IMO 2016 Solution Notes

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This is an compilation of solutions for the 2016 IMO. Some of the solutions are my own work, but many are from the official solutions provided by the organizers (for which they hold any copyrights), and others were found on the Art of Problem Solving forums.

Corrections and comments are welcome!

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§0 Problems

1. In convex pentagon $ABCDE$ with $\angle B > 90°$, let F be a point on $\overline{AC}$ such that $\angle FBC = 90°$. It is given that $FA = FB$, $DA = DC$, $EA = ED$, and rays $\overline{AC}$ and $\overline{AD}$ trisect $\angle BAE$. Let M be the midpoint of $\overline{CF}$. Let X be the point such that $AMXE$ is a parallelogram. Show that $\overline{FX}$, $\overline{EM}$, $\overline{BD}$ are concurrent.

2. Find all integers n for which each cell of $n \times n$ table can be filled with one of the letters I, M and O in such a way that:

   - In each row and column, one third of the entries are I, one third are M and one third are O; and

   - in any diagonal, if the number of entries on the diagonal is a multiple of three, then one third of the entries are I, one third are M and one third are O.

   Note that an $n \times n$ table has $4n - 2$ diagonals.

3. Let $P = A_1 A_2 \cdots A_k$ be a convex polygon in the plane. The vertices $A_1, A_2, \ldots, A_k$ have integral coordinates and lie on a circle. Let S be the area of P. An odd positive integer n is given such that the squares of the side lengths of P are integers divisible by n. Prove that $2S$ is an integer divisible by n.

4. A set of positive integers is called *fragrant* if it contains at least two elements and each of its elements has a prime factor in common with at least one of the other elements. Let $P(n) = n^2 + n + 1$. What is the smallest possible positive integer value of b such that there exists a non-negative integer a for which the set

$$\{P(a + 1), P(a + 2), \ldots, P(a + b)\}$$

   is fragrant?

5. The equation

$$(x - 1)(x - 2) \cdots (x - 2016) = (x - 1)(x - 2) \cdots (x - 2016)$$

   is written on the board, with 2016 linear factors on each side. What is the least possible value of k for which it is possible to erase exactly k of these 4032 linear factors so that at least one factor remains on each side and the resulting equation has no real solutions?

6. There are $n \geq 2$ line segments in the plane such that every two segments cross and no three segments meet at a point. Geoff has to choose an endpoint of each segment and place a frog on it facing the other endpoint. Then he will clap his hands $n - 1$ times. Every time he claps, each frog will immediately jump forward to the next intersection point on its segment. Frogs never change the direction of their jumps. Geoff wishes to place the frogs in such a way that no two of them will ever occupy the same intersection point at the same time.

   (a) Prove that Geoff can always fulfill his wish if n is odd.
   (b) Prove that Geoff can never fulfill his wish if n is even.

§1 IMO 2016/1, proposed by Art Waeterschoot (BEL)

In convex pentagon $ABCDE$ with $\angle B > 90°$, let F be a point on $\overline{AC}$ such that $\angle FBC = 90°$. It is given that $FA = FB$, $DA = DC$, $EA = ED$, and rays $\overline{AC}$ and $\overline{AD}$ trisect $\angle BAE$. Let M be the midpoint of $\overline{CF}$. Let X be the point such that $AMXE$ is a parallelogram. Show that $\overline{FX}, \overline{EM}, \overline{BD}$ are concurrent.

Here is a "long" solution which I think shows where the "power" in the configuration comes from (it should be possible to come up with shorter solutions by cutting more directly to the desired conclusion). Throughout the proof, we let

$$\theta = \angle FAB = \angle FBA = \angle DAC = \angle DCA = \angle EAD = \angle EDA.$$

We begin by focusing just on $ABCD$ with point F, ignoring for now the points E and X (and to some extent even point M). It turns out this is a very familiar configuration.

> **Lemma** (Central lemma)
>
> The points F and C are the incenter and A-excenter of $\triangle DAB$. Moreover, $\triangle DAB$ is isosceles with $DA = DB$.

