f(x) \Rightarrow$ because f increases, that:

  $x < m < \lceil x \rceil$ $\#$ $\Rightarrow f(m) < f(x) \Rightarrow$

$\Rightarrow \lceil f(x) \rceil = \lceil f(\lceil x \rceil) \rceil$ □

$\boxed{3}$

(a) Is it true that $\lfloor \sqrt{\lfloor x \rfloor} \rfloor = \lfloor \sqrt{x} \rfloor,\ \forall x \geq 0$?

We take the highest perfect square that is smaller than x, m^2. Since it is the highest, x must be smaller than the next perfect square (otherwise, contradiction first statement), $(m+1)^2$, so:

$m^2 \leq \lfloor x \rfloor < x < (m+1)^2 \Rightarrow m \leq \sqrt{\lfloor x \rfloor} < \sqrt{x} < m+1 \quad (1)$

(Note: we suppose $x \in \mathbb{R},\ x \notin \mathbb{Z}$, otherwise trivial).

$\boxed{2}$

$(1) \Rightarrow \lfloor \sqrt{\lfloor x \rfloor} \rfloor = \lfloor \sqrt{x} \rfloor = m \quad \square$

(b) Is it true that $\lceil \sqrt{\lceil x \rceil} \rceil = \lceil \sqrt{x} \rceil$, $\forall x \geq 0$?

We use a similar reasoning as before:

$$(m-1)^2 < x < \lceil x \rceil \leq (m)^2 \Rightarrow m-1 < \sqrt{x} < \sqrt{\lceil x \rceil} \leq m$$

$$\Rightarrow \lceil \sqrt{x} \rceil = \lceil \sqrt{\lceil x \rceil} \rceil = m \quad \square$$

$\boxed{7}$ (I)

Find a necessary and sufficient condition so that:

$$\lceil \sqrt{\lfloor x \rfloor} \rceil = \lceil \sqrt{x} \rceil, \quad \forall x \geq 0$$

Similarly to problem 3: ($x \in \mathbb{Z}$ trivial).

$$m^2 \leq \lfloor x \rfloor < x < \lceil x \rceil \leq (m+1)^2 \Rightarrow$$

$$\Rightarrow m \leq \sqrt{\lfloor x \rfloor} < \sqrt{x} < \sqrt{\lceil x \rceil} \leq m+1 \Rightarrow$$

$\Rightarrow$ If $m = \sqrt{\lfloor x \rfloor} \Rightarrow \lceil m \rceil = m = \lceil \sqrt{\lfloor x \rfloor} \rceil < \lceil \sqrt{x} \rceil = m+1$

If $m < \sqrt{\lfloor x \rfloor} \Rightarrow \lceil \sqrt{\lfloor x \rfloor} \rceil = \lceil \sqrt{x} \rceil = m+1$

So in conclusion: (being m^2 the highest perfect square before x)

Given that:

$$m^2 \leq \lfloor x \rfloor < x < \lceil x \rceil \leq (m+1)^2$$

Then:

If $m^2 < \lfloor x \rfloor \Rightarrow \lceil \sqrt{\lfloor x \rfloor} \rceil = \lceil \sqrt{x} \rceil = m+1$

If $(m+1)^2 > \lceil x \rceil \Rightarrow \lfloor \sqrt{\lceil x \rceil} \rfloor = \lfloor \sqrt{x} \rfloor = m$

$\boxed{6}$ ✓

Proven in exercise 7. (Not always true, they need 7s) conditions).

As an addition to 7, no matter what; for every $x \geq 0$

$\lfloor \sqrt{\lfloor x \rfloor} \rfloor = \lfloor \sqrt{x} \rfloor$ and $\lceil \sqrt{\lceil x \rceil} \rceil = \lceil \sqrt{x} \rceil$

$\boxed{7}$ (II) ✓

Necessary and sufficient condition that:

$$\lceil \sqrt{\lfloor x \rfloor} \rceil = \lceil \sqrt{x} \rceil, \quad \forall x \geq 0$$ (Note: $x \in \mathbb{Z}^+$ is trivial)

Let m^2 be the highest possible perfect square before x; then:

$$m^2 \leq \lfloor x \rfloor < x < (m+1)^2$$

- Suppose $\lfloor x \rfloor = m^2 \Rightarrow \sqrt{\lfloor x \rfloor} = m < \sqrt{x} < m+1 \Rightarrow$

$$\Rightarrow \lceil \sqrt{\lfloor x \rfloor} \rceil = m \neq m+1 = \lceil \sqrt{x} \rceil$$

- Suppose $\lfloor x \rfloor > m^2 \Rightarrow m < \sqrt{\lfloor x \rfloor} < \sqrt{x} < m+1 \Rightarrow$

$$\Rightarrow \lceil \sqrt{\lfloor x \rfloor} \rceil = \lceil \sqrt{x} \rceil = m+1$$ as we wanted

So for $x \in \mathbb{R}$, if $m^2 < \lfloor x \rfloor < x < (m+1)^2 \Rightarrow$

$$\Rightarrow \lceil \sqrt{\lfloor x \rfloor} \rceil = \lceil \sqrt{x} \rceil = m+1.$$

In other words, if $\lfloor x \rfloor \neq m^2$ with m^2 a perfect square, then: $\lceil \sqrt{\lfloor x \rfloor} \rceil = \lceil \sqrt{x} \rceil$

$\boxed{1}$

(a) $\forall x, y \in \mathbb{R}, \ \lfloor x+y \rfloor \geq \lfloor x \rfloor + \lfloor y \rfloor$ (superadditive)

$$\begin{matrix} x \geq \lfloor x \rfloor \\ y \geq \lfloor y \rfloor \end{matrix} \Rightarrow x+y \geq \lfloor x \rfloor + \lfloor y \rfloor$$

- $\lfloor x \rfloor + \{x\} + \lfloor y \rfloor + \{y\} = x+y$

If $0 \leq \{x\} + \{y\} < 1 \Rightarrow \lfloor x+y \rfloor = \lfloor \underbrace{\lfloor x \rfloor + \lfloor y \rfloor}_{\in \mathbb{Z}} \rfloor = \lfloor x \rfloor + \lfloor y \rfloor$ $\Uparrow$

If $1 \leq \{x\} + \{y\} < 2 \Rightarrow x+y = \lfloor x \rfloor + \lfloor y \rfloor + 1 + \{z\} \Rightarrow$

$\Rightarrow \lfloor x+y \rfloor = \lfloor \underbrace{\lfloor x \rfloor + \lfloor y \rfloor + 1}_{\in \mathbb{Z}} \rfloor = \lfloor x \rfloor + \lfloor y \rfloor + 1 > \lfloor x \rfloor + \lfloor y \rfloor$ $\Rightarrow$

