DISCRETE MATHEMATICS

Chapter 2: Combinatorial identities

2.1 Combinatorial identities

2.1.1 Combinatorial identities and proofs

Definition

A **combinatorial identity** is an equality where there are combinatorial numbers.

Example

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n, n \in \mathbb{N}^*.$$

An identity can be **proved** in different ways:

- **Analytically**: by induction, reduction to another known equality,...
- **Combinatorially**: by finding a combinatorial problem and two different counting methods whose results are each side of the equality.

2.1 Combinatorial identities

2.1.1 Combinatorial identities and proofs

Example

$$\boxed{\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n, n \in \mathbb{N}^*}$$

- **Combinatorial proof**: Let Ω be a set of n elements.
  - First counting method:
    Subsets of $\Omega \leftrightarrow$ sequences of length n formed with zeros and ones.
    There are 2^n.
  - Second counting method:
    Subsets of Ω of k elements: $\binom{n}{k}$, $k = 0, 1, \dots, n$. Therefore, there are $\sum_{k=0}^{n} \binom{n}{k}$ subsets of Ω.
- **Analytical proof**: By induction on n:
  For $n = 0$, $\binom{0}{0} = 1 = 2^0$.
  Assume that the formula is true for $n - 1$.
  $\sum_{k=0}^{n} \binom{n}{k} = \binom{n}{0} + [\binom{n-1}{1} + \binom{n-1}{0}] + [\binom{n-1}{2} + \binom{n-1}{1}] + \cdots + \left[\binom{n-1}{n-1} + \binom{n-1}{n-2}\right] + \binom{n}{n} =$
  $\binom{n-1}{0} + \binom{n-1}{1} + \binom{n-1}{0} + \binom{n-1}{2} + \binom{n-1}{1} + \cdots + \binom{n-1}{n-1} + \binom{n-1}{n-2} + \binom{n-1}{n-1} =$
  $2\binom{n-1}{0} + 2\binom{n-1}{1} + \cdots + 2\binom{n-1}{n-2} + 2\binom{n-1}{n-1} = 2\sum_{k=0}^{n-1} \binom{n-1}{k} = 2 \cdot 2^{n-1} = 2^n.$

2.1 Combinatorial identities

2.1.1 Combinatorial identities and proofs

Example

$$\boxed{\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots + (-1)^n \binom{n}{n} = 0, n \in \mathbb{N}}$$

- **Analytical proof**: Since $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$,
  $\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots + (-1)^n \binom{n}{n} = \binom{n}{0} - \binom{n-1}{1} - \binom{n-1}{0} + \binom{n-1}{2} + \binom{n-1}{1} - \cdots +$
  $(-1)^{n-1}\binom{n-1}{n-1} + (-1)^{n-1}\binom{n-1}{n-2} + (-1)^n \binom{n}{n} = 0.$
- **Combinatorial proof**: We have to prove $\binom{n}{0} + \binom{n}{2} + \cdots = \binom{n}{1} + \binom{n}{3} + \cdots$, that is, the number of subsets of $\{1, \dots, n\}$ with an even number of elements is equal to the number of subsets of $\{1, \dots, n\}$ with an odd number of elements.
  Let $\mathcal{E}$ = set of subsets of $\{1, \dots, n\}$ with an even number of elements and
  $\mathcal{O}$ = set of subsets of $\{1, \dots, n\}$ with an odd number of elements. Define the map

  $$\begin{array}{ccc} \mathcal{E} & \to & \mathcal{O} \\ A & \to & \begin{cases} A \setminus \{1\} & \text{if } 1 \in A \\ A \cup \{1\} & \text{if } 1 \notin A \end{cases} \end{array}$$

  Since it is a bijection, $|\mathcal{E}| = |\mathcal{O}|$.

