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Assignment on Laplace Transforms

Q.1.

(a) $f(t) = 1 \quad 0 \le t < 1$

$\sin t \quad 1 \le t < \pi$

$\cosh t \quad \pi \le t < 4$

$t+1 \quad t \geq 4$

$$L(f(t)) = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^1 e^{-st}\cdot 1\,dt + \int_1^\pi e^{-st}\sin t\,dt + \int_\pi^4 e^{-st}\cosh t\,dt + \int_4^\infty e^{-st}(t+1)\,dt$$

$$= \frac{\left[e^{-st}\right]_0^1}{s} + \left[\frac{e^{-st}}{s^2+1}\cdot(-s\sin t - 1\cdot\cos t)\right]_1^\pi$$

$$+ \frac{1}{2}\int_\pi^4 e^{-st}(e^t + e^{-t})\,dt + \left[\frac{(t+1)e^{-st}}{-s} - \frac{e^{-st}}{s^2}\right]_4^\infty$$

$$= \frac{(e^{-s}-1)}{s} + \left[\frac{e^{-\pi s}(1)}{s^2+1} + \frac{e^{-s}}{s^2+1}(s\cdot\sin 1 + \cos 1)\right]$$

$$+ \frac{1}{2}\int_\pi^4 e^{-(s-1)t} + e^{-(s+1)t}\,dt + \frac{e^{-4s}}{s^2} + \frac{e^{-4s}s}{s}$$

$$= \frac{e^{-s}-1}{s} + \frac{e^{-\pi s} + e^{-s}(s\sin 1 + \cos 1)}{s^2+1}$$

$$+ \frac{1}{2}\left[\frac{e^{-(s-1)t}}{-(s-1)} + \frac{e^{-(s+1)t}}{-(-s+1)}\right]_\pi^4 + \frac{5e^{-4s}}{s} + \frac{e^{-4s}}{s^2}$$

$$= \frac{e^{-s}-1}{s} + \frac{e^{-\pi s} + e^{-s}(s\cdot\sin 1 + \cos 1)}{s^2+1} + \frac{1}{2}\left(\frac{e^{-\pi(s-1)}}{s-1} + \right.$$

$$\frac{e^{-(s+1)\pi}}{s+1} - \frac{e^{-(s-1)4}}{s-1} - \frac{e^{-4(s+1)}}{s+1}\Bigg) + \frac{e^{-4s}}{s}\left[\frac{s+1}{s}\right]$$

$$\therefore L(f(t)) = \frac{e^{-s}-1}{s} + \frac{e^{-\pi s} + e^{-s}(s.\sin 1 + \cos 1)}{s^2+1} + \frac{e^{-4s}}{s}\left[\frac{s+1}{s}\right]$$
$$+\frac{1}{2}\left(\frac{e^{-\pi(s-1)} - e^{-4(s-1)}}{s-1} + \frac{e^{-\pi(s+1)} - e^{-4(s+1)}}{s+1}\right)$$

(b) $f(t) = \int_0^t e^{-3t}\, t \cos^2 2t\, dt$

Now, $L(\cos^2 2t) = L\left(\frac{1+\cos 4t}{2}\right)$

$$= \frac{1}{2}\left(\frac{1}{s} + \frac{1}{s^2+16}\right)$$

$$L(t \times \cos^2 2t) = -1 \times \frac{d}{ds}\frac{1}{2}\left(\frac{1}{s} + \frac{s}{s^2+16}\right)$$

$$= \frac{-1}{2}\left(\frac{d}{ds}\left(\frac{1}{s}\right) + \frac{d}{ds}\left(\frac{s}{s^2+16}\right)\right)$$

$$= \frac{-1}{2}\left[\frac{-1}{s^2} + \frac{s^2+16-2s^2}{(s^2+16)^2}\right]$$

$$= \frac{-1}{2}\left[\frac{-1}{s^2} + \frac{16-s^2}{(s^2+16)^2}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s^2} + \frac{s^2-16}{(s^2+16)^2}\right]$$

$$L(e^{-3t}.t.\cos^2 2t) = \frac{1}{2}\left[\frac{1}{(s+3)^2} + \frac{(s+3)^2-16}{((s+3)^2+16)^2}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s^2+6s+9} + \frac{s^2+6s-7}{(s^2+6s+25)^2}\right]$$

$$\therefore L\left(\int_0^t e^{-3t}.t.\cos^2 2t\right) = \frac{1}{2s}\left[\frac{1}{s^2+6s+9} + \frac{s^2+6s-7}{(s^2+6s+25)^2}\right]$$

