Chapter III

GENERATING FUNCTIONS AND RECURRENCE RELATIONS

List of exercises 8

Generating functions and recurrence relations

1. Find the generating functions of the following sequences:
   (a) $1,\ 5,\ 5^2,\ 5^3,\ \ldots$
   (b) $1,\ -1,\ 1,\ -1,\ \ldots$
   (c) $1,\ 0,\ 1,\ 0,\ \ldots$
   (d) $0,\ 2,\ 0,\ 4,\ 0,\ 6,\ 0,\ 8,\ \ldots$
   (e) $4,\ 8,\ 16,\ 32,\ 64,\ \ldots$
2. Find the general term (or nth term) of the sequences that generate the following functions:
   (a) $\left(\frac{1}{1-t}\right)^3$
   (b) $\frac{1}{1-t}\frac{1}{1+t}$
   (c) $\frac{1}{1+4t}$
   (d) $\frac{2t}{(1-t)(1-2t)}$
   (e) e^{2t}
   (f) e^{t^2}
   (g) $\sin t$
   (h) $\cos t$
3. Assume that $g(t)$ (defined on $(-r, r)$, with $r > 0$) is the generating function of the sequence $(a_n)_{n\geq 0}$. In each of the following cases, find (that is, express by $g(t)$) the generating function of the sequence $(b_n)_{n\geq 0}$.

(a) $b_n := (-1)^n a_n$
(b) $b_n := a_{n+1}$
(c) $b_n := a_{n-1},\ (a_{-1} = 0)$
(d) $b_n := a_n + a_{n+1}$
(e) $b_n := a_n - a_{n-1},\ (a_{-1} = 0)$
(f) $b_n := a_n$ or 0, depending on n even or odd.
(g) $b_n := a_n$ or 0, depending on n odd or even.
(h) $b_n := a_{n/2}$ or 0, depending on n even or odd.
(i) $b_n := a_0 a_n + a_1 a_{n-1} + \ldots + a_n a_0$
(j) $b_n := a_0 + a_1 + \ldots + a_n$
(k) $b_n := n a_n$
(l) $b_n := n(n-1)\cdots(n-k+1)a_n$, ($k$ fixed).

4. Find the generating function of the sequence $(a_n)_{n\geq 0}$.
   (a) $a_0 := 0,\ a_1 := 1,\ a_n := 2a_{n-1} - a_{n-2},\ n \geq 2$
   (b) $a_0 := 1,\ a_1 := 1,\ a_n := 3a_{n-1} + 4a_{n-2},\ n \geq 2$

5. Find the generating function of the sequence $a_0, a_1, \ldots$, with a_n:
   (a) The number of solutions, in non-negative integers, of the equation $x + y + z + 4u = n$.
   (b) The number of solutions, in non-negative integers, of the equation $2x+2y+3z+3u = n$.

6. Find the value of a_{10} in (a) and the value a_{15} in (b) of the previous exercise.

7. Let a_n be the number of ways to obtain a total of n points when a die is thrown 4 times. Find the generating function of (a_n) and the values of a_{12} and a_{20}.

8. Let a_n be the number of subsets of $\{1, 2, \ldots, n\}$ that do not contain two consecutive integers.
   (a) Find $a_0,\ a_1,\ a_2,\ a_3$.
   (b) Give a recurrence relation for, $n \geq 2$.
   (c) Find the generating function of $(a_n)_{n\geq 0}$.

9. A set of natural numbers is said fat if every element is at least the cardinality. For example, $\{6, 10, 11, 20, 33, 34\}$ is fat, but $\{2, 200, 300\}$ is not. Let a_n be the number of fat subsets of $\{1, 2, \ldots, n\}$ (the $\emptyset$ is considered fat). Solve the same questions as in the previous exercise.

10. A domino piece is a 1×2 rectangle. Let a_n be the number of ways to build an $n \times 2$ rectangle with n domino pieces. Solve the same questions as in the previous exercise.

11. Let a_n be the number of sequences of length n (that is, of n terms) formed by zeros and ones in which there are not two consecutive ones. Solve the same questions as in the previous exercise.

Solutions 8

1. (a) $(1-5t)^{-1}$; (b) $(1+t)^{-1}$; (c) $(1-t^2)^{-1}$; (d) $[(1-t^2)^{-1}]' = 2t(1-t^2)^{-2}$; (e) $4(1-2t)^{-1}$. **2.** (a) $\binom{n+2}{n}$; (b) $a_n = 1$ or 0, depending on n even or odd; (c) $(-4)^n$; (d) $2^{n+1}-2$; (e) $2^n/n!$; (f) $a_n = 1/(n/2)!$ or 0, depending on n even or odd; (g) $a_{2n-1} = (-1)^{n-1}/(2n-1)!$, $a_{2n} = 0$; (h) $a_{2n} = (-1)^n/(2n)!$, $a_{2n-1} = 0$. **3.** (a) $g(-t)$; (b) $[g(t)-g(0)]/t$; (c) $tg(t)$; (d) $[(1+t)g(t)-g(0)]/t$; (e) $(1-t)g(t)$; (f) $[g(t)+g(-t)]/2$; (g) $[g(t)-g(-t)]/2$; (h) $g(t^2)$; (i) $[g(t)]^2$; (j) $g(t)/(1-t)$; (k) $tg'(t)$; (l) $t^k g^{(k)}(t)$. **4.** (a) $t(1-t)^{-2}$; (b) $(1-2t)/(1-3t-4t^2)$. **5.** (a) $(1-t)^{-3}(1-t^4)^{-1}$; (b) $(1-t^2)^{-2}(1-t^3)^{-2}$. **6.** 100; 36. **7.** $(t+t^2+\ldots+t^6)^4 = t^4(1-t^6)^4/(1-t)^4$; 125; 35. **8.** (a) $1,2,3,5$; (b) $a_n = a_{n-1}+a_{n-2}$ $(n \geq 2)$; (c) $(1+t)/(1-t-t^2)$**9.** (a_n) the same as in the previous exercise. **10.** (a) $1,1,2,3$; (b) $a_n = a_{n-1}+a_{n-2}$ $(n \geq 2)$; (c) $1/(1-t-t^2)$**11.** (a_n) the same as in exercises 8 and 9.