DISCRETE MATHEMATICS

Chapter 4: Main families of numbers

4.1 Fibonacci numbers

Leonardo de Pisa (1175-1250): Suppose that we have a pair of rabbits. Every month, except for the first one, each pair gives birth to a new pair. Rabbits never die.
We call the number of pairs of rabbits alive at month n, F_n.
Thus,

$$\begin{array}{ll} F_1 = 1 & R_1 \\ F_2 = 1 & R_1 \\ F_3 = 2 & R_1, R_{11} \\ F_4 = 3 & R_1, R_{11}, R_{12} \\ F_5 = 5 & R_1, R_{11}, R_{12}, R_{13}, R_{111} \\ F_6 = 8 & R_1, R_{11}, R_{12}, R_{13}, R_{111}, R_{14}, R_{112}, R_{121} \end{array}$$

F_n is formed by starting with the F_{n-1} pairs alive last month and adding the baby pairs that can only come from the F_{n-2} pairs alive two months ago. Hence,

$$F_0 = 0, \quad F_1 = 1, \quad F_n = F_{n-1} + F_{n-2}, \quad n \geq 2.$$

The generating function is

$$F(x) = \frac{x}{1 - x - x^2}.$$

The explicit expression for F_n is

$$F_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right].$$

4.1 Fibonacci numbers

Some properties about partial sums:

1. $$F_0 + F_1 + \cdots + F_n = F_{n+2} - 1, \quad n \geq 0$$

2. $$(n+1)F_0 + nF_1 + \cdots + 2F_{n-1} + F_n = F_{n+4} - (n+3)$$

3. $$F_0 + F_2 + \cdots + F_{2n} = F_{2n+1} - 1$$

4. $$F_1 + F_3 + \cdots + F_{2n-1} = F_{2n}$$

5. $$\sum_{k=0}^{n} (n+1-k)F_{2k} = F_{2n+2} - (n+1)$$

4.1 Fibonacci numbers

Some properties about products, divisibility,...:

1. $$F_{n+m} = F_m \cdot F_{n+1} + F_{m-1} \cdot F_n, \quad n \geq 0, m \geq 1$$

2. $$F_{kn} \text{ is a multiple of } F_n, \quad k \geq 1$$

3. $$F_{2n+1} = F_{n+1}^2 + F_n^2$$

4.2 Catalan numbers

Eugène Charles Catalan (1814-1894) defined the Catalan numbers.

Definition

The **Catalan numbers** are

$$C_n = \frac{1}{n+1}\binom{2n}{n}, \quad n = 0, 1, 2, 3, \ldots .$$

Thus,

$C_0 = \frac{1}{1}\binom{0}{0} = 1,$

$C_1 = \frac{1}{2}\binom{2}{1} = \frac{2}{2} = 1,$

$C_2 = \frac{1}{3}\binom{4}{2} = \frac{6}{3} = 2,$

$C_3 = \frac{1}{4}\binom{6}{3} = \frac{6\cdot 5\cdot 4}{3!\cdot 4} = 5,$

$\vdots$

It turns out that $C_n = \binom{2n}{n} - \binom{2n}{n+1}$.

Thus, C_n is a natural number.

4.2 Catalan numbers

Recall $\Theta_{(0,0)}^{(p,q)}$= set of U-D trajectories from $(0,0)$ to (p,q), with $p,q \in \mathbb{N}$.
If $p \geq q \geq 0$ and p and q have the same parity then an U-D trajectory from $(0,0)$ to (p,q) can be seen as a sequence formed by $\frac{p+q}{2}$ zeros and $\frac{p-q}{2}$ ones.
Therefore, $|\Theta_{(0,0)}^{(p,q)}| = \binom{p}{\frac{p+q}{2}}$.
In particular, if $q=0$ and $p=2n$, $|\Theta_{(0,0)}^{(2n,0)}| = \binom{2n}{n}$.
Put $\Theta_n = \Theta_{(0,0)}^{(2n,0)}$,
Θ_n^* = set of "super" U-D trajectories, that is, set of U-D trajectories from $(0,0)$ to $(2n,0)$ with $y \geq 0$, and
Θ_n^{**} = set of "extra-super" U-D trajectories, that is, set of U-D trajectories from $(0,0)$ to $(2n,0)$ with $y > 0$ except for the initial and final points.
It turns out that $|\Theta_n^*| = C_n$ and $|\Theta_n^{**}| = C_{n-1}$.
How many "super" U-D trajectories from $(0,0)$ to $(2n,0)$ are there such that they touch OX for the first time at $(2k,0)$, $k = 1,2,\ldots,n$? $|\Theta_k^{**}|\cdot|\Theta_{n-k}^*| = C_{k-1}C_{n-k}$.
Thus,

