USAMO 2017 Solution Notes

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This is an compilation of solutions for the 2017 USAMO. Some of the solutions are my own work, but many are from the official solutions provided by the organizers (for which they hold any copyrights), and others were found on the Art of Problem Solving forums.

Corrections and comments are welcome!

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§0 Problems

1. Prove that there exist infinitely many pairs of relatively prime positive integers $a, b > 1$ for which $a + b$ divides $a^b + b^a$.

2. Let $m_1, m_2, \ldots, m_n$ be a collection of n positive integers, not necessarily distinct. For any sequence of integers $A = (a_1, \ldots, a_n)$ and any permutation $w = w_1, \ldots, w_n$ of $m_1, \ldots, m_n$, define an A-inversion of w to be a pair of entries w_i, w_j with $i < j$ for which one of the following conditions holds:

- $a_i \geq w_i > w_j$,

- $w_j > a_i \geq w_i$, or

- $w_i > w_j > a_i$.

Show that, for any two sequences of integers $A = (a_1, \ldots, a_n)$ and $B = (b_1, \ldots, b_n)$, and for any positive integer k, the number of permutations of $m_1, \ldots, m_n$ having exactly k A-inversions is equal to the number of permutations of $m_1, \ldots, m_n$ having exactly k B-inversions.

3. Let ABC be a scalene triangle with circumcircle Ω and incenter I. Ray AI meets $\overline{BC}$ at D and Ω again at M; the circle with diameter $\overline{DM}$ cuts Ω again at K. Lines MK and BC meet at S, and N is the midpoint of $\overline{IS}$. The circumcircles of $\triangle KID$ and $\triangle MAN$ intersect at points L_1 and L_2. Prove that Ω passes through the midpoint of either $\overline{IL_1}$ or $\overline{IL_2}$.

4. Let $P_1, P_2, \ldots, P_{2n}$ be $2n$ distinct points on the unit circle $x^2 + y^2 = 1$, other than $(1, 0)$. Each point is colored either red or blue, with exactly n red points and n blue points. Let $R_1, R_2, \ldots, R_n$ be any ordering of the red points. Let B_1 be the nearest blue point to R_1 traveling counterclockwise around the circle starting from R_1. Then let B_2 be the nearest of the remaining blue points to R_2 travelling counterclockwise around the circle from R_2, and so on, until we have labeled all of the blue points $B_1, \ldots, B_n$. Show that the number of counterclockwise arcs of the form $R_i \to B_i$ that contain the point $(1, 0)$ is independent of the way we chose the ordering $R_1, \ldots, R_n$ of the red points.

5. Find all real numbers $c > 0$ such that there exists a labeling of the lattice points in $\mathbb{Z}^2$ with positive integers for which:

- only finitely many distinct labels occur, and

- for each label i, the distance between any two points labeled i is at least c^i.

6. Find the minimum possible value of

$$\frac{a}{b^3 + 4} + \frac{b}{c^3 + 4} + \frac{c}{d^3 + 4} + \frac{d}{a^3 + 4}$$

given that a, b, c, d are nonnegative real numbers such that $a + b + c + d = 4$.

§1 USAMO 2017/1, proposed by Gregory Galperin

Prove that there exist infinitely many pairs of relatively prime positive integers $a, b > 1$ for which $a + b$ divides $a^b + b^a$.

One construction: let $d \equiv 1 \pmod 4$, $d > 1$. Let $x = \frac{d^d + 2^d}{d+2}$. Then set

$$a = \frac{x+d}{2}, \qquad b = \frac{x-d}{2}.$$

To see this works, first check that b is odd and a is even. Let $d = a - b$ be odd. Then:

$$
\begin{aligned}
a + b \mid a^b + b^a &\iff (-b)^b + b^a \equiv 0 \pmod{a+b} \\
&\iff b^{a-b} \equiv 1 \pmod{a+b} \\
&\iff b^d \equiv 1 \pmod{d+2b} \\
&\iff (-2)^d \equiv d^d \pmod{d+2b} \\
&\iff d + 2b \mid d^d + 2^d.
\end{aligned}
$$

So it would be enough that

$$d + 2b = \frac{d^d + 2^d}{d+2} \implies b = \frac{1}{2}\left(\frac{d^d + 2^d}{d+2} - d\right)$$

which is what we constructed. Also, since $\gcd(x, d) = 1$ it follows $\gcd(a, b) = \gcd(d, b) = 1$.

