DISCRETE MATHEMATICS

Chapter 1: Basic combinatorics

1.1 Combinatorics

Definition

Combinatorial theory is the study of methods of counting:

1. how many objects there are of a given description, or
2. in how many ways something can be done, or
3. in how many ways a certain event can occur.

Example

1. **How many objects are there of a given description?**
   - How many pairs of natural numbers (x, y) are there such that $x + y = 10$ or $x + y = n$?
   - How many tennis matches will be played in a tournament in which there are 137 tennis players?
2. **In how many different ways can something be done?**
   - In how many different ways can four people be seated at a circular table?
3. **In how many different ways can a certain event occur?**
   - In how many ways can 15 points be obtained after throwing a die four times?

1.1 Combinatorics

Let A be the finite set of elements that satisfy a property.

$$A = \{x : x \text{ satisfies a property } P\}.$$

Aim: to calculate the number of elements of A, i.e., the **cardinality** of A:

$$|A|$$

Notation:
$\mathbb{N} = \{1, 2, 3, \ldots\}$
$\mathbb{N}^* = \{0, 1, 2, \ldots\}$

1.2 Lists

In order to count the number of the elements of a set a first idea is to **make a list** of the elements.

Example

Solutions of an equation:

1. $A = \{(x, y) \in \mathbb{N}^* \times \mathbb{N}^* : x + y = 5\}$
   List: $\{(0, 5), (1, 4), (2, 3), (3, 2), (4, 1), (5, 0)\}$
   Therefore, $|A| = 6$.
2. **(Generalization)** $A = \{(x, y) \in \mathbb{N}^* \times \mathbb{N}^* : x + y = n, n \in \mathbb{N}^*\}$
   List: $\{(0, n), (1, n-1), (2, n-2), (3, n-3), \ldots, (n-1, 1), (n, 0)\}$
   Therefore, $|A| = n + 1$.
3. **(Variation)** $A = \{(x, y) \in \mathbb{N}^* \times \mathbb{N}^* : x + y = n, n \in \mathbb{N}^*, x, y \geq 2\}$
   If $n \leq 3$, $|A| = 0$.
   If $n \geq 4$, list: $\{(2, n-2), (3, n-3), \ldots, (n-2, 2)\} \Rightarrow |A| = n - 3$.

1.2 Lists

Example

- $A = \{(x,y,z) \in \mathbb{N}^* \times \mathbb{N}^* \times \mathbb{N}^* : x+y+z = n, n \in \mathbb{N}^*\}$
  List:

$$\begin{array}{c|cccccccccccc}
x & 0 & 0 & 0 & \cdots & 0 & 1 & 1 & 1 & \cdots & 1 & \cdots \\
\hline
y & 0 & 1 & 2 & \cdots & n & 0 & 1 & 2 & \cdots & n-1 & \cdots \\
\hline
z & n & n-1 & n-2 & \cdots & 0 & n-1 & n-2 & n-3 & \cdots & 0 & \cdots
\end{array}$$

$$\underbrace{\qquad\qquad\qquad}_{n+1}\quad\underbrace{\qquad\qquad\qquad}_{n}$$

$$\begin{array}{c|cccccccccc}
x & \cdots & k & k & k & \cdots & k & \cdots & n-1 & n-1 & n \\
\hline
y & \cdots & 0 & 1 & 2 & \cdots & n-k & \cdots & 0 & 1 & 0 \\
\hline
z & \cdots & n-k & n-k-1 & n-k-2 & \cdots & 0 & \cdots & 1 & 0 & 0
\end{array}$$

$$\underbrace{\qquad\qquad\qquad}_{n-k+1}\quad\underbrace{\qquad}_{2}\quad\underbrace{\quad}_{1}$$

Hence,

$$\begin{array}{rcccccccc}
|A| = & (n+1) & + & n & + & \cdots & + & 1 \\
|A| = & 1 & + & 2 & + & \cdots & + & (n+1) \\
\hline
2|A| = & \multicolumn{7}{l}{(n+1)(n+2)}
\end{array}$$

$$|A| = \frac{(n+1)(n+2)}{2}$$

1.2 Lists

Example

- $A = \{(x_1, \dots, x_m) \in \mathbb{N}^* \times \cdots \times \mathbb{N}^* : x_1 + x_2 + \cdots + x_m = n, n \in \mathbb{N}^*\}$

  Would you try to build the list in this case?

Remark

- *Building the list may be easy, less easy or impracticable.*
- *A list cannot have repetitions or absences (all the elements must be in the list and without repetitions).*
- *Before trying to solve a problem it may be convenient to analyze a particular case.*

1.2 Lists

Example

Multiples of an integer:

1. $A =$ set of multiples of 4 among the 1871 first natural numbers.
   $A = \{x : 1 \leq x \leq 1871, x \equiv 0 \pmod 4\} = \{4, 8, 12, 16, \ldots, 1868\}$
   $A = \{4 \cdot 1, 4 \cdot 2, 4 \cdot 3, 4 \cdot 4, \ldots, 4 \cdot 467\} \Rightarrow |A| = 467$
2. $A_n =$ set of multiples of 4 among the first n natural numbers.
   $A_n = \{4 \cdot 1, 4 \cdot 2, 4 \cdot 3, 4 \cdot 4, \ldots, 4 \cdot c\}$ where c is the quotient of dividing n by 4 $(n = 4c + r, 0 \leq r < 4)$.
   Thus, $|A_n| = c$, that is, c is the integer part of $\frac{n}{4}$ $(c = \left[\frac{n}{4}\right])$.
3. $a \in \mathbb{N}$, $A_{n,a} =$ set of multiples of a among the first n natural numbers.
   $A_{n,a} = \{a \cdot 1, a \cdot 2, a \cdot 3, a \cdot 4, \ldots, a \cdot c\}$ and $|A_{n,a}| = c = \left[\frac{n}{a}\right]$.

1.3 Floor and ceiling functions

Definition

Let $x \in \mathbb{R}$.
$\lfloor x \rfloor =$ **floor function** of $x =$ integer part of $x =$ the largest integer not greater than x.

Definition

Let $x \in \mathbb{R}$.
$\lceil x \rceil =$ **ceiling function** of $x =$ the smallest integer not less than x.

Remark

These names and notation were introduced by Keneth E. Iverson in 1962.

Properties:

i) $\lfloor x \rfloor = \lceil x \rceil \Leftrightarrow x \in \mathbb{Z}$
ii) $\lceil x \rceil = \lfloor x \rfloor + 1 \Leftrightarrow x \notin \mathbb{Z}$
iii) $\lfloor x \rfloor \leq x \leq \lceil x \rceil, x \in \mathbb{R}$
iv) $\lfloor -x \rfloor = -\lceil x \rceil, x \in \mathbb{R}$
v) $\lceil -x \rceil = -\lfloor x \rfloor, x \in \mathbb{R}$

1.3 Floor and ceiling functions

Definition

Let $x \in \mathbb{R}$.
$\{x\}$ = **fractional part** of $x = x - \lfloor x \rfloor$.

Definition

Let $x \in \mathbb{R}$.
$< x >=$ **pseudofractional part** of $x = \lceil x \rceil - x$.

Remark

$\{x\}, < x > \in [0,1)$.