$\Rightarrow \forall x, y \in \mathbb{R}, \ \lfloor x+y \rfloor \geq \lfloor x \rfloor + \lfloor y \rfloor$ $\square$

($n \leq x < n+1$; $m \leq y < m+1 \Rightarrow \lfloor x \rfloor + \lfloor y \rfloor = m+n$ but

$n+m \leq x+y < n+m+2 \Rightarrow$ if $n+m+1 \leq x+y < n+m+2 \Rightarrow$

$\Rightarrow \lfloor x+y \rfloor = n+m+1$ (or $m+n$ in the other case) $\Rightarrow$

$\Rightarrow \lfloor x+y \rfloor = \begin{cases} n+m+1 \\ n+m \end{cases} \geq \lfloor x \rfloor + \lfloor y \rfloor = n+m$ $\square$)

(b) $\forall x, y \in \mathbb{R}, \ \lceil x+y \rceil \leq \lceil x \rceil + \lceil y \rceil$ (subadditive)

$$\begin{matrix} n-1 < x \leq n \Rightarrow \lceil x \rceil = n \\ m-1 < y \leq m \Rightarrow \lceil y \rceil = m \end{matrix} \Rightarrow \lceil x \rceil + \lceil y \rceil = n+m$$

$$n+m-2 < x+y \leq n+m \begin{cases} \text{(i)} \ n+m+1 < x+y \leq n+m \Rightarrow \lceil x+y \rceil = n+m \checkmark \\ \text{(ii)} \ n+m-2 < x+y \leq n+m+1 \Rightarrow \lceil x+y \rceil = n+m-1 \leq n+m \end{cases}$$

$\Rightarrow \lceil x+y \rceil \leq \lceil x \rceil + \lceil y \rceil$ $\square$

Given $x, y \in \mathbb{R}$ so that;

$n \le x \le n+1$ and $m \le y \le m+1$

(1)

If $n+m \le x+y < n+m+1 \Rightarrow \lfloor x+y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor$

Else: $\lfloor x+y \rfloor > \lfloor x \rfloor + \lfloor y \rfloor$

(2)

If $n+m-1 < x+y \le n+m \Rightarrow \lceil x+y \rceil = \lceil x \rceil + \lceil y \rceil$

Else: $\lceil x+y \rceil < \lceil x \rceil + \lceil y \rceil$

Another proof for b using the floor and the $\underbrace{\text{pseudofractional part of } x}_{\langle x \rangle}$:

$x, y \in \mathbb{R}$

$$\begin{matrix} n \le x \le n+1 \\ m \le y \le m+1 \end{matrix} \Rightarrow x+y \le \lceil x \rceil + \lceil y \rceil \quad (x \le \lceil x \rceil \text{ and } y \le \lceil y \rceil)$$

$$x+y = \lceil x \rceil - \langle x \rangle + \lceil y \rceil - \langle y \rangle$$

- if $\langle x \rangle + \langle y \rangle \in [0, 1) \Rightarrow$

$\lceil \lceil x \rceil + \lceil y \rceil - (\langle x \rangle + \langle y \rangle) \rceil \Rightarrow$

$$\Rightarrow \lceil \underbrace{\lceil x \rceil + \lceil y \rceil - 0}_{\in \mathbb{Z}} \rceil \ge \lceil \lceil x \rceil + \lceil y \rceil - (\langle x \rangle + \langle y \rangle) \rceil > \lceil \underbrace{\lceil x \rceil + \lceil y \rceil - 1}_{\in \mathbb{Z}} \rceil$$

$= \lceil x \rceil + \lceil y \rceil$ $\qquad$ $= \lceil x \rceil + \lceil y \rceil - 1$

$$\Rightarrow \lceil x+y \rceil = \lceil x \rceil + \lceil y \rceil$$

- if $\langle x \rangle + \langle y \rangle \in [1, 2) \Rightarrow$

$$\lceil \underbrace{\lceil x \rceil + \lceil y \rceil - 1}_{\in \mathbb{Z} \Rightarrow = \lceil x \rceil + \lceil y \rceil - 1} \rceil \ge \lceil \underbrace{\lceil x \rceil + \lceil y \rceil - (\langle x \rangle + \langle y \rangle)}_{x+y} \rceil > \lceil \underbrace{\lceil x \rceil + \lceil y \rceil - 2}_{\in \mathbb{Z} \Rightarrow = \lceil x \rceil + \lceil y \rceil - 2} \rceil$$

$\Rightarrow \lceil x \rceil + \lceil y \rceil - 1 = \lceil x+y \rceil < \lceil x \rceil + \lceil y \rceil$

$\Rightarrow \lceil x+y \rceil \le \lceil x \rceil + \lceil y \rceil$

And the above rectangle can also be summarized to:

> Given $x, y \in \mathbb{R}$ so that:
> $n \le x \le n+1$, $m \le y \le m+1$
>
> (1)
> $\{x\} + \{y\} \in [0,1) \Rightarrow \lfloor x+y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor$
> otherwise $\lfloor x+y \rfloor > \lfloor x \rfloor + \lfloor y \rfloor$
>
> (2)
> $\langle x \rangle + \langle y \rangle \in [0,1) \Rightarrow \lceil x+y \rceil = \lceil x \rceil + \lceil y \rceil$
> otherwise $\lceil x+y \rceil < \lceil x \rceil + \lceil y \rceil$

8

$a, b \in \mathbb{Z}$; $a \le b$. How many integers in : ?

a) $(a, b]$

$\{A\} = \{a, a+1, \dots, b-1, b\} = \{a, \dots, (b-a)+a\}$ but since $\{a\}$ has 1 elements $\Rightarrow$ $\{a, \dots, a+k\}$ has $k+1$ $\Rightarrow$ $\{a, \dots, a+(b-a)\}$ has $|A| = b-a+1$ elements.

In our case, a is not included $\Rightarrow$

$\Rightarrow |A| - 1 = \boxed{b-a}$ elements ✓

b)

Similar to a) but this time we need to specify b is not included $\Rightarrow$

$|A| - 1 = \boxed{b - a}$ ✓

c)

None of a or b is included $\Rightarrow$

$|A| - 2 = \boxed{b - a - 1}$ ✓

d)

Both included $\Rightarrow$

$|A| = \boxed{b - a + 1}$ ✓

9

Same as 8, but now $a, b \in \mathbb{R}$

We are going to do general $|A|$ and then restrict. Suppose $a, b \notin \mathbb{Z}$ then:

$\lfloor a \rfloor$ would work $a \in \mathbb{Z}$, but not for real numbers because $\lfloor x \rfloor \notin [x, x+1]$

$A = \{z \in [a, b] \mid z \in \mathbb{Z}\} = \lceil a \rceil, \lceil a \rceil + 1, \dots, \lfloor b \rfloor - 1, \lfloor b \rfloor\}$ similar to before $\Rightarrow |A| = \lfloor b \rfloor - \lceil a \rceil + 1$,

