Seminario 4

Josu Pérez Zorraonandia

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P1 Demostrar que $z = y \ln(x^2-y^2)$ satisface:

$$\frac{1}{x}\cdot\frac{\partial z}{\partial x} + \frac{1}{y}\cdot\frac{\partial z}{\partial y} = \frac{z}{y^2}$$

- $$\frac{\partial z}{\partial x} = y\cdot\frac{2x}{x^2-y^2}$$

- $$\frac{\partial z}{\partial y} = \ln(x^2-y^2) - y\cdot\frac{2y}{x^2-y^2}$$

- $$z = y\ln(x^2-y^2) \Rightarrow \ln(x^2-y^2) = \frac{z}{y}$$

Así obtenemos:

$$\frac{1}{x}\cdot\frac{\partial z}{\partial x} + \frac{1}{y}\cdot\frac{\partial z}{\partial y} = \cancel{\frac{2y}{x^2-y^2}} + \frac{z}{y^2} - \cancel{\frac{2y}{x^2-y^2}} = \frac{z}{y^2} \;//$$

$$\Rightarrow \boxed{\frac{1}{x}\cdot\frac{\partial z}{\partial x} + \frac{1}{y}\cdot\frac{\partial z}{\partial y} = \frac{z}{y^2}}$$

P2 Sea $h = f \circ g$, donde $f: \mathbb{R}^2 \to \mathbb{R}$ es $f \in C^{(2)}$ y

$g(x, y, z) = (u, v)$ con $u = x^2 + y^2 + z^2$, $v = x + y + z$.

a) Probar que $\|\nabla h\|^2 = 4u(D_1 f)^2 + 4v\, D_2 f \cdot D_2 f + 3(D_2 f)^2$

$$\begin{cases} g: \mathbb{R}^3 \longrightarrow \mathbb{R}^2 \\ f: \mathbb{R}^2 \longrightarrow \mathbb{R} \\ (f \circ g)(x) = f(g(x)) : \mathbb{R}^3 \longrightarrow \mathbb{R} \end{cases}$$

$$\begin{tikzcd} \mathbb{R}^3 \arrow[r, "g"] \arrow[rr, bend right, "f \circ g"'] & \mathbb{R}^2 \arrow[r, "f"] & \mathbb{R} \\ (x,y,z) & (u,v) & t \end{tikzcd}$$

Sabemos que:

$\nabla h = (D_1 h, D_2 h, D_3 h) \Rightarrow \|\nabla h\|^2 = (D_1 h)^2 + (D_2 h)^2 + (D_3 h)^2$

Calculamos las derivadas parciales de h con la regla de la cadena.

$$df = (D_1 f, D_2 f)$$

$$dg = \begin{pmatrix} D_1 g_1, & D_2 g_1, & D_3 g_1 \\ D_1 g_2, & D_2 g_2, & D_3 g_2 \\ D_1 g_3, & D_2 g_3, & D_3 g_3 \end{pmatrix}$$

$$d(f \circ g) = df \circ dg = (D_1 f, D_2 f) \begin{pmatrix} D_1 g_1, & D_2 g_1, & D_3 g_1 \\ D_1 g_2, & D_2 g_2, & D_3 g_2 \end{pmatrix} =$$

$$= \left(D_1 f \cdot D_1 g_1 + D_2 f \cdot D_1 g_2 , \right.$$
$$D_1 f \cdot D_2 g_1 + D_2 f \cdot D_2 g_2 ,$$
$$\left. D_1 f \cdot D_3 g_1 + D_2 f \cdot D_3 g_2 \right) =$$

$$= (D_1 h , D_2 h , D_3 h) \Rightarrow$$

$$\begin{cases} D_1 h = D_1 f \cdot D_1 g_1 + D_2 f \cdot D_1 g_2 \\ D_2 h = D_1 f \cdot D_2 g_1 + D_2 f \cdot D_2 g_2 \\ D_3 h = D_1 f \cdot D_3 g_1 + D_2 f \cdot D_3 g_2 \end{cases}$$

