IMO 2011 Solution Notes

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This is an compilation of solutions for the 2011 IMO. Some of the solutions are my own work, but many are from the official solutions provided by the organizers (for which they hold any copyrights), and others were found on the Art of Problem Solving forums.

Corrections and comments are welcome!

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§0 Problems

1. Given any set $A = \{a_1, a_2, a_3, a_4\}$ of four distinct positive integers, we denote the sum $a_1 + a_2 + a_3 + a_4$ by s_A. Let n_A denote the number of pairs (i, j) with $1 \le i < j \le 4$ for which $a_i + a_j$ divides s_A. Find all sets A of four distinct positive integers which achieve the largest possible value of n_A.

2. Let $\mathcal{S}$ be a finite set of at least two points in the plane. Assume that no three points of $\mathcal{S}$ are collinear. A *windmill* is a process that starts with a line ℓ going through a single point $P \in \mathcal{S}$. The line rotates clockwise about the *pivot* P until the first time that the line meets some other point belonging to $\mathcal{S}$. This point, Q, takes over as the new pivot, and the line now rotates clockwise about Q, until it next meets a point of $\mathcal{S}$. This process continues indefinitely.

   Show that we can choose a point P in $\mathcal{S}$ and a line ℓ going through P such that the resulting windmill uses each point of $\mathcal{S}$ as a pivot infinitely many times.

3. Let $f \colon \mathbb{R} \to \mathbb{R}$ be a real-valued function defined on the set of real numbers that satisfies
$$f(x + y) \le y f(x) + f(f(x))$$
   for all real numbers x and y. Prove that $f(x) = 0$ for all $x \le 0$.

4. Let $n > 0$ be an integer. We are given a balance and n weights of weight $2^0, 2^1, \dots, 2^{n-1}$. We are to place each of the n weights on the balance, one after another, in such a way that the right pan is never heavier than the left pan. At each step we choose one of the weights that has not yet been placed on the balance, and place it on either the left pan or the right pan, until all of the weights have been placed. Determine the number of ways in which this can be done.

5. Let $f \colon \mathbb{Z} \to \mathbb{Z}_{>0}$ be a function such that $f(m - n) \mid f(m) - f(n)$ for $m, n \in \mathbb{Z}$. Prove that if $m, n \in \mathbb{Z}$ satisfy $f(m) \le f(n)$ then $f(m) \mid f(n)$.

6. Let ABC be an acute triangle with circumcircle Γ. Let ℓ be a tangent line to Γ, and let ℓ_a, ℓ_b, ℓ_c be the lines obtained by reflecting ℓ in the lines BC, CA, and AB, respectively. Show that the circumcircle of the triangle determined by the lines ℓ_a, ℓ_b, and ℓ_c is tangent to the circle Γ.

§1 IMO 2011/1, proposed by Fernando Campos, MEX

Given any set $A = \{a_1, a_2, a_3, a_4\}$ of four distinct positive integers, we denote the sum $a_1 + a_2 + a_3 + a_4$ by s_A. Let n_A denote the number of pairs (i, j) with $1 \le i < j \le 4$ for which $a_i + a_j$ divides s_A. Find all sets A of four distinct positive integers which achieve the largest possible value of n_A.

There are two curves of solutions, namely $\{x, 5x, 7x, 11x\}$ and $\{x, 11x, 19x, 29x\}$, for any positive integer x, achieving $n_A = 4$ (easy to check). We'll show that $n_A \le 4$ and equality holds only in one of the curves.

Let $A = \{a < b < c < d\}$.

Claim — We have $n_A \le 4$ with equality iff

$$a + b \mid c + d, \quad a + c \mid b + d, \quad a + d = b + c.$$

Proof. Note $a + b \mid s_A \iff a + b \mid c + d$ etc. Now $c + d \nmid a + b$ and $b + d \nmid a + c$ for size reasons, so we already have $n_A \le 4$; moreover $a + d \mid b + c$ and $b + c \mid a + d$ if and only if $a + d = b + c$. □

We now show the equality curve is the one above.

$$a + c \mid b + d \iff a + c \mid -a + 2b + c \iff a + c \mid 2(b - a).$$

Since $a + c > |b - a|$, so we must have $a + c = 2(b - a)$. So we now have

$$\begin{aligned} c &= 2b - 3a \\ d &= b + c - a = 3b + c - 4a. \end{aligned}$$

The last condition is

$$a + b \mid c + d = 5b - 7a \iff a + b \mid 12a.$$

Now, let $x = \gcd(a, b)$. The expressions for c and d above imply that $x \mid c, d$ so we may scale down so that $x = 1$. Then $\gcd(a + b, a) = \gcd(a, b) = 1$ and so $a + b \mid 12$.

