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Assignment

Q.1

$$f(t) = \begin{cases} 1 & 0 \le t < 1 \\ \sin t & 1 \le t \le \pi \\ \cosh t & \pi \le t < 4 \\ t+1 & t \ge 4 \end{cases}$$

To find Laplace transform

$$L(f(t)) = \int_0^\infty e^{-st} f(t)\, dt$$

$$= \int_0^\infty e^{-st} f(t)\, dt$$

$$= \int_0^1 e^{-st} \cdot 1\, dt + \int_1^\pi e^{-st} \sin t\, dt + \int_\pi^4 e^{-st} \cosh t\, dt$$

$$+ \int_4^\infty e^{-st}(t+1)\, dt$$

$$= \left[\frac{e^{-st}}{s}\right]_0^1 + \left[\frac{e^{-st}}{s^2+1}(-s \times \sin t - 1 \times \cos t)\right]_1^\pi$$

$$+ \frac{1}{2}\int_\pi^4 e^{-st}(e^t + e^{-t})\, dt + \left[\frac{(t+1)e^{-st}}{-s} - \frac{e^{-st}}{s^2}\right]_4^\infty$$

$$= \left(\frac{e^{-s} - 1}{s}\right) + \left[\frac{e^{-\pi s}(1)}{s^2+1} + \frac{e^{-s}}{s^2+1}(s \times \sin 1 + \cos 1)\right]$$

$$+ \frac{1}{2}\int_\pi^4 \left(e^{-(s-1)t} + e^{-(s+1)t}\right) dt + \frac{e^{-4s}}{s^2} + \frac{e^{-4s} \cdot 5}{s}$$

D	I
$t+1$	e^{-st}
1	$\frac{e^{-st}}{-s}$
0	$\frac{e^{-st}}{s^2}$

$$\frac{e^{-s}-1}{s} + \frac{e^{-\pi s} + e^{-s}(s\sin 1 + \cos 1)}{s^2+1}$$

$$+ \frac{1}{2}\left[\frac{e^{-(s-1)t}}{-(s-1)} + \frac{e^{-(s+1)t}}{-(s+1)}\right]_{\pi}^{4}$$

$$+ \frac{5e^{-4s}}{s} + \frac{e^{-4s}}{s^2}$$

$$= \frac{e^{-s}-1}{s} + \frac{e^{-\pi s} + e^{-s}(s \cdot \sin 1 + \cos 1)}{s^2+1}$$

$$+ \frac{1}{2}\left(\frac{e^{-\pi(s-1)}}{s-1} + \frac{e^{-(s+1)\pi}}{s+1} - \frac{e^{-(s-1)4}}{s-1} - \frac{e^{-(s+1)4}}{s+1}\right)$$

$$+ \frac{e^{-4s}}{s}\left[5 + \frac{1}{s}\right]$$

$$L(f(t)) = \frac{e^{-s}-1}{s} + \frac{e^{-\pi s} + e^{-s}(s \cdot \sin 1 + \cos 1)}{s^2+1} + \frac{e^{-4s}}{s}\left[5 + \frac{1}{s}\right]$$

$$+ \frac{1}{2}\left(\frac{e^{-\pi(s-1)} - e^{-4(s-1)}}{s-1} + \frac{e^{-\pi(s+1)} - e^{-4(s+1)}}{s+1}\right)$$

(b) $f(t) = \int_0^t e^{-3t} t \cos^2 2t \, dt$.

$$L(\cos^2 2t) = L\left(\frac{1 + \cos 4t}{2}\right)$$

$$= \frac{1}{2}\left(\frac{1}{s} + \frac{s}{s^2+16}\right)$$

$$L(t \times \cos^2 2t) = -1 \times \frac{d}{ds} \frac{1}{2}\left(\frac{1}{s} + \frac{s}{s^2+16}\right)$$

$$= \frac{-1}{2}\left(\frac{d}{ds}\left(\frac{1}{s}\right) + \frac{d}{ds}\left(\frac{s}{s^2+16}\right)\right)$$

