

IMO 2004 Solution Notes

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This is an compilation of solutions for the 2004 IMO. Some of the solutions are my own work, but many are from the official solutions provided by the organizers (for which they hold any copyrights), and others were found on the Art of Problem Solving forums.

Corrections and comments are welcome!

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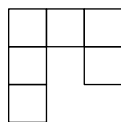
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§0 Problems

1. Let ABC be an acute-angled triangle with $AB \neq AC$. The circle with diameter BC intersects the sides AB and AC at M and N respectively. Denote by O the midpoint of the side BC . The bisectors of the angles $\angle BAC$ and $\angle MON$ intersect at R . Prove that the circumcircles of the triangles BMR and CNR have a common point lying on the side BC .
2. Find all polynomials P with real coefficients such that for all reals a, b, c such that $ab + bc + ca = 0$, we have

$$P(a - b) + P(b - c) + P(c - a) = 2P(a + b + c).$$

3. Define a “hook” to be a figure made up of six unit squares as shown below in the picture, or any of the figures obtained by applying rotations and reflections to this figure.



Which $m \times n$ rectangles can be tiled by hooks?

4. Let $n \geq 3$ be an integer and t_1, t_2, \dots, t_n positive real numbers such that

$$n^2 + 1 > (t_1 + t_2 + \dots + t_n) \left(\frac{1}{t_1} + \frac{1}{t_2} + \dots + \frac{1}{t_n} \right).$$

Show that t_i, t_j, t_k are the sides of a triangle for all i, j, k with $1 \leq i < j < k \leq n$.

5. In a convex quadrilateral $ABCD$, the diagonal BD bisects neither the angle ABC nor the angle CDA . The point P lies inside $ABCD$ and satisfies

$$\angle PBC = \angle DBA \quad \text{and} \quad \angle PDC = \angle BDA.$$

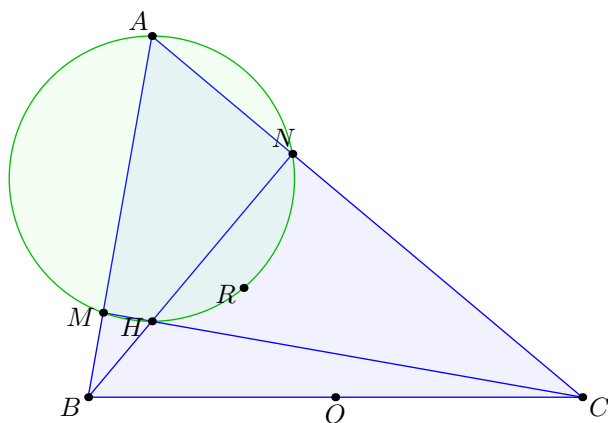
Prove that $ABCD$ is a cyclic quadrilateral if and only if $AP = CP$.

6. We call a positive integer *alternating* if every two consecutive digits in its decimal representation are of different parity. Find all positive integers n which have an alternating multiple.

§1 IMO 2004/1

Let ABC be an acute-angled triangle with $AB \neq AC$. The circle with diameter BC intersects the sides AB and AC at M and N respectively. Denote by O the midpoint of the side BC . The bisectors of the angles $\angle BAC$ and $\angle MON$ intersect at R . Prove that the circumcircles of the triangles BMR and CNR have a common point lying on the side BC .

By Miquel's theorem it's enough to show $AMRN$ is cyclic.



In fact, since the bisector of $\angle MON$ is just the perpendicular bisector of \widehat{MN} , the point R is actually just the arc midpoint of \widehat{MN} of (AMN) as desired.

§2 IMO 2004/2

Find all polynomials P with real coefficients such that for all reals a, b, c such that $ab+bc+ca=0$, we have

$$P(a-b) + P(b-c) + P(c-a) = 2P(a+b+c).$$

The answer is

$$P(x) = \alpha x^4 + \beta x^2$$

which can be checked to work, for any real numbers α and β .

It is easy to obtain that P is even and $P(0) = 0$. The trick is now to choose $(a, b, c) = (6x, 3x, -2x)$ and then compare the leading coefficients to get

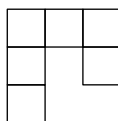
$$3^n + 5^n + 8^n = 2 \cdot 7^n$$

for $n = \deg f$ (which is even). As $n \geq 7 \implies (8/7)^n > 2$, this means that we must have $n \leq 6$. Moreover, taking modulo 7 we have $3^n + 5^n \equiv 6 \pmod{7}$ which gives $n \equiv 2, 4 \pmod{6}$.

Thus $\deg P \leq 4$, which (combined with P even) resolves the problem.

§3 IMO 2004/3

Define a “hook” to be a figure made up of six unit squares as shown below in the picture, or any of the figures obtained by applying rotations and reflections to this figure.



Which $m \times n$ rectangles can be tiled by hooks?