Proof. The proof uses three observations:

- We already know that $\overline{FAC}$ is the angle bisector of $\angle ABD$.

- We were given $\angle FBC = 90°$.

- Next, note that $\triangle AFB \sim \triangle ADC$ (they are similar isosceles triangles). From this it follows that $AF \cdot AC = AB \cdot AD$.

These three facts, together with F lying inside $\triangle ABD$, are enough to imply the result. □
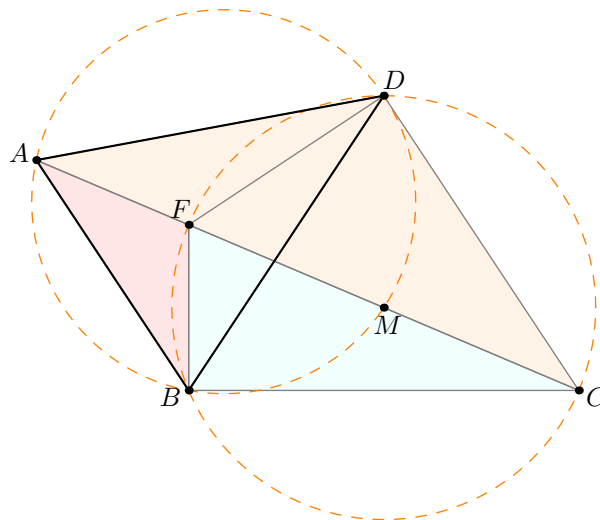
> **Corollary**
>
> The point M is the midpoint of arc $\overset{\frown}{BD}$ of (DAB), and the center of cyclic quadrilateral $FDCB$.

Proof. Fact 5.   □

Using these observations as the anchor for everything that follows, we now prove several claims about X and E in succession.

> **Claim —** Point E is the midpoint of arc $\widehat{AD}$ in $(ABMD)$, and hence lies on ray BF.

Proof. This follows from $\angle EDA = \theta = \angle EBA$.   □

> **Claim —** Points X is the second intersection of ray $\overline{ED}$ with $(BFDC)$.

Proof. First, $\overline{ED} \parallel \overline{AC}$ already since $\angle AED = 180° - 2\theta$ and $\angle CAE = 2\theta$.
  Now since $DB = DA$, we get $MB = MD = ED = EA$. Thus, $MX = AE = MB$, so X also lies on the circle $(BFDC)$ centered at M.   □

> **Claim —** The quadrilateral $EXMF$ is an isosceles trapezoid.

Proof. We already know $\overline{EX} \parallel \overline{FM}$. Since $\angle EFA = 180° - \angle AFB = 2\theta = \angle FAE$, we have $EF = EA$ as well (and $F \neq A$). As $EXMA$ was a parallelogram, it follows $EXMF$ is an isosceles trapezoid.   □

The problem then follows by radical axis theorem on the three circles $(AEDMB)$, $(BFDXC)$ and $(EXMF)$.

§2 IMO 2016/2, proposed by Trevor Tau (AUS)

Find all integers n for which each cell of $n \times n$ table can be filled with one of the letters I, M and O in such a way that:

- In each row and column, one third of the entries are I, one third are M and one third are O; and

- in any diagonal, if the number of entries on the diagonal is a multiple of three, then one third of the entries are I, one third are M and one third are O.

Note that an $n \times n$ table has $4n - 2$ diagonals.

The answer is n divisible by 9.

First we construct $n = 9$ and by extension every multiple of 9.

I	I	I	M	M	M	O	O	O
M	M	M	O	O	O	I	I	I
O	O	O	I	I	I	M	M	M
I	I	I	M	M	M	O	O	O
M	M	M	O	O	O	I	I	I
O	O	O	I	I	I	M	M	M
I	I	I	M	M	M	O	O	O
M	M	M	O	O	O	I	I	I
O	O	O	I	I	I	M	M	M

We now prove $9 \mid n$ is necessary.

