IMO 2005 Solution Notes

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This is an compilation of solutions for the 2005 IMO. Some of the solutions are my own work, but many are from the official solutions provided by the organizers (for which they hold any copyrights), and others were found on the Art of Problem Solving forums.

Corrections and comments are welcome!

Contents

§0 Problems

1. Six points are chosen on the sides of an equilateral triangle ABC: A_1, A_2 on BC, B_1, B_2 on CA and C_1, C_2 on AB, such that they are the vertices of a convex hexagon $A_1A_2B_1B_2C_1C_2$ with equal side lengths. Prove that the lines A_1B_2, B_1C_2 and C_1A_2 are concurrent.

2. Let a_1, a_2, $\ldots$ be a sequence of integers with infinitely many positive and negative terms. Suppose that for every positive integer n the numbers a_1, a_2, $\ldots$, a_n leave n different remainders upon division by n. Prove that every integer occurs exactly once in the sequence.

3. Let $x, y, z > 0$ satisfy $xyz \geq 1$. Prove that

$$\frac{x^5 - x^2}{x^5 + y^2 + z^2} + \frac{y^5 - y^2}{x^2 + y^5 + z^2} + \frac{z^5 - z^2}{x^2 + y^2 + z^5} \geq 0.$$

4. Determine all positive integers relatively prime to all the terms of the infinite sequence
$$a_n = 2^n + 3^n + 6^n - 1, \quad n \geq 1.$$

5. Let $ABCD$ be a fixed convex quadrilateral with $BC = DA$ and $\overline{BC} \nparallel \overline{DA}$. Let two variable points E and F lie on the sides BC and DA, respectively, and satisfy $BE = DF$. The lines AC and BD meet at P, the lines BD and EF meet at Q, the lines EF and AC meet at R. Prove that the circumcircles of the triangles PQR, as E and F vary, have a common point other than P.

6. In a mathematical competition 6 problems were posed to the contestants. Each pair of problems was solved by more than $\frac{2}{5}$ of the contestants. Nobody solved all 6 problems. Show that there were at least 2 contestants who each solved exactly 5 problems.

§1 IMO 2005/1

Six points are chosen on the sides of an equilateral triangle ABC: A_1, A_2 on BC, B_1, B_2 on CA and C_1, C_2 on AB, such that they are the vertices of a convex hexagon $A_1A_2B_1B_2C_1C_2$ with equal side lengths. Prove that the lines A_1B_2, B_1C_2 and C_1A_2 are concurrent.
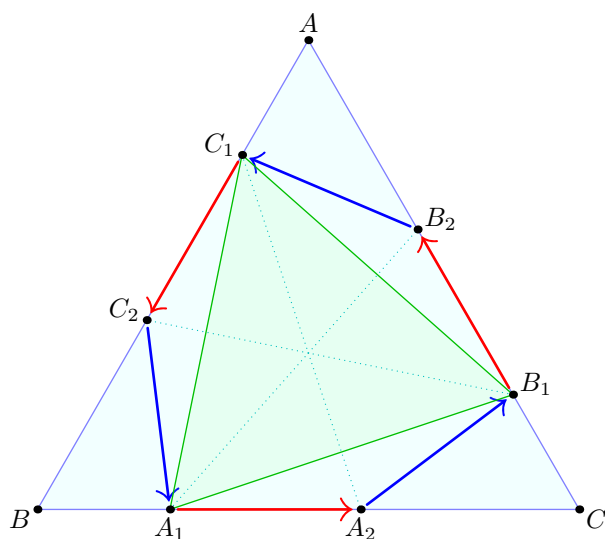

The six sides of the hexagon, when oriented, comprise six vectors with vanishing sum. However note that

$$\overrightarrow{A_1A_2} + \overrightarrow{B_1B_2} + \overrightarrow{C_1C_2} = 0.$$

Thus

$$\overrightarrow{A_2B_1} + \overrightarrow{B_2C_1} + \overrightarrow{C_2A_1} = 0$$

and since three unit vectors with vanishing sum must be rotations of each other by $120°$, it follows they must also form an equilateral triangle.