Example

Recall that $A_{n,a}$ = set of multiples of a among the first n natural numbers and $|A_{n,a}| = c = \left[\frac{n}{a}\right] = \lfloor \frac{n}{a} \rfloor$.
The ratio of multiples of a in $\{1, 2, \ldots, n\}$ is

$$\frac{|A_{n,a}|}{n} = \frac{\lfloor \frac{n}{a} \rfloor}{n} = \frac{\frac{n}{a} - \{\frac{n}{a}\}}{n} = \frac{1}{a} - \frac{\{\frac{n}{a}\}}{n} \xrightarrow[n\to\infty]{} \frac{1}{a}.$$

(Notice that $\{\frac{n}{a}\} \in [0,1)$).

1.3 Floor and ceiling functions

Example

Integer coordinates in a circle:
$A_n = \{(x, y) \in \mathbb{N} \times \mathbb{N} : x^2 + y^2 \leq n^2\}$. How much is $|A_n|$?

- First column: $(1,1), (1,2), \ldots, (1,k)$ where k is the largest integer such that $1^2 + k^2 \leq n^2 \Leftrightarrow k^2 \leq n^2 - 1 \Leftrightarrow k \leq \sqrt{n^2 - 1}$. Therefore, $k = \lfloor \sqrt{n^2 - 1} \rfloor$.
- Second column: $(2,1), (2,2), \ldots, (2,l)$ where l is the largest integer such that $2^2 + l^2 \leq n^2 \Leftrightarrow l^2 \leq n^2 - 2^2 \Leftrightarrow l \leq \sqrt{n^2 - 2^2}$. Therefore, $l = \lfloor \sqrt{n^2 - 2^2} \rfloor$.

$\vdots$

- jth column: $(j,1), (j,2), \ldots, (j,r)$ where r is the largest integer such that $j^2 + r^2 \leq n^2 \Leftrightarrow r^2 \leq n^2 - j^2 \Leftrightarrow r \leq \sqrt{n^2 - j^2}$. Therefore, $r = \lfloor \sqrt{n^2 - j^2} \rfloor$.

Thus, $|A_n| = \sum_{j=1}^{n} \lfloor \sqrt{n^2 - j^2} \rfloor$.

Example

Let A_n be the set of squares of area $1u^2$ in a quarter of a circle of radius n. How much is $|A_n|$?

Area covered by the squares$= |A_n|u^2 \leq$ area of the fourth of the circle $= \frac{\pi n^2}{4}$.

Hence, $|A_n| \leq \frac{\pi n^2}{4}$.

Move the squares one up and one to the right: $\frac{\pi n^2}{4} \leq |A_n| + 2n - 1$.

Thus, $\frac{\pi n^2}{4} - 2n + 1 \leq |A_n| \leq \frac{\pi n^2}{4} \Rightarrow 1 - \frac{2n-1}{\frac{\pi n^2}{4}} \leq \frac{|A_n|}{\frac{\pi n^2}{4}} \leq 1$.

Since $\frac{2n-1}{\frac{\pi n^2}{4}} \xrightarrow[n\to\infty]{} 0$, by the sandwich rule, $\frac{|A_n|}{\frac{\pi n^2}{4}} \xrightarrow[n\to\infty]{} 1$.

Thus, $|A_n| \sim \frac{\pi n^2}{4}$.

Example

Number of digits in the decimal system:

Given $n \in \mathbb{N}$ calculate the number of digits of n in the decimal system.

For example, $n = 1234 \Rightarrow d(n) = 4$; $n = 300 \Rightarrow d(n) = 3$.

$1000 \leq abcd \leq 9999 < 10000 \Rightarrow 10^3 \leq abcd < 10^4$

$10^{d(n)-1} \leq n < 10^{d(n)} \Rightarrow \log_{10}(10^{d(n)-1}) \leq \log_{10}(n) < \log_{10}(10^{d(n)}) \Rightarrow$
$d(n) - 1 \leq \log_{10}(n) < d(n) \Rightarrow d(n) - 1 = \lfloor \log_{10}(n) \rfloor \Rightarrow$

$$d(n) = \lfloor \log_{10}(n) \rfloor + 1.$$

1.3 Floor and ceiling functions

Example

Number of integers in the interval (α, β), $\alpha, \beta \in \mathbb{R}$:

Let $A_{\alpha,\beta}$ = set of integers in the interval (α, β), $\alpha, \beta \in \mathbb{R}$. How much is $|A_{\alpha,\beta}|$?

For example:

- $\alpha = 2.5, \beta = 4.7 \Rightarrow |A_{\alpha,\beta}| = 2$
- $\alpha = 2.5, \beta = 2.8 \Rightarrow |A_{\alpha,\beta}| = 0$

$$A_{\alpha,\beta} = \{\lfloor\alpha\rfloor+1, \lfloor\alpha\rfloor+2, \ldots, \lceil\beta\rceil-1\} = \{\lfloor\alpha\rfloor+1, \lfloor\alpha\rfloor+2, \ldots, \lfloor\alpha\rfloor+\lceil\beta\rceil-1-\lfloor\alpha\rfloor\}.$$

Remark: Notice that the first integer $\lfloor\alpha\rfloor + 1$ instead of $\lceil\alpha\rceil$ since if α were integer then $\lceil\alpha\rceil = \alpha$. But the interval is open in α!

Thus, $|A_{\alpha,\beta}| = \lceil\beta\rceil - 1 - \lfloor\alpha\rfloor$.

For example,
$\alpha = 2.5, \beta = 4.7 \Rightarrow |A_{\alpha,\beta}| = 5 - 2 - 1 = 2$
$\alpha = 2.5, \beta = 2.8 \Rightarrow |A_{\alpha,\beta}| = 3 - 2 - 1 = 0$

1.4 Tree diagrams

1.4.1 Rule of product

Definition

Rule of product: If an event X can happen in x ways and a distinct event Y in y ways then X and Y can happen in xy ways.

Example

If we can go from Boston to Chicago in 3 ways (x, y, z) and from Chicago to Dallas in 5 ways (1,2,3,4,5) then the number of ways in which we can go from Boston to Chicago to Dallas is 15:

$$\begin{array}{ccccc} x1 & x2 & x3 & x4 & x5 \\ y1 & y2 & y3 & y4 & y5 \\ z1 & z2 & z3 & z4 & z5 \end{array}$$

1.4 Tree diagrams

1.4.1 Rule of product

The **rule of product** can be **generalized** to situations involving more than two events: If X_1 can happen in x_1 ways, X_2 in x_2 ways, X_3 in x_3 ways... then X_1 and X_2 and X_3 and... can happen at once in $x_1 x_2 x_3 \cdots$ ways.

Example

In order to make up a menu we can select among two different first dishes: A or B, three different second dishes: a,b or c and two different desserts: α or β. How many different menus can we make up? $2 \cdot 3 \cdot 2 = 12$ menus.

Example

Pigeonhole

		$\cdots$	
1	2	$\cdots$	n

If there are c_1 ways of filling hole 1, c_2 ways of filling hole 2,..., c_n ways of filling hole n then there are $c_1 \cdot c_2 \cdot \ldots \cdot c_n$ ways of filling the pigeonhole.

1.4 Tree diagrams

1.4.2 Variations

Definition

A **set** is an unordered collection of distinct objects.
The objects are called **elements** of the set.
We use braces to denote a set.
For example, the set with elements 1, 2 and 3 is denoted $\{1, 2, 3\}$.
Since the elements are not ordered, $\{1, 2, 3\}$ and $\{2, 3, 1\}$ are the same set.