2.1 Combinatorial identities

2.1.1 Combinatorial identities and proofs

Example

$$\boxed{\binom{n}{0}+\binom{n}{2}+\cdots=2^{n-1}, n\in\mathbb{N}} \quad \boxed{\binom{n}{1}+\binom{n}{3}+\cdots=2^{n-1}, n\in\mathbb{N}}$$

Proof: Define
$a=\binom{n}{0}+\binom{n}{2}+\cdots$,
$b=\binom{n}{1}+\binom{n}{3}+\cdots$.
By the two previous examples: $a+b=2^n$, $a=b$.
Therefore, $2a=2^n$, $a=2^{n-1}$ and $b=2^{n-1}$.

2.1 Combinatorial identities

2.1.2 Basic combinatorial identities

Proposition

Recall:

1. $C_{n,k}=\binom{n}{k}=\frac{V_{n,k}}{k!}=\frac{\frac{n!}{(n-k)!}}{k!}=\frac{n!}{(n-k)!k!}, \quad k=0,\dots,n.$
2. $\binom{n}{k}=\frac{n!}{(n-k)!k!}=\frac{n!}{k!(n-k)!}=\frac{n!}{(n-(n-k))!(n-k)!}=\binom{n}{n-k}, \quad k=0,\dots,n.$
3. $\binom{n}{0}=\frac{n!}{(n-0)!0!}=\frac{n!}{n!\cdot 1}=1.$
4. $\binom{n}{n}=\frac{n!}{(n-n)!n!}=\frac{n!}{1\cdot n!}=1.$
5. $\binom{n}{k}=0, \quad k>n.$
6. $\binom{n-1}{k-1}+\binom{n-1}{k}=\binom{n}{k}, \quad n\geq 1, k=0,\dots,n.$
7. *If* $k\geq 1$, $\binom{n}{k}=\frac{n}{k}\binom{n-1}{k-1}$.
8. *If* $0\leq k\leq n-1$, $\binom{n}{k}=\frac{n}{n-k}\binom{n-1}{k}$.

2.1 Combinatorial identities

2.1.3 Pascal's triangle

$$\boxed{\binom{n}{0}+\binom{n+1}{1}+\binom{n+2}{2}+\cdots+\binom{n+r}{r}=\binom{n+r+1}{r}, \quad n,r\in\mathbb{N}^*}$$

$$\binom{0}{0}$$
$$\binom{1}{0} \quad \binom{1}{1}$$
$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$
$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$
$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$
$$\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5}$$
$$\binom{6}{0} \quad \binom{6}{1} \quad \binom{6}{2} \quad \binom{6}{3} \quad \binom{6}{4} \quad \binom{6}{5} \quad \binom{6}{6}$$
$$\binom{7}{0} \quad \binom{7}{1} \quad \binom{7}{2} \quad \binom{7}{3} \quad \binom{7}{4} \quad \binom{7}{5} \quad \binom{7}{6} \quad \binom{7}{7}$$

$$1$$
$$1 \quad 1$$
$$1 \quad 2 \quad 1$$
$$1 \quad 3 \quad 3 \quad 1$$
$$1 \quad 4 \quad 6 \quad 4 \quad 1$$
$$1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1$$
$$1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1$$
$$1 \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1$$

2.1 Combinatorial identities

2.1.4 Vandermonde's formula and other identities

Vandermonde's formula:

$$\boxed{\binom{n}{0}\binom{m}{r}+\binom{n}{1}\binom{m}{r-1}+\cdots+\binom{n}{r}\binom{m}{0}=\binom{n+m}{r}, \quad n,m,r\in\mathbb{N}^*}$$

or

$$\boxed{\sum_{k=0}^{r}\binom{n}{k}\binom{m}{r-k}=\binom{n+m}{r}, \quad n,m,r\in\mathbb{N}^*}$$

2.1 Combinatorial identities

2.1.4 Vandermonde's formula and other identities

Other identities:

$$\boxed{\binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}, n \in \mathbb{N}^*}$$

$$\boxed{\binom{n}{r}\binom{r}{k} = \binom{n}{k}\binom{n-k}{r-k}, \quad n \geq r \geq k \geq 0}$$

2.2 Binomial formula

Theorem

Let $x, y \in \mathbb{C}$ and $n \in \mathbb{N}^*$. Then

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

2.2 Binomial formula

Corollary

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k = \binom{n}{0} + \binom{n}{1} x + \cdots + \binom{n}{n} x^n.$$

