Problems 1.1

Solutions

4

$f: D \longrightarrow \mathbb{R}$ cont., increasing, $f(x) \in \mathbb{Z} \Rightarrow x \in \mathbb{Z}$

(a) $\lfloor f(x) \rfloor = \lfloor f(\lfloor x \rfloor) \rfloor$?

(i) $x \in \mathbb{Z} \rightarrow$ trivial.

$x \in \mathbb{Z} \Rightarrow x = \lfloor x \rfloor \Rightarrow f(x) = f(\lfloor x \rfloor) \Rightarrow \lfloor f(x) \rfloor = \lfloor f(\lfloor x \rfloor) \rfloor$ //

(ii) $x \notin \mathbb{Z} \rightarrow$.

$$x \in \mathbb{Z} \Rightarrow x \neq \lfloor x \rfloor \Leftrightarrow \lfloor x \rfloor < x \underset{\text{f. incr}}{\Rightarrow} \underset{\substack{\in \;\; \notin \\ \mathbb{Z} \;\; \mathbb{Z} \\ \text{we don't know}}}{f(\lfloor x \rfloor)} < \underset{\substack{\in \\ \mathbb{Z}}}{f(x)}$$

- Case 1:

$$\exists\, n \in \mathbb{Z} : \underbrace{f(n)}_{\text{highest integer before } f(x)} \leq f(\lfloor x \rfloor) < f(x) \Rightarrow \lfloor f(\lfloor x \rfloor) \rfloor = \lfloor f(x) \rfloor$$

- Case 2:

$$\exists\, n \in \mathbb{Z} : f(\lfloor x \rfloor) < f(n) < f(x) \Rightarrow \lfloor x \rfloor < n < x \;\#\; \Rightarrow$$

$\Rightarrow$ Case 1 that is true (Case 2 can't happen). $\square$

(Part (b) is analogous.)

$\boxed{3}$

a) $\lfloor \sqrt{x} \rfloor = \lfloor \sqrt{\lfloor x \rfloor} \rfloor, \quad \forall x \geq 0$

Let m^2 be the highest perfect square below x, then:

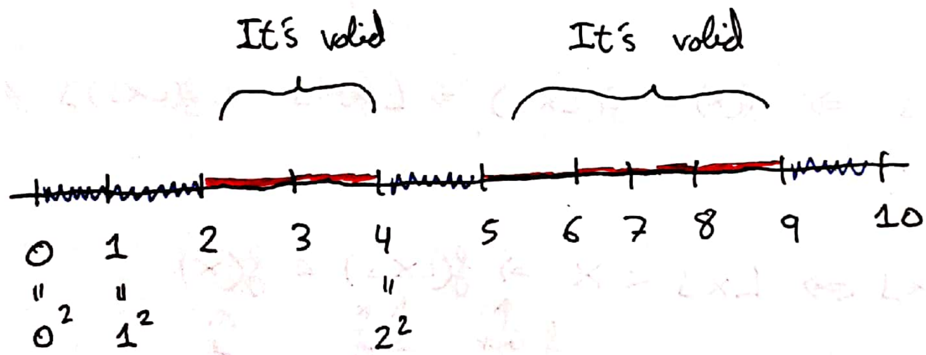
$m^2 \leq \lfloor x \rfloor < x < (m+1)^2 \Rightarrow m \leq \sqrt{\lfloor x \rfloor} < \sqrt{x} < m+1 \Rightarrow$

$\Rightarrow \lfloor \sqrt{\lfloor x \rfloor} \rfloor = \lfloor \sqrt{x} \rfloor = m$

$\boxed{7}$

$\lceil \sqrt{\lfloor x \rfloor} \rceil = \lceil \sqrt{x} \rceil, \quad \forall x \geq 0 \iff ¿? \iff$ Sol.: $n^2 + 1 \leq x \leq (n+1)^2, \; n = 0, 1, \ldots$