Let $n = 3k$, which divides the given grid into k^2 sub-boxes (of size 3×3 each). We say a multiset of squares S is *clean* if the letters distribute equally among them; note that unions of clean multisets are clean.

Consider the following clean sets (given to us by problem statement):

- All columns indexed 2 (mod 3),

- All rows indexed 2 (mod 3), and

- All $4k - 2$ diagonals mentioned in the problem.

Take their union. This covers the center of each box four times, and every other cell exactly once. We conclude the set of k^2 center squares are clean, hence $3 \mid k^2$ and so $9 \mid n$, as desired.

Shown below is the sums over all diagonals only, and of the entire union.

1		1	1		1	1		1
	2			2			2	
1		1	1		1	1		1
1		1	1		1	1		1
	2			2			2	
1		1	1		1	1		1
1		1	1		1	1		1
	2			2			2	
1		1	1		1	1		1

1	1	1	1	1	1	1	1	1
1	4	1	1	4	1	1	4	1
1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1
1	4	1	1	4	1	1	4	1
1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1
1	4	1	1	4	1	1	4	1
1	1	1	1	1	1	1	1	1

§3 IMO 2016/3, proposed by Russia

Let $P = A_1 A_2 \cdots A_k$ be a convex polygon in the plane. The vertices $A_1, A_2, \ldots, A_k$ have integral coordinates and lie on a circle. Let S be the area of P. An odd positive integer n is given such that the squares of the side lengths of P are integers divisible by n. Prove that $2S$ is an integer divisible by n.

Solution by Jeck Lim: We will prove the result just for $n = p^e$ where p is an odd prime and $e \geq 1$. The case $k = 3$ is resolved by Heron's formula directly: we have $S = \frac{1}{4}\sqrt{2(a^2b^2 + b^2c^2 + c^2a^2) - a^4 - b^4 - c^4}$, so if $p^e \mid \gcd(a^2, b^2, c^2)$ then $p^{2e} \mid S^2$.

Now we show we can pick a diagonal and induct down on k by using inversion.

Let the polygon be $A_1 A_2 \ldots A_{k+1}$ and suppose for contradiction that all sides are divisible by p^e but no diagonals are. Let $O = A_{k+1}$ for notational convenience. By applying inversion around O with radius 1, we get the "generalized Ptolemy theorem"

$$\frac{A_1 A_2}{OA_1 \cdot OA_2} + \frac{A_2 A_3}{OA_2 \cdot OA_3} + \cdots + \frac{A_{k-1} A_k}{OA_{k-1} \cdot OA_k} = \frac{A_1 A_k}{OA_1 \cdot OA_k}$$

or, making use of square roots,

$$\sqrt{\frac{A_1 A_2^2}{OA_1^2 \cdot OA_2^2}} + \sqrt{\frac{A_2 A_3^2}{OA_2^2 \cdot OA_3^2}} + \cdots + \sqrt{\frac{A_{k-1} A_k^2}{OA_{k-1}^2 \cdot OA_k^2}} = \sqrt{\frac{A_1 A_k^2}{OA_1^2 \cdot OA_k^2}}$$

Suppose ν_p of all diagonals is strictly less than e. Then the relation becomes

$$\sqrt{q_1} + \cdots + \sqrt{q_{k-1}} = \sqrt{q}$$

where q_i are positive rational numbers. Since there are no nontrivial relations between square roots (see this link) there is a positive rational number b such that $r_i = \sqrt{q_i/b}$ and $r = \sqrt{q/b}$ are all rational numbers. Then

$$\sum r_i = r.$$

However, the condition implies that $\nu_p(q_i^2) > \nu_p(q^2)$ for all i (check this for $i = 1$, $i = k-1$ and $2 \leq i \leq k-2$), and hence $\nu_p(r_i) > \nu_p(r)$. This is absurd.

> **Remark.** I think you basically have to use some Ptolemy-like geometric property, and also all correct solutions I know of $n = p^e$ depend on finding a diagonal and inducting down. (Actually, the case $k = 4$ is pretty motivating; Ptolemy implies one can cut in two.)