Corollary

$$(1-x)^n = \sum_{k=0}^{n} \binom{n}{k} (-x)^k = \sum_{k=0}^{n} \binom{n}{k} (-1)^k x^k = \binom{n}{0} - \binom{n}{1} x + \cdots + (-1)^n \binom{n}{n} x^n.$$

Corollary

$$\binom{n}{0} + \binom{n}{2} x^2 + \binom{n}{4} x^4 + \cdots = \frac{(1+x)^n + (1-x)^n}{2}.$$

Corollary

$$\binom{n}{1} x + \binom{n}{3} x^3 + \binom{n}{5} x^5 + \cdots = \frac{(1+x)^n - (1-x)^n}{2}.$$

2.2 Binomial formula

Some combinatorial identities obtained from the binomial formula:

- $x = 1$:
  - $2^n = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n}$.
  - $0 = \binom{n}{0} - \binom{n}{1} + \cdots + (-1)^n \binom{n}{n}, \quad n \geq 1.$
  - $2^{n-1} = \binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots$.
  - $2^{n-1} = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots$.
- $x = i$:
  - $\binom{n}{0} + \binom{n}{4} + \cdots = \frac{2^{n-1} + 2^{\frac{n}{2}} \cos \frac{n\pi}{4}}{2}$.
  - $\binom{n}{1} + \binom{n}{5} + \cdots = \frac{2^{n-1} + 2^{\frac{n}{2}} \sin \frac{n\pi}{4}}{2}$.
  - $\binom{n}{2} + \binom{n}{6} + \cdots = \frac{2^{n-1} - 2^{\frac{n}{2}} \cos \frac{n\pi}{4}}{2}$.
  - $\binom{n}{3} + \binom{n}{7} + \cdots = \frac{2^{n-1} - 2^{\frac{n}{2}} \sin \frac{n\pi}{4}}{2}$.

2.2 Binomial formula

Some combinatorial identities obtained from the binomial formula:

- By differentation:

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k \Rightarrow n(1+x)^{n-1} = \sum_{k=1}^{n} \binom{n}{k} k x^{k-1}, \quad n \geq 2.$$

For $x = 1$:

$$n2^{n-1} = \sum_{k=0}^{n} k \binom{n}{k}, \quad n \geq 2.$$

For $x = -1$:

$$0 = \sum_{k=0}^{n} (-1)^{k-1} k \binom{n}{k}, \quad n \geq 2.$$

2.2 Binomial formula

Some combinatorial identities obtained from the binomial formula:

- By integration:

$$\int_0^t (1+x)^n dx = \sum_{k=0}^{n} \binom{n}{k} \int_0^t x^k dx \Rightarrow \left[\frac{(1+x)^{n+1}}{n+1} \right]_0^t = \sum_{k=0}^{n} \binom{n}{k} \left[\frac{x^{k+1}}{k+1} \right]_0^t \Rightarrow$$

$$\frac{(1+t)^{n+1} - 1}{n+1} = \sum_{k=0}^{n} \binom{n}{k} \frac{t^{k+1}}{k+1}.$$

For $t = 1$:

$$\frac{2^{n+1} - 1}{n+1} = \sum_{k=0}^{n} \frac{1}{k+1} \binom{n}{k}.$$

For $t = -1$:

$$\frac{1}{n+1} = \sum_{k=0}^{n} \frac{(-1)^k}{k+1} \binom{n}{k}.$$

2.2 Binomial formula

Some combinatorial identities obtained from the binomial formula:

- By integration again:

$$\int_0^x \frac{(1+t)^{n+1}-1}{n+1}dt = \sum_{k=0}^{n}\binom{n}{k}\frac{1}{k+1}\int_0^x t^{k+1}dt \Rightarrow$$

$$\left[\frac{(1+t)^{n+2}}{(n+1)(n+2)} - \frac{t}{n+1}\right]_0^x = \sum_{k=0}^{n}\binom{n}{k}\left[\frac{t^{k+2}}{(k+1)(k+2)}\right]_0^x \Rightarrow$$

$$\frac{(1+x)^{n+2}-1}{(n+1)(n+2)} - \frac{x}{n+1} = \sum_{k=0}^{n}\binom{n}{k}\frac{x^{k+2}}{(k+1)(k+2)}.$$