(c) $f(t) = e^{-2t}\int_0^t \cosh t \cdot t^4 dt$

$$f(t) = \frac{e^{-2t}}{2}\int_0^t (e^t + e^{-t})\, t^4 dt$$

$$= \frac{e^{-2t}}{2}\int_0^t (e^t.t^4 + e^{-t}.t^4)\, dt$$

$$L(t^4) = \frac{24}{s^5}$$

$$L(e^t.t^4 + e^{-t}.t^4) = \frac{24}{(s-1)^5} + \frac{24}{(s+1)^5}$$

$$\therefore L\left(\int_0^t (\cosh t)\, t^4 dt\right) = \frac{1}{s}\left[\frac{24}{(s-1)^5} + \frac{24}{(s+1)^5}\right]$$

$$\therefore L\left(\frac{1}{2}e^{-2t}\int_0^t (e^t + e^{-t})\, t^4 dt\right) = \frac{12}{s+2}\left[\frac{1}{(s+1)^5} + \frac{1}{(s+3)^5}\right]$$

$$\therefore L(f(t)) = \frac{12}{s+2}\left[\frac{1}{(s+1)^5} + \frac{1}{(s+3)^5}\right]$$

(d) $\left(\frac{\sin 2t}{\sqrt{2t}}\right)^2 = \frac{\sin^2 2t}{2t} = f(t)$

$$L(\sin^2 2t) = \frac{1}{2} L(1-\cos 4t)$$

$$= \frac{1}{2}\left(\frac{1}{s} - \frac{s}{s^2+16}\right)$$

$$L\left(\frac{\sin^2 2t}{\sqrt{2t}}\right) = \frac{1}{4}\int_s^\infty \frac{1}{s} - \frac{s}{s^2+16}$$

$$= \frac{1}{4}\left[\log s - \frac{1}{2}\log(s^2+16)\right]_s^\infty$$

$$= \frac{1}{4}\left[\log\frac{s}{\sqrt{s^2+16}}\right]_s^\infty$$

$$\therefore L(f(t)) = \frac{-1}{4}\log\frac{s}{\sqrt{s^2+16}}$$

(e) $f(t) = e^{-t}\cos t \quad 0 \le t < \pi$

$e^{-t}\sin t \quad \pi \le t < 2\pi$

$f(t) = f(t+2\pi)$ for all t.

The laplace of periodic function of period T is

$$L(f(t)) = \frac{1}{1-e^{-Ts}}\int_0^T e^{-st} f(t)\,dt$$

$$\therefore L(f(t)) = \frac{1}{1-e^{-2\pi s}}\int_0^{2\pi} e^{-st} f(t)\,dt$$

$$L(f(t)) = \frac{1}{1-e^{-2\pi s}}\left[\int_0^\pi e^{-st}\cos t\,dt + \int_\pi^{2\pi} e^{-st}\sin t\,dt\right]$$

$$= \frac{1}{1-e^{-2\pi s}}\left(\left[\frac{e^{-st}}{s^2+1}(-s\cos t+\sin t)\right]_0^\pi + \left[\frac{e^{-st}}{s^2+1}(-s\cdot\sin t - \cos t)\right]_\pi^{2\pi}\right)$$

$$= \frac{1}{1-e^{-2\pi s}}\left(\frac{e^{-s\pi}}{s^2+1}(-s) + \frac{s}{s^2+1} + \frac{e^{-2\pi s}(-1)}{s^2+1} - \frac{e^{-\pi s}}{s^2+1}\right)$$

$$\therefore L(f(t)) = \frac{1}{1-e^{-2\pi s}}\left(\frac{s - se^{-\pi s} - e^{-2\pi s} - e^{-\pi s}}{s^2+1}\right)$$

Q.2.

(a) $\int_0^\infty e^{-2t}\frac{(\cos 3t - \cos 2t)}{t}\,dt$

$$= L\left(\frac{\cos 3t - \cos 2t}{t}\right)(s=2)$$

$$= L(\cos 3t - \cos 2t) = \frac{s}{s^2+9} - \frac{s}{s^2+4}$$

$$= L\left(\frac{\cos 3t - \cos 2t}{t}\right) = \int_s^\infty \frac{s}{s^2+9} - \frac{s}{s^2+4}\,ds$$

$$= \frac{1}{2}\left[\log\frac{s^2+9}{s^2+4}\right]_s^\infty = \frac{1}{2}\left\{\log 1 - \log\frac{s^2+9}{s^2+4}\right.$$

$$= \frac{-1}{2}\log\left(\frac{s^2+9}{s^2+4}\right).$$

$$= \frac{1}{2}\log\left(\frac{s^2+4}{s^2+9}\right)$$

Putting $s=2$,

$$\therefore \; = \frac{1}{2}\log\frac{8}{13}$$

(b) $$\int_0^\infty e^{-t/2}\left(\frac{1}{t}\int_0^t e^{-u}\sin u\,du\right)dt$$