Consequently, triangles $A_1A_2B_1$, $B_1B_2C_1$, $C_1C_2A_1$ are congruent, as $\angle A_2 = \angle B_2 = \angle C_2$. So triangle $A_1B_1C_1$ is equilateral and the diagonals are concurrent at the center.

§2 IMO 2005/2

Let $a_1, a_2, \ldots$ be a sequence of integers with infinitely many positive and negative terms. Suppose that for every positive integer n the numbers $a_1, a_2, \ldots, a_n$ leave n different remainders upon division by n. Prove that every integer occurs exactly once in the sequence.

Obviously every integer appears at most once (otherwise take n much larger). So we will prove every integer appears at least once.

> **Claim** — For any $i < j$ we have $|a_i - a_j| < j$.

Proof. Otherwise, let $n = |a_i - a_j| \neq 0$. Then $i, j \in [1, n]$ and $a_i \equiv a_j \pmod{n}$, contradiction. $\square$

> **Claim** — For any n, the set $\{a_1, \ldots, a_n\}$ is of the form $\{k+1, \ldots, k+n\}$ for some integer k.

Proof. By induction, with the base case $n = 1$ being vacuous. For the inductive step, suppose $\{a_1, \ldots, a_n\} = \{k+1, \ldots, k+n\}$ are determined. Then

$$a_{n+1} \equiv k \pmod{n+1}.$$

Moreover by the earlier claim we have

$$|a_{n+1} - a_1| < n + 1.$$

From this we deduce $a_{n+1} \in \{k, k+n+1\}$ as desired. $\square$

This gives us actually a complete description of all possible sequences satisfying the hypothesis: choose any value of a_1 to start. Then, for the nth term, the set $S = \{a_1, \ldots, a_{n-1}\}$ is (in some order) a set of $n - 1$ consecutive integers. We then let $a_n = \max S + 1$ or $a_n = \min S - 1$. A picture of six possible starting terms is shown below.
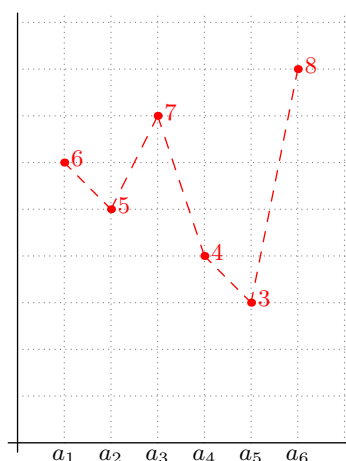
Finally, we observe that the condition that the sequence has infinitely many positive and negative terms (which we have not used until now) implies it is unbounded above and below. Thus it must contain every integer.

§3 IMO 2005/3

Let $x, y, z > 0$ satisfy $xyz \geq 1$. Prove that

$$\frac{x^5 - x^2}{x^5 + y^2 + z^2} + \frac{y^5 - y^2}{x^2 + y^5 + z^2} + \frac{z^5 - z^2}{x^2 + y^2 + z^5} \geq 0.$$

Negating both sides and adding 3 eliminates the minus signs:

$$\sum_{\text{cyc}} \frac{1}{x^5 + y^2 + z^2} \leq \frac{3}{x^2 + y^2 + z^2}.$$

Thus we only need to consider the case $xyz = 1$.

Direct expansion and Muirhead works now! As advertised, once we show it suffices to analyze if $xyz = 1$ the inequality becomes more economically written as

$$S = \sum_{\text{cyc}} x^2(x^2 - yz)(y^4 + x^3z + xz^3)(z^4 + x^3y + xy^3) \overset{?}{\geq} 0.$$