§4 IMO 2016/4, proposed by Luxembourg

A set of positive integers is called *fragrant* if it contains at least two elements and each of its elements has a prime factor in common with at least one of the other elements. Let $P(n) = n^2 + n + 1$. What is the smallest possible positive integer value of b such that there exists a non-negative integer a for which the set

$$\{P(a+1), P(a+2), \dots, P(a+b)\}$$

is fragrant?

The answer is $b = 6$.

First, we prove $b \geq 6$ must hold. It is not hard to prove the following divisibilities by Euclid:

$$\gcd(P(n), P(n+1)) \mid 1$$
$$\gcd(P(n), P(n+2)) \mid 7$$
$$\gcd(P(n), P(n+3)) \mid 3$$
$$\gcd(P(n), P(n+4)) \mid 19.$$

Now assume for contradiction $b \leq 5$. Then any GCD's among $P(a+1)$, …, $P(a+b)$ must be among $\{3, 7, 19\}$. Consider a multi-graph on $\{a+1, \dots, a+b\}$ where we join two elements with nontrivial GCD and label the edge with the corresponding prime. Then we readily see there is at most one edge each of $\{3, 7, 19\}$: id est at most one edge of gap 2, 3, 4 (and no edges of gap 1). (By the gap of an edge $e = \{u, v\}$ we mean $|u - v|$.) But one can see that it's now impossible for every vertex to have nonzero degree, contradiction.

To construct $b = 6$ we use the Chinese remainder theorem: select a with

$$a + 1 \equiv 7 \pmod{19}$$
$$a + 5 \equiv 11 \pmod{19}$$
$$a + 2 \equiv 2 \pmod{7}$$
$$a + 4 \equiv 4 \pmod{7}$$
$$a + 3 \equiv 1 \pmod{3}$$
$$a + 6 \equiv 1 \pmod{3}$$

which does the trick.

§5 IMO 2016/5, proposed by Russia

The equation
$$(x-1)(x-2)\cdots(x-2016) = (x-1)(x-2)\cdots(x-2016)$$
is written on the board, with 2016 linear factors on each side. What is the least possible value of k for which it is possible to erase exactly k of these 4032 linear factors so that at least one factor remains on each side and the resulting equation has no real solutions?

The answer is 2016. Obviously this is necessary in order to delete duplicated factors. We now prove it suffices to deleted 2 (mod 4) and 3 (mod 4) guys from the left-hand side, and 0 (mod 4), 1 (mod 4) from the right-hand side.

Consider the 1008 inequalities
$$(x-1)(x-4) < (x-2)(x-3)$$
$$(x-5)(x-8) < (x-6)(x-7)$$
$$(x-9)(x-12) < (x-10)(x-11)$$
$$\vdots$$
$$(x-2013)(x-2016) < (x-2014)(x-2015).$$

Notice that in all these inequalities, at most one of them has non-positive numbers in it, and we never have both zero. If there is exactly one negative term among the $1008 \cdot 2 = 2016$ sides, it is on the left and we can multiply all together. Thus the only case that remains is if $x \in (4m-2, 4m-1)$ for some m, say the mth inequality. In that case, the two sides of that inequality differ by a factor of at least 9.

> **Claim** — We have
> $$\prod_{k \geq 0} \frac{(4k+2)(4k+3)}{(4k+1)(4k+4)} < e.$$

Proof of claim using logarithms. To see this, note that it's equivalent to prove
$$\sum_{k \geq 0} \log\left(1 + \frac{2}{(4k+1)(4k+4)}\right) < 1.$$

To this end, we use the deep fact that $\log(1+t) \leq t$, and thus it follows from $\sum_{k \geq 0} \frac{1}{(4k+1)(4k+4)} < \frac{1}{2}$, which one can obtain for example by noticing it's less than $\frac{1}{4}\frac{\pi^2}{6}$. $\square$