For $x = 1$:

$$\frac{2^{n+2}-1}{(n+1)(n+2)} - \frac{1}{n+1} = \sum_{k=0}^{n}\binom{n}{k}\frac{1}{(k+1)(k+2)}.$$

For $x = -1$:

$$\frac{-1}{(n+1)(n+2)} + \frac{1}{n+1} = \sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^{k+2}}{(k+1)(k+2)}.$$

2.2 Binomial formula

Some combinatorial identities obtained from the binomial formula:

- Other operations:
  $(1+x)^n(1+x)^m = (1+x)^{n+m}$ is a polynomial identity, so both sides of the equality have the same coefficients. Therefore,

$$\sum_{i=0}^{r}\binom{n}{i}\binom{m}{r-i} = \binom{n+m}{r},$$

which is Vandermonde's formula.

2.2 Binomial formula

Some combinatorial identities obtained from the binomial formula:

- A little bit more:

$$\frac{\partial^2}{\partial x \partial y}(x+y)^n = n(n-1)(x+y)^{n-2} = \sum_{k=1}^{n-1} \binom{n}{k} k x^{k-1}(n-k) y^{n-k-1}.$$

For $x = y = 1$:

$$n(n-1)2^{n-2} = \sum_{k=0}^{n} k(n-k)\binom{n}{k}.$$

2.3 Multinomial coefficients

2.3.1 Multinomial coefficients

Definition

Let $n \in \mathbb{N}^*$ and $r_1, r_2, \ldots, r_m \in \mathbb{N}^*$ such that $r_1 + r_2 + \cdots + r_m = n$.

$$\binom{n}{r_1, r_2, \ldots, r_m} = \frac{n!}{r_1! r_2! \cdots r_m!}$$

Example

$$\binom{6}{3, 2, 1} = \frac{6!}{3!2!1!} = 5 \cdot 4 \cdot 3 = 60.$$

2.3 Multinomial coefficients

2.3.1 Multinomial coefficients

Proposition

1. $$\binom{n}{r_1, r_2} = \binom{n}{r_1}$$

2. $$\binom{n}{r_1, r_2, \ldots, r_m} = \binom{n}{r_1}\binom{n-r_1}{r_2}\binom{n-r_1-r_2}{r_3}\cdots\binom{n-r_1-\cdots-r_{m-2}}{r_{m-1}}$$

Proof.- 2.

$$\binom{n}{r_1}\binom{n-r_1}{r_2}\binom{n-r_1-r_2}{r_3}\cdots\binom{n-r_1-\cdots-r_{m-2}}{r_{m-1}} =$$

$$= \frac{n!}{r_1!(n-r_1)!}\frac{(n-r_1)!}{r_2!(n-r_1-r_2)!}\frac{(n-r_1-r_2)!}{r_3!(n-r_1-r_2-r_3)!}\cdots\frac{(n-r_1-\cdots-r_{m-2})!}{r_{m-1}!r_m!} =$$

$$= \frac{n!}{r_1!r_2!\cdots r_m!}.$$

2.3 Multinomial coefficients

2.3.2 Combinatorial meaning

$\binom{n}{r_1,r_2,\ldots,r_m} = \frac{n!}{r_1!r_2!\cdots r_m!}$

- is the number of sequences of length n that can be formed with r_1 symbols α_1,..., r_m symbols α_m,
- is also the number of ways to place n distinguishable balls in m numbered boxes so that there are r_1 balls in the first box, r_2 in the second,..., and r_m in the last one.