$$C_n = \sum_{k=1}^{n} C_{k-1}C_{n-k}.$$

That is,

$$C_0 = 1, \quad C_n = C_0C_{n-1} + C_1C_{n-2} + \cdots + C_{n-1}C_0, \quad n \geq 1.$$

4.2 Catalan numbers

The generating function is

$$c(x) = \sum_{n=0}^{\infty} C_n x^n, \quad |x| < \frac{1}{4}, x \neq 0.$$

$c(x) \cdot c(x) = \left(\sum_{n=0}^{\infty} C_n x^n\right)\left(\sum_{n=0}^{\infty} C_n x^n\right) =$
$(C_0 + C_1 x + C_2 x^2 + \cdots)(C_0 + C_1 x + C_2 x^2 + \cdots) = C_0 C_0 + (C_0 C_1 + C_1 C_0)x +$
$(C_0 C_2 + C_1 C_1 + C_2 C_0)x^2 + \cdots + (C_0 C_n + C_1 C_{n-1} + \cdots + C_n C_0)x^n + \cdots =$
$C_1 + C_2 x + C_3 x^2 + \cdots + C_{n+1} x^n + \cdots = \frac{c(x) - C_0}{x}$.

Thus, $c(x)^2 = \frac{c(x)-C_0}{x}$ and $xc(x)^2 - c(x) + 1 = 0$.

Therefore, $c(x) = \frac{1 \pm \sqrt{1-4x}}{2x}$.

Since $\lim_{x \to 0} \frac{1+\sqrt{1-4x}}{2x} = \pm\infty$ and $\lim_{x \to 0} \frac{1-\sqrt{1-4x}}{2x} = 1$

$$c(x) = \begin{cases} \frac{1-\sqrt{1-4x}}{2x} & |x| < \frac{1}{4}, x \neq 0 \\ 1 & x = 0 \end{cases}$$

4.2 Catalan numbers

Combinatorial problem related to the Catalan numbers: triangulation of convex polygons with numbered vertices.
Triangulation is a decomposition of a convex polygon in triangles with disjoint interiors and whose vertices are the vertices of the polygon.
Let T_n be the number of possible triangulations of a polygon with $n+2$ sides and numbered vertices $v_1, v_2, \ldots, v_{n+2}$.
Notice that two consecutive vertices are related in a triangle.
We can classify in terms of the third vertex: If v_1, v_2 are the two fixed vertices then we can get $v_1 v_2 v_{k+2}$, $k = 1, \ldots, n$.
For each $k = 1, \ldots, n$ we would get the number of triangulations that can be made with vertices $v_2, v_3, \ldots, v_{k+2}$ times the number of triangulations that can be made with vertices $v_{k+2}, \ldots, v_{n+2}, v_1$.
Therefore,

$$T_n = \sum_{k=1}^{n} T_{k-1} T_{n-k}.$$

If we set $T_0 = 1$ then $T_n = C_n$.

4.2 Catalan numbers

Another combinatorial problem related to the Catalan numbers: parenthesize a product.

To parenthesize a product means to insert enough parentheses so that every subproduct is the multiplication of exactly two factors.

For example, the product $x_1x_2x_3x_4$ can be parenthesized as:

$$(x_1((x_2x_3)x_4))$$

or

$$(x_1(x_2(x_3x_4)))$$

or

$$((x_1x_2)(x_3x_4))$$

or

$$(((x_1x_2)x_3)x_4)$$

or

$$((x_1(x_2x_3))x_4)$$

Let a_n be the number of ways in which $x_1x_2\cdots x_n$ can be parenthesized.

It turns out that $a_n = C_{n-1}$.