(d) $[a, b]$

$|A| = \boxed{\lfloor b \rfloor - \lceil a \rceil + 1}$ ✓ ($a, b \in \mathbb{Z}$ then $\lceil a \rceil = a$, so the formula its still valid (and $\lfloor b \rfloor = b$ if $b \in \mathbb{Z}$))

(c) (a, b)

Note we can't use $\lfloor b \rfloor$ or $\lceil a \rceil$ anymore, because it wouldn't be valid if $a, b \in \mathbb{Z}$, since $a \notin A$ and $b \notin B$, so we use:

$\lfloor a \rfloor + 1$ and $\lceil b \rceil - 1$, that would be valid for any $a, b \in \mathbb{R}$

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(c) So this way:

$$|A| = \lceil b \rceil - 1 - \lfloor a \rfloor - 1 + 1 = \lceil b \rceil - \lfloor a \rfloor - 1$$ ✓

(a) $(a, b]$ Here:

$$A = \{\lfloor a \rfloor + 1, \lfloor a \rfloor + 2, \dots, \lfloor b \rfloor\} =$$

$$= \{\lfloor a \rfloor + 1, \dots, \lfloor a \rfloor + 1 + (\lfloor b \rfloor - \lfloor a \rfloor - 1)$$

So we would have:

$$|A| = \lfloor b \rfloor - \lfloor a \rfloor - 1 + 1 = \boxed{\lfloor b \rfloor - \lfloor a \rfloor}$$ ✓

(b) $[a, b)$

$$A = \{\lceil a \rceil, \lceil a \rceil + 1, \dots, \lceil b \rceil - 1\} =$$

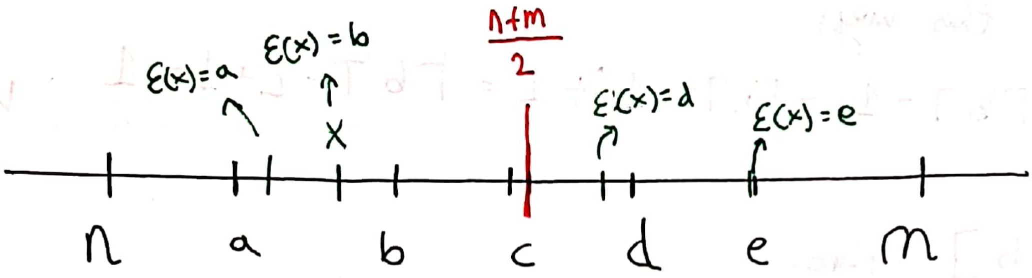
$$= \{\lceil a \rceil, \dots, \lceil a \rceil + \lceil b \rceil - 1 - \lceil a \rceil\}$$

And we add $+1$ because $\{\lceil a \rceil\}$ has 1 element, $\{\lceil a \rceil, \lceil a \rceil + 1\}$ has 2 elements etc $\Rightarrow$

$$\Rightarrow \quad |A| = \lceil b \rceil - 1 - \lceil a \rceil + 1 = \boxed{\lceil b \rceil - \lceil a \rceil}$$ ✓

11 Let $n, m \in \mathbb{Z}$. $\forall x \in [n, m]$ $(n < m)$ so that $x \in \mathbb{R}$ and $x \neq \frac{n+m}{2}$. Which integer $\varepsilon(x)$ it's is closest?

Next page.



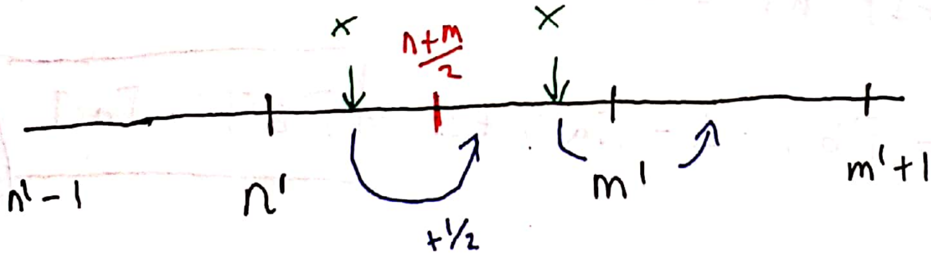
If $x \in \mathbb{Z}$ the solution is trivial ($\mathcal{E}(x) = x = \lfloor x \rfloor = \lceil x \rceil$).

Note that it will be either $\lfloor x \rfloor$ or $\lceil x \rceil$.

If $\{x\} < \langle x \rangle \Rightarrow \mathcal{E}(x) = \lfloor x \rfloor$

If $\langle x \rangle < \{x\} \Rightarrow \mathcal{E}(x) = \lceil x \rceil$

Since x is not the midpoint of any interval with integer endpoints $\Rightarrow \langle x \rangle \neq \{x\}$ (which can only be true for a real number when $\langle x \rangle = 0.5 = \{x\}$, the midpoint of a interval with integer endpoints). We notice that, thus:



If $\mathcal{E}(x) = m' \Rightarrow x > \frac{n'+m'}{2} \Rightarrow m' < x + \frac{1}{2} < m'+1 \Rightarrow \mathcal{E}(x) = \lfloor x + \frac{1}{2} \rfloor$
($\lceil x \rceil$ case)

If $\mathcal{E}(x) = n' \Rightarrow x < \frac{n'+m'}{2} \Rightarrow n' < x + \frac{1}{2} < m' \Rightarrow \mathcal{E}(x) = \lfloor x + \frac{1}{2} \rfloor$

So in any case: $\boxed{\mathcal{E}(x) = \lfloor x + \frac{1}{2} \rfloor}$

(Note that by the same logic, $\mathcal{E}(x) = \lceil x - \frac{1}{2} \rceil$).

$\boxed{2}$

Let's suppose:

$b^k \leq x_b < b^{k+1}$

Note that in this base: y has k digits

$y = d_k d_{k-1} \dots d_1 . 0 \rightarrow y = d_1 \cdot b^0 + \dots + d_k \cdot b^{k-1} \Rightarrow$

$\Rightarrow$ To express b^k in base b we need $k+1$ digits:

$\underbrace{1}_{d_{k+1}} \underbrace{0}_{d_k} \dots \underbrace{0}_{d_1} . 0 \rightarrow y = 0 \cdot b^0 + \dots + 1 \cdot b^k$

So, for x such that $b^k \leq x_b < b^{k+1}$, we will need $k+1$ digit. $\Rightarrow \log_b(b^k) \leq \log_b(x_b) < \log_b(b^{k+1}) \Rightarrow$

$\Rightarrow k \leq \log_b(x_b) < k+1$.