Elementary proof of claim, given by Espen Slettnes. For each $N \geq 0$, the partial product is bounded by
$$\prod_{k=0}^{N} \frac{(4k+2)(4k+3)}{(4k+1)(4k+4)} = \frac{2}{1} \cdot \left(\frac{3}{4} \cdot \frac{6}{5}\right) \cdot \left(\frac{7}{8} \cdot \frac{10}{9}\right) \cdot \cdots \cdot \frac{4N+3}{4N+4}$$
$$< 2 \cdot 1 \cdot 1 \cdot \cdots \cdot \frac{4N+3}{4N+4} < 2 < e. \qquad \square$$

This solves the problem, because then the factors being multiplied on by the positive inequalities before the mth one are both less than e, and $e^2 < 9$. In symbols, for $4m-2 < x < 4m-1$ we should have
$$\frac{(x-(4m-6))(x-(4m-5))}{(x-(4m-7))(x-(4m-4))} \times \cdots \times \frac{(x-2)(x-3)}{(x-1)(x-4)} < e$$

and

$$\frac{(x - (4m + 2))(x - (4m + 3))}{(x - (4m + 1))(x - (4m + 4))} \times \cdots \times \frac{(x - 2014)(x - 2015)}{(x - 2013)(x - 2016)} < e$$

because the $(k + 1)$st term of each left-hand side is at most $\frac{(4k+2)(4k+3)}{(4k+1)(4k+4)}$, for $k \geq 0$. As $e^2 < 9$, we're okay.

§6 IMO 2016/6, proposed by Josef Tkadlec (CZE)

There are $n \geq 2$ line segments in the plane such that every two segments cross and no three segments meet at a point. Geoff has to choose an endpoint of each segment and place a frog on it facing the other endpoint. Then he will clap his hands $n - 1$ times. Every time he claps, each frog will immediately jump forward to the next intersection point on its segment. Frogs never change the direction of their jumps. Geoff wishes to place the frogs in such a way that no two of them will ever occupy the same intersection point at the same time.

(a) Prove that Geoff can always fulfill his wish if n is odd.

(b) Prove that Geoff can never fulfill his wish if n is even.

The following solution was communicated to me by Yang Liu.

Imagine taking a larger circle ω encasing all $\binom{n}{2}$ intersection points. Denote by P_1, P_2, ..., P_{2n} the order of the points on ω in clockwise order; we imagine placing the frogs on P_i instead. Observe that, in order for every pair of segments to meet, each line segment must be of the form $P_i P_{i+n}$.
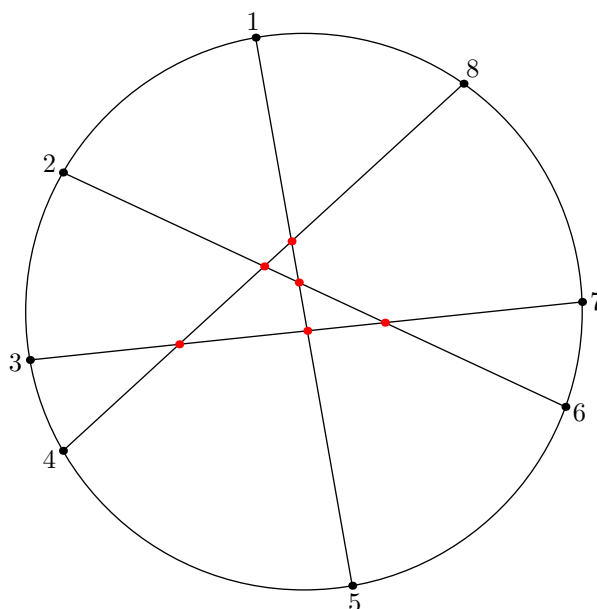
Then:

(a) Place the frogs on P_1, P_3, ..., P_{2n-1}. A simple parity arguments shows this works.

(b) Observe that we cannot place frogs on consecutive P_i, so the n frogs must be placed on alternating points. But since we also are supposed to not place frogs on diametrically opposite points, for even n we immediately get a contradiction.

> **Remark.** Yang says: this is easy to guess if you just do a few small cases and notice that the pairs of "violating points" just forms a large cycle around.