4.3 Partitions of natural numbers

4.3.1 Definition

Definition

Let $n \in \mathbb{N} = \{1, 2, \ldots\}$. A **partition of** n is an expression of n as sum of natural numbers, which are called **parts**.

Two partitions are considered identical if they only differ in the order of the parts.

For example, $2 + 1 \equiv 1 + 2$.

If two partitions that only differ in the order are considered different then they are **ordered partitions**.

4.3 Partitions of natural numbers

4.3.2 Ordered partitions

Let π_n be the number of ordered partitions of n.
$\pi_1 = 1 \quad (1)$,
$\pi_2 = 2 \quad (2, 1+1)$,
$\pi_3 = 4 \quad (3, 2+1, 1+2, 1+1+1)$,
$\pi_4 = 8 \quad (4, 3+1, 2+2, 2+1+1, 1+3, 1+2+1, 1+1+2, 1+1+1+1)$.
Conjecture: Is $\pi_n = 2^{n-1}$ for every $n \geq 1$?
We classify according to the number of parts:

- 1 part: 1 $\quad (n)$
- 2 parts: $n-1 \quad (1+(n-1), 2+(n-2), \dots, (n-1)+1)$
- k parts: number of solutions of $x_1 + x_2 + \cdots + x_k = n$ with $x_1, x_2, \dots, x_k \in \mathbb{N}$
- n parts: 1 $\quad (1+1+\cdots+1)$

How many are there?
For each k, the number of solutions of $x_1 + x_2 + \cdots + x_k = n$ with $x_i \in \mathbb{N} \equiv$
the number of solutions of $(1+y_1)+(1+y_2)+\cdots+(1+y_k) = n$ with $y_i \in \mathbb{N} \cup \{0\} \equiv$
the number of solutions of $y_1 + y_2 + \cdots + y_k = n-k$ with $y_i \in \mathbb{N} \cup \{0\} \equiv$
$CR_{k,n-k} = \binom{k+n-k-1}{n-k} = \binom{n-1}{k-1}$. Thus,

$$\pi_n = \sum_{k=1}^{n} \binom{n-1}{k-1} = 2^{n-1}, \quad n \geq 1.$$

4.3 Partitions of natural numbers

4.3.3 Partitions and restricted partitions

Let p_n be the number of partitions of n.
Let $p_n^{(k)}$ be the number of partitions of n with parts $\leq k$.
We set $p_0 = 1$ and $p_0^{(k)} = 1$.
It is clear that $p_n \leq \pi_n = 2^{n-1}$. Moreover, if $k \geq n$, $p_n^{(k)} = p_n$.

Theorem

The generating function of $(p_n^{(k)})_{n\geq 0}$ is

$$p^{(k)}(x) = \frac{1}{1-x}\frac{1}{1-x^2}\cdots\frac{1}{1-x^k} = \prod_{r=1}^{k}\frac{1}{1-x^r}, \quad |x| < 1.$$

Theorem

For $|x| < \frac{1}{2}$, $\quad \lim_{k\to\infty} |p(x) - p^{(k)}(x)| = 0$
with $p(x)$ the generating function of $(p_n)_{n\geq 0}$. That is,

$$p(x) = \lim_{k\to\infty} p^{(k)}(x) = \lim_{k\to\infty}\prod_{r=1}^{k}\frac{1}{1-x^r} = \prod_{r=1}^{\infty}\frac{1}{1-x^r}.$$

4.3 Partitions of natural numbers

4.3.4 Partitions into distinct parts

Let d_n be the number of partitions of n into distinct parts and $d_n^{(k)}$ be the number of partitions of n into distinct parts $\leq k$.
We set $d_0 = 1$ and $d_0^{(k)} = 1$.
Notice that $d_n^{(k)} \leq d_n \leq p_n \leq \pi_n = 2^{n-1}$ and if $n \leq k$ then $d_n^{(k)} = d_n$.

Theorem

The generating function of $(d_n^{(k)})_{n\geq 0}$ *is*

$$d^{(k)}(x) = (1+x)(1+x^2)\cdots(1+x^k) = \prod_{r=1}^{k}(1+x^r)$$

and $d_n^{(k)}$ *is the coefficient of* x^n *in* $d^{(k)}(x)$.