When it's $<$, $\lfloor \log_b(x_b) \rfloor + 1$ and $\lceil \log_b(x_b) \rceil$ both would work, but when it's equal as we've stated, the answer is $k+1$ so $\lceil \log_b(x_b) \rceil = \lceil k \rceil = k \neq k+1$.

That's why we use the other option:

$$\boxed{d_b(x) = \lfloor \log_b(x) \rfloor + 1}$$

Problems 1.3 (my attempts)

$\boxed{14}$

$n \in \mathbb{N}$ has a decomposition of $n = p_1^{m_1} \cdot \ldots \cdot p_k^{m_k}$. How many divisors does it have? Sum of all of them?

~~Let $D = \{p_1, \ldots, p_k\}$ be a set. Any multiset formed with D is a divisor of n, because it will be of the form $p_1^{m_1} \cdot \ldots \cdot p_k^{m_k} \leq n$. The maximum numbers D can have so that $d_D \leq n$ is~~

Let $n = p_1^{m_1} \cdot \ldots \cdot p_k^{m_k}$. Any divisor of n, let's call it d, will be of the form:

$$d = p_1^{m_1'} \cdot \ldots \cdot p_k^{m_k'}, \text{ with } m_i' \in \{0, \ldots, m_i\}$$

So we can represent any divisor just by choosing $m_i \; \forall i \in \{1, \ldots, k\}$. For each i: $\forall i \in \{1, \ldots, k\}$ there are $|\{0, \ldots, m_i\}| = m_i - 0 + 1 = m_i + 1$ options to choose from. Since they are independent events, we apply the product rule, and thus:

$D = \{m \in \mathbb{N} : m \text{ is divisor of } n\}$

(next page).

$$\Rightarrow |D| = (m_1+1) \cdot \ldots \cdot (m_k+1) \Rightarrow$$

$$\Rightarrow \boxed{|D| = \prod_{i=1}^{k} (m_i + i)}$$

For the sum, they ore asking:

$$\sum_{d \in D} d = \sum_{0 \leq i_1, \ldots, i_k \leq m_1, \ldots, m_k} \left(p_1^{i_1} \cdot \ldots \cdot p_k^{i_k} \right) \quad (1)$$

Let's call $\tilde{D} = \{0, \ldots, m_1\} \times \{0, \ldots, m_2\} \times \ldots \times \{0, \ldots, m_k\}$.

For each $\alpha \in \tilde{D}$, we define d_α as: $\prod_{j=1}^{k} p_j^{\alpha_j}$. Then: (1)

$$= \boxed{\sum_{\alpha \in \tilde{D}} d_\alpha = \sum_{\alpha \in \tilde{D}} \left(\prod_{j=1}^{k} (p_j^{\alpha_j}) \right) = \text{sum}(D)}$$

Not useful to compute the number.

(But it's not useful because we haven't said anything)

Note: for a geometric sum:

$$S_N = r^0 + r^1 + \ldots + r^n \Rightarrow$$

$$r \cdot S_N = r^1 + \ldots + r^n + r^{n+1}$$

$$\Rightarrow (1-r) S_n = r^0 - r^{n+1} \Rightarrow$$

$$\Rightarrow S_n = \frac{r^{ko} - r^{n+1}}{1-r}$$

Suppose our $n = p_1^{m_1}$. Then: sum(D)

$$\text{Sum}(D) = \sum_{i=0}^{m_1} p_1^{m_1} = S_{m_1}(p_1) = \frac{1 - p_1^{m_1+1}}{1 - p_1}$$

Suppose our $n = p_1^{m_1} \cdot p_2^{m_2}$. Then:

$$\text{Sum}(D) = p_1^0 \cdot p_2^0 + p_1^0 \cdot p_2^1 + \dots + p_1^0 \cdot p_2^{m_2} + p_1^1 \cdot p_2^0 + \dots$$

$$+ \dots + p_1^{m_1} \cdot p_2^{m_2} =$$

$$= \sum_{\substack{0 \le i \le m_1 \\ 0 \le j \le m_2}} p_1^i \cdot p_2^j = \sum_{i=0}^{m_1} p_1^i \cdot \left(\sum_{j=0}^{m_2} p_2^j \right) \overset{(1)}{=}$$

$$= \sum_{i=0}^{m_1} p_1^i \cdot \left(\frac{1 - p_2^{m_2+1}}{1 - p_2} \right) = \left(\frac{1 - p_2^{m_2+1}}{1 - p_2} \right) \cdot \sum_{i=0}^{m_1} \cdot p_1^i =$$

$$= \left(\frac{1 - p_2^{m_2+1}}{1 - p_2} \right) \left(\frac{1 - p_1^{m_1+1}}{1 - p_1} \right) \overset{(1)}{=} \sum_{i=0}^{m_1} (p_1^i) \cdot \sum_{j=0}^{m_2} (p_2^j).$$

In general:

$$\boxed{\text{Sum}(D)} = \sum_{i_1=0}^{m_1} \left(p_1^{i_1} \cdot \sum_{i_2=0}^{m_2} p_2^{i_2} \cdot \left(\dots \cdot \sum_{i_k=0}^{m_k} p_k^{i_k} \right) \right) \dots \Bigg) =$$

$$= \sum_{i_1=0}^{m_1} p_1^{i_1} \cdot \sum_{i_2=0}^{m_2} p_2^{i_2} \cdot \dots \cdot \sum_{i_k=0}^{m_k} p_k^{i_k} =$$

$$= \prod_{j=1}^{k} \left(\sum_{i_j=0}^{m_j} p_j^{i_j} \right) = \boxed{\prod_{j=1}^{k} \left(\frac{1 - p_j^{m_j+1}}{1 - p_j} \right)}$$

1

9 persons in 3 groups of 3 persons each. Order of groups and persons doesn't matter:

$$\underbrace{a_1 \quad a_2 \quad a_3}_{C_{9,3}} \quad \underbrace{a_4 \quad a_5 \quad a_6}_{C_{6,3}} \quad \underbrace{a_7 \quad a_8 \quad a_9}_{\cancel{C_{3,3}} \to 1}$$

It doesn't matter choosing $a_1a_2a_3$ first or second, here we are counting these options as different.

$$\boxed{\text{Total} = \underbrace{\binom{9}{3}\cdot\binom{6}{3}\cdot\cancel{\binom{3}{3}}^{1}}_{\text{Choosing the 3 groups}} \cdot \frac{1}{3!} = 280}$$

order of elements of groups don't matter ← Choosing the 3 groups

$\frac{1}{3!}$ ↓ we don't care about the order of the group

(This is the same as $\xrightarrow{\text{we don't care order inside groups}}$ $\dfrac{9!}{(3!\cdot 3!\cdot 3!)\cdot 3!}$)

permute all elements ↗ $9!$

$(3!\cdot3!\cdot3!)$: we don't care order inside groups

$3!$: we don't care order of groups.

5

n girls, n boys in a row?