Para poner $D_i h$ en términos de $D_k f$, necesitamos calcular $D_i g_j$.

$$g(x,y,z) = (x^2 + y^2 + z^2 , x + y + z)$$

$$D_1 g_1 = 2x \; ; \; D_2 g_1 = 2y \; ; \; D_3 g_1 = 2z$$
$$D_1 g_2 = 1 \; ; \; D_2 g_2 = 1 \; ; \; D_3 g_2 = 1$$

Por tanto:

$$(D_1 h)^2 = 4x^2 \cdot (D_1 f)^2 + (D_2 f)^2 + 4x \cdot D_1 f \cdot D_2 f$$

$$(D_2 h)^2 = 4y^2 \cdot (D_1 f)^2 + (D_2 f)^2 + 4y \cdot D_1 f \cdot D_2 f$$

$$(D_3 h)^2 = 4z^2 \cdot (D_1 f)^2 + (D_2 f)^2 + 4z \cdot D_1 f \cdot D_2 f$$

Sumando los tres:

$$(D_1 h)^2 + (D_2 h)^2 + (D_3 h)^2 =$$

$$= 4(x^2 + \underbrace{y^2 + z^2}_{u})(D_1 f)^2 + 4(x + \underbrace{y + z}_{v})(D_2 f)^2 + 3 \cdot D_1 f \cdot D_2 f =$$

$$= 4u(D_1 f)^2 + 4v(D_2 f)^2 + 3 \cdot D_1 f \cdot D_2 f$$

$$\Rightarrow \boxed{\|\nabla h\|^2 = 4u(D_1 f)^2 + 4v(D_2 f)^2 + 3 \cdot D_1 f \cdot D_2 f \quad \square}$$

b) Calcular $\dfrac{\partial (\|\nabla h\|^2)}{\partial x}$

Podemos tomar la siguiente interpretación de $\|\nabla h\|^2$ como función:

$$\begin{tikzcd} \underset{(x,y,z)}{\mathbb{R}^3} \arrow[r, "g"] \arrow[rr, bend right, "\|\nabla h\|^2 \circ g"'] & \underset{(u,v)}{\mathbb{R}^2} \arrow[r, "\|\nabla h\|^2"] & \underset{t}{\mathbb{R}} \end{tikzcd}$$

$\dfrac{\partial (\|\nabla h\|^2)}{\partial x}$ es en realidad $D_1(g \circ \|\nabla h^2\|)$:

$$d\,\|\nabla h^2\| = (D_1 \|\nabla h\|^2 \, , \, D_2 \|\nabla h\|^2)$$

$$dg = \begin{pmatrix} D_1 g_1 & D_2 g_1 & D_3 g_1 \\ D_1 g_2 & D_2 g_2 & D_3 g_2 \end{pmatrix}$$

$$d(\|\nabla h\|^2 \circ g) = (D_1 \|\nabla h\|^2, D_2 \|\nabla h\|^2) \begin{pmatrix} D_1 g_1 & D_2 g_1 & D_3 g_1 \\ D_1 g_2 & D_2 g_2 & D_3 g_2 \end{pmatrix}$$

$= \left(D_1 (\|\nabla h\|^2 \circ g\cdot), D_2 (g \circ \|\nabla h\|^2) \right) \Rightarrow$

$\Rightarrow D_1 (\|\nabla h\|^2 \circ g) = D_1 (\|\nabla h\|^2) \cdot D_1 g_1 + D_2 (\|\nabla h\|^2) D_1 g_2$