Theorem

For $|x| < \frac{1}{2}$, $\lim_{k\to\infty} |d(x) - d^{(k)}(x)| = 0$
with $d(x)$ *the generating function of* $(d_n)_{n\geq 0}$. *That is,*

$$d(x) = \lim_{k\to\infty} d^{(k)}(x) = \lim_{k\to\infty} \prod_{r=1}^{k}(1+x^r) = \prod_{r=1}^{\infty}(1+x^r).$$

4.3 Partitions of natural numbers

4.3.5 Partitions into odd parts

Let o_n be the number of partitions of n into odd parts and let $o_n^{(k)}$ be the number of partitions of n into odd parts $\leq k$.
We set $o_0 = 1$ and $o_0^{(k)} = 1$.
It is clear that $o_n^{(k)} \leq o_n \leq p_n \leq \pi_n = 2^{n-1}$. Moreover, $o_n^{(2l-1)} = o_n^{(2l)}$ and if $k \geq n$, $o_n^{(k)} = o_n$.

Theorem

The generating function of $(o_n^{(2l-1)})_{n\geq 0}$, $l \geq 1$, *is*

$$o^{(2l-1)}(x) = \frac{1}{1-x}\frac{1}{1-x^3}\cdots\frac{1}{1-x^{2l-1}} = \prod_{r=1}^{l}\frac{1}{1-x^{2r-1}}, \quad |x| < 1$$

and $o_n^{(2l-1)}$ *is the coefficient of* x^n *in* $o^{(2l-1)}(x)$. *Moreover, the generating function of* $(o_n^{(2l)})_{n\geq 0}$, $l \geq 1$, *is* $o^{(2l)}(x) = o^{(2l-1)}(x)$.

4.3 Partitions of natural numbers

4.3.5 Partitions into odd parts

Theorem

For $|x| < \frac{1}{2}$, $\quad \lim_{k\to\infty} |o(x) - o^{(2l-1)}(x)| = 0$
with $o(x)$ *the generating function of* $(o_n)_{n\geq 0}$. *That is,*

$$o(x) = \lim_{l\to\infty} o^{(2l-1)}(x) = \lim_{l\to\infty} \prod_{r=1}^{l} \frac{1}{1-x^{2r-1}} = \prod_{r=1}^{\infty} \frac{1}{1-x^{2r-1}}.$$

Corollary

For $|x| < \frac{1}{2}$, $d(x) = o(x)$ *and* $d_n = o_n$ *for all* n.

Remark

We could generalize the above taking $A = \{a_1 < a_2 < a_3 < \cdots\} \subseteq \mathbb{N}$, α_n *as the number of partitions of* n *with parts in* A *and* $\alpha_n^{(k)}$ *as the number of partitions of* n *with parts* $\leq k$ *in* A.

4.3 Partitions of natural numbers

4.3.6 Ferrers diagrams

Example

Ferrers diagram of the partition $13 = 5 + 3 + 2 + 2 + 1$:

$$\begin{array}{rrlllll}
13 = & 5 & \circ & \circ & \circ & \circ & \circ \\
 & +3 & \circ & \circ & \circ & & \\
 & +2 & \circ & \circ & & & \\
 & +2 & \circ & \circ & & & \\
 & +1 & \circ & & & &
\end{array}$$

In general,

$$\begin{array}{rrllllllll}
n = & x_1 & \circ & \circ & \cdots & \circ & \cdots & \circ & \circ & \circ \\
 & +x_2 & \circ & \circ & \cdots & \circ & \cdots & \circ & \circ & \\
 & +x_3 & \circ & \circ & \cdots & \circ & \cdots & \circ & & \\
 & \vdots & \vdots & & & & & & & \\
 & +x_k & \circ & \circ & \cdots & \circ & & & &
\end{array}$$

4.3 Partitions of natural numbers

4.3.6 Ferrers diagrams

Characteristics of a Ferrers diagram of a partition of n:

- The total number of circles is n.
- The number of rows is the number of parts.
- The number of columns is the greatest part.

Definition

Let π be a partition of n. The **conjugate of** π, π^t, is a partition whose Ferrers diagram is the transpose of the Ferrers diagram of π.