The answer is that one requires:

- $\{1, 2, 5\} \not\subset \{m, n\}$,
- $3 \mid m$ or $3 \mid n$,
- $4 \mid m$ or $4 \mid n$.

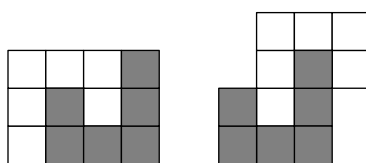
First, we check all of these work, in fact we claim:

Claim — Any rectangle satisfying these conditions can be tiled by 3×4 rectangles (and hence by hooks).

Proof. In fact it will be sufficient to tile with 3×4 rectangles. If $3 \mid m$ and $4 \mid n$, this is clear. Else suppose $12 \mid m$ but $3 \nmid n$, $4 \nmid n$. Then $n \geq 7$, so it can be written in the form $3a + 4b$ for nonnegative integers a and b , which permits a tiling. \square

We now prove these conditions are necessary. It is not hard to see that $m, n \in \{1, 2, 5\}$ is necessary.

We thus turn our attention to divisibility conditions. Each hook has a *hole*, and if we associate each hook with the one that fills its hole, we get a bijective pairing of hooks. Thus the number of cells is divisible by 12, and the cells come in two types of **tiles** shown below (rotations and reflections permitted).



In particular, the total number of cells is divisible by 12. Thus the problem is reduce to proving that:

Claim — if a $6a \times 2b$ rectangle is tiled by tiles, then at least one of a and b is even.

Proof. Note that the tiles come in two forms:

- **First type:** These tiles have exactly four columns, each with exactly three cells (an odd number). Moreover, all rows have an even number of cells (either 2 or 4).
- **Second type:** vice-versa. These tiles have exactly four rows, each with exactly three cells (an odd number). Moreover, all rows have an odd number of cells.

We claim that any tiling uses an even number of each type, which is enough.

By symmetry we prove an even number of first-type tiles. Color red every fourth column of the rectangle. The number of cells colored is red. The tiles of the second type cover an even number of red cells, and the tiles of the first type cover an odd number of red cells. Hence the number of tiles of the first type must be even. \square

Remark. This shows that a rectangle can be tiled by hooks iff it can be tiled by 3×4 rectangles. But there exists tilings which do not decompose into 3×4 ; see e.g. <https://aops.com/community/c6h14023p99881>.

§4 IMO 2004/4

Let $n \geq 3$ be an integer and t_1, t_2, \dots, t_n positive real numbers such that

$$n^2 + 1 > (t_1 + t_2 + \dots + t_n) \left(\frac{1}{t_1} + \frac{1}{t_2} + \dots + \frac{1}{t_n} \right).$$

Show that t_i, t_j, t_k are the sides of a triangle for all i, j, k with $1 \leq i < j < k \leq n$.

Let $a = t_1, b = t_2, c = t_3$. Expand:

$$\begin{aligned} n^2 + 1 &> (t_1 + t_2 + \dots + t_n) \left(\frac{1}{t_1} + \dots + \frac{1}{t_n} \right) \\ &= n + \sum_{1 \leq i < j \leq n} \left(\frac{t_i}{t_j} + \frac{t_j}{t_i} \right) \\ &= n + \sum_{1 \leq i < j \leq n} \left(\frac{t_i}{t_j} + \frac{t_j}{t_i} \right) \\ &\geq n + \sum_{1 \leq i < j \leq 3} \left(\frac{t_i}{t_j} + \frac{t_j}{t_i} \right) + \sum_{\substack{1 \leq i < j \leq n \\ j > 3}} 2 \\ &= n + 2 \left(\binom{n}{2} - 3 \right) + \left(\frac{a}{b} + \frac{b}{a} \right) + \frac{a+b}{c} + \frac{c}{b} + \frac{c}{a} \\ &\geq n + 2 \left(\binom{n}{2} - 3 \right) + 2 + \frac{a+b}{c} + c \cdot \frac{4}{a+b} \end{aligned}$$

So, we conclude that

$$\frac{a+b}{c} + \frac{4c}{a+b} < 5$$

which rearranges to

$$(4c - (a+b))(c - (a+b)) < 0.$$

This is enough to imply $c \leq a+b$.

Remark. A variant of the same argument allows one to improve the left-hand side to $(n + \sqrt{10} - 3)^2$. One does so by writing

$$\text{RHS} \geq \left(\sqrt{(a+b+c) \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right)} + (n-3) \right)^2$$

and estimating the square root as in the previous approach.

In addition, $(n + \sqrt{10} - 3)^2$ is best possible, as seen by taking $(a, b, c) = (2, 1, 1)$ and $t_4 = t_5 = \dots = \frac{2}{5}\sqrt{10}$.

§5 IMO 2004/5, proposed by Waldemar Pompe

In a convex quadrilateral $ABCD$, the diagonal BD bisects neither the angle ABC nor the angle CDA . The point P lies inside $ABCD$ and satisfies

$$\angle PBC = \angle DBA \quad \text{and} \quad \angle PDC = \angle BDA.$$

Prove that $ABCD$ is a cyclic quadrilateral if and only if $AP = CP$.