Reescribimos la información que tenemos hasta ahora:

$$\|\nabla h\|^2 = 4u \cdot \left(\frac{\partial f}{\partial u}\right)^2 + 4v \left(\frac{\partial f}{\partial v}\right)^2 + 3 \cdot \left(\frac{\partial f}{\partial u}\right) \cdot \left(\frac{\partial f}{\partial v}\right)$$

$$\frac{\partial}{\partial x} (\|\nabla h\|^2 \circ g) = \frac{\partial}{\partial u} (\|\nabla h\|^2) \cdot \frac{\partial u}{\partial x} + \frac{\partial}{\partial v} (\|\nabla h\|^2) \cdot \frac{\partial v}{\partial x}$$

(Pues $g = (u, v) \rightarrow g_1 = u$, $g_2 = v$ con u y v funciones de $\mathbb{R}^3$ a $\mathbb{R}^2$). Calculamos:

$\frac{\partial}{\partial u} (\|\nabla h\|^2) =$ (Interpretamos como función de una variable, u, y aplicamos las reglas de derivación conocidas.)

$$= 4 \cdot \left(\frac{\partial f}{\partial u}\right)^2 + 8u \cdot \left(\frac{\partial f}{\partial u}\right) \cdot \left(\frac{\partial^2 f}{\partial u^2}\right) + 8v \left(\frac{\partial f}{\partial v}\right) \left(\frac{\partial^2 f}{\partial v \cdot \partial u}\right) +$$

$$+ 3 \cdot \left(\frac{\partial f}{\partial u}\right) \left(\frac{\partial^2 f}{\partial v \cdot \partial u}\right) + 3 \cdot \left(\frac{\partial^2 f}{\partial u^2}\right) \left(\frac{\partial f}{\partial v}\right)$$

$\frac{\partial}{\partial v} (\|\nabla h\|^2) =$ (análogo al caso anterior debido a la simetría de la función).

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x} (x^2 + y^2 + z^2) = 2x$$

$$\frac{\partial v}{\partial x} = \frac{\partial}{\partial x} (x + y + z) = 1$$

Por tanto:

$$\boxed{\frac{\partial}{\partial x}(\|\nabla h\|^2 \circ g) =}$$

$$= 2x \cdot \left[4 \cdot \left(\frac{\partial f}{\partial u}\right)^2 + \left(\frac{\partial^2 f}{\partial u^2}\right)\left(8u\left(\frac{\partial f}{\partial u}\right) + 3\left(\frac{\partial f}{\partial v}\right)\right) + \right.$$

$$\left. + \left(\frac{\partial^2 f}{\partial v \partial u}\right)\left(3\left(\frac{\partial f}{\partial u}\right) + 8v\left(\frac{\partial f}{\partial v}\right)\right)\right] +$$

$$+ \left[4 \cdot \left(\frac{\partial f}{\partial v}\right)^2 + \left(\frac{\partial^2 f}{\partial v^2}\right)\left(8v\left(\frac{\partial f}{\partial v}\right) + 3\left(\frac{\partial f}{\partial u}\right)\right) + \right.$$

$$\left. + \left(\frac{\partial^2 f}{\partial u \partial v}\right)\left(3\left(\frac{\partial f}{\partial v}\right) + 8u\left(\frac{\partial f}{\partial u}\right)\right)\right]$$

P3 Hallar $u = f(x,y,z)$ t.q. $(f'_x, f'_y, f'_z) = F$, donde $F(x,y,z) = (6xy\cos(z),\ 3x^2\cos(z),\ -3x^2y\,\mathrm{sen}(z))$.

$$\mathbb{R}^3 \xrightarrow{F} \mathbb{R}^3 \quad y \quad \mathbb{R}^3 \xrightarrow{u} \mathbb{R}^2$$

$$F(x,y,z) = (6xy\cos(z),\ 3x^2\cos(z),\ -3x^2y\,\mathrm{sen}(z)) =$$
$$= \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = \nabla f$$