Example

$$\pi = 5+3+2+2+1: \qquad\qquad \pi^t = 5+4+2+1+1:$$

$$13 = \begin{array}{rccccc} 5 & \circ & \circ & \circ & \circ & \circ \\ +3 & \circ & \circ & \circ & & \\ +2 & \circ & \circ & & & \\ +2 & \circ & \circ & & & \\ +1 & \circ & & & & \end{array} \qquad 13 = \begin{array}{rccccc} 5 & \circ & \circ & \circ & \circ & \circ \\ +4 & \circ & \circ & \circ & \circ & \\ +2 & \circ & \circ & & & \\ +1 & \circ & & & & \\ +1 & \circ & & & & \end{array}$$

Theorem

The number of partitions of n into (at most) k parts is equal to the number of partitions of n in which the greatest part is (at most) k.

4.3 Partitions of natural numbers

4.3.6 Ferrers diagrams

Definition

A partition π of n is **self-conjugate** if $\pi^t = \pi$, that is, if the Ferrers diagram is symmetric with respect to the main diagonal.

How many self-conjugate partitions of n are there?

$$\begin{array}{rccc} f: & \{\text{Self-conjugate partitions of } n\} & \to & \{\text{Partitions of } n \text{ into odd distinct parts}\} \\ & x_1 + x_2 + \cdots + x_{x_1} & \to & (2x_1 - 1) + (2(x_2 - 1) - 1) + \cdots + (2(x_l - (l-1)) - 1) \end{array}$$

where $l = x_1$ or ($l \le x_l$ and $l + 1 > x_{l+1}$), is a bijection.

Example

$$15 = \begin{array}{rccccc} 5 & \circ & \circ & \circ & \circ & \circ \\ +4 & \circ & \circ & \circ & \circ & \\ +3 & \circ & \circ & \circ & & \\ +2 & \circ & \circ & & & \\ +1 & \circ & & & & \end{array} \quad \to \quad 15 = \begin{array}{rccccccccc} 9 & \circ & \circ & \circ & \circ & \circledcirc & \circ & \circ & \circ & \circ \\ +5 & \circ & \circ & \circledcirc & \circ & \circ & & & & \\ +1 & \circledcirc & & & & & & & & \end{array}$$

There are as many self-conjugate partitions of n as the number of partitions of n into odd distinct parts.

4.4 Bell numbers

Definition

Let Ω be a finite set. A **partition of** Ω is a collection of subsets of Ω, $\{A_1, A_2, \ldots, A_k\}$, such that:

1. $A_i \neq \emptyset$ for all i.
2. They are pairwise disjoint, i.e., $A_i \cap A_j = \emptyset$ if $i \neq j$.
3. Their union is Ω, i.e., $\Omega = \cup_{i=1}^{k} A_i$.

Definition (Eric Temple Bell (1883-1960))

Let $B_0 = 1$ and B_n be the number of partitions of a set of n elements, $n \geq 1$.
$B_0, B_1, B_2, \ldots$ are called the **Bell numbers**.

Example

- $n = 1$ $\quad \Omega = \{1\}$, $B_1 = 1$, $(\{\{1\}\})$
- $n = 2$ $\quad \Omega = \{1, 2\}$, $B_2 = 2$, $(\{\{1, 2\}\}, \{\{1\}, \{2\}\})$
- $n = 3$ $\quad \Omega = \{1, 2, 3\}$, $B_3 = 5$,
  $(\{\{1, 2, 3\}\}, \{\{1, 2\}, \{3\}\}, \{\{1, 3\}, \{2\}\}, \{\{2, 3\}, \{1\}\}, \{\{1\}, \{2\}, \{3\}\})$

4.4 Bell numbers

Recurrence relation for the Bell numbers:
Let $\Omega = \{1, 2, \ldots, n\}$.
Let A_1 be any subset of Ω that contains the element 1.
There are $\binom{n-1}{k-1}$ sets A_1 that have k elements, with $k = 1, \ldots, n$.
Once A_1 is fixed there are B_{n-k} partitions of $\Omega \setminus A_1$.
Therefore,

$$B_n = \sum_{k=1}^{n} \binom{n-1}{k-1} B_{n-k} = \sum_{k=1}^{n} \binom{n-1}{n-k} B_{n-k} = \sum_{k=0}^{n-1} \binom{n-1}{k} B_k.$$