$x = 0$



(Solution next page)

$\boxed{7}$

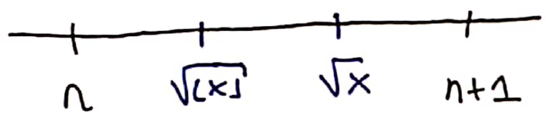
$\lceil \sqrt{\lfloor x \rfloor} \rceil = \lceil \sqrt{x} \rceil$, $x \geq 0 \iff x = 0$ or $n^2+1 \leq x \leq (n+1)^2$, $n = 0, \dots$

Proof:

- $x = 0$ trivial
- $n^2+1 \leq x \leq (n+1)^2 \iff n^2+1 \leq \lfloor x \rfloor \leq x \leq (n+1)^2 \iff$

$\iff n^2 < n^2+1 \leq \lfloor x \rfloor \leq (n+1)^2 \iff n < \sqrt{\lfloor x \rfloor} \leq \sqrt{x} \leq n+1$

$\iff n < \lceil \sqrt{\lfloor x \rfloor} \rceil \leq \lceil \sqrt{x} \rceil \leq n+1$



$\iff n < \lceil \sqrt{\lfloor x \rfloor} \rceil = \lceil \sqrt{x} \rceil = n+1$ $\square$

Problems 1.3 (solutions)

14

$$n = p_1^{m_1} \cdot p_2^{m_2} \cdot p_3^{m_3} \cdot \ldots \cdot p_k^{m_k}$$

$$\text{Divisors} = \{ p_1^{n_1} \cdot p_2^{n_2} \cdot \ldots \cdot p_k^{n_k} : \forall i \ \ 0 \leq n_i \leq m_i \} \Rightarrow$$

$$\Rightarrow \boxed{\left| \{ \text{Divisors} \} \right|} = (m_1 + 1) \cdot \ldots \cdot (m_k + 1) = \boxed{\prod_{i=1}^{k} (m_i + 1)}$$

I.E:

$$36 = 3^2 \cdot 2^2 \Rightarrow \begin{matrix} 3^0 \cdot 2^0, & 3^1 \cdot 2^0, & 3^2 \cdot 2^0 \\ 3^0 \cdot 2^1, & 3^1 \cdot 2^1, & 3^2 \cdot 2^1 \\ 3^0 \cdot 2^2, & 3^1 \cdot 2^2, & 3^2 \cdot 2^2 \end{matrix}$$

$$\rightarrow 3 \cdot 3 = 9 \ //$$

Sum of all divisors?

$$(p_1^0 + p_1^1 + \ldots + p_1^{m_1})(p_2^0 + p_2^1 + \ldots + p_2^{m_2}) \cdot \ldots \cdot (p_k^0 + p_k^1 + \ldots + p_k^{m_k}) =$$

$$= \left(\sum_{i=0}^{m_1} p_1^i \right) \cdot \ldots \cdot \left(\sum_{i=0}^{m_k} p_k^i \right) = \prod_{j=1}^{k} \left(\sum_{i_j=0}^{m_j} p_j^{i_j} \right) \overset{(1)}{=}$$

$$\overset{(1)}{=} \boxed{\prod_{j=1}^{k} \left(\frac{1 - p_1^{m_1+1}}{1 - p_1} \right)}$$

In (1) we've used that:

$$\sum_{i=k_0}^{n} r^i = \frac{r^{k_0} - r^{n+1}}{1 - r}$$

Put it in other words:

$$\sum_{n_1=0}^{m_1} \sum_{n_2=0}^{m_2} \dots \sum_{n_k=0}^{m_k} \left(p_1^{n_1} \cdot p_2^{n_2} \cdot \dots \cdot p_k^{n_k} \right) =$$

$$= \sum_{n_1=0}^{m_1} p_1^{n_1} \sum_{n_2=0}^{m_2} p_2^{n_2} \dots \sum_{n_k=0}^{m_k} p_k^{n_k} = \dots \checkmark$$