$$= \frac{-1}{2}\left[\frac{-1}{s^2} + \frac{s^2+16-2s^2}{(s^2+16)^2}\right]$$

$$= \frac{-1}{2}\left[\frac{-1}{s^2} + \frac{16-s^2}{(s^2+16)^2}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s^2} + \frac{s^2-16}{(s^2+16)^2}\right]$$

$$L(e^{-3t} . t \cdot \cos^2 2t) = \frac{1}{2}\left[\frac{1}{(s+3)^2} + \frac{(s+3)^2-16}{((s+3)^2+16)^2}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s^2+6s+9} + \frac{s^2+6s-7}{(s^2+6s+25)^2}\right]$$

$$L\left(\int_0^t e^{-3t} . t \cdot \cos^2 2t\right) = \frac{1}{2S} * \left[\frac{1}{s^2+6s+9} + \frac{s^2+6s-7}{(s^2+6s+25)^2}\right]$$

c) $f(t) = e^{-2t} \int_0^t \cosh t \times t^4 \, dt.$

$$f(t) = \frac{1 \times e^{-2t}}{2} \int_0^t (e^{+t} + e^{-t}) t^4 \, dt$$

$$= \frac{e^{-2t}}{2} \int_0^t (e^{t} \cdot t^4 + e^{-t} \cdot t^4) \, dt$$

$$L\{e^{t} \cdot 1(t^4) = \frac{24}{s^5}$$

$$L(e^{t} \times t^4 + e^{-t} \times t^4) = \frac{24}{(s-1)^5} + \frac{24}{(s+1)^5}$$

$$\Rightarrow L\left(\int_0^t (\cosh t) t^4\right) = \frac{1}{s}\left[\frac{24}{(s-1)^5} + \frac{24}{(s+1)^5}\right]$$

$$\cancel{L\left(\frac{e^{-2t}}{2}\int_0^t (\cosh t) t^4\right)} = \cancel{\frac{1}{s}(12 + 12}$$

$$L\left(\frac{1}{2} e^{-2t} \int_0^t (e^{t} + e^{-t}) t^4 \, dt\right) = \frac{12}{s+2}\left[\frac{1}{(s+1)^5} + \frac{1}{(s+3)^5}\right]$$

$$L(f(t)) = \frac{12}{s+2}\left[\frac{1}{(s+1)^5} + \frac{1}{(s+3)^5}\right]$$

(d) $\left(\frac{\sin 2t}{\sqrt{2t}}\right)^2 = \frac{\sin^2 2t}{2t} = f(t)$

$$L(\sin^2 2t) = \frac{1}{2} \times L(1-\cos 4t)$$

$$= \frac{1}{2}\left(\frac{1}{s} - \frac{s}{s^2+16}\right)$$

$$L\left(\frac{\sin^2 2t}{2t}\right) = \frac{1}{4}\int_s^\infty \frac{1}{s} - \frac{s}{s^2+16}$$

$$= \frac{1}{4}\left[\log s - \frac{1}{2}\log(s^2+16)\right]_s^\infty$$

$$= \frac{1}{4}\left[\log \frac{s}{\sqrt{s^2+16}}\right]_s^\infty$$

$$L(f(t)) = \frac{-1}{4}\log \frac{s}{\sqrt{s^2+16}}$$

(e) $f(t) = \begin{cases} e^{-t}\cos t & t \in [0, \pi) \\ e^{-t}\sin t & t \in [\pi, 2\pi) \end{cases}$

$f(t) = f(t+2\pi)$ Here $T = 2\pi$

Laplace of Periodic function of Period T is.

$$L(f(t)) = \frac{1}{1-e^{-Ts}}\int_0^T e^{-st} f(t)\, dt.$$

$$L(f(t)) = \frac{1}{1-e^{-2\pi s}}\int_0^{2\pi} e^{-st} f(t)\, dt.$$

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$$L(f(t)) = \frac{1}{1-e^{-2\pi s}}\left[\int_0^{\pi} e^{-st} * \cos t\,dt + \int_{\pi}^{2\pi} e^{-st}\sin t\,dt\right]$$

$$= \frac{1}{1-e^{-2\pi s}}\left(\left[\frac{e^{-st}}{s^2+1}(-s\cos t + \sin t)\right]_0^{\pi} + \left[\frac{e^{-st}}{s^2+1}(-s\sin t - \cos t)\right]_{\pi}^{2\pi}\right)$$

$$= \frac{1}{1-e^{-2\pi s}}\left(\frac{e^{-s\pi}}{s^2+1}(-s) + \frac{s}{s^2+1} + \frac{e^{-2\pi s}}{s^2+1}(-1) - \frac{e^{-\pi s}}{s^2+1}\right)$$

$$L(f(t)) = \frac{1}{1-e^{-2\pi s}}\left(\frac{s - se^{-\pi s} - e^{-2\pi s} - e^{-\pi s}}{s^2+1}\right)$$