- $\frac{\partial f}{\partial x} = 6xy\cos(z) \Rightarrow f = \int 6xy\cos(z)\,dx \Rightarrow$
  $\Rightarrow f = 3x^2 \cdot y \cdot \cos(z) + C$
- $\frac{\partial f}{\partial y} = 3x^2\cos(z) \Rightarrow f = \int 3x^2\cos(z)\,dy \Rightarrow$
  $\Rightarrow f = 3x^2y\cos(z) + C$
- $\frac{\partial f}{\partial z} = -3x^2y\,\mathrm{sen}(z) \Rightarrow f = \int -3x^2y\,\mathrm{sen}(z)\,dz \Rightarrow$
  $\Rightarrow f = 3x^2y\cos(z) + C$

Una de estas u podría ser tomando $C = 0$.

$$\boxed{u = f(x,y,z) = 3x^2y\cos(z)}$$

P4

a) $z = f(x,y)$ función diferenciable tal que $f(tx, ty) = t^m \cdot f(x,y)$ $\forall t>0$ y cierto $m>0$.

demostrar:

$$x \cdot \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = mz$$

Consideramos la siguiente composición de funciones:

$$\mathbb{R}^3 \xrightarrow{g} \mathbb{R}^2 \xrightarrow{f} \mathbb{R}$$
$$(x,y,t) \longmapsto (u,v) \longmapsto f(u,v) = t^m \cdot f(x,y)$$

con $g(x,y,t) = (tx, ty) = (u,v)$ $\begin{cases} u = tx \\ v = ty \end{cases}$

Por tanto, aplicando la regla de la cadena:

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial x}$$ con

$$\begin{cases} u = tx \rightarrow \dfrac{\partial u}{\partial x} = t \\ v = ty \rightarrow \dfrac{\partial v}{\partial x} = 0 \end{cases}$$

$$\Rightarrow \frac{\partial f}{\partial x} = t \cdot \frac{\partial f}{\partial u} \Rightarrow x \cdot \frac{\partial f}{\partial x} = xt \cdot \frac{\partial f}{\partial u}$$

Análogamente:

$$\frac{\partial f}{\partial y} = t \cdot \frac{\partial f}{\partial v} \Rightarrow y \cdot \frac{\partial f}{\partial y} = y \cdot t \cdot \frac{\partial f}{\partial v}$$

Calculamos además: $\frac{\partial f}{\partial t}$

$t>0$

Por una parte:

$$\frac{\partial}{\partial t}\left(f(tx,ty)\right) = \frac{\partial}{\partial t}\left(t^m \cdot \underbrace{f(x,y)}_{\text{cte respecto de } t}\right) = m \cdot t^{m-1} \cdot f(x,y) =$$

$$= m \cdot \frac{t^m}{t} \cdot f(x,y) = \frac{m}{t} \cdot f(tx,ty) = \frac{1}{t} \cdot m \cdot f(tx,ty) \quad (1)$$

Por otra parte:

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial t} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial t} \quad \text{con}$$

$$\begin{cases} u = tx \longrightarrow \frac{\partial u}{\partial t} = x \\ v = ty \longrightarrow \frac{\partial v}{\partial t} = y \end{cases}$$

$$\Rightarrow \frac{\partial f}{\partial t} = \frac{\partial f}{\partial u} \cdot x + \frac{\partial f}{\partial v} \cdot y = x \cdot t \cdot \frac{\partial f}{\partial u} \cdot \frac{1}{t} + y \cdot t \cdot \frac{\partial f}{\partial v} \cdot \frac{1}{t} =$$

$$= x \cdot \frac{\partial f}{\partial x} \cdot \frac{1}{t} + y \cdot \frac{\partial f}{\partial y} \cdot \frac{1}{t} \quad (2)$$

probado página anterior

Juntando (1) y (2):

$$\cancel{\frac{1}{t}} \cdot m \cdot f = x \cdot \frac{\partial f}{\partial x} \cdot \cancel{\frac{1}{t}} + y \cdot \frac{\partial f}{\partial y} \cdot \cancel{\frac{1}{t}} \Rightarrow$$

$$\Rightarrow \boxed{x \cdot \frac{\partial f}{\partial x} + y \cdot \frac{\partial f}{\partial y} = m \cdot f = m \cdot z}$$

b) $z = (x+y)\cdot f(y/x)$ con f función arbitraria.