We have $c > b$, so $3a < b$. The only pairs (a, b) with $3a < 2b$, $\gcd(a, b) = 1$ and $a + b \mid 12$ are $(a, b) \in \{(1, 5), (1, 11)\}$ which give the solutions earlier.

§2 IMO 2011/2, proposed by Geoff Smith, UNK

Let $\mathcal{S}$ be a finite set of at least two points in the plane. Assume that no three points of $\mathcal{S}$ are collinear. A *windmill* is a process that starts with a line ℓ going through a single point $P \in \mathcal{S}$. The line rotates clockwise about the *pivot* P until the first time that the line meets some other point belonging to $\mathcal{S}$. This point, Q, takes over as the new pivot, and the line now rotates clockwise about Q, until it next meets a point of $\mathcal{S}$. This process continues indefinitely.

Show that we can choose a point P in $\mathcal{S}$ and a line ℓ going through P such that the resulting windmill uses each point of $\mathcal{S}$ as a pivot infinitely many times.

Orient ℓ in some direction, and color the plane such that its left half is red and right half is blue. The critical observation is that:

> **Claim —** The number of points on the red side of ℓ does not change, nor does the number of points on the blue side (except at a moment when ℓ contains two points).

Thus, if $|\mathcal{S}| = n + 1$, it suffices to pick the initial configuration so that there are $\lfloor n/2 \rfloor$ red and $\lceil n/2 \rceil$ blue points. Then when the line ℓ does a full 180° rotation, the red and blue sides "switch", so the windmill has passed through every point.

(See official shortlist for verbose write-up; this is deliberately short to make a point.)

§3 IMO 2011/3, proposed by Igor Voronovich, BLR

Let $f\colon \mathbb{R} \to \mathbb{R}$ be a real-valued function defined on the set of real numbers that satisfies

$$f(x+y) \le yf(x) + f(f(x))$$

for all real numbers x and y. Prove that $f(x) = 0$ for all $x \le 0$.

We begin by rewriting the given as

$$f(z) \le (z-x)f(x) + f(f(x)) \quad \forall x, z \in \mathbb{R} \qquad (\heartsuit)$$

(which is better anyways since control over inputs to f is more valuable). We start by eliminating the double f: let $z = f(w)$ to get

$$f(f(w)) \le (f(w) - x)f(x) + f(f(x))$$

and then use the *symmetry trick* to write

$$f(f(x)) \le (f(x) - w)f(w) + f(f(w))$$

so that when we sum we get

$$wf(w) + xf(x) \le 2f(x)f(w).$$

Next we use *cancellation trick*: set $w = 2f(x)$ in the above to get

$$xf(x) \le 0 \quad \forall x \in \mathbb{R}. \qquad (\spadesuit)$$

Claim — For every $p \in \mathbb{R}$, we have $f(p) \le 0$.

Proof. Assume $f(p) > 0$ for some $p \in \mathbb{R}$. Then for any negative number z,

$$0 \overset{(\spadesuit)}{\le} f(z) \overset{(\heartsuit)}{\le} (z-p)f(p) + f(f(p)).$$

which is false if we let $z \to -\infty$. $\square$

Together with $(\spadesuit)$ we derive $f(x) = 0$ for $x < 0$. Finally, letting x and z be any negative numbers in $(\heartsuit)$, we get $f(0) \ge 0$, so $f(0) = 0$ too.

Remark. As another corollary of the claim, $f(f(x)) = 0$ for all x.

Remark. A nontrivial example of a working f is to take

$$f(x) = \begin{cases} -\exp(\exp(\exp(x))) & x > 0 \\ 0 & x \le 0. \end{cases}$$

or some other negative function growing rapidly in absolute value for $x > 0$.

§4 IMO 2011/4, proposed by Morteza Saghafian (IRN)

Let $n > 0$ be an integer. We are given a balance and n weights of weight $2^0, 2^1, \dots, 2^{n-1}$. We are to place each of the n weights on the balance, one after another, in such a way that the right pan is never heavier than the left pan. At each step we choose one of the weights that has not yet been placed on the balance, and place it on either the left pan or the right pan, until all of the weights have been placed. Determine the number of ways in which this can be done.

The answer is $a_n = (2n-1)!!$. We refer to what we're counting as a *valid n-sequence*: an order of which weights to place, and whether to place them on the left or right pan.

We use induction, with $n = 1$ being obvious. Now consider the weight $2^0 = 1$.

