IMO 2002 Solution Notes

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This is an compilation of solutions for the 2002 IMO. Some of the solutions are my own work, but many are from the official solutions provided by the organizers (for which they hold any copyrights), and others were found on the Art of Problem Solving forums.

Corrections and comments are welcome!

Contents

§0 Problems

1. Let n be a positive integer. Let T be the set of points (x, y) in the plane where x and y are non-negative integers with $x + y < n$. Each point of T is coloured red or blue, subject to the following condition: if a point (x, y) is red, then so are all points (x', y') of T with $x' \le x$ and $y' \le y$. Let A be the number of ways to choose n blue points with distinct x-coordinates, and let B be the number of ways to choose n blue points with distinct y-coordinates. Prove that $A = B$.

2. Let BC be a diameter of circle ω with center O. Let A be a point of circle ω such that $0^\circ < \angle AOB < 120^\circ$. Let D be the midpoint of arc AB not containing C. Line ℓ passes through O and is parallel to line AD. Line ℓ intersects line AC at J. The perpendicular bisector of segment OA intersects circle ω at E and F. Prove that J is the incenter of triangle CEF.

3. Find all pairs of positive integers $m, n \ge 3$ for which there exist infinitely many positive integers a such that
$$\frac{a^m + a - 1}{a^n + a^2 - 1}$$
is itself an integer.

4. Let $n \ge 2$ be a positive integer with divisors $1 = d_1 < d_2 < \dots < d_k = n$. Prove that $d_1 d_2 + d_2 d_3 + \dots + d_{k-1} d_k$ is always less than n^2, and determine when it is a divisor of n^2.

5. Find all functions $f \colon \mathbb{R} \to \mathbb{R}$ such that
$$(f(x) + f(z))\,(f(y) + f(t)) = f(xy - zt) + f(xt + yz)$$
for all real numbers x, y, z, t.

6. Let $n \ge 3$ be a positive integer. Let C_1, C_2, $\dots$, C_n be unit circles in the plane, with centers O_1, O_2, $\dots$, O_n respectively. If no line meets more than two of the circles, prove that
$$\sum_{1 \le i < j \le n} \frac{1}{O_i O_j} \le \frac{(n-1)\pi}{4}.$$

§1 IMO 2002/1

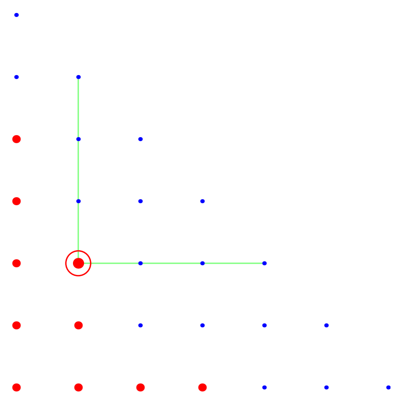
Let n be a positive integer. Let T be the set of points (x, y) in the plane where x and y are non-negative integers with $x + y < n$. Each point of T is coloured red or blue, subject to the following condition: if a point (x, y) is red, then so are all points (x', y') of T with $x' \le x$ and $y' \le y$. Let A be the number of ways to choose n blue points with distinct x-coordinates, and let B be the number of ways to choose n blue points with distinct y-coordinates. Prove that $A = B$.

Let a_x denote the number of blue points with a given x-coordinate. Define b_y to be the number of blue points with a given y-coordinate.

We actually claim that

Claim — The multisets $\mathcal{A} := \{a_x \mid x\}$ and $\mathcal{B} := \{b_y \mid y\}$ are equal.

Proof. By induction on the number of red points. If there are no red points at all, then $\mathcal{A} = \mathcal{B} = \{1, \dots, n\}$.



The proof consists of two main steps. First, suppose we color a single point $P = (x, y)$ from blue to red (while preserving the condition). Before the coloring, we have $a_x = b_y = n - (x + y)$; afterwards $a_x = b_y = n - (x + y) - 1$ and no other numbers change, as desired.