Definition

A **sequence** is an ordered collection of not necessarily distinct objects.
We use brackets to denote a sequence.
For example, $(1, 1, 2)$.
Since the entries are ordered, $(1, 1, 2)$ and $(1, 2, 1)$ are different sequences.

Sometimes the following notation will be used: $112 \equiv (1, 1, 2)$.

1.4 Tree diagrams

1.4.2 Variations

Definition

Let $\Omega = \{a_1, \dots, a_n\}$ be a set and $n \geq k \geq 1$.
A k**-variation without repetition of** Ω or k**-permutation without repetition of** n is an arrangement of the elements of Ω taken k at a time, where two arrangements are regarded as different if they differ in composition or in the order of their elements.
In other words, a k-variation without repetition of Ω is a sequence of length k that can be formed with the elements in Ω without repeating them.

Let $V_{n,k}$ denote the number of k-variations without repetition given n distinct objects.
How much is $V_{n,k}$?
We note that we can choose the first element in n ways, the second element in $n-1$ ways,..., the kth element in $n-k+1$ ways. By the rule of product,

$$\boxed{V_{n,k} = n(n-1)\cdots(n-k+1) = \frac{n!}{(n-k)!}}$$

1.4 Tree diagrams

1.4.2 Variations

$$\boxed{V_{n,k} = n(n-1)\cdots(n-k+1) = \frac{n!}{(n-k)!}}$$

Example

A certain society has 25 members. The members of the society are to elect a president, a vice president, a secretary, and a treasurer. In how many ways is it possible to select the 4 officers if no member of the society can hold more than one office at a time?
We are to find the number of variations (without repetitions) of 25 members taken 4 at a time. This number is equal to
$V_{25,4} = 25 \cdot 24 \cdot 23 \cdot 22 = 303,600$.

1.4 Tree diagrams

1.4.2 Variations

Example

In a draw there are 3 prizes: a car, a trip and a basket full of food. 100 tickets have been sold, numbered from 1 to 100. For each ticket a ball has been introduced into a drum. The first number drawn wins the basket, the second wins the trip and the third, the car.

- How many different possibilities are there?
  Basket: 100 possible numbers, trip: 99 possible numbers, car: 98 possible numbers
  $V_{100,3} = 100 \cdot 99 \cdot 98 = 970,200$.
- If I bought 5 tickets, how many possibilities would I have of winning the car?
  Basket: 99 possible numbers, trip: 98 possible numbers, car: 5 possible numbers
  $99 \cdot 98 \cdot 5 = 48,510$
- And of winning the three prizes?
  Basket: 5 possible numbers, trip: 4 possible numbers, car: 3 possible numbers
  $5 \cdot 4 \cdot 3 = 60$
- And of winning at least one prize?
  We study it later.

1.4 Tree diagrams

1.4.2 Variations

Definition

Let $\Omega = \{a_1, \ldots, a_n\}$ be a set and $k \geq 1$.
A k-**variation with repetition of** Ω or k-**permutation with repetition of** n is a sequence of length k that can be formed with the elements in Ω being able to repeat them.

Let $VR_{n,k}$ denote the number of k-variations with repetition given n distinct objects.
How much is $VR_{n,k}$?
We note that we can choose the first element in n ways, the second element in n ways,..., the kth element in n ways. By the rule of product,

$$\boxed{VR_{n,k} = n \cdot n \cdots n = n^k}$$

Notice that it is not necessary that $k \leq n$.

1.4 Tree diagrams

1.4.2 Variations

$$\boxed{VR_{n,k} = n \cdot n \cdots n = n^k}$$

Example

- How many numbers of 3 digits can be formed in the decimal system?

  $\overline{10}\quad \overline{10}\quad \overline{10} \quad \rightarrow \quad VR_{10,3} = 10^3 = 1,000$

- How many of them are palindromic?

  $\overline{10}\quad \overline{10}\quad \overline{1} \quad \rightarrow \quad VR_{10,2} = 10^2 = 100$

- How many are even?

  $\overline{10}\quad \overline{10}\quad \overline{5} \quad \rightarrow \quad 5 \cdot VR_{10,2} = 5 \cdot 10^2 = 500$

Example

How many football pools coupons must we fill in order to be absolutely sure we win?
Notice that there are 15 matches and 3 possible results.
$VR_{3,15} = 3^{15} = 14,348,907$

1.4 Tree diagrams

1.4.2 Variations

Example

Let $\Omega = \{a, e, o, u, b, c, d\}$.

- A is the set of words of 4 letters that can be formed with the alphabet in Ω in such a way that the first is a consonant and the last a vowel.

  $\overline{3}\quad \overline{7}\quad \overline{7}\quad \overline{4} \quad \rightarrow \quad |A| = 3 \cdot 7 \cdot 7 \cdot 4 = 12 \cdot 49 = 588$

- A is the set of words of 5 letters that can be formed with the alphabet in Ω in such a way that there are not equal consecutive letters.

  $\overline{7}\quad \overline{6}\quad \overline{6}\quad \overline{6}\quad \overline{6} \quad \rightarrow \quad |A| = 7 \cdot 6 \cdot 6 \cdot 6 \cdot 6 = 9,072$

Example

A is the set of numbers of 5 different digits starting and finishing in an even digit.

$\overline{5}\quad \overline{8}\quad \overline{7}\quad \overline{6}\quad \overline{4} \quad \rightarrow \quad |A| = 5 \cdot 8 \cdot 7 \cdot 6 \cdot 4 = 6,720$

1.4 Tree diagrams

1.4.2 Variations

Example

Calculate the number of ways to place 4 distinguishable balls in 3 numbered boxes in such a way that none of the boxes are empty.

First idea: Place balls B_1, B_2, B_3 one in each box and then ball B_4. This is not a good idea because $\boxed{B_1, B_2} \boxed{B_3} \boxed{B_4}$ wouldn't be counted.

Second idea: Select three balls, place them one in each box and place the fourth ball. This is not a good idea because we are counting $\boxed{B_1, B_4} \boxed{B_2} \boxed{B_3}$ and $\boxed{B_4, B_1} \boxed{B_2} \boxed{B_3}$ but they are the same.

Good idea: Choose the box with two elements (3), select the balls in that box (?), place the remaining balls one in each box (2). We study it later.

1.4 Tree diagrams

1.4.3 Combinations

Definition

Let $\Omega = \{a_1, \ldots, a_n\}$ be a set of elements and $n \geq k \geq 1$.
A k-**combination without repetition of** Ω is an arrangement of the elements of Ω taken k at a time, where two arrangements are regarded as different only if they differ in composition.
In other words, a k-combination without repetition of Ω is a subset of k elements of Ω.

The number of k-combinations without repetition of n elements is denoted by the symbol $C_{n,k}$.

There is a simple relation between $C_{n,k}$ and $V_{n,k}$:

Example

Let $\Omega = \{a, b, c, d, e\}$. Count the 3-variations without repetition.