Demostrar que $x \cdot \dfrac{\partial z}{\partial x} + y \cdot \dfrac{\partial z}{\partial y} = z$.

Consideramos:

$$\begin{tikzcd}
\mathbb{R}^2 & \mathbb{R} & \mathbb{R}^m \\
(x,y) \arrow[r, "h"] \arrow[rr, bend right, "f \circ h"'] & u \arrow[r, "f"] & f(u)
\end{tikzcd}$$

$\left(\;''\; \right)$

y

$$\begin{tikzcd}
\mathbb{R}^2 & \mathbb{R}^m \\
(x,y) \arrow[r, "z"] & (x+y)\cdot f(y/x)
\end{tikzcd}$$

Calculamos las derivadas parciales de z.

regla de la cadena

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}((x+y)\cdot f) = f + \frac{\partial f}{\partial x} =$$

$$= f + \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x} = f - \frac{\partial f}{\partial u}\cdot y/x^2 \Rightarrow$$

$$\left(u = \frac{y}{x} \Rightarrow \frac{\partial u}{\partial x} = - y/x^2 \right)$$

$$\Rightarrow \frac{\partial z}{\partial x}\cdot x = x\cdot f - \frac{\partial f}{\partial u}\cdot y/x$$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}\left[(x+y)\cdot f\right] = f + \frac{\partial f}{\partial y} =$$

$$= f + \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial y} = f + \frac{\partial f}{\partial u}\cdot\frac{1}{x}$$

$$\left(u = y/x \Rightarrow \frac{\partial u}{\partial y} = \frac{1}{x} \right)$$

$$\Rightarrow \frac{\partial z}{\partial y}\cdot y = y\cdot f + \frac{\partial f}{\partial u}\cdot\frac{y}{x}$$

Por tanto:

$$\frac{\partial z}{\partial x}\cdot x + \frac{\partial z}{\partial y}\cdot y = x\cdot f \underbrace{- \cancel{\frac{\partial f}{\partial u}\cdot\frac{y}{x}} + \cancel{\frac{\partial f}{\partial u}\cdot\frac{y}{x}}}_{0} + y\cdot f$$

$$= (x+y)\, f = z \Rightarrow$$

$$\Rightarrow \boxed{\frac{\partial z}{\partial x}\cdot x + \frac{\partial z}{\partial y}\cdot y = z}$$

P5 Si $z = \dfrac{y}{f(x^2-y^2)}$ demostrar que es:

$$\frac{1}{x}\cdot\frac{\partial z}{\partial x} + \frac{1}{y}\cdot\frac{\partial z}{\partial y} = \frac{z}{y^2}$$

Consideramos:

$$\begin{tikzcd} \mathbb{R}^2 \arrow[r, "h"] \arrow[rr, bend right, "f \circ h"'] & \mathbb{R} \arrow[r, "f"] & \mathbb{R} \end{tikzcd}$$

$$(x,y) \qquad (x^2-y^2) = u \qquad f(u) \qquad y$$

$$\mathbb{R}^2 \xrightarrow{z} \mathbb{R}$$

$$(x,y) \qquad z(x,y) = y\cdot\left[(f\circ h)(x,y)\right]^{-1}$$

Calculamos las derivadas parciales de z. Pero antes las de u.