So, clearing all the denominators gives

$$S = \sum_{\text{cyc}} x^2(x^2 - yz)\left[y^4z^4 + x^3y^5 + xy^7 + x^3z^5 + x^6yz + x^4y^3z + xz^7 + x^4yz^3 + x^2y^3z^3\right]$$

$$= \sum_{\text{cyc}}\left[x^4y^4z^4 + x^7y^5 + x^5y^7 + x^7z^5 + x^{10}yz + x^8y^3z + x^5z^7 + x^8yz^3 + x^6y^3z^3\right]$$

$$- \sum_{\text{cyc}}\left[x^2y^5z^5 + x^5y^6z + x^3y^8z + x^5yz^6 + x^8y^2z^2 + x^6y^4z^2 + x^3yz^8 + x^6y^2z^4 + x^4y^4z^4\right]$$

$$= \sum_{\text{cyc}}\left[x^7y^5 + x^5y^7 + x^7z^5 + x^{10}yz + x^5z^7 + x^6y^3z^3\right]$$

$$- \sum_{\text{cyc}}\left[x^2y^5z^5 + x^5y^6z + x^5yz^6 + x^8y^2z^2 + x^6y^4z^2 + x^6y^2z^4\right]$$

In other words we need to show

$$\sum_{\text{sym}}\left(2x^7y^5 + \frac{1}{2}x^{10}yz + \frac{1}{2}x^6y^3z^3\right) \geq \sum_{\text{sym}}\left(\frac{1}{2}x^8y^2z^2 + \frac{1}{2}x^5y^5z^2 + x^6y^4z^2 + x^6y^5z\right).$$

which follows by summing

$$\sum_{\text{sym}} \frac{x^{10}yz + x^6y^3z^3}{2} \geq \sum_{\text{sym}} x^8y^2z^2$$

$$\frac{1}{2}\sum_{\text{sym}} x^8y^2z^2 \geq \frac{1}{2}\sum_{\text{sym}} x^6y^4z^2$$

$$\frac{1}{2}\sum_{\text{sym}} x^7y^5 \geq \frac{1}{2}\sum_{\text{sym}} x^5y^5z^2$$

$$\frac{1}{2}\sum_{\text{sym}} x^7y^5 \geq \frac{1}{2}\sum_{\text{sym}} x^6y^4z^2$$

$$\sum_{\text{sym}} x^7y^5 \geq \sum_{\text{sym}} x^6y^5z.$$

The first line here comes from AM-GM, the rest come from Muirhead.

Remark. More elegant approach is to use Cauchy in the form

$$\frac{1}{x^5 + y^2 + z^2} \le \frac{x^{-1} + y^2 + z^2}{(x^2 + y^2 + z^2)^2}.$$

§4 IMO 2005/4

Determine all positive integers relatively prime to all the terms of the infinite sequence

$$a_n = 2^n + 3^n + 6^n - 1, \quad n \geq 1.$$

The answer is 1 only (which works).

It suffices to show there are no primes. For the primes $p = 2$ and $p = 3$, take $a_2 = 48$. For any prime $p \geq 5$ notice that

$$
\begin{aligned}
a_{p-2} &= 2^{p-2} + 3^{p-2} + 6^{p-2} - 1 \\
&\equiv \frac{1}{2} + \frac{1}{3} + \frac{1}{6} - 1 \pmod{p} \\
&\equiv 0 \pmod{p}
\end{aligned}
$$

so no other larger prime works.

§5 IMO 2005/5

Let $ABCD$ be a fixed convex quadrilateral with $BC = DA$ and $\overline{BC} \nparallel \overline{DA}$. Let two variable points E and F lie on the sides BC and DA, respectively, and satisfy $BE = DF$. The lines AC and BD meet at P, the lines BD and EF meet at Q, the lines EF and AC meet at R. Prove that the circumcircles of the triangles PQR, as E and F vary, have a common point other than P.