> **Remark.** Ryan Kim points out that in fact, $(a, b) = (2n - 1, 2n + 1)$ is always a solution.

§2 USAMO 2017/2, proposed by Maria Monks

Let $m_1, m_2, \ldots, m_n$ be a collection of n positive integers, not necessarily distinct. For any sequence of integers $A = (a_1, \ldots, a_n)$ and any permutation $w = w_1, \ldots, w_n$ of $m_1, \ldots, m_n$, define an A-inversion of w to be a pair of entries w_i, w_j with $i < j$ for which one of the following conditions holds:

- $a_i \geq w_i > w_j$,

- $w_j > a_i \geq w_i$, or

- $w_i > w_j > a_i$.

Show that, for any two sequences of integers $A = (a_1, \ldots, a_n)$ and $B = (b_1, \ldots, b_n)$, and for any positive integer k, the number of permutations of $m_1, \ldots, m_n$ having exactly k A-inversions is equal to the number of permutations of $m_1, \ldots, m_n$ having exactly k B-inversions.

The following solution was posted by Michael Ren, and I think it is the most natural one (since it captures all the combinatorial ideas using a q-generating function that is easier to think about, and thus makes the problem essentially a long computation).

Denote by M our multiset of n positive integers. Define an *inversion* of a permutation to be pair $i < j$ with $w_i < w_j$ (which is a $(0, \ldots, 0)$-inversion in the problem statement); this is the usual definition (see https://en.wikipedia.org/wiki/Inversion_(discrete_mathematics)). So we want to show the number of A-inversions is equal to the number of usual inversions. In what follows we count permutations on M with multiplicity: so $M = \{1, 1, 2\}$ still has $3! = 6$ permutations.

We are going to do what is essentially recursion, but using generating functions in a variable q to do our book-keeping. (Motivation: there's no good closed form for the number of inversions, but there's a great generating function known — which is even better for us, since we're only trying to show two numbers are equal!) First, we prove two claims.

> **Claim —** For any positive integer n, the generating function for the number of permutations of $(1, 2, \ldots, n)$ with exactly k inversions is
> $$n!_q \coloneqq 1 \cdot (1 + q) \cdot (1 + q + q^2) \cdot \ldots (1 + q + \cdots + q^{n-1}).$$

Here we mean that the coefficient of q^s above gives the number of permutations with exactly s inversions.

Proof. This is an induction on n, with $n = 1$ being trivial. Suppose we choose the first element to be i, with $1 \leq i \leq n$. Then there will always be exactly $i - 1$ inversions using the first element, so this contributes $q^i \cdot (n-1)!_q$. Summing $1 \leq i \leq n$ gives the result. $\square$

Unfortunately, the main difficulty of the problem is that there are repeated elements, which makes our notation much more horrific.

Let us define the following. We take our given multiset M of n positive integers, we suppose the distinct numbers are $\theta_1 < \theta_2 < \cdots < \theta_m$. We let e_i be the number of times θ_i appears. Therefore the multiplicities e_i should have sums

$$e_1 + \cdots + e_m = n$$

and m denotes the number of distinct elements. Finally, we let

$$F(e_1, \ldots, e_m) = \sum_{\text{permutations } \sigma} q^{\text{number inversions of } \sigma}$$

be the associated generating function for the number of inversions. For example, the first claim we proved says that $F(1, \ldots, 1) = n!_q$.