One way:

$$\underset{5}{-} \quad \underset{4}{-} \quad \underset{3}{-} \quad \rightarrow \quad V_{5,3} = 5 \cdot 4 \cdot 3 = 60$$

Another way:

Choose the three elements and then order the elements:

$V_{5,3} = C_{5,3} \cdot V_{3,3} = C_{5,3} \cdot 3! \Rightarrow C_{5,3} = \frac{V_{5,3}}{3!} = \frac{5 \cdot 4 \cdot 3}{3!} = \frac{\frac{5!}{2!}}{3!} = \frac{5!}{2!3!} = \binom{5}{3}$.

In general,

$$\boxed{C_{n,k} = \frac{V_{n,k}}{k!} = \binom{n}{k}}$$

Example

Calculate the number of ways to place 4 distinguishable balls in 3 numbered boxes in such a way that none of the boxes are empty.

We can choose the box with two elements (3), select the balls in that box ($C_{4,2}$), place the remaining balls one in each box (2). Therefore, $3 \cdot \binom{4}{2} \cdot 2 = 3\frac{4!}{2!2!} \cdot 2 = 36$.

Example (Generalization)

Calculate the number of ways to place $n+1$ distinguishable balls in n numbered boxes in such a way that none of the boxes are empty.

First we choose the box with two elements (n), select the balls in that box ($C_{n+1,2}$), place the remaining $n-1$ balls one in each box ($V_{n-1,n-1}$). Therefore, $nC_{n+1,2}V_{n-1,n-1}$.

1.4 Tree diagrams

1.4.3 Combinations

Example (Generalization)

Calculate the number of ways to place $n+2$ distinguishable balls in n numbered boxes in such a way that none of the boxes are empty.

Two ways:

1) one box has three balls and the remaining boxes have one ball each
2) two boxes have 2 balls and the remaining boxes have one ball each

1) Choose the box with tree elements (n), select the balls in that box ($C_{n+2,3}$), place the remaining $n-1$ balls one in each box ($V_{n-1,n-1}$). Therefore, $nC_{n+2,3}V_{n-1,n-1}$.
2) Choose the double boxes ($C_{n,2}$), select the balls in the first double box ($C_{n+2,2}$), select the balls in the second double box ($C_{n,2}$), place the remaining $n-2$ balls one in each box ($V_{n-2,n-2}$). Therefore, $C_{n,2}C_{n+2,2}C_{n,2}V_{n-2,n-2}$.
Thus, the solution is $nC_{n+2,3}V_{n-1,n-1} + C_{n,2}C_{n+2,2}C_{n,2}V_{n-2,n-2}$.

1.4 Tree diagrams

1.4.3 Combinations

Proposition (Basic properties of the binomial coefficients)

1. $C_{n,k} = \binom{n}{k} = \frac{V_{n,k}}{k!} = \frac{\frac{n!}{(n-k)!}}{k!} = \frac{n!}{(n-k)!k!}, \quad k = 0, \ldots, n.$
2. $\binom{n}{k} = \frac{n!}{(n-k)!k!} = \frac{n!}{k!(n-k)!} = \frac{n!}{(n-(n-k))!(n-k)!} = \binom{n}{n-k}, \quad k = 0, \ldots, n.$
   Choosing k elements among n is the same as rejecting $n-k$ among n.
3. $\binom{n}{0} = \frac{n!}{(n-0)!0!} = \frac{n!}{n!\cdot 1} = 1.$
4. $\binom{n}{n} = \frac{n!}{(n-n)!n!} = \frac{n!}{1\cdot n!} = 1.$
5. $\binom{n}{k} = 0, \quad k > n.$
6. $\binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}, \quad n \geq 1, k = 0, \ldots, n.$
7. *If* $k \geq 1$, $\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}$.
8. *If* $0 \leq k \leq n-1$, $\binom{n}{k} = \frac{n}{n-k}\binom{n-1}{k}$.

Definition

A **permutation of n elements** or **n-permutation** is an n-variation without repetition of n elements, that is, a variation without repetition of n elements which contains all the n elements.
In other words, permutations of n elements are all the possible n-arrangements each of which contains every element once, with two such arrangements differing only in the order of their elements.

The number of n-permutations is denoted by P_n.

$$\boxed{P_n = V_{n,n} = n(n-1)\cdots 2 \cdot 1 = n!}$$

Definition

A **multiset** is a generalization of the notion of set in which elements are allowed to appear more than once.

Definition

Let M be a multiset with n_1 elements of the first type, n_2 elements of the second type,..., n_k elements of the kth type, i. e.,

$$M = \{a_1, \overset{n_1}{\ldots}, a_1, a_2, \overset{n_2}{\ldots}, a_2, \ldots, a_k, \overset{n_k}{\ldots}, a_k\}.$$

The different arrangements of M are the **permutations with repetition**.
In other words, a permutation with n_1 elements of the first type, n_2 elements of the second type,..., n_k elements of the kth type is a sequence formed with n_1 elements of the first type, n_2 elements of the second type,..., n_k elements of the kth type.

$P(n_1, n_2, \ldots, n_k)$ denotes the number of such permutations.
How much is $P(n_1, n_2, \ldots, n_k)$?

Example

Calculate the number of sequences that can be formed with p zeros and q ones.

$$\underbrace{\_\ \_\ \cdots\ \_}_{p+q}$$

Choose where to place the zeros: $C_{p+q,p} = \binom{p+q}{p} = \frac{(p+q)!}{p!q!}$.

Remark: We get the same choosing the places for the ones: $C_{p+q,q} = \binom{p+q}{q}$.

Example

Calculate the number of sequences that can be formed with p zeros, q ones, and r twos.

Choose the places for the zeros and then choose the places for the ones:

$C_{p+q+r,p} \cdot C_{q+r,q} = \binom{p+q+r}{p} \cdot \binom{q+r}{q} = \frac{(p+q+r)!}{(p+q+r-p)!p!} \frac{(q+r)!}{(q+r-q)!q!} = \frac{(p+q+r)!}{(q+r)!p!} \frac{(q+r)!}{r!q!} = \frac{(p+q+r)!}{p!q!r!}$.

Let $n = n_1 + n_2 + \cdots + n_k$.

$$P(n_1, n_2, \ldots, n_k) = \binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\cdots\binom{n-n_1-\cdots-n_{k-2}}{n_{k-1}} =$$

$$\frac{n!}{n_1!(n-n_1)!}\frac{(n-n_1)!}{n_2!(n-n_1-n_2)!}\frac{(n-n_1-n_2)!}{n_3!(n-n_1-n_2-n_3)!}\cdots\frac{(n-n_1-\cdots-n_{k-2})!}{n_{k-1}!(n-n_1-\cdots-n_{k-2}-n_{k-1})!} =$$

$$\frac{n!}{n_1!n_2!\cdots n_{k-1}!n_k!}$$

$$\boxed{P(n_1, n_2, \ldots, n_k) = \frac{n!}{n_1!n_2!\cdots n_{k-1}!n_k!}}$$

Definition

Let $\Omega = \{a_1, \ldots, a_n\}$ be a set of elements and $n, k \geq 1$.
A k-**combination with repetition of** Ω is a multiset of k elements of Ω.

The number of k-combinations with repetition given n distinct objects is denoted by the symbol $CR_{n,k}$.
How much is $CR_{n,k}$?

