Student Information :-

Name - Rohan Kumar Pandey
Program - MCA (B3)
SAP ID - 590010160
Instructor - Saurabh Shanu
Date - 04.10.2024

Section 1

1) Transition matrix calculation:

$P(A \to A) = 0.7$
$P(A \to B) = 0.3$
$P(B \to A) = 0.4$
$P(B \to B) = 0.6$

The transition matrix P :-

$$P = \begin{pmatrix} P(A \to A) & P(A \to B) \\ P(B \to A) & P(B \to B) \end{pmatrix}$$

$$P = \begin{pmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{pmatrix}$$

2) Next State Probability.

$$P = \begin{pmatrix} 0.6 & 0.4 \\ 0.3 & 0.7 \end{pmatrix}$$

$P(A \to B) = 0.4$

3) Two step Transition Probability.

$$P = \begin{pmatrix} 0.5 & 0.5 \\ 0.4 & 0.6 \end{pmatrix}$$

$$P^2 = \begin{pmatrix} 0.45 & 0.55 \\ 0.44 & 0.56 \end{pmatrix}$$

$P(A \to B) = 0.55$

55% prob.

4. State Probability after one step.

$\pi_0 = (1, 0)$

$$P = \begin{pmatrix} 0.8 & 0.2 \\ 0.5 & 0.5 \end{pmatrix}$$

$$\pi_1 = \pi_0 \times P = (1, 0) \times \begin{pmatrix} 0.8 & 0.2 \\ 0.5 & 0.5 \end{pmatrix}$$

$\pi_1 = (0.8 \quad 0.2)$

This means there are 80% prob. of being in State A and 20% prob. of being in State B.

5. Three-state Transition matrix.

Given

$P(A \rightarrow B) = 0.2$
$P(A \rightarrow C) = 0.3$
$P(B \rightarrow A) = 0.4$
$P(C \rightarrow A) = 0.5$
$P(C \rightarrow B) = 0.3$

For state A

$P(A \rightarrow A) = 1 - P(A \rightarrow B) - P(A \rightarrow C)$

$= 1 - 0.2 - 0.3 = 0.5$

For state B

$P(B \rightarrow B) = 1 - P(B \rightarrow A) - P(B \rightarrow C)$

$P(B \rightarrow C) = 1 - 0.4 = 0.6$

For state C

$P(C \rightarrow C) = 1 - P(C \rightarrow A) - P(C \rightarrow B)$

$= 1 - 0.5 - 0.3 = 0.2$

$$P = \begin{pmatrix} 0.5 & 0.2 & 0.3 \\ 0.4 & 0.0 & 0.6 \\ 0.5 & 0.3 & 0.2 \end{pmatrix}$$

6) Absorbing State Identification.

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0.2 & 0.5 & 0.3 \\ 0.3 & 0.3 & 0.4 \end{pmatrix}$$

$P(A \to A) = 1$

$P(B \to B) = 0.5$

$P(C \to C) = 0.4$

since $P(C \to C)$ is not equal to 1 State C is not an absorbing state.

The only absorbing stat in this transition matrix is state 1 (state A)

7) Equilibrium Distribution Check.

$\pi P = \pi$

↳ FOR eqb. distribution. We need to prove that $\pi P = \pi$

$\pi = (0.4 \quad 0.6)$

$$P = \begin{pmatrix} 0.6 & 0.4 \\ 0.3 & 0.7 \end{pmatrix}$$

$$\pi P = (0.4 \quad 0.6) \times \begin{pmatrix} 0.6 & 0.4 \\ 0.3 & 0.7 \end{pmatrix}$$

$$\pi P = (0.42 \quad 0.58)$$

$$\pi P = \pi$$

$$(0.42 \quad 0.58) \neq (0.4 \quad 0.6)$$

So, the candidate distribution $(0.4 \quad 0.6)$ is not an eqb. distribution for the given ~~st~~ transition matrix.

(8) Transition Prob. Summation.

$$P = \begin{pmatrix} 0.5 & 0.5 \\ 0.2 & 0.8 \end{pmatrix}$$

First row

$$0.5 + 0.5 = 1.0$$

Second row $0.2 + 0.8 = 1.0$

Since both row to 1, the row sums of the transition matrix do indeed add up to 1.

(9) State prob. cal.