We also must show that this operation (repeatedly adding a single point P) reaches all possible shapes of red points. This is well-known as the red points form a Young tableaux; for example, one way is to add all the points with $x = 0$ first one by one, then all the points with $x = 1$, and so on. So the induction implies the result. $\square$

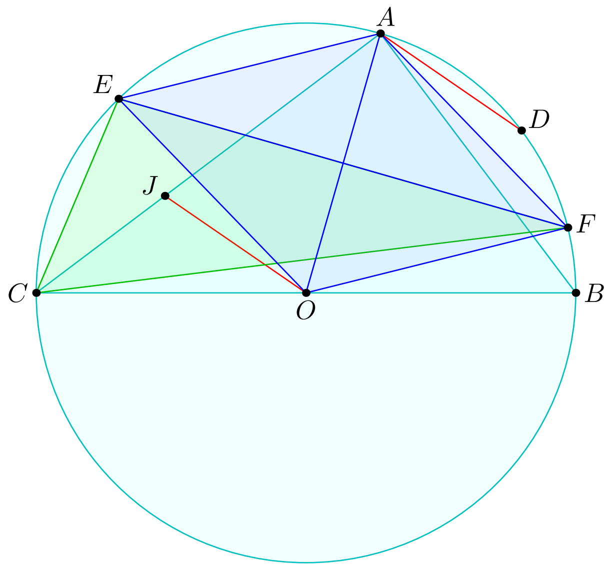
Finally,

$$A = \prod_{x=0}^{n-1} a_x = \prod_{y=0}^{n-1} b_y = B.$$

§2 IMO 2002/2

Let BC be a diameter of circle ω with center O. Let A be a point of circle ω such that $0^\circ < \angle AOB < 120^\circ$. Let D be the midpoint of arc AB not containing C. Line ℓ passes through O and is parallel to line AD. Line ℓ intersects line AC at J. The perpendicular bisector of segment OA intersects circle ω at E and F. Prove that J is the incenter of triangle CEF.

By construction, $AEOF$ is a rhombus with 60°-120° angles. Consequently, we may set $s = AO = AE = AF = EO = EF$.



Claim — We have $AJ = s$ too.

Proof. It suffices to show $AJ = AO$ which is angle chasing. Let $\theta = \angle BOD = \angle DOA$, so $\angle BOA = 2\theta$. Thus $\angle CAO = \frac{1}{2}\angle BOA = \theta$. However $\angle AOJ = \angle OAD = 90^\circ - \frac{1}{2}\theta$, as desired. □

Then, since $AE = AJ = AF$, we are done by the infamous Fact 5.

§3 IMO 2002/3, proposed by Laurentiu Panaitopol (ROM)

Find all pairs of positive integers $m, n \geq 3$ for which there exist infinitely many positive integers a such that

$$\frac{a^m + a - 1}{a^n + a^2 - 1}$$

is itself an integer.

The condition is equivalent to $a^n + a^2 - 1$ dividing $a^m + a - 1$ as polynomials. The big step is the following analytic one.

Claim — We must have $m \leq 2n$.

Proof. Assume on contrary $m > 2n$ and let $0 < r < 1$ be the unique real number with $r^n + r^2 = 1$, hence $r^m + r = 1$. But now

$$\begin{aligned} 0 = r^m + r - 1 &< r(r^n)^2 + r - 1 = r\left((1-r^2)^2 + 1\right) - 1 \\ &= -(1-r)\left(r^4 + r^3 - r^2 - r + 1\right). \end{aligned}$$

As $1 - r > 0$ and $r^4 + r^3 - r^2 - r + 1 > 0$, this is a contradiction □

Now for the algebraic part. Obviously $m > n$.

$$\begin{aligned} & a^n + a^2 - 1 \mid a^m + a - 1 \\ \iff & a^n + a^2 - 1 \mid (a^m + a - 1)(a + 1) = a^m(a+1) + (a^2 - 1) \\ \iff & a^n + a^2 - 1 \mid a^m(a+1) - a^n \\ \iff & a^n + a^2 - 1 \mid a^{m-n}(a+1) - 1. \end{aligned}$$

The right-hand side has degree $m - n + 1 \leq n + 1$, and the leading coefficients are both $+1$. So the only possible situations are

$$\begin{aligned} a^{m-n}(a+1) - 1 &= (a+1)\left(a^n + a^2 - 1\right) \\ a^{m-n}(a+1) + 1 &= a^n + a^2 - 1. \end{aligned}$$

The former fails by just taking $a = -1$; the latter implies $(m, n) = (5, 3)$. As our work was reversible, this also implies $(m, n) = (5, 3)$ works, done.

§4 IMO 2002/4

Let $n \geq 2$ be a positive integer with divisors $1 = d_1 < d_2 < \dots < d_k = n$. Prove that $d_1 d_2 + d_2 d_3 + \dots + d_{k-1} d_k$ is always less than n^2, and determine when it is a divisor of n^2.

We always have

$$\begin{aligned} d_k d_{k-1} + d_{k-1} d_{k-2} + \dots + d_2 d_1 &< n \cdot \frac{n}{2} + \frac{n}{2} \cdot \frac{n}{3} + \dots \\ &= \left(\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \dots \right) n^2 = n^2. \end{aligned}$$

This proves the first part.