Strategy:
1) choose the places for the α_1's or choose the balls in the first box: $\binom{n}{r_1}$
2) choose the places for the α_2's or choose the balls in the second box: $\binom{n-r_1}{r_2}$
3) ...
4) choose the places for the α_{m-1}'s or choose the balls in the penultimate box: $\binom{n-r_1-\cdots-r_{m-2}}{r_{m-1}}$
5) choose the places for the α_m's or choose the balls in the last box: 1

2.3 Multinomial coefficients

2.3.3 Multinomial formula

Theorem

Let $x_1, x_2, \ldots, x_m \in \mathbb{C}$ and $n \in \mathbb{N}^*$. Then,

$$(x_1+x_2+\cdots+x_m)^n = \sum_{\substack{r_i \in \mathbb{N}^* \\ r_1+r_2+\cdots+r_m=n}} \binom{n}{r_1, r_2, \ldots, r_m} x_1^{r_1} x_2^{r_2} \cdots x_m^{r_m}.$$

Proof.-

$$(x_1+x_2+\cdots+x_m)^n = (x_1+x_2+\cdots+x_m)\cdots(x_1+x_2+\cdots+x_m) = \sum_{i_1=1}^{m}\sum_{i_2=1}^{m}\cdots\sum_{i_n=1}^{m} x_{i_1}x_{i_2}\cdots x_{i_n} =$$

$$= \sum (\text{ number of sequences } x_{i_1}x_{i_2}\cdots x_{i_n} \text{ in which there are } r_1 \text{ symbols } x_1, \ldots, r_m \text{ symbols } x_m) x_1^{r_1} x_2^{r_2} \cdots x_m^{r_m} =$$

$$= \sum_{\substack{r_i \in \mathbb{N}^* \\ r_1+r_2+\cdots+r_m=n}} \binom{n}{r_1, r_2, \ldots, r_m} x_1^{r_1} x_2^{r_2} \cdots x_m^{r_m}.$$

2.3 Multinomial coefficients

2.3.4 Some applications

1. $$m^n = \sum_{\substack{r_i \in \mathbb{N}^* \\ r_1+r_2+\cdots+r_m=n}} \binom{n}{r_1, r_2, \ldots, r_m}.$$
2. $$\sum_{\substack{r_i \in \mathbb{N}^* \\ r_1+r_2+r_3+r_4=n}} (-1)^{r_3+r_4} \binom{n}{r_1, r_2, r_3, r_4} = 0, \quad n \geq 1.$$
3. If $r_1+\cdots+r_m = n$ then $r_1!\cdots r_m!$ is a divisor of $n!$.
4. $[[(m-1)!]!]^m$ divides $[m!]!$

Definition

$$m!! = \begin{cases} m(m-2)(m-4)\cdots 2 & \text{if } m \equiv 0 \pmod 2 \\ m(m-2)(m-4)\cdots 1 & \text{if } m \equiv 1 \pmod 2 \end{cases}$$

is called **semifactorial of** m.

2.4 Generalized binomial formula

Definition

Let $\alpha \in \mathbb{R}$, $k \in \mathbb{N}^*$.

$$\binom{\alpha}{k} = \frac{\alpha^{\underline{k}}}{k!} = \begin{cases} 1 & k = 0 \\ \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!} & k \geq 1 \end{cases}$$

Proposition

1. *If $\alpha = n \in \mathbb{N}^*$ then $\binom{n}{k} = C_{n,k}$; otherwise, if $\alpha \notin \mathbb{N}^*$ then $\binom{\alpha}{k}$ has no combinatorial meaning.*
2. $$\binom{\alpha}{k-1} + \binom{\alpha}{k} = \binom{\alpha+1}{k}, \quad \alpha \in \mathbb{R}, k \in \mathbb{N}^*.$$
3. $$\binom{-\alpha}{k} = (-1)^k \binom{\alpha+k-1}{k}$$

2.4 Generalized binomial formula

Theorem (Taylor's formula for a polynomial)

Given a polynomial of degree at most n,

$$p(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n = \sum_{k=0}^{n} c_k x^k$$

it turns out that

$$c_k = \frac{p^{(k)}(0)}{k!}, k = 0, \ldots, n, \text{ and } p(x) = \sum_{k=0}^{n} \frac{p^{(k)}(0)}{k!} x^k.$$

For $p(x) = (1+x)^n$, note that
$p^{(r)}(x) = n(n-1)\cdots(n-r+1)(1+x)^{n-r} = n^{\underline{r}}(1+x)^{n-r}$.
Thus, $c_r = \frac{p^{(r)}(0)}{r!} = \frac{n^{\underline{r}}}{r!} = \binom{n}{r}$.
Therefore,

$$(1+x)^n = \sum_{r=0}^{n} \binom{n}{r} x^r.$$

which is a new proof of Newton's binomial formula.