$\pi_0 = (0.7 \quad 0.3)$

$P = \begin{pmatrix} 0.8 & 0.2 \\ 0.5 & 0.5 \end{pmatrix}$

$\pi_1 = \pi_0 \times P$

$= (0.7 \quad 0.3) \times \begin{pmatrix} 0.8 & 0.2 \\ 0.5 & 0.5 \end{pmatrix}$

$\pi_1 = (0.71 \quad 0.29)$

After one step, the state distri-bution is $(0.71 \quad 0.29)$.

(10) Irreducibility check:-

$P = \begin{pmatrix} 0.5 & 0.5 \\ 0.2 & 0.8 \end{pmatrix}$

$P(1 \to 1) = 0.5$
$P(1 \to 2) = 0.5$
$P(2 \to 1) = 0.2$
$P(2 \to 2) = 0.8$

Let's consider the squared matrix

$p2 = p \times p$

$$= \begin{pmatrix} 0.5 & 0.5 \\ 0.2 & 0.8 \end{pmatrix} \times \begin{pmatrix} 0.5 & 0.5 \\ 0.2 & 0.8 \end{pmatrix}$$

$$p^2 = \begin{pmatrix} 0.35 & 0.65 \\ 0.26 & 0.74 \end{pmatrix}$$

The squared matrix shows that it is still possible to move between all states (even indirectly)

Since It is possible to move from any state to any other state, the markov chain discribed by the given transition matrix is irreducible.

Section (2)

(1) Stationary Distribution calculation.

$$p = \begin{pmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{pmatrix}$$

$\pi p = \pi$

Where $\pi = (\pi_1 \ \pi_2)$ is the stationary distribution vector.

$$(\pi_1 \;\; \pi_2) \begin{pmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{pmatrix} = (\pi_1 \;\; \pi_2)$$

$$0.7\pi_1 + 0.4\pi_2 = \pi_1$$
$$0.3\pi_1 + 0.6\pi_2 = \pi_2$$

$$0.7\pi_1 + 0.4\pi_2 = \pi_1 = 3/4$$
$$0.3\pi_1 + 0.6\pi_2 = \pi_2 = 3/4$$

Additionally, since the sum of stationary must equal 1

$$\pi_1 + \pi_2 = 1$$

$$\pi_1 + \frac{3}{4}\pi_1 = 1 \Rightarrow \frac{7}{4}\pi_1 = 1$$

$$\Rightarrow \pi_1 = \frac{4}{7}$$

$$\pi_2 = \frac{3}{4}\pi_1 = \frac{3}{7}$$

$$\left(\frac{4}{7}, \frac{3}{4}\right).$$

2 Two step Transition Matrix.

$$P^2 = P \times P$$

$$P = \begin{bmatrix} 0.8 & 0.2 \\ 0.4 & 0.6 \end{bmatrix}$$

$$P^2 = \begin{bmatrix} 0.8 & 0.2 \\ 0.4 & 0.6 \end{bmatrix} \times \begin{bmatrix} 0.8 & 0.2 \\ 0.4 & 0.6 \end{bmatrix}$$

$$= \begin{bmatrix} 0.72 & 0.28 \\ 0.56 & 0.44 \end{bmatrix}$$

3 Expected Number of steps to Absorption.

$$P = \begin{pmatrix} 1 & 0 \\ 0.5 & 0.5 \end{pmatrix}$$

Idetify the fundamental matrix N :-

$$Q = 0.5$$

$$N = (I - Q)^{-1}$$

$$I - Q = 1 - 0.5 = 0.5$$

$$N = \frac{1}{0.5} = 2$$

(4) Mixing Time calculation.

The stationary distribution $\pi = (\pi_1, \pi_2)$

$\pi P = \pi$

$$P = \begin{pmatrix} 0.6 & 0.4 \\ 0.3 & 0.7 \end{pmatrix}$$

$\pi P = (\pi_1 \; \pi_2)$

$$(\pi_1 \; \pi_2)\begin{pmatrix} 0.6 & 0.4 \\ 0.3 & 0.7 \end{pmatrix} = (\pi_1 \; \pi_2)$$

$\pi_1(0.6) + \pi_2(0.3) = \pi_1$
$\pi_1(0.4) + \pi_2(0.7) = \pi_2$

$(\pi_1 + \pi_2 = 1)$

using first eq:

$0.6\pi_1 + 0.3\pi_2 = \pi_1$

$0.3\pi_1 = 0.3\pi_2$
$\pi_1 = \pi_2$

we have,
~~Since~~ $(\pi_1 + \pi_2) = 1$

$\pi_1 = \pi_2 = 0.5$

thus the stationary distribution is:

$$\pi = \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix}$$

n step = Initial distb. $\times P^n$

$= 0.01 \times [0.5 \ 0.5]$

(5) Long term Prob.

$$(\pi_A \quad \pi_B) \begin{pmatrix} 0.9 & 0.1 \\ 0.4 & 0.6 \end{pmatrix} = (\pi_A \quad \pi_B)$$

eq (I) $\pi_A \cdot 0.9 + \pi_B \cdot 0.4 = \pi_A$

eq (II) $\pi_A \cdot 0.1 + \pi_B \cdot 0.6 = \pi_B$

eq (I)
Rearrange :- $0.9\pi_A + 0.4\pi_B = \pi_A$

$$\pi_B = \frac{0.1}{0.4} \pi_A$$

$$\pi_B = 0.25\,\pi_A$$

Eq (ii) Rearrange:-

$$0.1\,\pi_A - 0.4\,\pi_B = 0$$

Substitute $\pi_B = 0.25\,\pi_A$ into this eq.

$$0.1\pi_A - 0.4 \cdot 0.25\,\pi_A = 0$$

$$0.1\pi_A - 0.1\pi_A = 0$$

$$0 = 0$$

using $\pi_A + \pi_B = 1$:

$$\pi_A = 0.8.$$

$$\pi_B = 0.25\,\pi_A$$

$$= 0.25 \cdot 0.8 = 0.2$$

The long term prob. of being in state A (π_A) is 0.8 or 80%

(6) $P = \begin{bmatrix} 0.6 & 0.4 \\ 0.4 & 0.6 \end{bmatrix}$

$$E[T_A] = \frac{1}{\pi_A}$$

$$\pi_A + \pi_A = 1$$

$$2\pi_A = 1 \Rightarrow \pi_A = \frac{1}{2} = 0.5$$

$$E[T_A] = \frac{1}{0.5}$$

$$= 2$$

(7) Entropy of Markov chain

~~$\log_2 0.5 = -1$~~

~~0 1 0 0 0 0~~

$$P = \begin{bmatrix} 0.5 & 0.5 \\ 0.3 & 0.7 \end{bmatrix}$$

$$H(P) = -\sum_i P_i \log_2 P_i$$

For first Row: $P_1 = 0.5$
$P_2 = 0.5$

$$H(\text{Row1}) = -(0.5\log_2 0.5 + 0.5\log_2 0.5)$$

$$= -(0.5.(-1) + 0.5.(-1)) = 1$$

For second Row: $P_1 = 0.3$
$P_2 = 0.7$

$$H(\text{Row 2}) = -(0.3\log_2 0.3 + 0.7\log_2 0.7)$$

using the values.

$$\log_2 0.3 \approx -1.737$$
$$\log_2 0.7 \approx -0.514$$

therefore,

$$H(\text{ROW } 2) = -(0.3 \cdot (-1.737) + 0.7(-0.514))$$

$$\approx 0.3 \cdot 1.737 + 0.7 \cdot 0.514 \approx 0$$

$$H(\text{average}) = \frac{H(\text{ROW } 1) + H(\text{ROW } 2)}{2}$$

$$= \frac{1 + 0.881}{2}$$

$$= 0.941 \text{ bits.}$$

Appx. 94 %

⑧ Eigenvalues and Eigenvectors:

$$P = \begin{pmatrix} 0.7 & 0.3 \\ 0.2 & 0.8 \end{pmatrix}$$

The eigenvalues of the transition matrix A are:

$\cancel{A =}$

$$\lambda_1 = 0.5 \quad \lambda_2 = 1.0$$

/B/

The corresponding eigenvectors are:-

$$v_1 = \begin{pmatrix} -0.8321 \\ 0.5547 \end{pmatrix}$$

$$v_2 = \begin{pmatrix} -0.7071 \\ -0.7071 \end{pmatrix}$$

(9) Expected Number of visit to a state

$$P = \begin{pmatrix} 0.8 & 0.2 \\ 0.5 & 0.5 \end{pmatrix}$$

State A corrosponds to state 1
State B corrosponds to state 2.

since this ~~matter~~ matrix does not include an absorbing state (a state where, once entered, you can't leave) calculateing the expected no. of visit to state B before absorption isn't applicable.