Let $$f(t) = \frac{1}{t}\int_0^t e^{-u}\sin u\,du$$

$$\therefore \int_0^\infty e^{-t/2} f(t)\,dt = L(f(t)) \quad \left(s=\frac{1}{2}\right)$$

$$L(\sin u) = \frac{1}{s^2+1}$$

$$\therefore L(e^{-u}\sin u) = \frac{1}{(s+1)^2+1}$$

$$\therefore L\left[\int_0^t e^{-u}\sin u\,du\right] = \frac{1}{s}\cdot\frac{1}{(s+1)^2+1}$$

$$\therefore L\left(\frac{1}{t}\int_0^t e^{-u}\sin u\,du\right) = \int_0^\infty \frac{1}{s((s+1)^2+1)}$$

Let

$$\therefore \frac{1}{s(s^2+2s+2)} = \frac{A}{s} + \frac{Bs+C}{s^2+2s+2}$$

$$\therefore 1 = A(s^2+2s+2) + Bs^2 + Cs$$

Comparing coefficients,

$$2A = 1$$
$$\therefore A = \frac{1}{2}$$

$$A+B=0$$
$$\therefore B = -A = \frac{-1}{2}$$

and $$2A+C=0$$
$$\therefore C = -2A = -1.$$

$$\therefore \frac{1}{s(s^2+2s+2)} = \frac{1}{2s} + \frac{\left(\frac{-1}{2}\right)s-1}{s^2+2s+2}$$

$$= \frac{1}{2s} - \frac{1}{2}\frac{(s+2)}{s^2+2s+2}$$

$$= \frac{1}{2s} - \frac{s+1}{s^2+2s+2} - \frac{1}{(s+1)^2+1^2}$$

$$= \left[\frac{1}{2}\log s - \frac{1}{4}\log(s^2+2s+2) - \tan^{-1}(s+1)\right]_s^\infty$$

$$= \left[\frac{1}{2}\log\frac{s}{\sqrt{s^2+2s+2}} - \tan^{-1}(s+1)\right]_s^\infty$$

$$= \frac{-1}{2}\log\frac{s}{\sqrt{s^2+2s+2}} - \frac{\pi}{2} + \tan^{-1}(s+1)$$

putting $s = \frac{1}{2}$,

$= \frac{1}{2} \log \sqrt{13} - \frac{\pi}{2} + \tan^{-1}\frac{3}{2}$

$= \frac{1}{4} \log 13 - \frac{\pi}{2} + \tan^{-1}\frac{3}{2}$

Q.3

(a) $f(s) = e^{-3s} \cdot \frac{s+1}{s^2+2s+2}$

Let $f_1(s) = \frac{s+1}{s^2+2s+2}$

$\therefore L^{-1}(f_1(s)) = L^{-1}\left(\frac{s+1}{(s+1)^2+1^2}\right)$

$= e^{-t} L^{-1}\left(\frac{s}{s^2+1}\right)$

$= e^{-t} \cos t$

$L^{-1}(e^{-3s} f_1(s)) = e^{-(t-3)} \cos(t-3) u(t-3)$

$\therefore L^{-1}(f(s)) = e^{-(t-3)} \cos(t-3) u(t-3)$

(b) $F(s) = \frac{1}{s^4-64}$

$F(s) = \frac{1}{(s^2-8)(s^2+8)}$

$= \frac{1}{16}\left(\frac{1}{s^2-8} - \frac{1}{s^2+8}\right)$

$\therefore L^{-1}(F(s)) = \frac{1}{16} \cdot \frac{1}{2\sqrt{2}} \cdot L^{-1}\left(\frac{2\sqrt{2}}{s^2-8} - \frac{2\sqrt{2}}{s^2+8}\right)$

$\therefore L^{-1}\left(\frac{1}{s^4-64}\right) = \frac{1}{32\sqrt{2}}\left(\sinh 2\sqrt{2}t - \sin 2\sqrt{2}t\right)$