2.4 Generalized binomial formula

Theorem (Generalized binomial formula)

For every $\alpha \in \mathbb{R}$,

$$(1+x)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k \text{ for } |x| < 1.$$

- $\alpha = n \in \mathbb{N}^*$:
  $(1+x)^n = \sum_{k=0}^{\infty} \binom{n}{k} x^k = \sum_{k=0}^{n} \binom{n}{k} x^k.$
- $\alpha = -1$ if $|x| < 1$: since $(1+x)^{-1} = \sum_{k=0}^{\infty} \binom{-1}{k} x^k$ for $|x| < 1$,

$$\frac{1}{1+x} = \sum_{k=0}^{\infty} (-1)^k \binom{1+k-1}{k} x^k = \sum_{k=0}^{\infty} (-1)^k x^k = 1 - x + x^2 - x^3 + x^4 - \cdots$$

Consequently,

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots$$
$$\frac{1}{1-x^2} = 1 + x^2 + x^4 + x^6 + x^8 + \cdots$$
$$\frac{1}{1-x^r} = 1 + x^r + x^{2r} + x^{3r} + x^{4r} + \cdots$$

2.4 Generalized binomial formula

- $\alpha = -2$, $|x| < 1$:
  $(1+x)^{-2} = \frac{1}{(1+x)^2} = \sum_{k=0}^{\infty} \binom{-2}{k} x^k = \sum_{k=0}^{\infty} (-1)^k \binom{2+k-1}{k} x^k = \sum_{k=0}^{\infty} (-1)^k (k+1) x^k.$
  Thus,

$$\frac{1}{(1+x)^2} = 1 - 2x + 3x^2 - 4x^3 + 5x^4 - \cdots$$

Consequently,

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \cdots$$
$$\frac{1}{(1-x^r)^2} = 1 + 2x^r + 3x^{2r} + 4x^{3r} + 5x^{4r} + \cdots$$

2.4 Generalized binomial formula

- $\alpha = -n, n \in \mathbb{N}^*$, $|x| < 1$:

$$(1+x)^{-n} = \frac{1}{(1+x)^n} = \sum_{k=0}^{\infty} \binom{-n}{k} x^k.$$

$$(1-x)^{-n} = \frac{1}{(1-x)^n} = \sum_{k=0}^{\infty} \binom{-n}{k} (-x)^k = \sum_{k=0}^{\infty} (-1)^k \binom{n+k-1}{k} (-x)^k =$$

$$= \sum_{k=0}^{\infty} \binom{n+k-1}{k} x^k = \sum_{k=0}^{\infty} CR_{n,k} x^k.$$

Notice that
$C_{n,k}$ is the coefficient of x^k in the series $(1+x)^n = \sum_{k=0}^{\infty} C_{n,k} x^k$ and
$CR_{n,k}$ is the coefficient of x^k in the series $(1-x)^{-n} = \sum_{k=0}^{\infty} CR_{n,k} x^k$.