(10) conditional probability calculation.

$$P = \begin{pmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{pmatrix}$$

~~09/11/2~~

16

$P = \begin{pmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{pmatrix}$

$P(A \to A) = 0.7$ and $P(A \to B) = 0.3$

$v_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$V_2 = v_0 \times P \times P$

$V_1 = v_0 \times P$

$= (1 \;\; 0) \times \begin{pmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{pmatrix} = (0.7 \;\; 0.3)$

$v_2 = v_1 \times P = (0.7 \;\; 0.3) \times \begin{pmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{pmatrix}$

Section 3

1. Stationary Distribution for a Large matrix

$P = \begin{pmatrix} 0.4 & 0.3 & 0.3 \\ 0.2 & 0.5 & 0.3 \\ 0.3 & 0.2 & 0.5 \end{pmatrix}$

$\pi = (\pi_1 \;\; \pi_2 \;\; \pi_3)$

$\pi P = \pi$

$0.4\pi_1 + 0.2\pi_2 + 0.3\pi_3 = \pi_1$
$0.3\pi_1 + 0.5\pi_2 + 0.2\pi_3 = \pi_2$
$0.3\pi_1 + 0.3\pi_2 + 0.5\pi_3 = \pi_3$

Aditionally, the sum of the probability must equal 1:

$$\pi_1 + \pi_2 + \pi_3 = 1$$

(2) First Passage Time calculation.

$$P = \begin{bmatrix} 0.8 & 0.2 \\ 0.4 & 0.6 \end{bmatrix}$$

Let state A be denoted as 1 and state B as 2

~~E_1 is~~

Let E_1 be the expected time to go ~~to~~ from state 1 to state 2.

$$E_1 = 1\cdot(1-0.2) + (0.8\cdot E_1) + (0.2\cdot 0)$$

$$E_1 = 1 + 0.8E_1$$

$$0.2E_1 = 1$$
$$E_1 = \frac{1}{0.2} = 5$$

(3) $p = 0.5$
$x_0 = 3$ units
$A = 5$ units.
$B = 0$ units.

$$P_B(x_0) = \frac{B - x_0}{B - A}$$

$$P_B(3) = \frac{5-3}{5-0} = 0.4$$

The prob. of going bankrupt before reaching 5 units is 0.4 or 40%

(4) Inverse of Transition matrix :-

$$P = \begin{bmatrix} 0.6 & 0.4 \\ 0.3 & 0.7 \end{bmatrix}$$

$$\det(P) = (0.6 \times 0.7) - (0.4 \times 0.3)$$

$$= 0.42 - 0.12$$
$$= 0.3.$$

2×2 matrix P

$$P^{-1} = \frac{1}{\det(P)} \times \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

where the element a, b, c, d are from the original matrix :-

$$P = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$P^{-1} = \frac{1}{0.3} \times \begin{pmatrix} 0.7 & -0.4 \\ -0.3 & 0.6 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{0.7}{0.3} & \frac{-0.4}{0.3} \\ \frac{-0.3}{0.3} & \frac{0.6}{0.3} \end{pmatrix}$$

$$= \begin{pmatrix} 2.\bar{3} & -1.\bar{3} \\ -1 & 2 \end{pmatrix}$$

(5) Analyze a Continous - Time markov chain.

$$Q = \begin{pmatrix} -0.3 & 0.3 \\ 0.2 & -0.2 \end{pmatrix}$$

$$\pi Q = 0$$

$$(\pi_1 \ \pi_2) \begin{pmatrix} -0.3 & 0.3 \\ 0.2 & -0.2 \end{pmatrix} = (0 \ 0)$$

$-0.3\pi_1 + 0.2\pi_2 = 0$
$0.3\pi_1 - 0.2\pi_2 = 0$

$\pi_1 + \pi_2 = 1$

$-0.3\pi_1 + 0.2\pi_2 = 0 \Rightarrow 0.3\pi_1 = 0.2\pi_2$

$\Rightarrow \frac{\pi_1}{\pi_2} = 2/3$

$\pi_1 = 2/3\,\pi_2$

using the normalization condition

$\pi_1 + \pi_2 = 1$

$2/3\,\pi_2 + \pi_2 = 1$

$\pi_2 = 0.6$

$\pi_1 = 0.4$

$\pi = (0.4 \quad 0.6)$

This stationary distribution means that in the long run, the process will spend 40% of the time in state 1 and 60% of the time in state 2

(6) Solve for the Fundamental Matrix.