Q.2. (a) $\int_0^{\infty} e^{-2t}\left(\frac{\cos 3t - \cos 2t}{t}\right)dt$.

$$= L\left(\frac{\cos 3t - \cos 2t}{t}\right)_{s=2}$$

$$= L(\cos 3t - \cos 2t) = \frac{s}{s^2+9} - \frac{s}{s^2+4}$$

$$= L\left(\frac{\cos 3t - \cos 2t}{t}\right) = \int_s^{\infty} \frac{s}{s^2+9} - \frac{s}{s^2+4}\,ds$$

$$= \frac{1}{2}\left[\log\frac{s^2+9}{s^2+4}\right]_s^{\infty} = \frac{1}{2}\left[\log 1 - \log\frac{s^2+9}{s^2+4}\right]$$

$$= \frac{-1}{2}\log\frac{s^2+9}{s^2+4}$$

$$= \frac{1}{2} \log \frac{s^2+4}{s^2+9}$$

Now put $s = 2$

$$\text{Ans} = \frac{1}{2} \log \frac{8}{13}$$

b) $$\int_0^\infty e^{-t/2} \left(\frac{1}{t} \int_0^t e^{-u} \sin u \, du \right) dt.$$

let $$f(t) = \frac{1}{t} \int_0^t e^{-u} \sin u \, du$$

$$\therefore \int_0^\infty e^{-t/2} . f(t) \, dt.$$

$$= L(f(t)) \Rightarrow s = +\frac{1}{2}.$$

$$L(\sin u) = \frac{1}{s^2+1}$$

$$L(e^{-u} \sin u) = \frac{1}{(s+1)^2+1}$$

$$L\left(\int_0^t e^{-u} \sin u \, du\right) = \frac{1}{s} . \frac{1}{(s+1)^2+1}$$

$$L\left(\frac{1}{t} \int_0^t e^{-u} \sin u \, du\right) = \int_s^\infty \frac{1}{s((s+1)^2+1)}$$

$$\frac{1}{s(s^2+2s+2)} = \frac{A}{s} + \frac{Bs+C}{s^2+2s+2}$$

$$1 = A(s^2+2s+2) + Bs^2 + Cs$$

$$2A = 1$$
$$A = \frac{1}{2}$$

$$A + B = 0$$
$$B = -A = -\frac{1}{2}$$

$$2A + C = 0$$
$$C = -2A = -1.$$

$$\frac{1}{s(s^2+2s+2)} = \frac{1}{2s} + \frac{\left(-\frac{1}{2}\right)s - 1}{s^2+2s+2}$$

$$= \frac{1}{2s} - \frac{1}{2}\cdot\frac{s+2}{s^2+2s+2}$$

$$= \frac{1}{2s} - \frac{s+1}{s^2+2s+2} - \frac{1}{(s+1)^2+1^2}$$

$$= \left[\frac{1}{2}\log s - \frac{1}{4}\log(s^2+2s+2) - \tan^{-1}(s+1)\right]_s^{\infty}$$

$$= \left[\frac{1}{2}\log\frac{s}{\sqrt{s^2+2s+2}} - \tan^{-1}(s+1)\right]_s^{\infty}$$

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$$= \frac{-1}{2} \log \frac{s}{\sqrt{s^2+2s+2}} - \frac{\pi}{2} + \tan^{-1} s+1$$