> **Claim** — We have the explicit formula
>
> $$F(e_1, \ldots, e_m) = n!_q \cdot \prod_{i=1}^{m} \frac{e_i!}{e_i!_q}.$$

Proof. First suppose we perturb all the elements slightly, so that they are no longer equal. Then the generating function would just be $n!_q$.

Then, we undo the perturbations for each group, one at a time, and claim that we get the above $e_i!_q$ factor each time. Indeed, put the permutations into classes of $e_1!$ each where permutations in the same classes differ only in the order of the perturbed θ_1's (with the other $n - e_1$ elements being fixed). Then there is a factor of $e_1!_q$ from each class, owing to the slightly perturbed inversions we added within each class. So we remove that factor and add $e_1! \cdot q^0$ instead. This accounts for the first term of the product.

Repeating this now with each term of the product implies the claim. $\qquad\square$

Thus we have the formula for the number of inversions in general. We wish to show this also equals the generating function the number of A-inversions, for any fixed choice of A. This will be an induction by n, with the base case being immediate.

For the inductive step, fix A, and assume the first element satisfies $\theta_k \leq a_1 < \theta_{k+1}$ (so $0 \leq k \leq m$; we for convenience set $\theta_0 = -\infty$ and $\theta_m = +\infty$). We count the permutations based on what the first element θ_i of the permutation is. Then:

- Consider permutations starting with $\theta_i \in \{\theta_1, \ldots, \theta_k\}$. Then the number of inversions which will use this first term is $(e_1 + \cdots + e_{i-1}) + (e_{k+1} + \cdots + e_m)$. Also, there are e_i ways to pick which θ_i gets used as the first term. So we get a contribution of

  $$q^{e_1 + \cdots + e_{i-1} + (e_{k+1} + \cdots + e_m)} \cdot e_i \cdot F(e_1, \ldots, e_i - 1, \ldots, e_m)$$

  in this case (with inductive hypothesis to get the last F-term).

- Now suppose $\theta_i \in \{\theta_{k+1}, \ldots, \theta_m\}$. Then the number of inversions which will use this first term is $e_{k+1} + \cdots + e_{i-1}$. Thus by a similar argument the contribution is

  $$q^{e_{k+1} + \cdots + e_{i-1}} \cdot e_i \cdot F(e_1, \ldots, e_i - 1, \ldots, e_m).$$

Therefore, to complete the problem it suffices to prove

$$\sum_{i=1}^{k} q^{(e_1 + \cdots + e_{i-1}) + (e_{k+1} + \cdots + e_m)} \cdot e_i \cdot F(e_1, \ldots, e_i - 1, \ldots, e_m)$$

$$+ \sum_{i=k+1}^{m} q^{e_{k+1} + \cdots + e_{i-1}} \cdot e_i \cdot F(e_1, \ldots, e_i - 1, \ldots, e_m)$$

$$= F(e_1, \ldots, e_m).$$

Now, we see that

$$\frac{e_i \cdot F(e_1, \ldots, e_i - 1, \ldots, e_m)}{F(e_1, \ldots, e_m)} = \frac{1 + \cdots + q^{e_i - 1}}{1 + q + \cdots + q^{n-1}} = \frac{1 - q^{e_i}}{1 - q^n}$$

so it's equivalent to show

$$1 - q^n = q^{e_{k+1} + \cdots + e_m} \sum_{i=1}^{k} q^{e_1 + \cdots + e_{i-1}} (1 - q^{e_i}) + \sum_{i=k+1}^{m} q^{e_{k+1} + \cdots + e_{i-1}} (1 - q^{e_i})$$

which is clear, since the left summand telescopes to $q^{e_{k+1} + \cdots + e_m} - q^n$ and the right summand telescopes to $1 - q^{e_{k+1} + \cdots + e_m}$.

> **Remark.** Technically, we could have skipped straight to the induction, without proving the first two claims. However I think the solution reads more naturally this way.