For the second, we claim that this only happens when n is prime (in which case we get $d_1 d_2 = n$). Assume n is not prime (equivalently $k \geq 2$) and let p be the smallest prime dividing n. Then

$$d_k d_{k-1} + d_{k-1} d_{k-2} + \dots + d_2 d_1 > d_k d_{k-1} = \frac{n^2}{p}$$

exceeds the largest proper divisor of n^2, but is less than n^2, so does not divide n^2.

§5 IMO 2002/5

Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$(f(x)+f(z))\,(f(y)+f(t)) = f(xy-zt)+f(xt+yz)$$

for all real numbers x, y, z, t.

The answer is $f(x) \equiv 0$, $f(x) \equiv 1/2$ and $f(x) \equiv x^2$ which are easily seen to work. Let's prove they are the only ones; we show two solutions.

First solution (multiplicativity) Let $P(x,y,z,t)$ denote the given statement.

- By comparing $P(x,1,0,0)$ and $P(0,0,1,x)$ we get $\boxed{f \text{ even}}$.
- By $P(0,y,0,t)$ we get for nonconstant f that $f(0)=0$. If f is constant we get the solutions earlier, so in the sequel assume $\boxed{f(0)=0}$.
- By $P(x,y,0,0)$ we get $\boxed{f(xy)=f(x)f(y)}$. Note in particular that for any real number x we now have

  $$f(x) = f(|x|) = f\left(\sqrt{|x|}\right)^2 \ge 0$$

  that is, $f \ge 0$.

From $P(x,y,y,x)$ we now have

$$f(x^2+y^2) = (f(x)+f(y))^2 = f(x^2)+2f(x)f(y)+f(y^2) \ge f(x^2)$$

so f is weakly increasing. Combined with f multiplicative and nonconstant, this implies $f(x)=|x|^r$ for some real number r.

Finally, $P(1,1,1,1)$ gives $f(2)=4f(1)$, so $f(x) \equiv x^2$.

Second solution (ELMO) Let $P(x,y,z,t)$ denote the statement. Assume f is nonconstant, as before we derive that f is even, $f(0)=0$, and $f(x) \ge 0$ for all x.

Now comparing $P(x,y,z,t)$ and $P(z,y,x,t)$ we obtain

$$f(xy-zt)+f(xt+yz) = (f(x)+f(z))\,(f(y)+f(t)) = f(xy+zt)+f(xt-yz)$$

which in particular implies that

$$f(a-d)+f(b+c) = f(a+d)+f(b-c) \qquad \text{if } ad=bc \text{ and } a,b,c,d>0.$$

Thus the restriction of f to $(0,\infty)$ satisfies **ELMO 2011, problem 4** which implies that $f(x)=kx^2+\ell$ for constants k and ℓ. From here we recover the original.

(Minor note: technically ELMO 2011/4 is $f\colon (0,\infty)\to(0,\infty)$ but we only have $f \ge 0$, however the proof for the ELMO problem works as long as f is bounded below; we could also just apply the ELMO problem to $f+0.01$ instead.)

§6 IMO 2002/6

Let $n \geq 3$ be a positive integer. Let C_1, C_2, $\dots$, C_n be unit circles in the plane, with centers O_1, O_2, $\dots$, O_n respectively. If no line meets more than two of the circles, prove that

$$\sum_{1 \leq i < j \leq n} \frac{1}{O_i O_j} \leq \frac{(n-1)\pi}{4}.$$

For brevity, let d_{ij} be the length of O_{ij} and let $\angle(ijk)$ be shorthand for $\angle O_i O_j O_k$ (or its measure in radians).

First, we eliminate the circles completely and reduce the problem to angles using the following fact (which is in part motivated by the mysterious presence of π on right-hand side, and also brings d_{ij}^{-1} into the picture).

Lemma

For any indices i, j, m we have the inequalities

$$\angle(imj) \geq \max\left(\frac{2}{d_{mi}}, \frac{2}{d_{mj}}\right) \quad \text{and} \quad \pi - \angle(imj) \geq \max\left(\frac{2}{d_{mi}}, \frac{2}{d_{mj}}\right).$$

Proof. We first prove the former line. Consider the altitude from O_i to $O_m O_j$. The altitude must have length at least 2, otherwise its perpendicular bisector passes intersects all of C_i , C_m, C_j. Thus

$$2 \leq d_{mi} \sin \angle(imj) \leq \angle(imj)$$

proving the first line. The second line follows by considering the external angle formed by lines $O_m O_i$ and $O_m O_j$ instead of the internal one. $\square$

Our idea now is for any index m we will make an estimate on $\sum_{\substack{1 \leq i \leq n \\ i \neq b}} \frac{1}{d_{bi}}$ for each index b. If the centers formed a convex polygon, this would be much simpler, but because we do not have this assumption some more care is needed.