Now put $s = +\frac{1}{2}$

$$\text{Ans} = \frac{+1}{2} \log \sqrt{13} - \frac{\pi}{2} + \tan^{-1} \frac{3}{2}$$

$$= \frac{1}{4} \log 13 - \frac{\pi}{2} + \tan^{-1} \frac{3}{2}$$

Q.3

(a) $F(s) = e^{-3s} \cdot \frac{s+1}{s^2+2s+2}$

let $F_1(s) = \frac{s+1}{s^2+2s+2}$

$$L^{-1}(F_1(s)) = L^{-1}\left(\frac{s+1}{(s+1)^2+1^2}\right)$$

$$= e^{-t} L^{-1} \frac{s}{s^2+1}$$

$$= e^{-t} \cos t$$

$$L^{-1}(e^{-3s} F_1(s)) = e^{-(t-3)} \cos(t-3) u(t-3)$$

$$L^{-1}(F(s)) = e^{-(t-3)} \cos(t-3)\, u(t-3)$$

(b) $F(s) = \dfrac{1}{s^4-64} = \dfrac{1}{(s^2-8)(s^2+8)} = \cancel{\dfrac{1 * 1}{(s+\sqrt{8})(s-\sqrt{8})(s^2+8)}}$

$\cancel{\dfrac{1}{(s+\sqrt{8})(s-\sqrt{8})(s^2+8)} = \dfrac{1}{(s+2\sqrt{2})(s-2\sqrt{2})(s^2+8)} = \dfrac{A}{s+2\sqrt{2}} + \dfrac{B}{s-2\sqrt{2}} + (s+1)}$

$F(s) = \dfrac{1}{16}\left(\dfrac{1}{s^2-8} - \dfrac{1}{s^2+8}\right)$

$L^{-1}(F(s)) = \dfrac{1}{16} * \dfrac{1}{2\sqrt{2}} L^{-1}\left(\dfrac{2\sqrt{2}}{s^2-8} - \dfrac{2\sqrt{2}}{s^2+8}\right)$

$L^{-1}\left(\dfrac{1}{s^4-64}\right) = \dfrac{1}{32\sqrt{2}}\left(\sinh(2\sqrt{2}\,t) - \sin(2\sqrt{2}\,t)\right)$

(c) $F(s) = \dfrac{s}{s^2+2s-3} = \dfrac{s}{(s+1)^2-2^2} = \dfrac{s+1}{(s+1)^2-2^2} - \dfrac{1}{(s+1)^2-2^2}$

$L^{-1}(F(s)) = L^{-1}\left(\dfrac{s+1}{(s+1)^2-2^2} - \dfrac{1}{(s+1)^2-2^2}\right)$

$= e^{-t} L^{-1}\left(\dfrac{s}{s^2-2^2} - \dfrac{1}{2}\cdot\dfrac{2}{s^2-2^2}\right)$

$L^{-1}(F(s)) = e^{-t}\left(\cosh 2t - \dfrac{1}{2}\sinh 2t\right)$

(d) $\dfrac{1}{s(s^2+6s-7)} = \dfrac{1}{s} \cdot \dfrac{1}{(s-6)(s-1)}$

$$F_1(S) = \frac{1}{(s-6)(s-1)} = \frac{1}{5}\left[\frac{1}{s-6} - \frac{1}{s-1}\right]$$

$$L^{-1}(F_1(S)) = \frac{1}{5} L^{-1}\left(\frac{1}{s-6} - \frac{1}{s-1}\right)$$

$$= \frac{1}{5}\left(e^{6t} - e^{t}\right)$$

$$L^{-1}\left(\frac{1}{s} \cdot \frac{1}{(s-6)(s-1)}\right) = \frac{1}{5}\int_0^t \left(e^{6t} - e^{t}\right) dt$$

$$= \frac{1}{5}\left[\frac{e^{6t}}{6} - e^{t} - \frac{1}{6} + 1\right]$$

$$= \frac{1}{5}\left(\frac{e^{6t} - 6e^{t} - 5}{6}\right)$$

$$L^{-1}(F(S)) = \frac{1}{30}\left(e^{6t} - 6e^{t} - 5\right)$$

e) $\log\left(\dfrac{s^2-4}{s^2}\right)^{1/3} = \dfrac{1}{3}\left(\log(s^2-4) - \log(s^2)\right)$