Example

How much is $CR_{5,3}$?
Let $\Omega = \{a_1, a_2, a_3, a_4, a_5\}$. We want to form multisets of 3 elements. Define
$A = \{a_{\alpha_1} a_{\alpha_2} a_{\alpha_3} : 1 \leq \alpha_1 \leq \alpha_2 \leq \alpha_3 \leq 5\}$ (set of multisets of 3 elements of Ω).
Thus, $|A| = CR_{5,3}$.
Define $\Omega' = \{b_1, b_2, b_3, b_4, b_5, b_6, b_7\}$,
$B = \{b_{\beta_1} b_{\beta_2} b_{\beta_3} : 1 \leq \beta_1 < \beta_2 < \beta_3 \leq 7\}$ (set of sets of 3 elements of Ω').
Thus, $|B| = C_{7,3}$.
Define the map

$$\begin{array}{cccc} \phi : & A & \to & B \\ & a_{\alpha_1} a_{\alpha_2} a_{\alpha_3} & \to & b_{\alpha_1} b_{\alpha_2+1} b_{\alpha_3+2} \end{array}.$$

This is a bijection:

- Injective: Let $\phi(a_{\alpha_1} a_{\alpha_2} a_{\alpha_3}) = \phi(a_{\gamma_1} a_{\gamma_2} a_{\gamma_3})$. Then $b_{\alpha_1} b_{\alpha_2+1} b_{\alpha_3+2} = b_{\gamma_1} b_{\gamma_2+1} b_{\gamma_3+2} \Rightarrow \alpha_1 = \gamma_1, \alpha_2 + 1 = \gamma_2 + 1, \alpha_3 + 2 = \gamma_3 + 2 \Rightarrow \alpha_1 = \gamma_1, \alpha_2 = \gamma_2, \alpha_3 = \gamma_3 \Rightarrow a_{\alpha_1} a_{\alpha_2} a_{\alpha_3} = a_{\gamma_1} a_{\gamma_2} a_{\gamma_3}$.
- Surjective: Let $b_{\beta_1} b_{\beta_2} b_{\beta_3}$ with $1 \leq \beta_1 < \beta_2 < \beta_3 \leq 7$. Let $\alpha_1 = \beta_1, \alpha_2 = \beta_2 - 1, \alpha_3 = \beta_3 - 2$. Thus, $1 \leq \alpha_1 \leq \alpha_2 \leq \alpha_3 \leq 5$. Hence, there exists $a_{\alpha_1} a_{\alpha_2} a_{\alpha_3} \in A$ such that $\phi(a_{\alpha_1} a_{\alpha_2} a_{\alpha_3}) = b_{\beta_1} b_{\beta_2} b_{\beta_3}$.

Therefore, A and B both have the same number of elements. $CR_{5,3} = |A| = |B| = C_{7,3}$.

Example

Let $\Omega = \{a, b, c, d\}$. How much is $CR_{4,3}$?

		a	b	c	d		
$\{a,a,a\}$	$\leftrightarrow$	000				$\leftrightarrow$	000111
$\{a,a,b\}$	$\leftrightarrow$	00	0			$\leftrightarrow$	001011
$\{a,a,c\}$	$\leftrightarrow$	00		0		$\leftrightarrow$	001101
$\{a,b,b\}$	$\leftrightarrow$	0	00			$\leftrightarrow$	010011
$\{a,b,c\}$	$\leftrightarrow$	0	0	0		$\leftrightarrow$	010101
$\{c,c,d\}$	$\leftrightarrow$			00	0	$\leftrightarrow$	110010

Therefore, $CR_{4,3}$ = number of sequences that can be formed with 3 zeros and 3 ones.

This can be calculated choosing three places out of 6, which is $\binom{6}{3}$.

For each multiset of k among n elements there is a sequence formed with k zeros and $n-1$ ones, and conversely.
Therefore, $CR_{n,k}$ is the number of sequences formed with k zeros and $n-1$ ones. Thus,

$$\boxed{CR_{n,k} = C_{n+k-1,k} = \frac{V_{n+k-1,k}}{k!} = \frac{(n+k-1)!}{(n-1)!k!} = \binom{n+k-1}{k} = P(k, n-1)}$$

Example

Calculate the number of sequences that can be formed with p zeros and q ones.
$CR_{q+1,p} = \binom{q+1+p-1}{p} = \binom{p+q}{p}$.

Example

Calculate the number of sequences that can be formed with 2 zeros and 3 ones such that there are not consecutive zeros.
$\sim \; 1 \; \sim \; 1 \; \sim \; 1 \; \sim$
Choose 2 of the 4 possible places for the zeros: $C_{4,2} = \binom{4}{2} = \frac{4!}{2!2!} = 6$.

1.4 Tree diagrams

1.4.5 Combinations with repetitions

Example

Calculate the number of sequences that can be formed with p zeros and q ones such that there are not two consecutive zeros.

$\sim \quad 1 \quad \sim \quad 1 \quad \sim \quad \cdots \quad \sim \quad 1 \quad \sim$

Choose p of the possible $q+1$ places for the zeros: $C_{q+1,p} = \binom{q+1}{p}$.

Example

Calculate the number of sequences that can be formed with 2 zeros, 3 ones and 4 twos such that the zeros are not consecutive.

Choose a sequence formed by 1s and 2s and then choose 2 of the possible 8 places for the zeros: $\sim \quad x \quad \sim \quad x \quad \sim \quad \cdots \quad \sim \quad x \quad \sim \qquad x = 1$ or 2: $C_{7,3}C_{8,2} = \binom{7}{3}\binom{8}{2}$.

Example (Generalization)

Calculate the number of sequences that can be formed with p zeros, q ones and r twos such that there are not two consecutive zeros.

Choose a sequence of 1s and 2s and then choose p of the possible $q+r+1$ places for the zeros: $C_{q+r,q}C_{q+r+1,p} = \binom{q+r}{q}\binom{q+r+1}{p}$.

1.5 Factorial powers

Definition

Let $a \in \mathbb{C}$. Its **descending** or **lower** or **falling factorial (power) of order** k, $k \in \mathbb{N}^*$, is

$$a^{\underline{k}} = \begin{cases} a(a-1)\cdots(a-k+1) & k \geq 1 \\ 1 & k = 0 \end{cases}$$

Definition

Let $a \in \mathbb{C}$. Its **ascending** or **upper** or **rising factorial (power) of order** k, , $k \in \mathbb{N}^*$, is

$$a^{\overline{k}} = \begin{cases} a(a+1)\cdots(a+k-1) & k \geq 1 \\ 1 & k = 0 \end{cases}$$

1.5 Factorial powers

Proposition

Some properties are the following:

1. 1. $(-a)^k = (-1)^k a^k$
   2. $(-a)^{\underline{k}} = (-1)^k a^{\overline{k}}$
   3. $(-a)^{\overline{k}} = (-1)^k a^{\underline{k}}$

2. 1. $a^{k+l} = a^k \cdot a^l$
   2. $a^{\underline{k+l}} = a^{\underline{k}} \cdot (a-k)^{\underline{l}}$
   3. $a^{\overline{k+l}} = a^{\overline{k}} \cdot (a+k)^{\overline{l}}$

3. $a = n \in \mathbb{N}^*$
   1. *If* $0 \leq k \leq n$
      $n^{\underline{k}} = n(n-1)\cdots(n-k+1) = \frac{n(n-1)\cdots(n-k+1)(n-k)(n-k-1)\cdots 1}{(n-k)(n-k-1)\cdots 1} = \frac{n!}{(n-k)!}$
   2. *If* $k > n$
      $n^{\underline{k}} = 0$

1.6 Classifications

Definition

Rule of sum: Let Ω be a set. If $\{A_1, \ldots, A_k\}$ is a partition of Ω, that is, $\cup_{i=1}^k A_i = \Omega$ and $A_i \cap A_j = \emptyset$ for all $i \neq j$ then

$$|\Omega| = |A_1| + \cdots + |A_k|.$$

Classification is useful:

- Sometimes there is no other solution but to classify.
- We may find interesting identities.
- Classification is the main idea to get recurrence relations.