Proof for:

$$\sum_{i=k}^{n} p^i = p^k + p^{k+1} + \dots + p^{k+n}$$

$$p \cdot \sum_{i=k}^{n} p^i = p^{k+1} + \dots + p^{k+n} + p^{k+n+1}$$

$$\Rightarrow \left(\sum_{i=k}^{n} p^i \right) \cdot (1 - p) = p^k - p^{k+n+1} \Rightarrow$$

$$\Rightarrow \sum_{i=k}^{n} (p^i) = \frac{p^k - p^{k+n+1}}{1-p} \quad \square$$

$\boxed{18}$

(Using the hint)

Number of Us: $x \Rightarrow$ Number of Ds: $y \Rightarrow$ $\begin{cases} x+y = p \\ x-y = q+2 \end{cases} \Rightarrow \begin{cases} x = \frac{p+q}{2} + 1 \\ y = \frac{p-q}{2} + 1 \end{cases}$

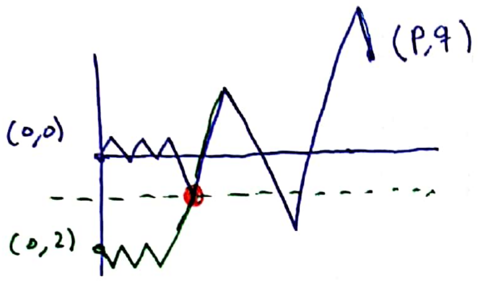
$(p, q$ same parity$) \Rightarrow$ $\begin{matrix} U\,D\,D \dots U\,D \\ 0\,1\,1 \dots 1\,0 \end{matrix}$ $\rightarrow C_{x+y,\,x} = \boxed{\binom{p}{\frac{p+q}{2}+1}}$

(choose U or D, all those type of sequences).

Why is the hint true?

Hint: Number of trajectories that cross $y=-1$ from $(0,0)$ to (p,q) without any (p,q) same as those from $(0,-2)$ to (p,q)

contains:

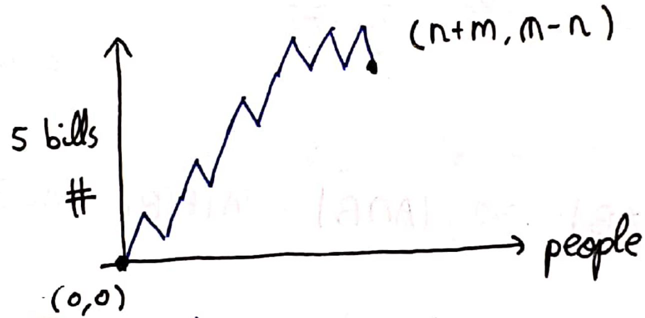


Point where $y=-1$ it's first reached.
We produce a symmetric respect to $y=-1$. There is a bijection

$$\Rightarrow \left|\Theta^{*(p,q)}_{(0,0)}\right| = \left|\Theta^{(p,q)}_{(0,-2)}\right| \rightarrow (\ldots)$$

$\boxed{19}$

$m \geq n$
$5 \rightarrow U$
$10 \rightarrow D$



$A = \{$paths from $(0,0) \nearrow (n+m, m-n)$ that do not reach $y=-1\}$

$\Omega = \Theta^{(n+m,m-n)}_{(0,0)}$

$B = \{$paths from $(0,0) \nearrow (n+m,m-n)$ that do reach $y=-1\} = A^c$

$$|\Omega| = |A| + |A^c| \Rightarrow |A| = |\Omega| - |A^c|$$

$$|\Omega| = \left|\Theta^{(p,q)}_{(0,0)}\right| = \binom{p}{\frac{p+q}{2}} \Rightarrow |A| = \binom{m+n}{n} - \binom{m+n}{n-1} \Rightarrow$$