- If we delete it from any valid n-sequence, we get a valid $(n-1)$-sequence with all weights doubled.
- Given a valid $(n-1)$-sequence with all weights doubled, we may insert $2^0 = 1$ it into $2n - 1$ ways. Indeed, we may insert it anywhere, and designate it either left or right, except we may not designate right if we choose to insert $2^0 = 1$ at the very beginning.

Consequently, we have that

$$a_n = (2n-1) \cdot a_{n-1}.$$

Since $a_1 = 1$, the conclusion follows.

§5 IMO 2011/5, proposed by Mahyar Sefidgaran, IRN

Let $f \colon \mathbb{Z} \to \mathbb{Z}_{>0}$ be a function such that $f(m-n) \mid f(m) - f(n)$ for $m, n \in \mathbb{Z}$. Prove that if $m, n \in \mathbb{Z}$ satisfy $f(m) \le f(n)$ then $f(m) \mid f(n)$.

Let $P(m, n)$ denote the given assertion. First, we claim f is even. This is straight calculation:

- $P(x, 0) \implies f(x) \mid f(x) - f(0) \implies f(x) \mid M := f(0)$.
- $P(0, x) \implies f(-x) \mid M - f(x) \implies f(-x) \mid f(x)$. Analogously, $f(x) \mid f(-x)$. So $f(x) = f(-x)$ and f is even.

Claim — Let x, y, z be integers with $x + y + z = 0$. Then among $f(x)$, $f(y)$, $f(z)$, two of them are equal and divide the third.

Proof. Let $a = f(\pm x)$, $b = f(\pm y)$, $c = f(\pm z)$ be positive integers. Note that

$$\begin{aligned} a &\mid b - c \\ b &\mid c - a \\ c &\mid a - b \end{aligned}$$

from $P(y, -z)$ and similarly. WLOG $c = \max(a, b, c)$; then $c > |a - b|$ so $a = b$. Thus $a = b \mid c$ from the first two. $\square$

This implies the problem, by taking x and y in the previous claim to be the integers m and n.

Remark. At https://aops.com/community/c6h418981p2381909, Davi Medeiros gives the following characterization of functions f satisfying the hypothesis.

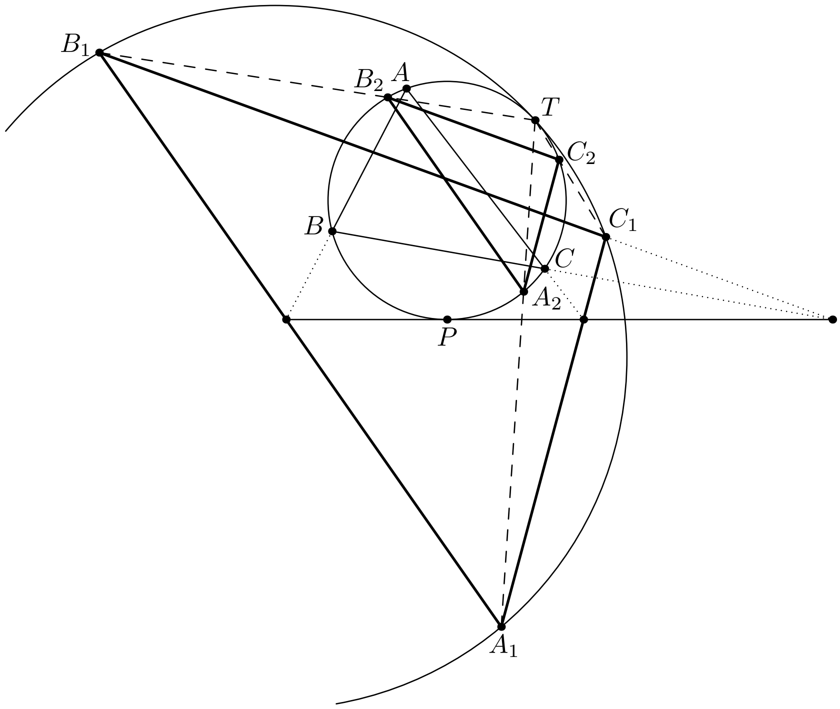
Pick $f(0)$, k positive integers, a chain $d_1 \mid d_2 \mid \dots \mid d_k$ of divisors of $f(0)$, and positive integers $a_1, a_2, \dots, a_{k-1}$, greater than 1 (if $k = 1$, a_i doesn't exist, for every i). We'll define f as follows:

- $f(n) = d_1$, for every integer n that is not divisible by a_1;
- $f(a_1 n) = d_2$, for every integer n that is not divisible by a_2;
- $f(a_1 a_2 n) = d_3$, for every integer n that is not divisible by a_3;
- $f(a_1 a_2 a_3 n) = d_4$, for every integer n that is not divisible by a_4;
- ...
- $f(a_1 a_2 \dots a_{k-1} n) = d_k$, for every integer n;

§6 IMO 2011/6, proposed by Japan

Let ABC be an acute triangle with circumcircle Γ. Let ℓ be a tangent line to Γ, and let ℓ_a, ℓ_b, ℓ_c be the lines obtained by reflecting ℓ in the lines BC, CA, and AB, respectively. Show that the circumcircle of the triangle determined by the lines ℓ_a, ℓ_b, and ℓ_c is tangent to the circle Γ.