$$\Rightarrow L^{-1}(F(s)) = \frac{1}{3} L^{-1}\left(\log(s^2-4) - \log(s^2)\right)$$

$$= \frac{1}{3} \cdot \left(\frac{-1}{t}\right) L^{-1}\left(\frac{2s}{s^2-4} - \frac{2}{s}\right) = \frac{-2}{3t}\left(\cosh 2t - 1\right)$$

$$L^{-1}(FS) = \frac{-4}{3t}\sinh^2 t$$

(d) $F(s) = \dfrac{5s+2}{(s+1)(s^2-2s+5)}$

$$\frac{5s+2}{(s+1)(s^2-2s+5)} = \frac{A}{s+1} + \frac{Bs+C}{s^2-2s+5}$$

$$5s+2 = A(s^2-2s+5) + (Bs+C)(s+1)$$

let $s=-1$

$$-3 = 8A$$

$$A = \frac{-3}{8}$$

let $s=0$

$$2 = 5A + C$$

$$2 = \frac{-15}{8} + C$$

$$C = \frac{31}{8}$$

let $s=1$

$$7 = 4A + (B+C)*2$$

$$7 = \frac{-3}{2} + 2\left(B+\frac{31}{8}\right)$$

$$\frac{17}{2 \times 2} = B + \frac{31}{8}$$

$$B = \frac{3}{8}$$

$$F(s) = \frac{-1}{s+1} + \frac{1}{8} \frac{3s+31}{s^2-2s+5}$$

$$\cancel{= \frac{-1}{s+1} + \frac{1}{8}\left[\frac{3(s+1)}{s^2-2s+5} + \frac{28}{s^2-2s+5}\right]}$$

$$\cancel{= -\frac{1}{s+1} + \frac{1}{8}\frac{3(s+1)}{s}}$$

$$= \frac{-1}{s+1} + \frac{1}{8}\left[\frac{3(s-1)}{(s-1)^2+2^2} + \frac{34}{(s+1)^2+2^2}\right]$$

$$L^{-1}(F(s)) = L^{-1}\left(\frac{-1}{s+1} + \frac{1}{8}\left[\frac{3(s-1)}{(s-1)^2+2^2} + 17 * \frac{2}{(s-1)^2+2^2}\right]\right)$$

$$= -e^{-t} + \frac{1}{8}\left[3e^{t} L^{-1}\left(\frac{s}{s^2+2^2}\right) + 17e^{t} L^{-1}\frac{2}{s^2+2^2}\right]$$

$$\cancel{= -e^{-t} + \frac{1}{8}\left[\frac{e^{t}}{2} \cdot \sin 2t + 17 e^{t} \sin 2t\right]}$$

$$= -e^{-t} + \frac{1}{8}\left[3e^{t}\cos 2t + 17e^{t}\sin 2t\right.$$

$$L^{-1}(F(s)) = -e^{-t} + \frac{e^{t}}{8}(3\cos 2t + 17\sin 2t)$$

Q.4

$$y'' + 2y' + y = 3te^{-t} \quad , y(0) = 0 , y'(0) = 2$$

Taking laplace on both sides.

$$L(y'') + 2L(y') + L(y) = 3L(te^{-t})$$

$$s^2 Y(s) - sy(0) - y'(0) + 2(sY(s) - y(0))$$

$$+ Y(s) = 3 * \frac{1}{(1+s)^2}$$

$$Y(s)(s^2+2s+1) - 2 = \frac{3}{(s+1)^2}$$

$$Y(s)(s+1)^2 = \frac{3}{(s+1)^2} + 2$$

$$Y(s) = \frac{3}{(s+1)^4} + \frac{2}{(s+1)^2}$$

Taking inverse laplace at both sides.

$$y(t) = L^{-1}\left(\frac{3}{(s+1)^4} + \frac{2}{(s+1)^2}\right)$$

$$= e^{-t}\left(L^{-1}\left(\frac{3}{s^4} + \frac{2}{s^2}\right)\right)$$

$$y(t) = e^{-t}\left(\frac{3t^3}{6} + 2t\right)$$

$$\boxed{y(t) = e^{-t}\left(\frac{t^3}{2} + 2t\right)}$$