1.6 Classifications

Example

$A =$ ways of placing 5 distinguishable balls in 3 numbered boxes such that none of them is empty.
Define $A_1 =$ ways in which there are two boxes with two balls each, $A_2 =$ ways in which there is a box with three balls. Therefore, $|A| = |A_1| + |A_2|$.

Example

$\Omega = \{a_1, \dots, a_n\}$.
$\mathcal{P}(\Omega) =$ collection of subsets of $\Omega =$ power set of Ω. How much is $|\mathcal{P}(\Omega)|$?
Subsets of Ω:

- with 0 elements: $\emptyset$
- with 1 element: $\{a_1\}, \{a_2\}, \dots, \{a_n\}$

  $\vdots$
- with k elements

  $\vdots$
- with n elements: Ω

$|\mathcal{P}(\Omega)| = C_{n,0} + C_{n,1} + \cdots + C_{n,n} = \sum_{i=0}^{n} C_{n,i} = \sum_{i=0}^{n} \binom{n}{i}$.

1.6 Classifications

Example

Let $\Omega = \{a_1, \dots, a_n\}$. How much is $|\mathcal{P}(\Omega)|$?
Set $x_n = |\mathcal{P}(\Omega)|$.
Subsets that contain a_n: there are $x_{n-1} = |\mathcal{P}(\{a_1, \dots, a_{n-1}\})|$.
Subsets that do not contain a_n: there are $x_{n-1} = |\mathcal{P}(\{a_1, \dots, a_{n-1}\})|$.
Therefore,
$x_n = x_{n-1} + x_{n-1} = 2x_{n-1} = 2 \cdot 2x_{n-2} = 2^3 x_{n-3} = \cdots = 2^{n-1}x_1 = 2^n$.

$$\boxed{2^n = \sum_{i=0}^{n} \binom{n}{i}}$$

1.6 Classifications

Example (Recurrence relation of the binomial numbers)

Let $\Omega = \{a_1, \dots, a_n\}$ and $n \geq k \geq 1$.
How many k-combinations without repetition of Ω are there? $C_{n,k}$

- Subsets with k elements that contain a_n: there are $C_{n-1,k-1}$.
- Subsets with k elements that do not contain a_n: there are $C_{n-1,k}$.

Therefore, $C_{n,k} = C_{n-1,k-1} + C_{n-1,k} \Rightarrow \binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$.
We get Pascal's triangle of the binomial numbers or combinations:

$$\begin{array}{ccccccccc}
 & & & & \binom{0}{0} & & & & \\
 & & & \binom{1}{0} & & \binom{1}{1} & & & \\
 & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & \\
 & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & \\
\binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4}
\end{array}$$

Thus,

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1
\end{array}$$

1.6 Classifications

Example

a_n=number of sequences of length n formed with zeros and ones in which there are not consecutive zeros.

$a_1 = 2 \quad (0), (1)$

$a_2 = 3 \quad (0,1), (1,0), (1,1)$

$a_3 = 5 \quad (0,1,0), (0,1,1), (1,0,1), (1,1,0), (1,1,1)$

We classify by the first element:

- Starting with zero: the second number must be 1 so there are a_{n-2}.
- Starting with one: there are a_{n-1}.

Hence,

$$a_n = a_{n-1} + a_{n-2}, \quad n \geq 3.$$

1.6 Classifications

Example

a_n=number of sequences of length n formed with zeros, ones and twos in which there are not consecutive ones.

$a_1 = 3 \quad (0), (1), (2)$

$a_2 = 8 \quad (0,0), (0,1), (0,2), (1,0), (1,2), (2,0), (2,1), (2,2)$

For $n \geq 3$:

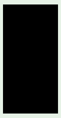
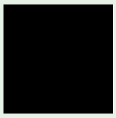
- Starting with 0: there are a_{n-1}
- Starting with 10: there are a_{n-2}
- Starting with 12: there are a_{n-2}
- Starting with 2: there are a_{n-1}

Therefore,

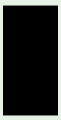
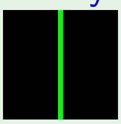
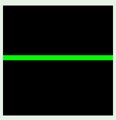
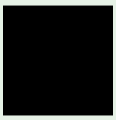
$$a_n = 2a_{n-1} + 2a_{n-2}, \quad n \geq 3.$$

1.6 Classifications

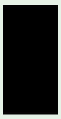
Example

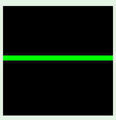
There are two types of floor tiles: 2x1 , 2x2 

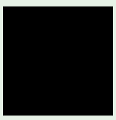
Let a_n be the number of ways to tile a rectangular floor of dimension 2xn.

$a_1 = 1$:  ; $a_2 = 3$: , , 

We classify the ways to tile by the first element(s):

: there are a_{n-1}

: there are a_{n-2}

: there are a_{n-2}

Therefore, $a_n = a_{n-1} + 2a_{n-2}, \quad n \geq 3.$

1.7 The principle of inclusion and exclusion

Notation: $A \cap B = AB$ for simplicity.

Example

Let $A_n = \{k : 1 \leq k \leq n$ and k is divisible by 2 or $3\}=$
$\{k : 1 \leq k \leq n, k \equiv 0 \pmod 2\} \cup \{k : 1 \leq k \leq n, k \equiv 0 \pmod 3\}$.

Define $B_n = \{k : 1 \leq k \leq n, k \equiv 0 \pmod 2\}$,
$C_n = \{k : 1 \leq k \leq n, k \equiv 0 \pmod 3\}$.

Is $\{B_n, C_n\}$ a partition of A_n?

It is true that $B_n \cup C_n = A_n$, but $B_n \cap C_n \neq \emptyset$.
For example $6 \in B_n \cap C_n, n \geq 6$.

So we cannot deduce that $|A_n| = |B_n| + |C_n|$.
But, is there any relationship?

Considering the Venn diagram:
$|A_n| = |B_n \cup C_n| = |B_n| + |C_n| - |B_nC_n|$
$|B_n| = \lfloor \frac{n}{2} \rfloor$,
$|C_n| = \lfloor \frac{n}{3} \rfloor$,
$|B_nC_n| = \lfloor \frac{n}{6} \rfloor$.

Therefore, $|A_n| = \lfloor \frac{n}{2} \rfloor + \lfloor \frac{n}{3} \rfloor - \lfloor \frac{n}{6} \rfloor$.

1.7 The principle of inclusion and exclusion

Example

$A_n = \{k : 1 \leq k \leq n, k \equiv 0 \pmod 2\}$
$B_n = \{k : 1 \leq k \leq n, k \equiv 0 \pmod 3\}$
$C_n = \{k : 1 \leq k \leq n, k \equiv 0 \pmod 5\}$

How much is $|D_n| = |A_n \cup B_n \cup C_n|$?