Thus,

$$B_n = \binom{n-1}{0} B_0 + \binom{n-1}{1} B_1 + \cdots + \binom{n-1}{n-1} B_{n-1}.$$

4.4 Bell numbers

We can make use of the Pascal's triangle to calculate the Bell numbers:

$$
\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & & & & 1 & & 1 & & & & & & \\
 & & & & & 1 & & 2 & & 1 & & & & & \\
 & & & & 1 & & 3 & & 3 & & 1 & & & & \\
 & & & 1 & & 4 & & 6 & & 4 & & 1 & & & \\
 & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & \\
 & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & \\
1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1
\end{array}
$$

$B_0 = 1,$
$B_1 = 1B_0 = 1,$
$B_2 = 1B_0 + 1B_1 = 2,$
$B_3 = 1B_0 + 2B_1 + 1B_2 = 5,$
$B_4 = 1B_0 + 3B_1 + 3B_2 + 1B_3 = 15,$
$B_5 = 1B_0 + 4B_1 + 6B_2 + 4B_3 + 1B_4 = 52, \ldots$

4.4 Bell numbers

Theorem

$$\left(\frac{B_{n+1}}{n!}\right)_{n\geq 0} = \left(\frac{B_n}{n!}\right)_{n\geq 0} * \left(\frac{1}{n!}\right)_{n\geq 0}$$

The exponential generating function of $(B_n)_{n\geq 0}$ is the generating function of $\left(\frac{B_n}{n!}\right)_{n\geq 0}$:

$$B(x) = \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n.$$

Hence, $B'(x) = \sum_{n=1}^{\infty} \frac{B_n}{n!} n x^{n-1} = \sum_{n=1}^{\infty} \frac{B_n}{(n-1)!} x^{n-1} = \sum_{n=0}^{\infty} \frac{B_{n+1}}{n!} x^n.$
Therefore, $B'(x)$ is the generating function of $\left(\frac{B_{n+1}}{n!}\right)_{n\geq 0}$.
Since e^x is the generating function of $\left(\frac{1}{n!}\right)_{n\geq 0}$,

$$B'(x) = B(x)e^x \qquad \text{(linear differential equation)}$$

As $\frac{B'(x)}{B(x)} = e^x$, $\ln B(x) = e^x + C$ and $B(x) = e^{e^x + C}$.
Moreover $B(0) = \frac{B_0}{0!} = 1$. So, $1 = e^{1+C}$, $0 = 1 + C$ and $C = -1$. Thus,

$$B(x) = e^{e^x - 1}.$$

4.5 Stirling numbers of the first kind

Recall

$$x^{\overline{n}} = \begin{cases} x(x+1)\cdots(x+n-1) & n \geq 1 \\ 1 & n = 0 \end{cases}$$

$x^{\overline{n}}$ is a polynomial of degree n.
Let $\mathbb{R}[x]$ be the vector space of polynomials in the indeterminate x with real coefficients and $\mathbb{R}_n[x]$ the vector subspace of polynomials of degree at most n.
Then $\mathcal{B} = \{1, x, x^2, \ldots\}$, $\overline{\mathcal{B}} = \{1, x^{\overline{1}}, x^{\overline{2}}, \ldots\}$ and $\underline{\mathcal{B}} = \{1, x^{\underline{1}}, x^{\underline{2}}, \ldots\}$ are bases of $\mathbb{R}[x]$ and $\mathcal{B}_n = \{1, x, x^2, \ldots, x^n\}$, $\overline{\mathcal{B}}_n = \{1, x^{\overline{1}}, x^{\overline{2}}, \ldots, x^{\overline{n}}\}$ and $\underline{\mathcal{B}}_n = \{1, x^{\underline{1}}, x^{\underline{2}}, \ldots, x^{\underline{n}}\}$ are bases of $\mathbb{R}_n[x]$.

Definition

For $n, k = 0, 1, 2, \ldots$, the **Stirling numbers of the first kind**, denoted by $\left[{n \atop k}\right]$, are the coordinates of $x^{\overline{n}}$ with respect to the canonical basis $\mathcal{B}_n$, in other words, $\left[{n \atop k}\right]$ is the coefficient of x^k in $x^{\overline{n}}$.