$$\frac{\partial u}{\partial x} = 2x \quad ; \quad \frac{\partial u}{\partial y} = -2y$$

$$\frac{\partial z}{\partial x} = y\cdot\frac{\partial\left((f\circ h)^{-1}\right)}{\partial x} = -y\cdot\frac{1}{(f\circ h)^2}\cdot\frac{\partial(f\circ h)}{\partial x} =$$

$$= -y\cdot\frac{1}{(f\circ h)^2}\cdot\frac{\partial(f\circ h)}{\partial u}\cdot\frac{\partial u}{\partial x} = -\frac{2xy}{(f\circ h)^2}\cdot\frac{\partial(f\circ h)}{\partial u}$$

$$\Rightarrow \frac{1}{x}\cdot\frac{\partial z}{\partial x} = -\frac{2y}{(f\circ h)^2}\cdot\frac{\partial(f\circ h)}{\partial u}$$

$$\frac{\partial z}{\partial y} = (f \circ h)^{-1} + y \cdot \frac{\partial((f\circ h)^{-1})}{\partial y} =$$

$$= (f\circ h)^{-1} + y \cdot \frac{-1}{(f\circ h)^2} \cdot \frac{\partial(f\circ h)}{\partial y} =$$

$$= (f\circ h)^{-1} - \frac{y}{(f\circ h)^2} \cdot \frac{\partial(f\circ h)}{\partial u} \cdot \frac{\partial u}{\partial y} =$$

$$= (f\circ h)^{-1} + \frac{2y^2}{(f\circ h)^2} \cdot \frac{\partial(f\circ h)}{\partial u} \Rightarrow$$

$$\Rightarrow \frac{1}{y} \cdot \frac{\partial z}{\partial y} = \frac{1}{y} \cdot (f\circ h)^{-1} + \frac{2y}{(f\circ h)^2} \cdot \frac{\partial(f\circ h)}{\partial u}$$

Por tanto:

$$\frac{1}{x} \cdot \frac{\partial z}{\partial x} + \frac{1}{y} \cdot \frac{\partial z}{\partial y} =$$

$$= \underbrace{-\frac{2y}{(f\circ h)^2} \cdot \frac{\partial(f\circ h)}{\partial u} + \frac{2y}{(f\circ h)^2} \cdot \frac{\partial(f\circ h)}{\partial u}}_{0} + \frac{1}{y} \cdot (f\circ h)^{-1} =$$

$$= y \cdot (f\circ h)^{-1} \cdot \frac{1}{y^2} = z \cdot \frac{1}{y^2} = \frac{z}{y^2}$$

Es decir:

$$\boxed{\frac{1}{x} \cdot \frac{\partial z}{\partial x} + \frac{1}{y} \cdot \frac{\partial z}{\partial y} = \frac{z}{y^2}}$$

P6 $G(x,y) = \int_{xy}^{x+y} \frac{\operatorname{sen}((x+y)\cdot t)}{t}\,dt.$

Hallar: $\frac{\partial G(x,y)}{\partial x}$.

Dado que para calcular $\frac{\partial G(x,y)}{\partial x}$ suponemos $y = cte$, con esta suposición y teniendo en mente lo que vamos a hacer reescribimos:

$$\begin{cases} h_1(x) = xy \\ h_2(x) = x+y \\ f(x,t) = \frac{\operatorname{sen}((x+y)\cdot t)}{t} \end{cases}$$

$$G(x) = \int_{h_1(x)}^{h_2(x)} f(x,t)\,dt \quad \Rightarrow$$

$$\Rightarrow \frac{\partial G(x,y)}{\partial x} = G'(x) = f(x, h_2(x))\cdot h_2'(x) +$$
$$- f(x, h_1(x))\cdot h_1'(x) +$$
$$+ \int_{h_1(x)}^{h_2(x)} \left(\frac{\partial f}{\partial x}\right) dy$$

Calculamos los distintos términos:

- $f(x, h_2(x)) = \frac{\operatorname{sen}((x+y)^2)}{(x+y)}$;

- $h_2'(x) = 1$;