2.4 Generalized binomial formula

- $\alpha = -\frac{1}{2}$, $|x| < 1$:
$(1+x)^{-\frac{1}{2}} = \frac{1}{\sqrt{1+x}} = \sum_{k=0}^{\infty} \binom{-\frac{1}{2}}{k} x^k$.
Now,
$\binom{-\frac{1}{2}}{k} = \frac{(-\frac{1}{2})^{\underline{k}}}{k!} = \frac{(-1)^k (\frac{1}{2})^{\overline{k}}}{k!} = (-1)^k \frac{\frac{1}{2}(\frac{1}{2}+1)(\frac{1}{2}+2)\cdots(\frac{1}{2}+k-1)}{k!} = (-1)^k \frac{\frac{1}{2}\frac{3}{2}\frac{5}{2}\cdots\frac{2k-1}{2}}{k!} =$
$= (-1)^k \frac{1\cdot 3\cdot 5\cdots(2k-1)}{2^k k!} = \left(-\frac{1}{2}\right)^k \frac{(2k-1)!!}{k!} = (-1)^k \frac{1\cdot 2\cdot 3\cdots(2k-1)(2k)}{2^k k! 2\cdot 4\cdot 6\cdots(2k)} =$
$= (-1)^k \frac{(2k)!}{2^k k!(2\cdot 1)(2\cdot 2)(2\cdot 3)\cdots(2k)} = (-1)^k \frac{(2k)!}{(2^k k!)^2} = \left(-\frac{1}{4}\right)^k \frac{(2k)!}{k!k!} = \left(-\frac{1}{4}\right)^k \binom{2k}{k}$.
Thus,

$$(1+x)^{-\frac{1}{2}} = \frac{1}{\sqrt{1+x}} = \sum_{k=0}^{\infty} \binom{2k}{k} \left(-\frac{x}{4}\right)^k.$$

2.4 Generalized binomial formula

Some applications of the generalized binomial formula:

- Since for $\alpha, \beta \in \mathbb{R}$ and $|x| < 1$, $(1+x)^\alpha(1+x)^\beta = (1+x)^{\alpha+\beta}$ then

$$\left[\binom{\alpha}{0} + \binom{\alpha}{1}x + \binom{\alpha}{2}x^2 + \cdots\right]\left[\binom{\beta}{0} + \binom{\beta}{1}x + \binom{\beta}{2}x^2 + \cdots\right] =$$

$$\left[\binom{\alpha+\beta}{0} + \binom{\alpha+\beta}{1}x + \binom{\alpha+\beta}{2}x^2 + \cdots\right].$$

The coefficients of x^r on the left and right hand sides must be equal. Therefore, $\binom{\alpha}{0}\binom{\beta}{r} + \binom{\alpha}{1}\binom{\beta}{r-1} + \cdots + \binom{\alpha}{r}\binom{\beta}{0} = \binom{\alpha+\beta}{r}$ and

$$\binom{\alpha+\beta}{r} = \sum_{k=0}^{r}\binom{\alpha}{k}\binom{\beta}{r-k}$$

which is the generalized Vandermonde's formula.

2.4 Generalized binomial formula

Some applications of the generalized binomial formula:

- Since for $\alpha \in \mathbb{R}$ and $|x| < 1$, $(1+x)^\alpha(1-x)^\alpha = (1-x^2)^\alpha$ then

$$\left[\binom{\alpha}{0} + \binom{\alpha}{1}x + \binom{\alpha}{2}x^2 + \cdots\right]\left[\binom{\alpha}{0} - \binom{\alpha}{1}x + \binom{\alpha}{2}x^2 - \cdots\right] =$$

$$\left[\binom{\alpha}{0} - \binom{\alpha}{1}x^2 + \binom{\alpha}{2}x^4 - \cdots\right].$$

Therefore,

$$\binom{\alpha}{0}(-1)^r\binom{\alpha}{r} + \binom{\alpha}{1}(-1)^{r-1}\binom{\alpha}{r-1} + \cdots + \binom{\alpha}{r}\binom{\alpha}{0} = \begin{cases} 0 & r \equiv 1 \pmod 2 \\ (-1)^{\frac{r}{2}}\binom{\alpha}{\frac{r}{2}} & r \equiv 0 \pmod 2 \end{cases}$$

Thus,

$$\sum_{k=0}^{r}(-1)^{r-k}\binom{\alpha}{k}\binom{\alpha}{r-k} = \begin{cases} 0 & r \equiv 1 \pmod 2 \\ (-1)^{\frac{r}{2}}\binom{\alpha}{\frac{r}{2}} & r \equiv 0 \pmod 2 \end{cases}.$$