Claim — Suppose O_a, O_b, O_c are consecutive vertices of the convex hull. Then

$$\frac{n-1}{n-2} \measuredangle(abc) \geq \frac{2}{d_{1b}} + \frac{2}{d_{2b}} + \dots + \frac{2}{d_{nb}}$$

where the term $\frac{2}{d_{bb}}$ does not appear (obviously).

Proof. WLOG let's suppose $(a, b, c) = (2, 1, n)$ and that rotating ray $O_2 O_1$ hits O_3, O_4,

..., O_n in that order. We have

$$\begin{aligned}
\frac{2}{d_{12}} &\le \angle(213) \\
\frac{2}{d_{13}} &\le \min\left\{\angle(213), \angle(314)\right\} \\
\frac{2}{d_{14}} &\le \min\left\{\angle(314), \angle(415)\right\} \\
&\vdots \\
\frac{2}{d_{1(n-1)}} &\le \min\left\{\angle((n-2)1(n-1)), \angle((n-1)1n)\right\} \\
\frac{2}{d_{1n}} &\le \angle\left((n-1)1n\right).
\end{aligned}$$

Of the $n-1$ distinct angles appearing on the right-hand side, we let κ denote the smallest of them. We have $\kappa \le \frac{1}{n-2}\angle(21n)$ by pigeonhole principle. Then we pick the minimums on the right-hand side in the unique way such that summing gives

$$\begin{aligned}
\sum_{i=2}^{n} \frac{2}{d_{1i}} &\ge \left(\angle(213) + \angle(314) + \dots + \angle((n-1)1n)\right) + \kappa \\
&\ge \angle(21n) + \frac{1}{n-2}\angle(21n) = \frac{n-1}{n-2}\angle(21n)
\end{aligned}$$

as desired. $\square$

Next we show a bound that works for any center, even if it does not lie on the convex hull $\mathcal{H}$.

Claim — For any index b we have

$$\frac{n-1}{n-2}\pi \ge \frac{2}{d_{1b}} + \frac{2}{d_{2b}} + \dots + \frac{2}{d_{nb}}$$

where the term $\frac{2}{d_{bb}}$ does not appear (obviously).

Proof. This is the same argument as in the previous proof, with the modification that because O_b could lie inside the convex hull now, our rotation argument should use lines instead of rays (in order for the angle to be π rather than 2π). This is why the first lemma is stated with two cases; we need it now.

Again WLOG $b = 1$. Consider line O_1O_2 (rather than just the ray) and imagine rotating it counterclockwise through O_2; suppose that this line passes through O_3, O_4, ..., O_n in that order before returning to O_2 again. We let $\measuredangle(i1j) \in \{\angle(i1j), \pi - \angle(i1j)\} \in [0, \pi)$ be the counterclockwise rotations obtained in this way, so that

$$\measuredangle(21n) = \measuredangle(213) + \measuredangle(314) + + \dots + \measuredangle((n-1)1n).$$

(This is not "directed angles", but related.)

Then we get bounds

$$\begin{aligned}
\frac{2}{d_{12}} &\le \measuredangle(213) \\
\frac{2}{d_{13}} &\le \min\left\{\measuredangle(213), \measuredangle(314)\right\} \\
&\vdots \\
\frac{2}{d_{1(n-1)}} &\le \min\left\{\measuredangle((n-2)1(n-1)), \measuredangle((n-1)1n)\right\} \\
\frac{2}{d_{1n}} &\le \measuredangle\left\{(n-1)1n\right\}
\end{aligned}$$

as in the last proof, and so as before we get

$$\sum_{i=1}^{n} \frac{2}{d_{1i}} \le \frac{n-1}{n-2} \measuredangle(21n)$$

which is certainly less than $\frac{n-1}{n-2}\pi$. $\square$

Now suppose there were r vertices in the convex hull. If we sum the first claim across all b on the hull, and the second across all b not on the hull (inside it), we get

$$\begin{aligned}
\sum_{1 \le i < j \le n} \frac{2}{d_{ij}} &= \frac{1}{2} \sum_{b} \sum_{i \ne b} \frac{2}{d_{bi}} \\
&\le \frac{1}{2} \cdot \frac{n-1}{n-2} \left((r-2)\pi + (n-r)\pi\right) \\
&= \frac{(n-1)\pi}{4}
\end{aligned}$$

as needed (with $(r-2)\pi$ being the sum of angles in the hull).