Remark

Notice that $(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$ *and* $x^{\overline{n}} = \sum_{k=0}^{n} \left[{n \atop k}\right] x^k$.

4.5 Stirling numbers of the first kind

Proposition

1. *If* $k > n$ *then* $\left[{n \atop k}\right] = 0$.
2. $\left[{0 \atop 0}\right] = 1$.
3. $\left[{n \atop n}\right] = 1, \quad \left[{n \atop 0}\right] = 0 \quad n \geq 1$.
4. $n! = \left[{n \atop 0}\right] + \left[{n \atop 1}\right] + \cdots + \left[{n \atop n}\right]$.
5. *If* $n \geq 1$ *then* $\left[{n \atop 1}\right] = (n-1)!$
6. $\left[{n \atop n-1}\right] = \binom{n}{2}$.

Recurrence relation:
For $n \geq 2$ and $1 \leq k \leq n-1$,

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} + (n-1) \begin{bmatrix} n-1 \\ k \end{bmatrix}.$$

4.5 Stirling numbers of the first kind

Pascal's triangle:

$$\begin{array}{ccccccc} & & & \left[{0 \atop 0}\right] & & & \\ & & \left[{1 \atop 0}\right] & & \left[{1 \atop 1}\right] & & \\ & \left[{2 \atop 0}\right] & & \left[{2 \atop 1}\right] & & \left[{2 \atop 2}\right] & \\ \left[{3 \atop 0}\right] & & \left[{3 \atop 1}\right] & & \left[{3 \atop 2}\right] & & \left[{3 \atop 3}\right] \end{array}$$

$$\begin{array}{ccccccccccccc} & & & & & & 1 & & & & & & \\ & & & & & 0 & & 1 & & & & & \\ & & & & 0 & & 1 & & 1 & & & & \\ & & & 0 & & 2 & & 3 & & 1 & & & \\ & & 0 & & 6 & & 11 & & 6 & & 1 & & \\ & 0 & & 24 & & 50 & & 35 & & 10 & & 1 & \\ 0 & & 120 & & 274 & & 225 & & 85 & & 15 & & 1 \end{array}$$

Ex: $\left[{6 \atop 2}\right] = 274 = 24 + 5 \cdot 50 = \left[{5 \atop 1}\right] + 5\left[{5 \atop 2}\right].$

4.5 Stirling numbers of the first kind

Combinatorial meaning:
$\left[{n \atop k}\right]$ can be seen as the number of permutations of $1, 2, \ldots, n$ that can be expressed as a product of k cycles with disjoint orbits.

Example:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 6 & 3 & 5 & 4 & 2 & 7 & 1 \end{pmatrix} = (\,1 \quad 6 \quad 7\,)(\,2 \quad 3 \quad 5\,)(\,4\,)$$

4.5 Stirling numbers of the second kind

Recall

$$x^{\underline{n}} = \begin{cases} x(x-1)\cdots(x-n+1) & n \geq 1 \\ 1 & n = 0 \end{cases}$$

$x^{\underline{n}}$ is a polynomial of degree n.

Definition

For $n, k = 0, 1, 2, \ldots$, the **Stirling numbers of the second kind**, denoted by $\left\{ {n \atop k} \right\}$, are the coordinates of x^n with respect to $\underline{\mathcal{B}} = \{1, x^{\underline{1}}, x^{\underline{2}}, \ldots\}$, in other words, $\left\{ {n \atop k} \right\}$ is the coefficient of $x^{\underline{k}}$ in x^n.

Remark

Notice that $x^n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} x^{\underline{k}}$.

Proposition

1. *If* $k > n$ *then* $\left\{ {n \atop k} \right\} = 0$.
2. $\left\{ {0 \atop 0} \right\} = 1, \quad \left\{ {n \atop 0} \right\} = 0, \quad \left\{ {n \atop 1} \right\} = 1, \quad \left\{ {n \atop 2} \right\} = 2^{n-1} - 1, \quad \left\{ {n \atop 3} \right\} = \frac{3^{n-1}+1}{2} - 2^{n-1}$.
3. $\left\{ {n \atop n} \right\} = 1$.