Let M be the Miquel point of complete quadrilateral $ADBC$; in other words, let M be the second intersection point of the circumcircles of $\triangle APD$ and $\triangle BPC$. (A good diagram should betray this secret; all the points are given in the picture.) This makes lots of sense since we know E and F will be sent to each other under the spiral similarity too.
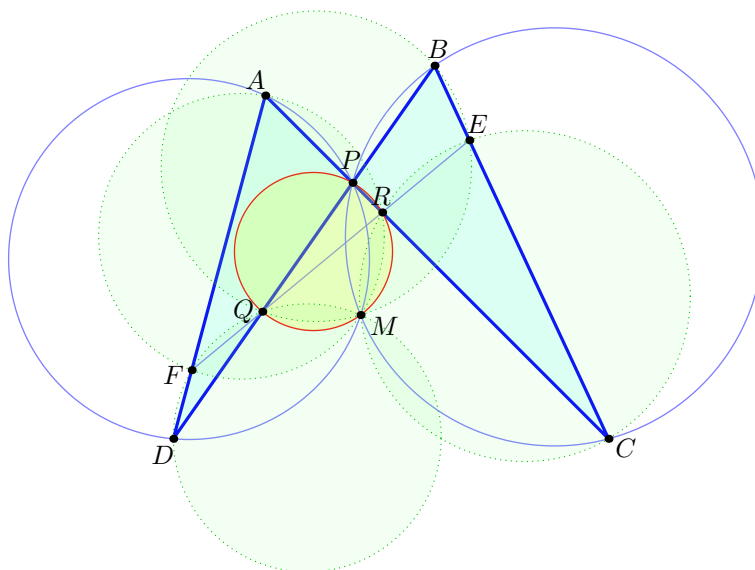
Thus M is the Miquel point of complete quadrilateral $FACE$. As $R = \overline{FE} \cap \overline{AC}$ we deduce $FARM$ is a cyclic quadrilateral (among many others, but we'll only need one).

Now look at complete quadrilateral $AFQP$. Since M lies on (DFQ) and (RAF), it follows that M is in fact the Miquel point of $AFQP$ as well. So M lies on (PQR).

Thus M is the fixed point that we wanted.

Remark. Naturally, the congruent length condition can be relaxed to $DF/DA = BE/BC$.

§6 IMO 2005/6, proposed by Radu Gologan and Dan Schwartz

In a mathematical competition 6 problems were posed to the contestants. Each pair of problems was solved by more than $\frac{2}{5}$ of the contestants. Nobody solved all 6 problems. Show that there were at least 2 contestants who each solved exactly 5 problems.

Assume not and at most one contestant solved five problems. By adding in solves, we can assume WLOG that one contestant solved problems one through five, and every other contestant solved four of the six problems.

We split the remaining contestants based on whether they solved P6. Let a_i denote the number of contestants who solved $\{1, 2, \ldots, 5\} \setminus \{i\}$ (and missed P6). Let b_{ij} denote the number of contestants who solved $\{1, 2, \ldots, 5, 6\} \setminus \{i, j\}$, for $1 \le i < j \le 5$ (thus in particular they solved P6). Thus

$$n = 1 + \sum_{1 \le i \le 5} a_i + \sum_{1 \le i < j \le 5} b_{ij}$$

denotes the total number of contestants.

Considering contestants who solved P1/P6 we have

$$t_1 := b_{23} + b_{24} + b_{25} + b_{34} + b_{35} + b_{45} \ge \frac{2}{5}n + \frac{1}{5}$$

and we similarly define t_2, t_3, t_4, t_5. (We have written $\frac{2}{5}n + \frac{1}{5}$ since we know the left-hand side is an integer strictly larger than $\frac{2}{5}n$.) Also, by considering contestants who solved P1/P2 we have

$$t_{12} = 1 + a_3 + a_4 + a_5 + b_{34} + b_{35} + b_{45} \ge \frac{2}{5}n + \frac{1}{5}$$

and we similarly define t_{ij} for $1 \le i < j \le 5$.

> **Claim** — The number $\frac{2n+1}{5}$ is equal to some integer k, fourteen of the t's are equal to k, and the last one is equal to $k + 1$.