§3 USAMO 2017/3, proposed by Evan Chen

Let ABC be a scalene triangle with circumcircle Ω and incenter I. Ray AI meets $\overline{BC}$ at D and Ω again at M; the circle with diameter $\overline{DM}$ cuts Ω again at K. Lines MK and BC meet at S, and N is the midpoint of $\overline{IS}$. The circumcircles of $\triangle KID$ and $\triangle MAN$ intersect at points L_1 and L_2. Prove that Ω passes through the midpoint of either $\overline{IL_1}$ or $\overline{IL_2}$.

Let W be the midpoint of $\overline{BC}$, let X be the point on Ω opposite M. Observe that $\overline{KD}$ passes through X, and thus lines BC, MK, XA concur at the orthocenter of $\triangle DMX$, which we call S. Denote by I_A the A-excenter of ABC.

Next, let E be the foot of the altitude from I to $\overline{XI_A}$; observe that E lies on the circle centered at M through I, B, C, I_A. Then, S is the radical center of Ω and the circles with diameter $\overline{IX}$ and $\overline{II_A}$; hence line SI passes through E; accordingly I is the orthocenter of $\triangle XSI_A$; denote by L the foot from X to $\overline{SI_A}$.
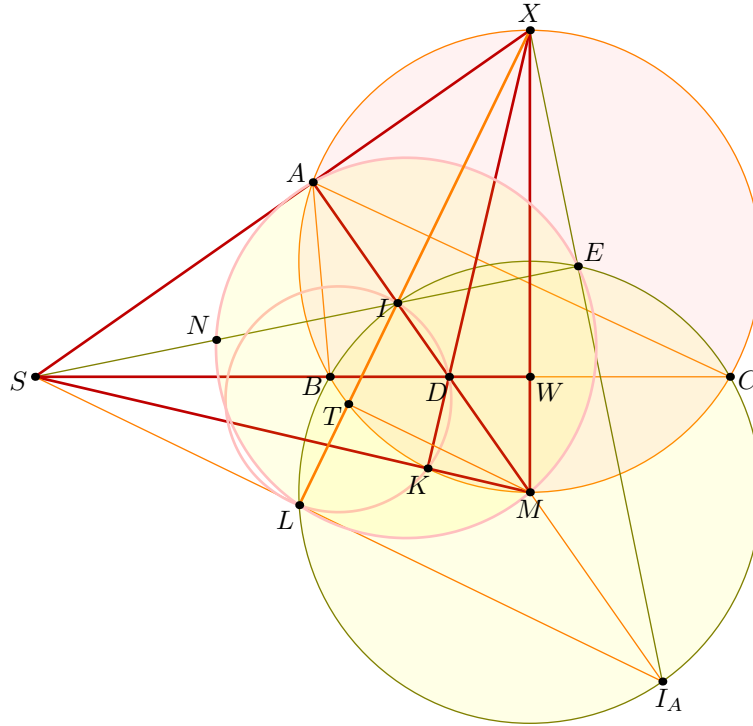
We claim that this L lies on both the circumcircle of $\triangle KID$ and $\triangle MAN$. It lies on the circumcircle of $\triangle MAN$ since this circle is the nine-point circle of $\triangle XSI_A$. Also, $XD \cdot XK = XW \cdot XM = XA \cdot XS = XI \cdot XL$, so $KDIL$ are concyclic.

All that remains to show is that the midpoint T of $\overline{IL}$ lies on Ω. But this follows from the fact that $\overline{TM} \parallel \overline{LI_A} \implies \angle MTX = 90°$, thus the problem is solved.