4.5 Stirling numbers of the second kind

Recurrence relation:
For $n \geq 2$ and $1 \leq k \leq n-1$,

$$\left\{ {n \atop k} \right\} = \left\{ {n-1 \atop k-1} \right\} + k \left\{ {n-1 \atop k} \right\}.$$

4.5 Stirling numbers of the second kind

Pascal's triangle:

$$\begin{array}{ccccccc} & & & \left\{ {0 \atop 0} \right\} & & & \\ & & \left\{ {1 \atop 0} \right\} & & \left\{ {1 \atop 1} \right\} & & \\ & \left\{ {2 \atop 0} \right\} & & \left\{ {2 \atop 1} \right\} & & \left\{ {2 \atop 2} \right\} & \\ \left\{ {3 \atop 0} \right\} & & \left\{ {3 \atop 1} \right\} & & \left\{ {3 \atop 2} \right\} & & \left\{ {3 \atop 3} \right\} \end{array}$$

$$\begin{array}{ccccccccccc} & & & & & 1 & & & & & \\ & & & & 0 & & 1 & & & & \\ & & & 0 & & 1 & & 1 & & & \\ & & 0 & & 1 & & 3 & & 1 & & \\ & 0 & & 1 & & 7 & & 6 & & 1 & \\ 0 & & 1 & & 15 & & 25 & & 10 & & 1 \end{array}$$

Ex: $\left\{ {5 \atop 2} \right\} = 15 = 1 + 2 \cdot 7 = \left\{ {4 \atop 1} \right\} + 2\left\{ {4 \atop 2} \right\}$.

4.5 Stirling numbers of the second kind

Combinatorial meaning:
Define $b(n, k)$ as the number of partitions of $\{1, 2, \ldots, n\}$ in k (nonempty) subsets.
Example: $b(4, 2) = 7$ because if $\{1, 2, 3, 4\}$:
$\{\{1\}, \{2, 3, 4\}\}$, $\{\{2\}, \{1, 3, 4\}\}$, $\{\{3\}, \{1, 2, 4\}\}$, $\{\{4\}, \{1, 2, 3\}\}$, $\{\{1, 2\}, \{3, 4\}\}$, $\{\{1, 3\}, \{2, 4\}\}$, $\{\{1, 4\}, \{2, 3\}\}$.
Notice that $b(n, 0) = 0$ for $n \geq 1$ and $b(n, n) = 1$ for $n \geq 1$. We can set $b(0, 0) = 1$.
We can classify the partitions according to element 1:

- Those in which $\{1\}$ is a subset: there are $b(n-1, k-1)$.
- Those in which $\{1\}$ is not a subset.- we choose a partition of $\{2, \ldots, n\}$ in k (nonempty) subsets and insert element 1: there are $kb(n-1, k)$.

Therefore, $b(n, k) = b(n-1, k-1) + kb(n-1, k)$.
Thus,

$$\left\{ {n \atop k} \right\} = b(n, k) = \text{ number of partitions of } \{1, 2, \ldots, n\} \text{ in } k \text{ (nonempty) subsets.}$$

4.5 Stirling numbers of the second kind

Relation with the Bell numbers:
B_n = number of partitions of $\{1, 2, \ldots, n\}$.

$$B_n = \left\{ {n \atop 0} \right\} + \left\{ {n \atop 1} \right\} + \cdots + \left\{ {n \atop n} \right\}.$$

Notice that in the Pascal's triangle the sum of the elements of the nth row is B_n.

4.5 Stirling numbers of the second kind

Another combinatorial meaning:
Let $c(n, k)$ be the number of ways to put n distinguishable balls in k indistinguishable boxes with no empty boxes.
Idea: Choose a partition of $\{1, 2, \ldots, n\}$ in k subsets and put the k (nonempty) subsets of balls in the k boxes (one per box):

$$c(n, k) = \left\{ {n \atop k} \right\}$$

Moreover,

$$k! \left\{ {n \atop k} \right\}$$

- is the number of ways to put n distinguishable balls in k numbered boxes with no empty boxes.
- is the number of surjective mappings from $\{1, \ldots, n\}$ to $\{1, \ldots, k\}$.