- $f(x, h_1(x)) = \dfrac{\text{sen}\,((x+y)xy)}{(xy)}$

- $h_1'(x) = y$

- $\dfrac{\partial f}{\partial x} = \cos((x+y)\cdot t) \Rightarrow \displaystyle\int \frac{\partial f}{\partial x}\, dy = \frac{\text{sen}((x+y)\cdot t)}{t} = f(x,$

- $\displaystyle\int_{h_1(x)}^{h_2(x)} \frac{\partial f}{\partial x}\, dy = f(x, h_2(x)) - f(x, h_1(x))$ (ya calculados)

Es decir:

$$G'(x) = f(x, h_2(x))\cdot(h_2'(x)+1) - f(x, h_1(x))\cdot(h_1'(x)+1)$$

Obteniendo:

$$\boxed{\frac{\partial G(x,y)}{\partial x} = \frac{2\,\text{sen}((x+y)^2)}{(x+y)} - \frac{\text{sen}((x+y)xy)}{xy}(y+1)}$$

P7

$$f(x,y,z) = \begin{cases} yz + \dfrac{x^2+xz}{x^2+y^2} & \text{si } x\neq 0 \text{ ó } y\neq 0 \\ z^2 & \text{si } x=0 \text{ e } y=0 \end{cases}$$

Hallar:

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial z}\right)\right)(0,0,0)$$

Primero hallamos $\frac{\partial f}{\partial z}(x,y,z)$.

Si $x\neq 0$ ó $y\neq 0$, no tenemos problemas:

$$\frac{\partial f}{\partial z}(x,y,z) = \frac{\partial}{\partial z}\left(yz + \frac{x^2+xz}{x^2+y^2}\right) = y + \frac{x}{x^2+y^2}$$

$(x\neq 0$ ó $y\neq 0)$

Si $x=0$, $y=0$:

$$\frac{\partial f}{\partial z}(x,y,z) = \lim_{h\to 0}\frac{f(0,0,z+h)-f(0,0,z)}{h} = \frac{\partial f}{\partial z}(z^2) = 2z$$

$$\Rightarrow \quad \frac{\partial f}{\partial z}(x,y,z) = \begin{cases} y + \dfrac{x}{x^2+y^2} & \text{si } x\neq 0 \text{ ó } y\neq 0 \\ 2z & \text{si } x=0 \text{ e } y=0 \end{cases}$$

Hallemos ahora $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial z}\right)(x,y,z)$.

Si $x\neq 0$ ó $y\neq 0$, no tenemos problemas.

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial z}\right)(x,y,z) = \frac{\partial}{\partial y}\left(y + \frac{x}{x^2+y^2}\right) = 1 - \frac{2xy}{(x^2+y^2)^2}$$

Si $x=0$, $y=0$:

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial z}\right)(x,y,z) = \lim_{h\to 0}\frac{\frac{\partial f}{\partial z}(0,h,z) - \frac{\partial f}{\partial z}(0,0,z)}{h} =$$

$$= \lim_{h\to 0}\frac{h+0-2z}{h} \quad \begin{cases} \nexists & \text{si } z\neq 0 \\ 1 & \text{si } z=0 \end{cases}$$

Es decir:

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial z}\right)(x,y,z) = \begin{cases} 1-\dfrac{2xy}{(x^2+y^2)^2} & \text{si } x\neq 0 \text{ ó } y\neq 0 \\ 1 & \text{si } x=0,\ y=0,\ z=0 \\ \nexists & \text{si } x=0,\ y=0,\ z\neq 0 \end{cases}$$

Veamos si podemos calcular:

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial z}\right)\right)(0,0,0) = \lim_{h\to 0}\frac{\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial z}\right)(h,0,0) - \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial z}\right)(0,0,0)}{h}$$

$$= \lim_{h\to 0}\frac{1-0-1}{h} = \lim_{h\to 0}\cancel{\frac{0}{h}}^{\,0} = 0$$

Por tanto:

$$\boxed{\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial z}\right)\right)(0,0,0) = 0}$$