(c) $F(s) = \frac{s}{s^2+2s-3}$

$\therefore F(s) = \frac{s}{(s+1)^2-s^2} = \frac{s+1}{(s+1)^2-2^2} = \frac{s+1}{(s+1)^2-2^2} - \frac{1}{(s+1)^2-2^2}$

$L^{-1}(f(s)) = L^{-1}\left(\frac{s+1}{(s+1)^2-2^2} - \frac{1}{(s+1)^2-2^2}\right)$

$= e^{-t} L^{-1}\left(\frac{s}{s^2-2^2} - \frac{1}{2} \cdot \frac{2}{s^2-2^2}\right)$

$L^{-1}(f(s)) = e^{-t}\left(\cosh 2t - \frac{1}{2}\sinh 2t\right)$

(d) $\log\left(\frac{s^2-4}{s^2}\right)^{1/3} = \frac{1}{3}\left(\log(s^2-4) - \log(s^2)\right)$

$L^{-1}(f(s)) = \frac{1}{3} L^{-1}\left(\log(s^2-4) - \log(s^2)\right)$

$= \frac{1}{3}\left(\frac{-1}{t}\right) L^{-1}\left(\frac{2s}{s^2-4} - \frac{2}{s}\right)$

$= \frac{-2}{3t}\left(\cancel{\cos t}\ \cosh 2t - 2\right)$

$\therefore L^{-1}(f(s)) = \frac{-4}{3t} \sinh^2 t$

(f) $f(s) = \frac{5s+2}{(s+1)(s^2-2s+5)}$

Let $\frac{5s+2}{(s+1)(s^2-2s+5)} = \frac{A}{s+1} + \frac{Bs+C}{s^2-2s+5}$

$\therefore 5s+2 = A(s^2-2s+5) + (Bs+C)(s+1)$

Let $s=-1$,

$\therefore -3 = 8A$

$\therefore A = \frac{-3}{8}$

Let $s=0$,

$2 = 5A + C$

$\therefore 2 = \frac{-15}{8} + C$

$\therefore C = \frac{31}{8}$.

Let $s=1$,

$7 = 4A + (B+C)2$

$\therefore 7 = \frac{-3}{2} + 2B + \frac{31}{8}$

$\therefore \frac{17}{4} = B + \frac{31}{8}$

$\therefore B = \frac{3}{8}$

$\therefore f(s) = \frac{-1}{s+1} + \frac{1}{8}\left(\frac{31+3s}{s^2-2s+5}\right)$

$= \frac{-1}{s+1} + \frac{1}{8}\left[\frac{3(s-1)}{(s-1)^2+2^2} + \frac{34}{(s-1)^2+2^2}\right]$

$L^{-1}(f(s)) = L^{-1}\left(\frac{-1}{(s+1)} + \frac{1}{8}\left(\frac{3(s-1)}{(s-1)^2+2^2} + \frac{17.2}{(s-1)^2+2^2}\right)\right)$

$= -e^{-t} + \frac{1}{8}\left[3e^{t} L^{-1}\left(\frac{s}{s^2+2^2}\right) + 17e^{t} L^{-1}\left(\frac{2}{s^2+2^2}\right)\right]$

$\therefore L^{-1}(f(s)) = -e^{-t} + \frac{1}{8}\left[3e^{t}\cos 2t + 17e^{t}\sin 2t\right]$

Q.4. $y'' + 2y' + y = 3te^{-t}$, $y(0)=0$, $y'(0)=2$.

$L(y'') + 2L(y') + L(y) = 3L(te^{-t})$

~~$s^2y(s) - s(y\ s\ y(0) - y'(0) + 2\ sy(s)$~~

$s^2 Y(s) - sy(0) - y'(0) + 2(sY(s) - y(0)) + Y(s) = \frac{3}{(1+s)^2}$

$\therefore Y(s)(s^2+2s+1) - 2 = \frac{3}{(s+1)^2}$

$Y(s)(s+1)^2 = \frac{3}{(s+1)^2} + 2$

$\therefore Y(s) = \frac{3}{(s+1)^4} + \frac{2}{(s+1)^2}$

Taking Inverse laplace on both sides.

$$y(t) = L^{-1}\left(\frac{3}{(s+1)^4} + \frac{2}{(s+1)^2}\right)$$

$$= e^{-t}\left(L^{-1}\left(\frac{3}{s^4} + \frac{2}{s^2}\right)\right)$$

$$\therefore y(t) = e^{-t}\left(\frac{3t^3}{6} + 2t\right)$$

$$\therefore y(t) = e^{-t}\left(\frac{t^3}{2} + 2t\right).$$