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0.3 & 0.4 & 0.3 \\ 0.2 & 0.5 & 0.3 \end{pmatrix}$$

$$P = \begin{pmatrix} I & 0 \\ R & Q \end{pmatrix}$$

$$I = (1)$$

$$Q = \begin{pmatrix} 0.4 & 0.3 \\ 0.5 & 0.3 \end{pmatrix}$$

$$R = \begin{pmatrix} 0.3 \\ 0.2 \end{pmatrix}$$

$$N = (I - Q)^{-1}$$

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$I - Q$:-

$$I - Q = \begin{pmatrix} 0.6 & -0.3 \\ -0.5 & 0.7 \end{pmatrix}$$

$$A^{-1} = \frac{1}{ab - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$(I-Q)^{-1} = \frac{1}{0.27}\begin{pmatrix} 0.7 & 0.3 \\ 0.5 & 0.6 \end{pmatrix}$$

$$(I-Q)^{-1} = \begin{pmatrix} 2.\overline{59} & 1.\overline{11} \\ 1.\overline{85} & 2.\overline{22} \end{pmatrix}$$

$$N = \begin{pmatrix} 2.\overline{59} & 1.\overline{11} \\ 1.\overline{85} & 2.\overline{22} \end{pmatrix}$$

(7) variable of time to Absorption.

$$P = \begin{pmatrix} 1 & 0 \\ 0.2 & 0.8 \end{pmatrix}$$

$$N = (I-Q)^{-1}$$

$$N = \frac{1}{1-Q} = \frac{1}{1-0.8} = \frac{1}{0.2} = 5$$

$$Var(T) = (2N-I).t - t.t$$

$$= 2N.N^2 = 2\times5\times5 - 5^2$$
$$= 25.$$

(23)

(8) Entropy Rate calculation.

$$P = \begin{pmatrix} 0.4 & 0.6 \\ 0.6 & 0.4 \end{pmatrix}$$

$\pi P = \pi$

$\pi_1 \times 0.4 + \pi_2 \times 0.6 = \pi_1$

$\pi_1 \times 0.6 + \pi_2 \times 0.4 = \pi_2$

$\pi_1 + \pi_2 = 1$

$0.4\pi_1 + 0.6\pi_2 = \pi_1$

$\Rightarrow 0.6\pi_2 = 0.6\pi_1 \Rightarrow \pi_2 = \pi_1$

$\pi_1 = 0.5$

$\pi_1 = 0.5$ and $\pi_2 = 0.5$.

$\pi = (0.5 \;\; 0.5)$

$$H(P) = -\sum_{i=1}^{n} \pi_i \sum_{j=1}^{n} p_{ij} \log p_{ij}$$

$H(P) = -(0.4 \log 0.4 + 0.6 \log 0.6)$

$\log_2 0.4 \simeq -1.32193$

$\log_2 0.6 \simeq -0.73697$

$H(P) \simeq 0.970954$.

24

9 $A = \begin{pmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{pmatrix}$

$B = \begin{pmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{pmatrix}$

$\pi = \begin{pmatrix} 0.6 & 0.4 \end{pmatrix}$

$O = [O_1, O_2, O_1]$:

$\delta_1(1) = \pi_1 \times B_1 O_2 = 0.6 \times 0.9 = 0.54$

$= \pi_2 \times B_2 O_1 = 0.4 \times 0.2 = 0.08$

$\psi(1) = 0$

$\psi(2) = 0$

10 Lyapunov Function for markov chain:-

$P = \begin{pmatrix} 0.8 & 0.2 \\ 0.5 & 0.5 \end{pmatrix}$

$\pi \cdot P = \pi$

$\pi_1 + \pi_2 = 1$

$\pi_1 \times 0.8 + \pi_2 \times 0.5 = \pi_1$

$\pi_1 \times 0.2 + \pi_2 \times 0.5 = \pi_2$

$$0.8\pi_1 + 0.5\pi_2 = \pi_1$$

$$0.5\pi_2 = 0.2\pi_1$$

$$\pi_2 = 0.4\pi_1$$

$$\pi_1 + 0.4\pi_1 = 1$$

$$1.4\pi_1 = 1$$

$$\pi_1 \simeq 0.714$$

$$\pi_2 \simeq 0.286.$$

$$\pi = \left(\frac{5}{7}, \frac{2}{7}\right)$$