Apply barycentric coordinates to $\triangle PBD$ with $P = (1, 0, 0)$, $B = (0, 1, 0)$ and $D = (0, 0, 1)$. Define $a = BD$, $b = DP$ and $c = PB$.

Since A and C are isogonal conjugates with respect to $\triangle PBD$, we set

$$A = (au : bv : cw) \quad \text{and} \quad C = \left(\frac{a}{u} : \frac{b}{v} : \frac{c}{w} \right).$$

For brevity define $M = au + bv + cw$ and $N = avw + bwu + cuv$.

We now compute each condition.

Claim — Quadrilateral $ABCD$ is cyclic if and only if $N^2 = u^2 M^2$.

Proof. We know a circle through B and D is a locus of points with

$$\frac{a^2 yz + b^2 zx + c^2 xy}{x(x + y + z)}$$

is equal to some constant. Therefore quadrilateral $ABCD$ is cyclic if and only if $\frac{abc \cdot N}{au \cdot M}$ is equal to $\frac{abc \cdot uvw \cdot M}{avw \cdot N}$ which rearranges to $N^2 = u^2 M^2$. \square

Claim — We have $PA = PC$ if and only if $N^2 = u^2 M^2$.

Proof. We have the displacement vector $\overrightarrow{PA} = \frac{1}{M}(bv + cw, -bv, -cw)$. Therefore,

$$\begin{aligned} M^2 \cdot |PA|^2 &= -a^2(bv)(cw) + b^2(cw)(bv + cw) + c^2(bv)(bv + cw) \\ &= bc(-a^2 vw + (bw + cv)(bv + cw)). \end{aligned}$$

In a similar way (by replacing u, v, w with their inverses) we have

$$\begin{aligned} \left(\frac{N}{uvw} \right)^2 \cdot |PC|^2 &= (vw)^{-2} \cdot bc(-a^2 vw + (bv + cw)(bw + cv)) \\ \iff N^2 \cdot |PC|^2 &= u^2 bc(-a^2 vw + (bw + cv)(bv + cw)) \end{aligned}$$

These are equal if and only if $N^2 = u^2 M^2$, as desired. \square

§6 IMO 2004/6

We call a positive integer *alternating* if every two consecutive digits in its decimal representation are of different parity. Find all positive integers n which have an alternating multiple.

If $20 \mid n$, then clearly n has no alternating multiple since the last two digits are both even. We will show the other values of n all work.

The construction is just rush-down do-it. The meat of the solution is the two following steps.

Claim (Tail construction) — For every even integer $w \geq 2$,

- there exists an even alternating multiple $g(w)$ of 2^{w+1} with exactly w digits, and
- there exists an even alternating multiple $h(w)$ of 5^w with exactly w digits.

(One might note this claim is implied by the problem, too.)

Proof. We prove the first point by induction on w . If $w = 2$, take $g(2) = 32$. In general, we can construct $g(w+2)$ from $g(w)$ by adding some element in

$$10^w \cdot \{10, 12, 14, 16, 18, 30, \dots, 98\}$$

to $g(w)$, corresponding to the digits we want to append to the start. This multiple is automatically divisible by 2^{w+1} , and also can take any of the four possible values modulo 2^{w+3} .

The second point is a similar induction, with base case $h(2) = 50$. The same set above consists of numbers divisible by 5^w , and covers all residues modulo 5^{w+2} . Careful readers might recognize the second part as essentially USAMO 2003/1. \square

Claim (Head construction) — If $\gcd(n, 10) = 1$, then for any b , there exists an even alternating number $f(b \bmod n)$ which is $b \pmod{n}$.

Proof. A standard argument shows that

$$10 \cdot \frac{100^m - 1}{99} = \underbrace{1010 \dots 10}_{m \text{ 10's}} \equiv 0 \pmod{n}$$

for any m divisible by $\varphi(99n)$. Take a very large such m , and then add on b distinct numbers of the form $10^{\varphi(n)r}$ for various even values of r ; these all are $1 \pmod{n}$ and change some of the 1's to 3's. \square

Now, we can solve the problem. Consider three cases:

- If $n = 2^k m$ where $\gcd(m, 10) = 1$ and $k \geq 2$ is even, then the concatenated number

$$10^k f\left(-\frac{g(k)}{10^k} \bmod m\right) + g(k)$$

works fine.

- If $n = 5^k m$ where $\gcd(m, 10) = 1$ and $k \geq 2$ is even, then the concatenated number

$$10^k f\left(-\frac{h(k)}{10^k} \bmod m\right) + h(k)$$

works fine.

- If $n = 50m$ where $\gcd(m, 10) = 1$, then the concatenated number

$$100f\left(-\frac{1}{2} \bmod m\right) + 50$$

works fine.

Since every non-multiple of 20 divides such a number, we are done.