This is a hard problem with many beautiful solutions. The following solution is not very beautiful but not too hard to find during an olympiad, as the only major insight it requires is the construction of A_2, B_2, and C_2.



We apply complex numbers with ω the unit circle and $p = 1$. Let $A_1 = \ell_B \cap \ell_C$, and let $a_2 = a^2$ (in other words, A_2 is the reflection of P across the diameter of ω through A). Define the points B_1, C_1, B_2, C_2 similarly.

We claim that $\overline{A_1A_2}$, $\overline{B_1B_2}$, $\overline{C_1C_2}$ concur at a point on Γ.

We begin by finding A_1. If we reflect the points $1+i$ and $1-i$ over $\overline{AB}$, then we get two points Z_1, Z_2 with

$$\begin{aligned} z_1 &= a + b - ab(1-i) = a + b - ab + abi \\ z_2 &= a + b - ab(1+i) = a + b - ab - abi. \end{aligned}$$

Therefore,

$$\begin{aligned} z_1 - z_2 &= 2abi \\ \overline{z_1}z_2 - \overline{z_2}z_1 &= -2i\left(a + b + \frac{1}{a} + \frac{1}{b} - 2\right). \end{aligned}$$

Now ℓ_C is the line $\overline{Z_1Z_2}$, so with the analogous equation ℓ_B we obtain:

$$\begin{aligned} a_1 &= \frac{-2i\left(a+b+\frac{1}{a}+\frac{1}{b}-2\right)(2aci)+2i\left(a+c+\frac{1}{a}+\frac{1}{c}-2\right)(2abi)}{\left(-\frac{2}{ab}i\right)(2aci)-\left(-\frac{2}{ac}i\right)(2abi)} \\ &= \frac{[c-b]\,a^2+\left[\frac{c}{b}-\frac{b}{c}-2c+2b\right]a+(c-b)}{\frac{c}{b}-\frac{b}{c}} \\ &= a+\frac{(c-b)\left[a^2-2a+1\right]}{(c-b)(c+b)/bc} \\ &= a+\frac{bc}{b+c}(a-1)^2. \end{aligned}$$

Then the second intersection of $\overline{A_1A_2}$ with ω is given by

$$\begin{aligned} \frac{a_1-a_2}{1-a_2\overline{a_1}} &= \frac{a+\frac{bc}{b+c}(a-1)^2-a^2}{1-a-a^2\cdot\frac{(1-1/a)^2}{b+c}} \\ &= \frac{a+\frac{bc}{b+c}(1-a)}{1-\frac{1}{b+c}(1-a)} \\ &= \frac{ab+bc+ca-abc}{a+b+c-1}. \end{aligned}$$

Thus, the claim is proved.

Finally, it suffices to show $\overline{A_1B_1} \parallel \overline{A_2B_2}$. One can also do this with complex numbers; it amounts to showing a^2-b^2, $a-b$, i (corresponding to $\overline{A_2B_2}$, $\overline{A_1B_1}$, $\overline{PP}$) have their arguments an arithmetic progression, equivalently

$$\frac{(a-b)^2}{i(a^2-b^2)} \in \mathbb{R} \iff \frac{(a-b)^2}{i(a^2-b^2)} = \frac{\left(\frac{1}{a}-\frac{1}{b}\right)^2}{\frac{1}{i}\left(\frac{1}{a^2}-\frac{1}{b^2}\right)}$$

which is obvious.

Remark. One can use directed angle chasing for this last part too. Let $\overline{BC}$ meet ℓ at K and $\overline{B_2C_2}$ meet ℓ at L. Evidently

$$\begin{aligned} -\measuredangle B_2LP &= \measuredangle LPB_2 + \measuredangle PB_2L \\ &= 2\measuredangle KPB + \measuredangle PB_2C_2 \\ &= 2\measuredangle KPB + 2\measuredangle PBC \\ &= -2\measuredangle PKB \\ &= \measuredangle PKB_1 \end{aligned}$$

as required.