> **Remark.** Some additional facts about this picture: the point T is the contact point of the A-mixtilinear incircle (since it is collinear with X and I), while the point K is such that $\overline{AK}$ is an A-symmedian (since $\overline{KD}$ and $\overline{AD}$ bisect $\angle A$ and $\angle K$, say).
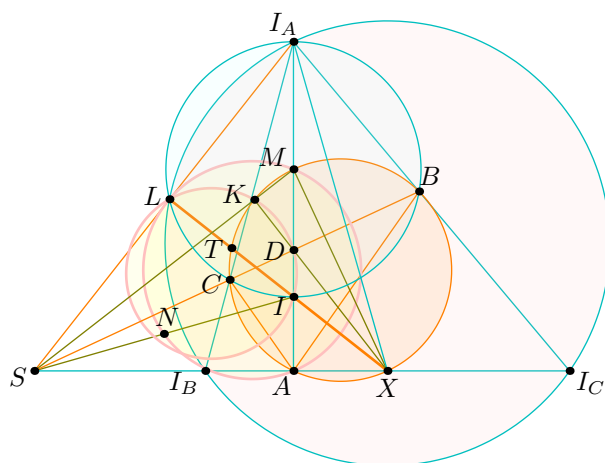
Remark. In fact, the point L is the Miquel point of cyclic quadrilateral $I_B I_C BC$ (inscribed in the circle with diameter $\overline{I_B I_C}$). This implies many of the properties that L has above. For example, it directly implies that L lies on the circumcircles of triangles $I_A I_B I_C$ and BCI_A, and that the point L lies on $\overline{SI_A}$ (since $S = \overline{BC} \cap \overline{I_B I_C}$). For this reason, many students found it easier to think about the problem in terms of $\triangle I_A I_B I_C$ rather than $\triangle ABC$.

§4 USAMO 2017/4, proposed by Maria Monks

Let P_1, P_2, ..., P_{2n} be $2n$ distinct points on the unit circle $x^2 + y^2 = 1$, other than $(1,0)$. Each point is colored either red or blue, with exactly n red points and n blue points. Let R_1, R_2, ..., R_n be any ordering of the red points. Let B_1 be the nearest blue point to R_1 traveling counterclockwise around the circle starting from R_1. Then let B_2 be the nearest of the remaining blue points to R_2 travelling counterclockwise around the circle from R_2, and so on, until we have labeled all of the blue points B_1, ..., B_n. Show that the number of counterclockwise arcs of the form $R_i \to B_i$ that contain the point $(1,0)$ is independent of the way we chose the ordering R_1, ..., R_n of the red points.

We present two solutions, one based on swapping and one based on an invariant.

First "local" solution by swapping two points　　Let $1 \le i < n$ be any index and consider the two red points R_i and R_{i+1}. There are two blue points B_i and B_{i+1} associated with them.

> **Claim** — If we swap the locations of points R_i and R_{i+1} then the new arcs $R_i \to B_i$ and $R_{i+1} \to B_{i+1}$ will cover the same points.

Proof. Delete all the points R_1, ..., R_{i-1} and B_1, ..., B_{i-1}; instead focus on the positions of R_i and R_{i+1}.

The two blue points can then be located in three possible ways: either 0, 1, or 2 of them lie on the arc $R_i \to R_{i+1}$. For each of the cases below, we illustrate on the left the locations of B_i and B_{i+1} and the corresponding arcs in green; then on the right we show the modified picture where R_i and R_{i+1} have swapped. (Note that by hypothesis there are no other blue points in the green arcs).
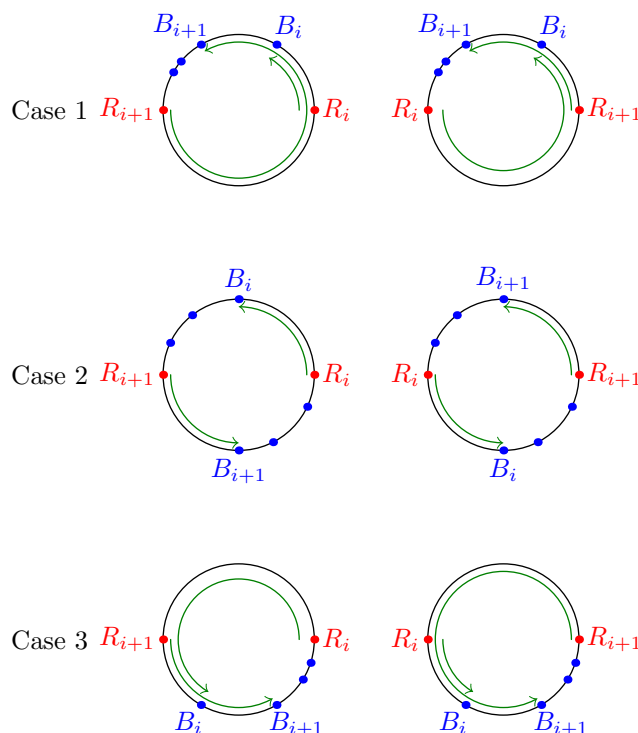