Proof. First, summing all fifteen equations gives

$$6n + 4 = 10 + 6(n - 1) = 10 + \sum_{1 \le i \le 5} 6a_i + \sum_{1 \le i < j \le 5} 6b_{ij}$$
$$= \sum_{1 \le i \le 5} t_i + \sum_{1 \le i < j \le 5} t_{ij}.$$

Thus the sum of the 15 t's is $6n + 4$. But since all the t's are integers at least $\frac{2n+1}{5} = \frac{6n+3}{15}$, the conclusion follows. $\square$

However, we will also manipulate the equations to get the following.

> **Claim** — We have
> $$t_{45} \equiv 1 + t_1 + t_2 + t_3 + t_{12} + t_{23} + t_{31} \pmod{3}.$$

Proof. This follows directly by computing the coefficient of the a's and b's. We will nonetheless write out a derivation of this equation, to motivate it, but the proof stands without it.

Let $B = \sum_{1 \leq i < j \leq 5} b_{ij}$ be the sum of all b's. First, note that

$$
\begin{aligned}
t_1 + t_2 &= B + b_{34} + b_{45} + b_{35} - b_{12} \\
&= B + (t_{12} - 1 - a_3 - a_4 - a_5) - b_{12} \\
\implies b_{12} &= B - (t_1 + t_2) + t_{12} - 1 - (a_3 + a_4 + a_5).
\end{aligned}
$$

This means we have more or less solved for each b_{ij} in terms of only t and a variables. Now

$$
\begin{aligned}
t_{45} &= 1 + a_1 + a_2 + a_3 + b_{12} + b_{23} + b_{31} \\
&= 1 + a_1 + a_2 + a_3 \\
&\quad + [B - (t_1 + t_2) + t_{12} - 1 - (a_3 + a_4 + a_5)] \\
&\quad + [B - (t_2 + t_3) + t_{23} - 1 - (a_1 + a_4 + a_5)] \\
&\quad + [B - (t_3 + t_1) + t_{13} - 1 - (a_2 + a_4 + a_5)] \\
&\equiv 1 + t_1 + t_2 + t_3 + t_{12} + t_{23} + t_{31} \pmod{3}
\end{aligned}
$$

as desired. $\qquad\square$

However, we now show the two claims are incompatible (and this is easy, many ways to do this). There are two cases.

- Say $t_5 = k + 1$ and the others are k. Then the equation for t_{45} gives that $k \equiv 6k + 1$ (mod 3). But now the equation for t_{12} give $k \equiv 6k$ (mod 3).

- Say $t_{45} = k + 1$ and the others are k. Then the equation for t_{45} gives that $k + 1 \equiv 6k$ (mod 3). But now the equation for t_{12} give $k \equiv 6k + 1$ (mod 3).

> **Remark.** It is significantly easier to prove that there is at least one contestant who solved five problems. One can see it by dropping the $+10$ in the proof of the claim, and arrives at a contradiction. In this situation it is not even necessary to set up the many a and b variables; just note that the expected number of contestants solving any particular pair of problems is $\frac{\binom{4}{2}n}{\binom{6}{2}} = \frac{2}{5}n$.
>
> The fact that $\frac{2n+1}{5}$ should be an integer also follows quickly, since if not one can improve the bound to $\frac{2n+2}{5}$ and quickly run into a contradiction. Again one can get here without setting up a and b.
>
> The main difficulty seems to be the precision required in order to nail down the second 5-problem solve.

> **Remark.** The second claim may look miraculous, but the proof shows that it is not too unnatural to consider $t_1 + t_2 - t_{12}$ to isolate b_{12} in terms of a's and t's. The main trick is: why mod 3?
>
> The reason is that if one looks closely, for a fixed k we have a system of 15 equations in 15 variables. Unless the determinant D of that system happens to be zero, this means there will be a rational solution in a and b, whose denominators are bounded by D. However if $p \mid D$ then we may conceivably run into mod p issues.
>
> This motivates the choice $p = 3$, since it is easy to see the determinant is divisible by 3, since constant shifts of $\vec{a}$ and $\vec{b}$ are also solutions mod 3. (The choice $p = 2$ is a possible

guess as well for this reason, but the problem seems to have better 3-symmetry.)