$$|A^c| = \left|\Theta^{*(p,q)}_{(0,0)}\right| = \binom{p}{\frac{p+q}{2}+1}$$

$\downarrow$ exercise 18

But for the people, order doesn't matter $\Rightarrow$

Answer: $\left[\binom{m+n}{m} - \binom{m+n}{m+1}\right] m! \cdot n!$ //

4

$A = \{$ cubes with at least one side red $\}$; $|A| = 80$
$B = \{$ " " " " " " blue $\}$; $|B| = 85$
$C = \{$ " " " " " " green $\}$; $|C| = 75$

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| \Rightarrow$$

$$\Rightarrow |A \cap B \cap C| = \underbrace{|A \cup B \cup C|}_{100} \underbrace{- |A|}_{-80} \underbrace{- |B|}_{-85} \underbrace{- |C|}_{-75} + |A \cap B| + |A \cap C| + |B \cap C|$$

≤ 100 (total)

By this principle:

$|A \cup B| = |A| + |B| - |A \cap B| \Rightarrow |A \cap B| = |A| + |B| - |A \cup B| \geq 80 + 85 - 100$

Thus:

$|A \cap B| \geq 65$

$|A \cap C| \geq 55$

$|B \cap C| \geq 60$

$$\Rightarrow |A \cap B \cap C| \geq 100 - 80 - 85 - 75 + 60 + 55 + 60 = 40$$

$\longrightarrow \boxed{|A \cap B \cap C| \geq 40}$

$\boxed{16}$

Let p be a prime number and $n \in \mathbb{N}$. How many times does p divide $n!$?

$n! = n(n-1)\cdot(n-2)\cdot \ldots \cdot 1 \Rightarrow 2 \leq p \leq n$

n decomposition $= p_1^{m_1} \cdot p_2^{m_2} \cdot \ldots \cdot p_j^{m_j}$

Example:

$n = 6 \rightarrow n! = 6! = 720 = 2^4 \cdot 3^2 \cdot 5$

$p = 2$	1	2	3	4	5	6
multiples of 2		X		X		X
multiples of 2^2				X		

$\longrightarrow f(2) = 4 = \lfloor \frac{6}{2} \rfloor + \lfloor \frac{6}{4} \rfloor$

$p = 3$	1	2	3	4	5	6
multiples of 3			x			x

$\longrightarrow f(3) = 2 = \lfloor \frac{6}{3} \rfloor$

$p = 5$	1	2	3	4	5	6
multiples of 5					x	

$f(p_j) = m_j$ and if p_j doesn't belong in the n decomposition then $(p_j) = 0$. With m_j = number of numbers (n_j) $2 \leq n_j \leq n$ that p^i divides n_j and p^{i+1} does not.

$\boxed{5}$

Then:

$$f_p(n!) = \sum_{i=1}^{x} \lfloor \frac{n}{p^i} \rfloor$$

With: $p^i \leq n \leq p^{i+1} \Rightarrow i \leq \log_p(n) \Rightarrow i = \lfloor \log_p(n) \rfloor = x$

$$\Rightarrow \boxed{f_p(n) = \sum_{i=1}^{\lfloor \log_p(n) \rfloor} \lfloor \frac{n}{p^i} \rfloor}$$

More formally: $n! = n(n-1)(n-2)\cdot\ldots\cdot 3\cdot 2\cdot 1$

$A_0 = \{k \in \{1, \ldots, n\}$ s.t. p^0 divides k and $p^1 \nmid k\}$

Ex.
$n = 6$
$p = 2$
$A_0 = \{1, 3, 5\}$

$A_1 = \{k \in \{1, \ldots, n\}$ s.t. p divides k and $p^2 \nmid k\}$

Ex.
$A_1 = \{2, 6\}$

$A_2 = \{k \in \{1, \ldots, n\}$ s.t. p^2 divides k and $p^3 \nmid k\}$

Ex.
$A_2 = \{4\}$

$\vdots$

$A_x = \{k \in \{1, \ldots, n\}$ s.t. p^x divides k and $p^{x+1} \nmid k\}$

Observe: $p^x \leq n < p^{x+1} \Rightarrow x \leq \log_p(n) < x+1$

$$\Rightarrow x = \lfloor \log_p(n) \rfloor$$

Problems 1.4 (solutions)