BGBGBG or GBGBGB → 2 options.

For each option, permute all the boys between them $(n!)$ and the girls $(n!)$. By the rule of product:

$$\boxed{\text{Total} = 2\cdot n!\cdot n! = 2\cdot(n!)^2}$$

4

$\boxed{2}$

$\left\{ \begin{array}{l} \left.\begin{array}{l} 3 \text{ boys} \\ 6 \text{ girls} \end{array}\right\} n = 9 \\ \text{Groups of } 3 \rightarrow 1 \text{ boy in each group} \\ \text{(order of selection doesn't matter).} \end{array} \right.$

We choose 3 pair of girls, one for each group:

(G, G)	(G, G)	(G, G)
↓	↓	↓
Group 1	Group 2	Group 3
(Boy 1)	(Boy 2)	(Boy 3)

} For us order matters because we distinguish each group by the boy in it.

$$\Rightarrow \binom{6}{2} \cdot \binom{4}{2} = \boxed{90 \text{ distinct groups}}$$

$\boxed{4}$ ✓

100 cubes. Their sides can be Red, Blue, Green.
80 have at least 1 R, 85 1 B and 75 1 G.
What can we say about the number of cubes that have 1R, 1G and 1B?

R = { cubes with at least one side Red }
B = { " " " " " Blue }
G = { " " " " " Green }

✓

We know:

$$\underbrace{|R \cup B \cup G|}_{100} = \underbrace{|R|}_{80} + \underbrace{|B|}_{85} + \underbrace{|G|}_{75} - \underbrace{|R \cap B|}_{¿?} - \underbrace{|R \cap G|}_{¿?} - \underbrace{|B \cap G|}_{¿?} + \underbrace{|R \cap B \cap G|}_{\text{We want to know this.}}$$

We observe that:

$$\underbrace{|R\cup B\cup G|}_{100} \geq |R\cup B| = |R| + |B| - |R\cap B| \Rightarrow$$

$$\Rightarrow -|R\cap B| \leq 100 - 80 - 85 = -65 \Rightarrow |R\cap B| \geq 65$$

Similarly:

$|R\cap G| \geq 55$ ✓

$|G\cap B| \geq 60$

We go back to the previous expression:

$$100 - 80 - 85 - 75 + |R\cap B| + |R\cap G| + |B\cap G| = |R\cap B\cap G|$$

$$\Rightarrow |R\cap B\cap G| = \underbrace{|R\cap B|}_{\geq 65} + \underbrace{|R\cap G|}_{\geq 55} + \underbrace{|B\cap G|}_{\geq 60} - 140 \geq$$

$$\geq 65 + 55 + 60 - 140 = 40$$

So we can affirm that: $\boxed{|R\cap B\cap G| \geq 40}$ ✓

$\boxed{16}$

Let p be prime $p\in\mathbb{N}$ and $n\in\mathbb{N}$. Find how many times p divides $n!$

$n! = p_1^{m_1}\cdot\ldots\cdot p_n^{m_n} \longrightarrow$ The number we are looking for is m_1.

Note that:

$n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot \ldots \cdot n$

So: $m_n(P_1) = \sum_{i=1}^{n} m_i(P_1)$

Example: $p = 2$, $n = 12$

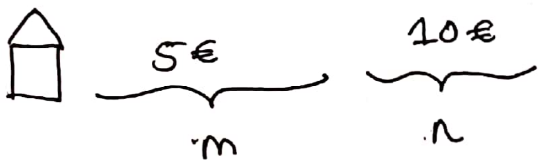
$1 \cdot 2^1 \cdot 3 \cdot \underbrace{2^2}_{4} \cdot 5 \cdot \underbrace{2^1 \cdot 3}_{6} \cdot 7 \cdot \underbrace{2^3}_{8} \cdot 9 \cdot \underbrace{2 \cdot 5^1}_{10} \cdot 11 \cdot \underbrace{2^2 \cdot 3}_{12}$

$\rightarrow 1+2+1+3+1+2 = 10$

We observe: $p^k < n \Rightarrow k < \log_p(n) \Rightarrow k = \lfloor \log_p(n) \rfloor$ is the maximum multiplicity.

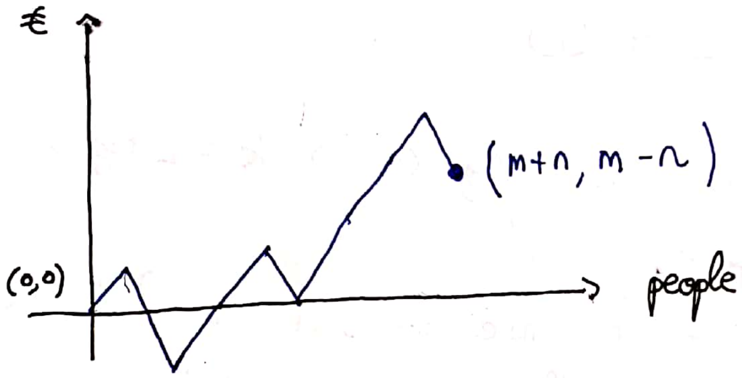
Given n, how many $x < n$ have multiplicity m_i? As many as there are multiples of p^{m_i}. Note that that number is $\lfloor \frac{n}{p^{m_i}} \rfloor$ and that adds a total of $m_i \cdot \lfloor \frac{n}{p^{m_i}} \rfloor$ to our count, but for each multiple of p^{m_i}, it also is a multiple of $p^{m_i - 1}, p^{m_i - 2}, \ldots, p^{m_i - (m_i - 1)}$ $\rightarrow$ total of m_i multiples. So by counting the distinct multiples we arrive to the solution:

$$\boxed{L_n(p) = \sum_{i=1}^{\lfloor \log_p(n) \rfloor} \left\lfloor \frac{n}{p^i} \right\rfloor}$$



$(n \leq m)$

Note for each person with 10€, we need at least one with 5€ before him. We represent the exchange of the ticket office as a function with respect to the number of people that have passed through the office:



For each person of [m], we gain 5€, a exchange. Similarly, for each person of [n], we loose 5€, a exchange. This motivates us to define U and D steps:

$U \to +5€ \longrightarrow (x,y) \longmapsto (x+1, y+1) \to$ number of Us $=a$

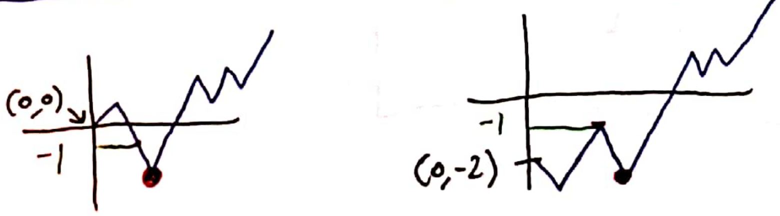
$D \to -5€ \longrightarrow (x,y) \longmapsto (x+1, y-1) \to$ number of Ds $=b$