$|D_n| = |A_n \cup B_n \cup C_n| = |(A_n \cup B_n) \cup C_n| = |(A_n \cup B_n)| + |C_n| - |(A_n \cup B_n)C_n| =$
$|A_n| + |B_n| - |A_nB_n| + |C_n| - |A_nC_n \cup B_nC_n| =$
$|A_n| + |B_n| - |A_nB_n| + |C_n| - (|A_nC_n| + |B_nC_n| - |A_nB_nC_n|) =$
$|A_n| + |B_n| + |C_n| - |A_nB_n| - |A_nC_n| - |B_nC_n| + |A_nB_nC_n|$.

Thus, $|D_n| = \lfloor \frac{n}{2} \rfloor + \lfloor \frac{n}{3} \rfloor + \lfloor \frac{n}{5} \rfloor - \lfloor \frac{n}{6} \rfloor - \lfloor \frac{n}{10} \rfloor - \lfloor \frac{n}{15} \rfloor + \lfloor \frac{n}{30} \rfloor$.

1.7 The principle of inclusion and exclusion

Theorem (The Principle of Inclusion-Exclusion)

Let $A_1, \ldots, A_n$ be subsets of a finite set Ω. Then

$$|\cup_{i=1}^n A_i| = \sum_{i=1}^n |A_i| - \sum_{1\leq i<j\leq n} |A_iA_j| + \sum_{1\leq i<j<k\leq n} |A_iA_jA_k| - \cdots +$$

$$+(-1)^{r-1} \sum_{1\leq i_1<i_2<\cdots<i_r\leq n} |A_{i_1}A_{i_2}\cdots A_{i_r}| + \cdots + (-1)^{n-1}|A_1\cdots A_n|$$

1.7 The principle of inclusion and exclusion

Example

Suppose that we have n married couples (a man and a woman) in a ball: $(M_1, W_1), \ldots, (M_n, W_n)$.

In how many different ways can we pair the men off with the women so that no man dances with his wife?

Let $\Omega = \{$ways in which all the dancing couples can be done$\}$,
$\Omega^* = \{$ways in which no man dances with his wife$\}$.
Notice that $|\Omega| = P_n = n!$.
We want to calculate $|\Omega^*|$ but it is easier to calculate $|\Omega^{*c}|$.
$\Omega^{*c} = \{$ways in which there is at least one dancing couple that is a married couple$\}$.
Hence, $|\Omega^*| = |\Omega| - |\Omega^{*c}|$.
Define $A_k = \{$ways in which married couple k dances together$\}, k = 1, \ldots, n$.
Fix married couple i: $|A_i| = \#$ of ways to arrange the remainder couples $= P_{n-1} = (n-1)!$
Fix married couples i, j: $|A_iA_j| = \#$ of ways to arrange the remainder couples $= P_{n-2} = (n-2)!$
Fix married couple $i_1, \ldots, i_r$: $|A_{i_1} \ldots A_{i_r}| = \#$ of ways to arrange the remainder couples $= P_{n-r} = (n-r)!$
$|A_1 \ldots A_n| = P_0 = 1$.
Thus,
$|\Omega^{*c}| = \sum_{i=1}^n |A_i| - \sum_{1\leq i<j\leq n}^n |A_iA_j| + \cdots + (-1)^{r-1} \sum_{1\leq i_1<\cdots<i_r\leq n}^n |A_{i_1} \ldots A_{i_r}| + \cdots + (-1)^n |A_1 \ldots A_n| = n\cdot(n-1)! - C_{n,2}\cdot(n-2)! + \cdots + (-1)^{r-1} C_{n,r}\cdot(n-r)! + \cdots + (-1)^n \cdot 1 = \sum_{i=1}^n (-1)^{i-1} C_{n,i}\cdot(n-i)!$
Therefore,
$|\Omega^*| = |\Omega| - |\Omega^{*c}| = n! - \sum_{i=1}^n (-1)^{i-1} C_{n,i}\cdot(n-i)! = \sum_{i=0}^n (-1)^i C_{n,i}\cdot(n-i)!$

1.7 The principle of inclusion and exclusion

Remark

A simpler expression for the principle of inclusion-exclusion:

Given $\emptyset \neq I \subseteq \{1, \ldots, n\}$ *define* $A_I = \cap_{i \in I} A_i$. *Then,*

$$| \cup_{i=1}^{n} A_i | = \sum_{\emptyset \neq I \subseteq \{1,\ldots,n\}} (-1)^{|I|-1} |A_I|.$$

1.7 The principle of inclusion and exclusion
1.7.1 Euler's totient or ϕ function

Definition

Let $n \in \mathbb{N}$. Define
$A_n = \{k \in \{1, \ldots, n\} : k \text{ is prime with } n\} = \{k \in \{1, \ldots, n\} : \gcd(k, n) = 1\}$.
Euler's totient or *phi* function is defined as

$$\phi(n) = |A_n|.$$

If $n = 1$, $\phi(1) = 1$.
If $n > 1$ and $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_l^{\alpha_l}$ is its prime decomposition then

$$\boxed{\phi(n) = n \left(1 - \frac{1}{p_1}\right) \left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_l}\right) = n \prod_{i=1}^{l} \left(1 - \frac{1}{p_i}\right)}$$

Example

$\phi(12) = \phi(2^2 \cdot 3) = 12 \left(1 - \frac{1}{2}\right) \left(1 - \frac{1}{3}\right) = 12 \cdot \frac{1}{2} \cdot \frac{2}{3} = 4.$

Remark

Notice that $\phi(n)$ *is an integer.*

1.8 Translations

Definition

To **translate** a problem is to state an equivalent problem in other terms.

Example

- Point in $\mathbb{R}^2 \Leftrightarrow (x, y)$ pair of points in $\mathbb{R}$.
- Line $\Leftrightarrow ax + by = c$
- To intersect two lines in $\mathbb{R}^2$ is to solve a linear system of 2 equations in 2 variables.

This is a translation of a geometrical problem in an algebraic one.

1.8 Translations

Example (Placements of distinguishable balls $\leftrightarrow$ mappings)

Placement of m distinguishable balls in n numbered boxes:

- Placement without exclusion (several balls can be placed in the same box) $\Leftrightarrow$ mapping from $\{1, \ldots, m\}$ to $\{1, \ldots, n\}$.
- Placement with exclusion (each box has at most one ball) $\Leftrightarrow$ injective mapping from $\{1, \ldots, m\}$ to $\{1, \ldots, n\}$.
- Placement without exclusion in which none box is empty $\Leftrightarrow$ surjective mapping from $\{1, \ldots, m\}$ to $\{1, \ldots, n\}$.

1.8 Translations

Example (Mappings $\leftrightarrow$ sequences)

$$f : \{1, \ldots, m\} \to \{1, \ldots, n\}$$

- Mapping from $\{1, \ldots, m\}$ to $\{1, \ldots, n\}$ $\Leftrightarrow$ sequence of length m formed by elements of $\{1, \ldots, n\}$ (repetitions are allowed).
- Injective mapping from $\{1, \ldots, m\}$ to $\{1, \ldots, n\}$ $\Leftrightarrow$ sequence of length m formed by elements of $\{1, \ldots, n\}$ without repetitions.
- Surjective mapping from $\{1, \ldots, m\}$ to $\{1, \ldots, n\}$ $\Leftrightarrow$ sequence of length m formed by elements of $\{1, \ldots, n\}$ in which each element of $\{1, \ldots, n\}$ is at least once.
- Bijective mapping from $\{1, \ldots, m\}$ to $\{1, \ldots, n\}$ $\Leftrightarrow$ permutation of $\{1, \ldots, n\}$.