3

15 couples:

1 = selected

0 = non-selected } → _ _ _ _ _ _ _ _ _ _ _ _ _ _ _

There can't be two consecutive 1s so: $\binom{11}{5}$ possible tides, choose five ones

$\sim 0 \sim 0 \sim \dots \sim 0 \sim \longrightarrow \binom{11}{5} = 462$ //

Very useful to translate the problem to similar scenarios.

7

$f(n) = \left\lfloor \frac{n}{\lfloor \sqrt{n} \rfloor} \right\rfloor \quad n \in \mathbb{N}$

a)

$f(3) = 3 > 2 = f(4)$

$f(8) = 4 > 3 = f(9)$

$f(15) = 5 > 4 = f(16)$

b) $f(n) > f(n+1) \iff n+1 = a^2, \; a \in \mathbb{N} \; (a > 1)$

($\Leftarrow$)

$f(n) = f(a^2 - 1) = \left\lfloor \frac{a^2 - 1}{\lfloor \sqrt{a^2 - 1} \rfloor} \right\rfloor = \left\lfloor \frac{(a+1)(a-1)}{(a-1)} \right\rfloor = a + 1$

$$q(a^2) = \left\lfloor \frac{a^2}{\lfloor\sqrt{a^2}\rfloor} \right\rfloor = \left\lfloor \frac{a^2}{a} \right\rfloor = a$$

$\Rightarrow \quad a < a+1 \Rightarrow q(n+1) < q(n)$

Remains to be proven:

$\lfloor\sqrt{m^2-1}\rfloor = (m-1)$

Other implication:

$\Rightarrow)$

$$c = \left\lfloor \frac{n}{\lfloor n \rfloor} \right\rfloor > \left\lfloor \frac{n+1}{\lfloor n+1 \rfloor} \right\rfloor = b \Rightarrow$$

$$b \leq \frac{n+1}{\lfloor\sqrt{n+1}\rfloor} < c \leq \frac{n}{\lfloor\sqrt{n}\rfloor} \Rightarrow$$

$$b \leq \frac{n+1}{\sqrt{n+1}} < \frac{n+1}{\lfloor\sqrt{n+1}\rfloor} < c < \frac{n}{\sqrt{n}} < \frac{n}{\lfloor\sqrt{n}\rfloor} \Rightarrow$$

$$b^2 \leq n+1 \leq \left(\frac{(n+1)}{\lfloor\sqrt{n+1}\rfloor}\right)^2 < c^2 \leq n \leq \left(\frac{n}{\lfloor\sqrt{n}\rfloor}\right)^2$$

$$b^2 \leq n+1$$

From $q(n) > q(n+1) \Rightarrow \left\lfloor \frac{n}{\lfloor\sqrt{n}\rfloor} \right\rfloor > \left\lfloor \frac{n+1}{\lfloor\sqrt{n+1}\rfloor} \right\rfloor \Rightarrow$

$\lfloor\sqrt{n}\rfloor < \lfloor\sqrt{n+1}\rfloor \Rightarrow \sqrt{n} < k \in \mathbb{N} \leq \sqrt{n+1} \Rightarrow n^2 < \underbrace{k^2}_{\in \mathbb{N}^2} \leq n^2+1$

Since n^2 and n^2+1 are consecutive naturals $\Rightarrow n^2+1 \in \mathbb{N}$

$\rightarrow n^2+1 \in \mathbb{N}$, is a perfect square. $\square$