Total paths:

$$\begin{cases} a+b = m+n \\ a-b = m-n \end{cases} \Rightarrow \begin{cases} a=m \\ b=n \end{cases} \Rightarrow \binom{a+b}{a} = \binom{m+n}{n}$$

Paths that cross OX, $y=-1$, no exchange in certain point:

By symmetry:



We assign to each of these paths one that goes from $(0,-2)$ to $(m+n, m-n)$. Bijection. Thus:

$$\begin{cases} a+b = m+n \\ a-b = m-n+2 \end{cases} \Rightarrow \begin{matrix} (+) \\ (-) \end{matrix}\begin{cases} a = m+1 \\ b = n-1 \end{cases} \Rightarrow \binom{a+b}{a} = \binom{m+n}{n-1}$$

So: total = cross + (no cross) $\Rightarrow$ (no cross) = total − cross

Always with exchange, what we are looking for!

$$S = \binom{m+n}{n} - \binom{m+n}{n-1} = \frac{(m+n)!}{n!\cdot m!} - \frac{(m+n)!}{(n-1)!\cdot(m+1)!} =$$

$$= \frac{(m+n)!}{n!\cdot m!} - \frac{(m+n)!\cdot n}{n!\cdot m!}\cdot\frac{1}{(m+1)} = \frac{(m+n)!}{n!\cdot m!}\left[1 - \frac{n}{m+1}\right] \Rightarrow$$

$\Rightarrow$ There are $\boxed{\binom{m+n}{n}\left[1-\frac{n}{m+1}\right]}$ possible ways to arrange them.

True, but since they are people, the way we arrange between them doesn't matter $\Rightarrow$

$$\Rightarrow \left(\binom{m+n}{n}\left[1-\frac{n}{m+1}\right]\right)\cdot m!\cdot n!$$

$\boxed{6}$ $A, B, C \in \{1, \dots, n\}$. How many ways can we arrange them so that they are always together in order A, B, C?

We suppose $1, \dots, n-2$ people. This is, we have as before $n-3$ people and now we consider A, B, C (in that order) as a block.

So:

$$\underbrace{\underline{(n-2)}\ \underline{(n-3)}\ \underline{(n-4)}\ \underline{(n-5)}\ \dots\ \underline{1}}_{n-2} \rightarrow \boxed{(n-2)! \text{ ways to do it.}}$$

$\boxed{10}$ Subsets of $\{a_1, \dots, a_m, b_1, \dots, b_n\}$ have at least one a_i and one b_j?

We define:

$\Omega = \{a_1, \dots, a_m, b_1, \dots, b_n\}$

$A = \{a_1, \dots, a_m\}$

$B = \{b_1, \dots, b_n\}$

possible pair of subsets (A', B')

The answer is to take every $(A' \neq 0,\ A' \subset A$ and $B' \neq 0,\ B' \subset B$ and join them.

$|A| = m$ (we substract one, we discard the empty set).

$\rightarrow |P(A)| = 2^m \rightarrow |P(A) \setminus \emptyset| = 2^m - 1$ similarly:

$|P(B) \setminus \emptyset| = 2^n - 1$

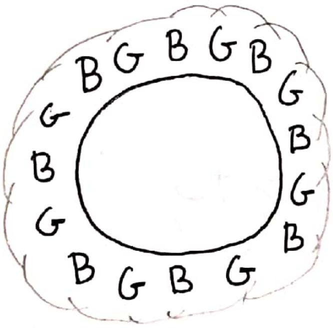
Total $\rightarrow \boxed{(2^m - 1)(2^n - 1)}$

By product rule.

$\boxed{10}$

Problems 1.4 (my attempts)

$\boxed{3}$ How many? (8B, 8G)



$\rightsquigarrow$ BG ; GB ; GB ; BG ; BG ;

If we knew how many pairs we culd make, then the answer we are looking for can be found from that number.

We start from B_1 in the circle and make a pair with the one to its right, move on to its right and repeat. This way we make every single possible pair. Which is one pair for each person sitting in the circle $\Rightarrow$ 16 pairs. Now we take groups of 5 pairs from this 16, but note: once a pair is chosen we can't choose any other pair with G_1 and B_1 again, so we discard another two more pairs: G_1B_1, ~~G_xB_x~~, ~~G_xB_1~~. So the number of ways to do this is:

$$\binom{16}{1}\binom{13}{1}\binom{10}{1}\binom{7}{1}\binom{4}{1} = 16 \cdot 13 \cdot 10 \cdot 7 \cdot 4 = 58240$$

$\boxed{3}$ (II)

B G B G B G B G B G B G B G B G

Total of 15 couples (same as before but since it's not a closed loop we loose a couple).

If we discard the extremes, choosing a couple discards two more (same argument as before) so:

choose a not extreme

$\binom{13}{1} \cdot \binom{12}{1} \Big($ $= 13 \cdot 10 \cdot 7 \cdot 4 \cdot 1 = 3640$

Choosing one of the extremes (2 extremes):

$2 \cdot \binom{1}{1} \cdot \binom{14}{1} \cdot \binom{11}{1} \cdot \binom{8}{1} \cdot \binom{5}{1} = 2 \cdot 14 \cdot 11 \cdot 8 \cdot 5 =$

Choosing both extremes: (we discard 2 and 2 contiguous).

$1 \cdot 1 \cdot \binom{11}{1} \cdot (1) \cdot (\ ) = (\qquad =$

A total of: $3640 + 3520 +$ $\qquad =$

But note that here we are considering this scenarios as different options: $(B_1 G_1), (B_2 G_2)$ and $(B_2 G_2), (B_1 G_1)$. But since the order in which the elements are chosen doesn't matter, we divide by the corresponding factorial of elements chosen:

Total =

// No

$\boxed{2}$

$\boxed{3}$ (III) ~ ✗

BG .... GBG

We choose the first couple:

$\binom{15}{1}$

If we chose an extreme, we discard the chosen couple and another one: $B_1G_1B_2G_2 \to B_1G_1$; ~~G_1B_2~~. If it wasn't, we discard itself and another two: $B_1G_1B_2G_2 \to$ $\to G_1B_2$; ~~B_1G_1~~ ; ~~B_2G_2~~. Now he have to choose again from either 13 or 12 options: repeat until 5 are chosen:

$$\binom{15}{1} \xrightarrow{\text{is extreme?}} \begin{cases} \text{yes} \to \binom{13}{1} \\ \text{no} \to \binom{12}{1} \end{cases}$$

We discard options I.E:

$\underline{1}\times\underline{\ }\ \underline{\ }\ \underline{\ }\ \underline{\ } \to 13$

$\Rightarrow$ $\underline{1}\times\underline{1}\times\underline{\ }\ \underline{\ }\ \underline{\ } \to 11$