Observe that in all cases, the number of arcs covering any given point on the circumference is not changed. Consequently, this proves the claim.　　□

Finally, it is enough to recall that any permutation of the red points can be achieved by swapping consecutive points (put another way: $(i\ i+1)$ generates the permutation group S_n). This solves the problem.

> **Remark.** This proof does *not* work if one tries to swap R_i and R_j if $|i-j| \neq 1$. For example if we swapped R_i and R_{i+2} then there are some issues caused by the possible presence of the blue point B_{i+1} in the green arc $R_{i+2} \to B_{i+2}$.

Second longer solution using an invariant   Visually, if we draw all the segments $R_i \to B_i$ then we obtain a set of n chords. Say a chord is *inverted* if satisfies the problem condition, and *stable* otherwise. The problem contends that the number of stable/inverted chords depends only on the layout of the points and not on the choice of chords.
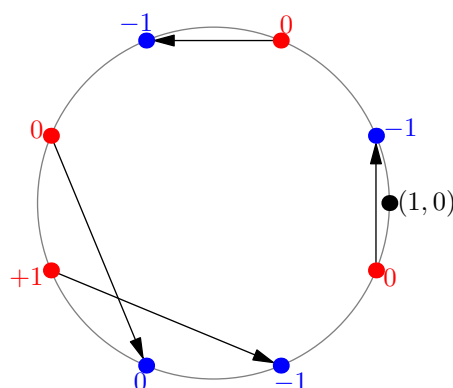
In fact we'll describe the number of inverted chords explicitly. Starting from $(1,0)$ we keep a running tally of $R - B$; in other words we start the counter at 0 and decrement by 1 at each blue point and increment by 1 at each red point. Let $x \leq 0$ be the lowest number ever recorded. Then:

> **Claim** — The number of inverted chords is $-x$ (and hence independent of the choice of chords).

This is by induction on n. I think the easiest thing is to delete chord $R_1 B_1$; note that the arc cut out by this chord contains no blue points. So if the chord was stable certainly no change to x. On the other hand, if the chord is inverted, then in particular the last point before $(1,0)$ was red, and so $x < 0$. In this situation one sees that deleting the chord changes x to $x + 1$, as desired.

§5 USAMO 2017/5, proposed by Ricky Liu

Find all real numbers $c > 0$ such that there exists a labeling of the lattice points in $\mathbb{Z}^2$ with positive integers for which:

- only finitely many distinct labels occur, and
- for each label i, the distance between any two points labeled i is at least c^i.


The answer is $c < \sqrt{2}$. Here is a solution with Calvin Deng.

The construction for any $c < \sqrt{2}$ can be done as follows. Checkerboard color the lattice points and label the black ones with 1. The white points then form a copy of $\mathbb{Z}^2$ again scaled up by $\sqrt{2}$ so we can repeat the procedure with 2 on half the resulting points. Continue this dyadic construction until a large N for which $c^N < 2^{\frac{1}{2}(N-1)}$, at which point we can just label all the points with N.

