Algorithmic Graph Theory
Winter Semester 2021/22
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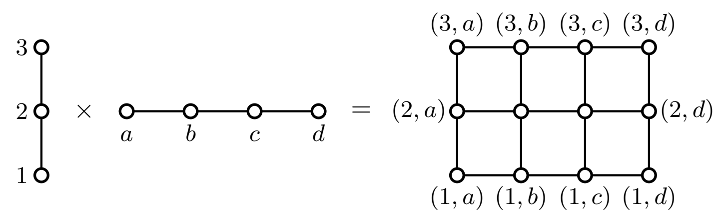
Exercise Sheet 2

Discussion during the exercise on 16.11.2021, 12:00 – 14:00 in IM SR 034.
For optional feedback: Hand in until 15.11.2021 12:00
via E-Mail to `simon.fink@uni-passau.de` or via upload to StudIP.

Question 1: *Graph Coloring and Independent Sets* ★★

Let G be a graph with n vertices. Show that $\chi(G) \leq r$ if and only if $\alpha(G \times K_r) = n$.

The *cartesian product* $G_1 \times G_2$ of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is defined as $G_1 \times G_2 = (V_1 \times V_2, E)$ with $E = \{\{(v_1, v_2), (v_1', v_2')\} \mid$ either $v_1 = v_1'$ and $\{v_2, v_2'\} \in E_2$ or $v_2 = v_2'$ and $\{v_1, v_1'\} \in E_1\}$ (see example below).



In what way does this show that Independent Set is NP-hard (given that Graph Coloring is NP-hard)?

Question 2: *Commutating Operations* ★

Let x and y be distinct vertices in a graph G. Show that $(G \circ x) - y = (G - y) \circ x$.

Question 3: *Algorithmic Vertex Multiplication* ★★

Let $x_1, \ldots, x_n$ be the vertices of graph G and let $h = (h_1, \ldots, h_n)$ be a vector with $h_i \in \mathbb{N}_0$. Observe that the algorithm shown on the right yields the graph $H = G \circ h$. Determine the runtime of the algorithm. Is this optimal and if not, how could the algorithm be improved?
Hint: How does the runtime relate to the output size?

```
H ← G;
for i ← 1 to n do
    if h_i = 0 then
        H ← H − x_i;
    else
        while h_i > 1 do
            H ← H ∘ x_i;
            h_i ← h_i − 1;
```

Question 4: *A little bit perfect?* ⋆⋆

Find a graph G with $\alpha(G) = k(G)$ and $\omega(G) < \chi(G)$.
Why doesn't this contradict the Perfect Graph Theorem?

Question 5: *Clique Cover and Independent Sets* ⋆

Let G be a graph with $\alpha(G) = k(G)$. Furthermore, let $\mathcal{K}$ be a clique cover with $|\mathcal{K}| = k(G)$ and let $\mathcal{U}$ be the set of all independent sets of size $\alpha(G)$. Show that $|U \cap K| = 1$ for all $U \in \mathcal{U}$ and $K \in \mathcal{K}$.

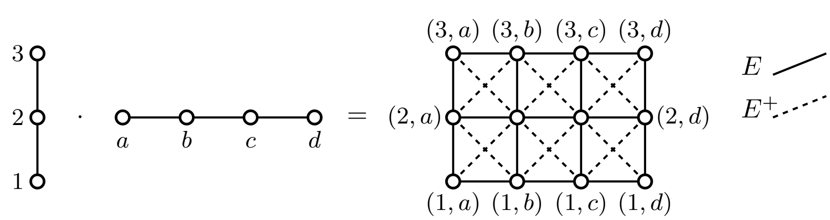
Also give the dual statement for graphs with $\omega(G) = \chi(G)$.

Question 6: *Not perfect at all* ⋆⋆⋆

Show that the difference between the clique number and the chromatic number can grow arbitrarily big. To do so, show that for all $k \in \mathbb{N}$ with $k \geq 2$ there exists a graph G with $\omega(G) = 2$ and $\chi(G) = k$.

Question 7: *Graph Parameters and the Normal Product* ⋆⋆

The *normal product* $G_1 \cdot G_2$ of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is defined as $G_1 \cdot G_2 = (V_1 \times V_2, E \cup E^+)$ with $E = \{\{(v_1, v_2), (v_1', v_2')\} \mid \text{ either } v_1 = v_1' \text{ and } \{v_2, v_2'\} \in E_2 \text{ or } v_2 = v_2' \text{ and } \{v_1, v_1'\} \in E_1\}$ and $E^+ = \{\{(v_1, v_2), (v_1', v_2')\} \mid \{v_1, v_1'\} \in E_1 \text{ and } \{v_2, v_2'\} \in E_2\}$ (see example below).



Show the following statements:

(a) $\chi(G_1 \cdot G_2) \geq \max\{\chi(G_1), \chi(G_2)\}$

(b) $\omega(G_1 \cdot G_2) = \omega(G_1) \cdot \omega(G_2)$

(c) $\alpha(G_1 \cdot G_2) \geq \alpha(G_1) \cdot \alpha(G_2)$

(d) $k(G_1 \cdot G_2) \leq k(G_1) \cdot k(G_2)$

For (a), (c) and (d) also give examples for both equality and inequality of both sides of the relation.