(But it is not considered)

$$\binom{15}{1}\begin{cases}\binom{13}{1}\begin{cases}\binom{11}{1}\begin{cases}\binom{9}{1}\begin{cases}\binom{7}{1}\\ \binom{6}{1}\end{cases}\\ \binom{8}{1}\begin{cases}\binom{6}{1}\\ \binom{5}{1}\end{cases}\end{cases}\\ \binom{10}{1}\begin{cases}\binom{8}{1}\begin{cases}\binom{6}{1}\\ \binom{5}{1}\end{cases}\\ \binom{7}{1}\begin{cases}\binom{5}{1}\\ \binom{4}{1}\end{cases}\end{cases}\end{cases}\\ \binom{12}{1} \\ \vdots\end{cases}$$

(2^4 sumandos)

Total ~~X~~ $(15\cdot13\cdot11\cdot9\cdot7 + \ldots\ldots + 15\cdot12\cdot9\cdot6\cdot3)\cdot\underbrace{\frac{1}{5!}}_{\text{we don't care order}}\cdot\underbrace{\frac{1}{32}}_{\text{we don't care the branch}}$

(Checked with Matlab)

Better way in solutions!! (Much better)

$\boxed{7}$

$q(n) : \left\lfloor \dfrac{n}{\lfloor \sqrt{n} \rfloor} \right\rfloor \qquad n = 1, 2, \dots$

Missing a lot of them still!

$\lfloor\sqrt{n}\rfloor$	n		$q(n)$
$= 1$	1	$\lfloor 1 \rfloor \cdot 1$	1
	2	$\lfloor \sqrt{2} \rfloor \cdot 2$	2
	3	$\lfloor \sqrt{3} \rfloor \cdot 3$	3
$= 2$	4		2
	5		2
	6	$3 \cdot (\lfloor \sqrt{2} \rfloor)$	3
	7		3
	8	$(4 \cdot \lfloor \sqrt{2} \rfloor)$	4
$= 3$	9		3
	10		3
	11		3
	12	$\lfloor \sqrt{3} \rfloor \cdot 4$	4
	13		4
	14		4
	15	$\lfloor \sqrt{3} \rfloor \cdot 5$	5
$= 4$	16		4
	17		4
	18		4
	19		4
	20		5
	21		5
	22		5
	23		5
	24		6
4	25		5

- Conjecture: if $n+1 = m^2$ with $n, m \in \mathbb{N}$

$$\Rightarrow q(n) > q(n+1); \quad q(n) = q(n+1) + 1$$

- Proof:

Let's suppose $n + 1 = m^2$. Thus:

$$q(n+1) = q(m^2) = \left\lfloor \frac{m^2}{\lfloor \sqrt{m^2} \rfloor} \right\rfloor = \left\lfloor \frac{m^2}{m} \right\rfloor = m$$

and

$$q(n) = q(m^2 - 1) = \left\lfloor \frac{m^2 - 1}{\lfloor \sqrt{m^2 - 1} \rfloor} \right\rfloor \overset{(1)}{=} \left\lfloor \frac{m^2 - 1}{(m-1)} \right\rfloor = \frac{(m+1)\cancel{(m-1)}}{\cancel{(m-1)}} = m+1$$

(1) remains to be proven

$$\Rightarrow q(n) > q(n+1) \text{ if } n+1 = m^2 \;//$$

- Also: $q(n) > q(n-1)$ if $n = p \cdot \lfloor \sqrt{n} \rfloor$

if and only if:

$$q(n) = \left\lfloor \frac{p \cdot \lfloor \sqrt{n} \rfloor}{\lfloor \sqrt{n} \rfloor} \right\rfloor = p$$

$$q(n+1) = \left\lfloor \frac{p \cdot \lfloor \sqrt{n} \rfloor}{\lfloor \sqrt{n} \rfloor} + \frac{1}{\sqrt{n}} \right\rfloor = p$$

$$q(n-1) = \left\lfloor \frac{p \cdot \cancel{\lfloor \sqrt{n} \rfloor}}{\cancel{\lfloor \sqrt{n} \rfloor}} - \frac{1}{\sqrt{n}} \right\rfloor = p - 1$$

$$(m-1)^2 < m^2 - 1 < x < m^2 \Rightarrow \lfloor \sqrt{m^2 - 1} \rfloor = (m-1)$$

6 $A \subset M : \{1, \dots, 11\}$ is nice if:

$2k \in A \Rightarrow 2k-1 \in A \wedge 2k+1 \in A$.

How many nice subsets are there?

- Nice subsets with 0 elements like $2k$, $k \in \mathbb{N}$.

  $\{1, 3, 5, 7, 9, 11\} \longrightarrow$ all possible subsets $\Rightarrow$

  $\Rightarrow 2^6 = 64$ //

- With 1 element like $2k$. (As $\{3,4,5\}$)

  $\{2, 4, 6, 8, 10\} \rightarrow$ choose 1 ($\Rightarrow$ we choose $2k \pm 1$)

  $\{1, 3, 5, 7, 9, 11\} \rightarrow$ we already chose 2, choose from 4 anything.

  $\Rightarrow \underbrace{\binom{5}{1}} \cdot \underbrace{2^4} = 80$ // (Note: we counted the empty subset!)

Ex.: $|\{1,2,3\}| \cdot |P(\{5,7,9,11\})|$

- With 2 elements like $2k$.

  We could have to options, they are $2k$, $2(k+1)$ (i) or they are not (ii).

  (i)

  $\{(2,4), (4,6), (6,8), (8,10)\} \rightarrow$ Choose 1 pair and the corresponding 3 odd numbers (a, b, c).

  $\{1, 3, 5, 7, 9, 11\} - \{a, b, c\} \rightarrow$ Choose any possible subset.

  $\Rightarrow \binom{4}{1} \cdot 2^3 = 32$

  (ii)

  $\{(2,6), (2,8), (2,10), (4,8), (4,10), (6,10)\} \rightarrow$ Choose 1 pair and the corresponding 4 odd numbers (a, b, c, d).

  $\{1, 3, 5, 7, 9, 11\} - \{a, b, c, d\} \rightarrow$ Choose any possible subset.

$\Rightarrow \binom{6}{1} \cdot 2^2 = 24$

So for this case (rule of sum) a total of $32+24 =$
$= 56$ //

- With 3 elements like $2k$.