I'll now prove that $c = \sqrt{2}$ (and hence $c \geq \sqrt{2}$) can't be done.

> **Claim** — It is impossible to fill a $2^n \times 2^n$ square with labels not exceeding $2n$.

The case $n = 1$ is clear. So now assume it's true up to $n-1$; and assume for contradiction a $2^n \times 2^n$ square S only contains labels up to $2n$. (Of course every $2^{n-1} \times 2^{n-1}$ square contains an instance of a label at least $2n-1$.)
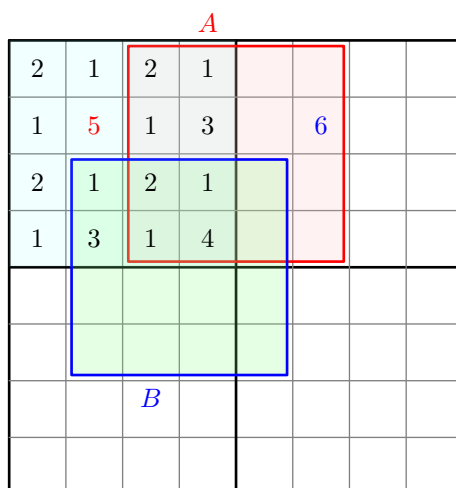
Now, we contend there are fewer than four copies of $2n$:

> **Lemma**
>
> In a unit square, among any four points, two of these points have distance ≤ 1 apart.

Proof. Look at the four rays emanating from the origin and note that two of them have included angle $\leq 90°$. □

So WLOG the northwest quadrant has no $2n$'s. Take a $2n - 1$ in the northwest and draw a square of size $2^{n-1} \times 2^{n-1}$ directly right of it (with its top edge coinciding with the top of S). Then A can't contain $2n - 1$, so it must contain a $2n$ label; that $2n$ label must be in the northeast quadrant.

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Then we define a square B of size $2^{n-1} \times 2^{n-1}$ as follows. If $2n - 1$ is at least as high $2n$, let B be a $2^{n-1} \times 2^{n-1}$ square which touches $2n - 1$ north and is bounded east by $2n$. Otherwise let B be the square that touches $2n - 1$ west and is bounded north by $2n$. We then observe B can neither have $2n - 1$ nor $2n$ in it, contradiction.

Remark. To my knowledge, essentially all density arguments fail because of hexagonal lattice packing.

§6 USAMO 2017/6, proposed by Titu Andreescu

Find the minimum possible value of

$$\frac{a}{b^3+4} + \frac{b}{c^3+4} + \frac{c}{d^3+4} + \frac{d}{a^3+4}$$

given that a, b, c, d are nonnegative real numbers such that $a+b+c+d=4$.

The minimum $\frac{2}{3}$ is achieved at $(a,b,c,d) = (2,2,0,0)$ and cyclic permutations.

The problem is an application of the tangent line trick: we observe the miraculous identity

$$\frac{1}{b^3+4} \geq \frac{1}{4} - \frac{b}{12}$$

since $12 - (3-b)(b^3+4) = b(b+1)(b-2)^2 \geq 0$. Moreover,

$$ab + bc + cd + da = (a+c)(b+d) \leq \left(\frac{(a+c)+(b+d)}{2}\right)^2 = 4.$$

Thus

$$\sum_{\text{cyc}} \frac{a}{b^3+4} \geq \frac{a+b+c+d}{4} - \frac{ab+bc+cd+da}{12} \geq 1 - \frac{1}{3} = \frac{2}{3}.$$

> **Remark.** The main interesting bit is the equality at $(a,b,c,d) = (2,2,0,0)$. This is the main motivation for trying tangent line trick, since a lower bound of the form $\sum a(1 - \lambda b)$ preserves the unusual equality case above. Thus one takes the tangent at $b=2$ which miraculously passes through the point $(0, 1/4)$ as well.