Chapter IV

MAIN FAMILIES OF NUMBERS

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Fibonacci numbers

Fibonacci numbers have a lot of properties that have been discovered and investigated by many authors (there is a research journal called *The Fibonacci Quarterly* exclusively dedicated to this (see `http://www.fq.math.ca/`)). Some of these properties are proposed here as exercises:

1. (Lucas)
$$F_1 + F_2 + \ldots + F_n = F_{n+2} - 1.$$

2. Show that
$$nF_1 + (n-1)F_2 + (n-2)F_3 + \ldots + 2F_{n-1} + F_n = F_{n+4} - (n+3).$$

3. (Lucas)
$$F_1 + F_3 + F_5 + \ldots + F_{2n-1} = F_{2n}.$$

4. (Lucas)
$$F_2 + F_4 + F_6 + \ldots + F_{2n} = F_{2n+1} - 1.$$

5. (Lucas)
$$F_1 - F_2 + F_3 - F_4 + \ldots + F_{2n-1} - F_{2n} = -F_{2n-1} + 1.$$

6. Show that
$$nF_2 + (n-1)F_4 + \ldots + 2F_{2n-2} + F_{2n} = F_{2n+2} - (n+1).$$

7. Show that
$$nF_1 + (n-1)F_3 + \ldots + 2F_{2n-3} + F_{2n-1} = F_{2n+1} - 1.$$

8. Show that
$$F_3 + F_6 + F_9 + \ldots + F_{3n} = \frac{1}{2}(F_{3n+2} - 1).$$

9. Simplify the following sums:
$$F_1 + 2F_2 + \ldots + nF_n$$
$$2F_1 + 3F_2 + \ldots + (n+1)F_n$$

10. (Lucas)
$$\binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} + \ldots = F_{n+1}.$$
Which is the last nonzero summand on the left?

11. (Cassini and Simson)
$$F_n F_{n+2} - F_{n+1}^2 = \pm 1.$$
For what values of n is a $+$ and for what is a $-$ on the right hand side?

12. For what values of n is F_n even?

13. Show that
$$F_n = 5F_{n-4} + 3F_{n-5},$$
and deduce from it that for each n, F_{5n} is a multiple of 5.

14. The above formula is only one of a series of relations:
$$F_{n+3} = 2F_{n+1} + F_n$$
$$F_{n+4} = 3F_{n+1} + 2F_n$$
$$F_{n+5} = 5F_{n+1} + 3F_n$$
$$\ldots$$
Prove the following general formula:
$$F_{n+m} = F_m F_{n+1} + F_{m-1} F_n,$$
and deduce from it that for each n, F_{kn} is a multiple of F_n.

15. (Lucas and Catalan)
$$F_{n-1}^2 + F_n^2 = F_{2n-1}$$

16. (Lucas and Catalan)
$$F_{n+1}^2 - F_{n-1}^2 = F_{2n}$$

17. (Lucas)
$$F_n F_{n+1} - F_{n-1} F_{n-2} = F_{2n-1}$$

18. (Lucas)
$$F_n^3 + F_{n+1}^3 - F_{n-1}^3 = F_{3n}$$

19. (Lucas)
$$F_1^2 + F_2^2 + \ldots + F_n^2 = F_n F_{n+1}$$

20. Show that

$$F_1F_2 + F_2F_3 + F_3F_4 + \ldots + F_{2n-1}F_{2n} = F_{2n}^2$$

21. Show that for $n \geq 3$:

$$\left(\frac{1+\sqrt{5}}{2}\right)^{n-2} < F_n < \left(\frac{1+\sqrt{5}}{2}\right)^{n-1}$$

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Partitions of numbers

1. Using Ferrers diagrams show that:

   (a) (Euler) The number of partitions of n in k parts or less is equal to the number of partitions in which each part is $\leq k$. We denote that number by $p_n^{(k)}$.

   (b) (Euler) The number of partitions of n in k parts is equal to the number of partitions in which the greatest part is k, and is $p_n^{(k)} - p_n^{(k-1)}$.

   (c) (Sylvester) The number of self-conjugate partitions of n is equal to the number of partitions in which all the parts are unequal and odd.

2. (Andrews) Let $p_n^{(k)}$ be as in the previous exercises. Prove:

   (a) $p_n^{(k)} \leq (n+1)^k$

   (b) $p_n \leq p_{n-1} + p_n^{(k)} + p_{n-k}$

3. Show that
$$p_n^{(2)} = \left\lfloor \frac{n}{2} \right\rfloor + 1.$$

4. (Cayley) Solve the previous exercise using the formula
$$\frac{1}{(1-x)(1-x^2)} = \frac{1/2}{(1-x)^2} + \frac{1/2}{1-x^2}.$$

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Stirling and Catalan numbers

1. The number of ways to place n distinguishable balls in m numbered boxes so that no box is empty is
$$m!\left\{ n \atop m \right\}.$$

2. The number of ways to place n distinguishable balls in m indistinguishable boxes (some of them can be empty) is
$$\left\{ n \atop 1 \right\} + \left\{ n \atop 2 \right\} + \ldots + \left\{ n \atop m \right\}.$$

3. Show that
$$\sum \binom{n}{n_1, n_2, \ldots, n_m} = m!\left\{ n \atop m \right\},$$
where the sum is over all m-tuples $(n_1, n_2, \ldots, n_m)$ of strictly positive integers such that $n_1 + n_2 + \ldots + n_m = n$.

4. Show that for $n \geq 3$
$$\left\{ n \atop n-2 \right\} = \binom{n}{3} + 3\binom{n}{4} = \frac{1}{4}\binom{n}{3}(3n-5).$$

5. Show that for $n \geq 4$
$$\left\{ n \atop n-3 \right\} = \binom{n}{4} + 10\binom{n}{5} + 15\binom{n}{6} = \frac{1}{2}\binom{n}{4}(n-2)(n-3).$$

6. Show that for $m \geq 1$
$$(e^x - 1)^m = \left\{ m \atop m \right\}\frac{m!}{m!}x^m + \left\{ m+1 \atop m \right\}\frac{m!}{(m+1)!}x^{m+1} + \ldots = \sum_{n=m}^{\infty} \left\{ n \atop m \right\}\frac{m!}{n!}x^n.$$

7. Show that for $n \geq m \geq 1$

$$m! \left\{ {n \atop m} \right\} = \binom{m}{m} m^n - \binom{m}{m-1}(m-1)^n + \ldots + (-1)^m \binom{m}{0}(m-m)^n = \sum_{k=0}^{m} (-1)^k \binom{m}{m-k}(m-k)^n.$$

8. Show that for $m \geq 1$

$$\frac{1}{(1-x)(1-2x)\cdots(1-mx)} = \left\{ {m \atop m} \right\} + \left\{ {m+1 \atop m} \right\} x + \left\{ {m+2 \atop m} \right\} x^2 + \ldots$$

9. The number of H-V trajectories from $(0,0)$ to (n,n) that do not cross the diagonal is $2C_n$ (C_n the n Catalan number).

10. The number of H-V trajectories from $(0,0)$ to (n,n) that do not touch the diagonal (except for the endpoints) is:

$$\frac{1}{2n-1}\binom{2n}{n}.$$

11. The number of sequences $a_1, a_2, \ldots, a_n$ of non-negative integers that satisfy

$$a_1 + a_2 + \ldots + a_k \geq k,\ \forall k = 1, 2, \ldots, n-1 \quad \text{and} \quad a_1 + a_2 + \ldots + a_n = n$$

is the Catalan number C_n.