  We make a partition. There can be 0, 2 or 3 consecutive numbers:

(i)
$\{(2,6,10)\} \rightarrow$ Choose 1 and their $3 \cdot 2 = 6$ odds $\Rightarrow$
$\Rightarrow$ only 1 option $\rightarrow 1$

(ii)
$\{(2,4,8),(2,4,10),(2,6,8),(2,8,10),(4,6,10),(4,8,10)\} \rightarrow \binom{6}{1}$
and $2 \cdot 2 - 1 + 2 = 5$ odds $\Rightarrow$ subsets from the last odd $\{a\}$
$\rightarrow 2^1 \Rightarrow 6 \cdot 2^1 = 12$

(iii)
$\{(2,4,6),(4,6,8),(6,8,10)\} \rightarrow \binom{3}{1}$ and $3 \cdot 2 - 2 = 4$
odd numbers $\Rightarrow \{a, b\}$ remaining to choose from (any subset)
$\Rightarrow \binom{3}{1} \cdot 2^2 = 12$

$\Rightarrow$ Total of $1 + 12 + 12 = 25$ //

- With 4 elements like $2k$.

  Another partition; we can have 4 consecutive, 3 or 2 and 2 (any other option can't happen).

(i)

$\{(2,4,6,8),(4,6,8,10)\} \to \binom{2}{1}$ and 1 remaining odd $\Rightarrow$

$\Rightarrow 2 \cdot 2^1 = 4$

(ii)

$\{(2,4,6,10),(2,6,8,10)\} \to \binom{2}{1}$ and 0 remaining odds $\Rightarrow$

$\Rightarrow 2 \cdot 1 = 2$

(iii)

$\{(2,4,8,10)\} \to \binom{1}{1}$ and 0 remaining odds $\Rightarrow$

$\Rightarrow 1 \cdot 1 = 1$

Total of $4+2+1 = 7$ //

- With 5 of $2k$.

  We need to take every element of $M \Rightarrow 1$ // option.

We made a partition, so by the rule of sum we obtain:

$S = \{A \subset M : 2k \in A \Rightarrow 2k+1 \wedge 2k-1 \in A\}$

$|S| = 64 + 80 + 56 + 25 + 7 + 1 =$ $\boxed{233 \text{ nice subsets.}}$

$\boxed{2}$

$\Omega = \{x \in \mathbb{Z} : 100 \leq x \leq 999\}$; $|\Omega| = 14$

Prove: $\exists A, B \neq \emptyset$: $A \subset \Omega$, $B \subset \Omega$, $A \cap B = \emptyset$ s.t.

$$\sum_{a \in A} a = \sum_{b \in B} b$$

Proof:

We define the following function f, that takes each nonempty subsets of Ω to the sum of its elements:

$$f: A \in (\mathcal{P}(\Omega) - \Omega - \emptyset) \longrightarrow x \in \mathbb{Z}$$
$$\emptyset \neq A \subsetneqq \Omega \longmapsto \sum_{a \in A} (a)$$

✓

$$|(\mathcal{P}(\Omega) - \Omega - \emptyset)| = 2^n - 2 = 2^{14} - 2 = 16382 = m$$

Since $A \subset \Omega \Rightarrow 100 \cdot 1 \leq \sum_{a \in A} (a) \leq 999 \cdot 14 = 13986 \Rightarrow$

$$\Rightarrow \left| \left\{ \sum_{a \in A} (a) : A \neq \emptyset, A \subsetneqq \Omega \right\} \right| \leq 13986 = n$$

✓

We observe:

$16382 = m > n = 13986 \Rightarrow$ We can apply the pigeonhole principle and affirm that for any f of that kind, f cannot be injective:

$f(A) = f(B) \not\Rightarrow A = B \Rightarrow \exists A \neq B$ s.t. $f(A) = f(B)$ ✓

$$\Rightarrow \sum_{a \in A} (a) = \sum_{b \in B} (b) \quad (A \not\subset B \text{ and } A \not\supset B)$$

$\boxed{8}$

If $A \cap B \neq \emptyset$, we create $A' = (A - A\cap B) \neq \emptyset$
($A \not\subset B \Rightarrow \exists a \in A : a \notin B$) and $B' = (B - A \cap B) \neq \emptyset$
($B \not\subset A \Rightarrow \exists b \in B : b \notin A$). So:

$f(A') = f(A) - f(A\cap B)$ (by definition)
$f(B') = f(B) - f(A \cap B)$ (" ")

Since $f(A) = f(B) \Rightarrow$

$$f(A') = f(A) - f(A\cap B) = f(B) - f(A \cap B) = f(B')$$

So: $\exists A', B' \neq \emptyset,\ A' \subset \Omega,\ B' \subseteq \Omega,\ A' \cap B' = \emptyset$ s.t.

$$\sum_{a \in A'} a = \sum_{b \in B'} b \quad \square$$

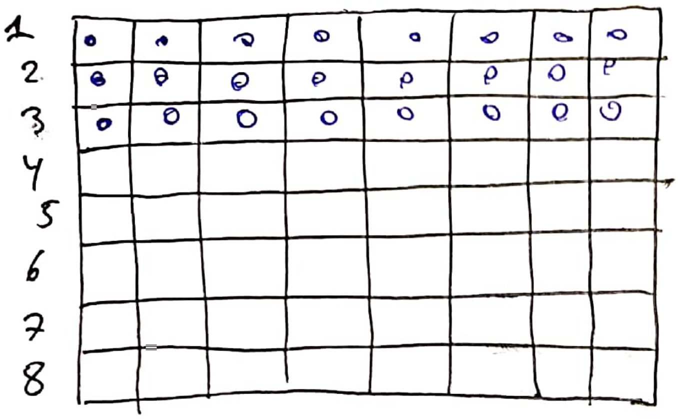
$\boxed{4}$

201 people, 5 different nationalities. (There are males and females).
In each 6 people group, at least two have the same age.
Prove there are at least 5 people of same nationality, age and gender. (G = gender, N = nationality, A = age)

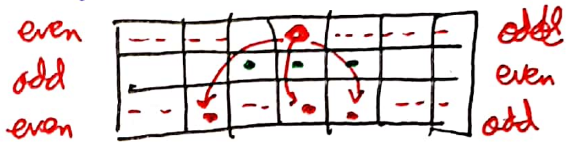
There are two genders: M and F. Applying the pigeonhole principle, there is at least a group, G, s.t. $|G|$ is at least 101 ($|G| \geq 101$). There are 5 nationalities. In group G, we apply the pigeonhole principle, there is at least a group $G_N \subset G$, s.t. $|G_N| \geq 21$ (at least 21). Suppose there could be 6 different ages. Then, $\exists G$ such that no one has the same age $\Rightarrow$ There are at most five different ages $\Rightarrow \exists G_{N,A} \subset G$ such that 5 people have the same age ($21/5 \to$ Pigeonhole to 21 people, 5 ages) $\Rightarrow$ We found a group of at least 5 people with same G, N, A $\square$

9

9



Note that if a piece is on a row with odd or even number, since it has to jump another piece to go down, it will keep its parity. I.E.:



Since the first three numbers are odd, even, odd (1, 2, 3) the last three should be too, but they are even odd even (6,7,8).
Thus, there is no solution.

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