1.8 Translations

$[m] = \{1, \ldots, m\}$

m dist. balls n numb. boxes	Mappings from $[m]$ to $[n]$	Sequences of length m formed by $[n]$	Quantity
Placements without exclusion	All	repetitions allowed	$VR_{n,m}$
Placements with exclusion	Injective	without repetitions	$V_{n,m}$
Plac. without exc., nonempty boxes	Surjective	repetitions allowed, each element at least once	$n!\left\{ {m \atop n} \right\}$

$\left\{ {m \atop n} \right\}$: Stirling number of the second kind

1.8 Translations

Let $\Omega = \{1, \ldots, n\}$.

Example (Subsets $\leftrightarrow$ sequences of 0s and 1s)

For example, if $\Omega = \{1, 2, 3, 4, 5\}$:

$\emptyset \leftrightarrow 00000$
$\{2\} \leftrightarrow 01000$
$\{2, 5\} \leftrightarrow 01001$

Therefore, subset of $\Omega \Leftrightarrow$ sequence of length n formed by zeros and ones.

Example (Subsets $\leftrightarrow$ sequences of 0s and 1s)

Subset of m elements of $\Omega \Leftrightarrow$ sequence of length n formed by m ones and $n - m$ zeros.

Example (Multisets $\leftrightarrow$ sequences of 0s and 1s)

Multiset of m elements of $\Omega \Leftrightarrow$ sequence of length $m + n - 1$ formed by m zeros and $n - 1$ ones.

1.8 Translations

Example (Placements of indistinguishable balls $\leftrightarrow$ sequences of 0s and 1s)

Placement of m indistinguishable balls in n numbered boxes:

$$\begin{array}{|c|c|c|c|}\hline 00 & & 0 & 000 \\ \hline\end{array} \leftrightarrow 001101000$$

- Placement without exclusion (several balls can be placed in the same box) $\Leftrightarrow$ sequence formed by m zeros and $n - 1$ ones.
- Placement with exclusion (each box has at most one ball) $\Leftrightarrow$ sequence formed by m zeros and $n - 1$ ones in which there are not consecutive zeros.
- Placement without exclusion in which none box is empty $\Leftrightarrow$ sequence formed by m zeros and $n - 1$ ones in which there are not consecutive ones and starts and finishes with 0.

1.8 Translations

Example (Placements of indistinguishable balls ↔ solutions of an equation)

Placement of m indistinguishable balls in n numbered boxes:
x_i = number of balls in box i

- Placement without exclusion (several balls can be placed in the same box) ⇔ solutions of $x_1 + \cdots + x_n = m$, $x_i \in \mathbb{N}^*$.
- Placement with exclusion (each box has at most one ball) ⇔ solutions of $x_1 + \cdots + x_n = m$, $x_i \in \{0, 1\}$.
- Placement without exclusion in which none box is empty ⇔ solutions of $x_1 + \cdots + x_n = m$, $x_i \in \mathbb{N}$.

1.8 Translations

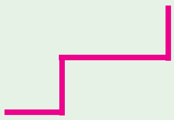
$[n] = \{1, \ldots, n\}$

m indist. balls n numb. boxes	Sequences of m 0s, $(n-1)$ 1s	Solutions of $x_1 + \cdots + x_n = m$	m elements of $[n]$	Quantity
Placements without exclusion	All	$x_i \in \mathbb{N}^*$	Multisets	$CR_{n,m}$
Placements with exclusion	no consec. 0s	$x_i \in \{0, 1\}$	Subsets	$C_{n,m}$
Plac. without exc., nonempty boxes	no consec. 1s, start/finish by 0	$x_i \in \mathbb{N}$	Multisets with all $[n]$	$CR_{n,m-n}$

1.8 Translations

Example (H-V trajectories $\leftrightarrow$ sequences of 0s and 1s)

H-V trajectory: continuous line formed by steps of the following types:
H-step: $(x,y) \to (x+1,y)$
V-step: $(x,y) \to (x,y+1)$

Ex: 

Let $T_{(0,0)}^{(p,q)}$ = set of H-V trajectories from $(0,0)$ to (p,q), with $p,q \in \mathbb{N}$.

Put 0 for H and 1 for V, so we get a sequence of p zeros and q ones.
Ex: 01001

H-V trajectory from $(0,0)$ to (p,q) $\Leftrightarrow$ sequence of p zeros and q ones.

$|T_{(0,0)}^{(p,q)}| = CR_{q+1,p} = C_{p+q,p} = P(p,q)$.

1.8 Translations

Example (U-D trajectories $\leftrightarrow$ sequences formed by 0s and 1s)

U-D trajectory: continuous line formed by steps of the following types:
U-step $\nearrow$: $(x,y) \to (x+1,y+1)$
D-step $\searrow$: $(x,y) \to (x+1,y-1)$

Let $\Theta_{(0,0)}^{(p,q)}$ = set of U-D trajectories from $(0,0)$ to (p,q), with $p,q \in \mathbb{N}$.

Let x = number of U, y = number of D $\Rightarrow x+y=p,\ x-y=q \Rightarrow x=\frac{p+q}{2}$, $y=\frac{p-q}{2}$.

- If $p<q$ then $y<0$, so there isn't any U-D trajectory from $(0,0)$ to (p,q).
- If p and q have different parity then x and y will not be integers, so there isn't any U-D trajectory from $(0,0)$ to (p,q).
- If $p \geq q > 0$ and p and q have the same parity then
  U-D trajectory from $(0,0)$ to (p,q) $\Leftrightarrow$ sequence formed by $\frac{p+q}{2}$ zeros (Us) and $\frac{p-q}{2}$ ones (Ds). Therefore,
  $|\Theta_{(0,0)}^{(p,q)}| = P(\frac{p+q}{2}, \frac{p-q}{2}) = C_{p,\frac{p+q}{2}} = CR_{\frac{p-q+2}{2},\frac{p+q}{2}}$.

1.9 The Dirichlet pigeonhole principle and the handshake lemma

Definition

Dirichlet Pigeonhole Principle: If m pigeons occupy n pigeonholes and $m > n$ then at least one will house at least two pigeons.
In mathematical terms, if $m > n$ there is not any injective mapping from $\{a_1, \ldots, a_m\}$ to $\{b_1, \ldots, b_n\}$.

Example

Let's prove that in New York there are at least two persons with the same number of hairs on the head.
Suppose that a person has at most $6 \cdot 10^6$ hairs on the head and that in New York more than $6 \cdot 10^6$ persons live. By the Dirichlet Pigeonhole Principle it follows.

1.9 The Dirichlet pigeonhole principle and the handshake lemma

Lemma

Handshake Lemma: *Let n be the number of guests in a party. The number of persons that shake hands with an odd number of persons is even.*

Therefore, if the number of guests is odd then there is at least one person that shakes hands with an even number of guests.