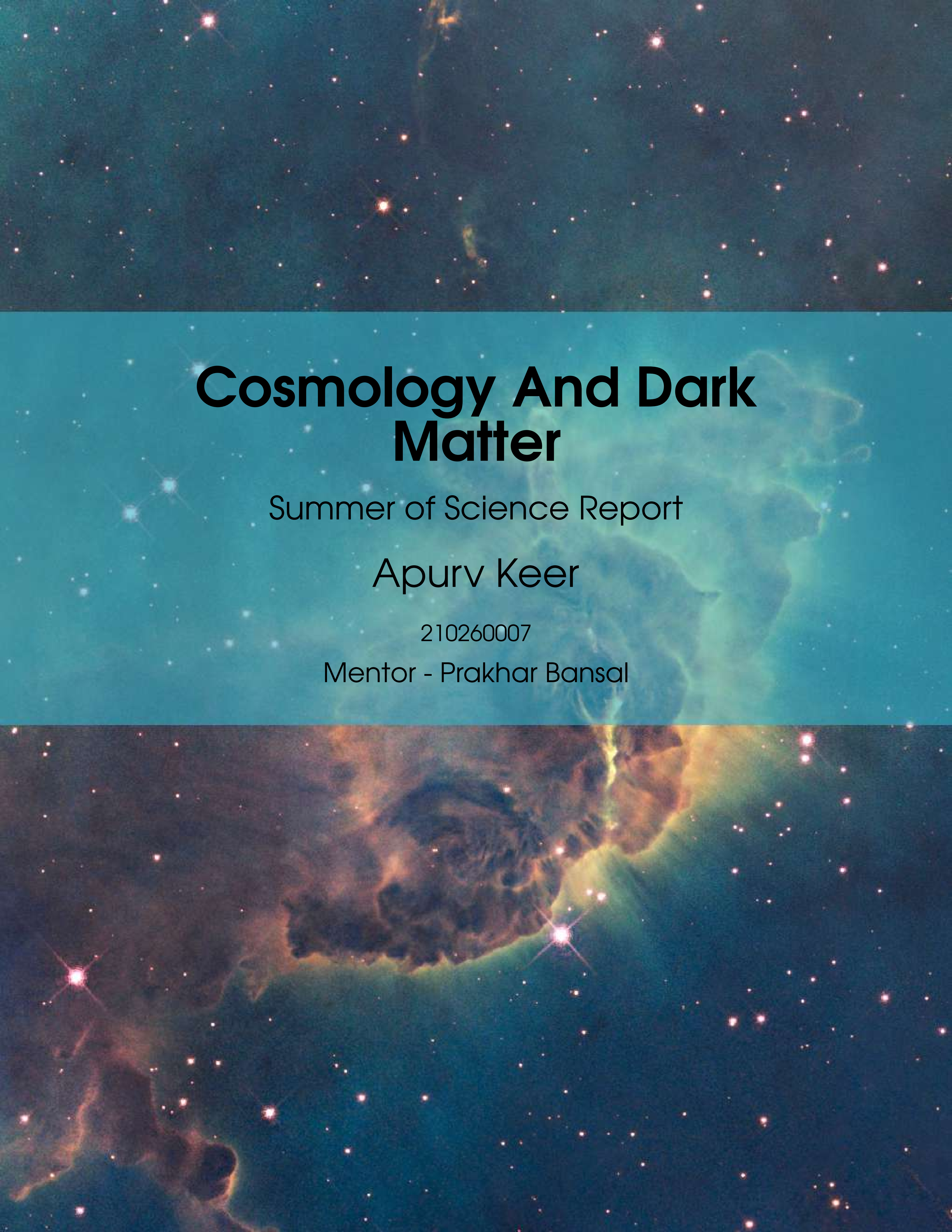

Cosmology And Dark Matter

Summer of Science Report

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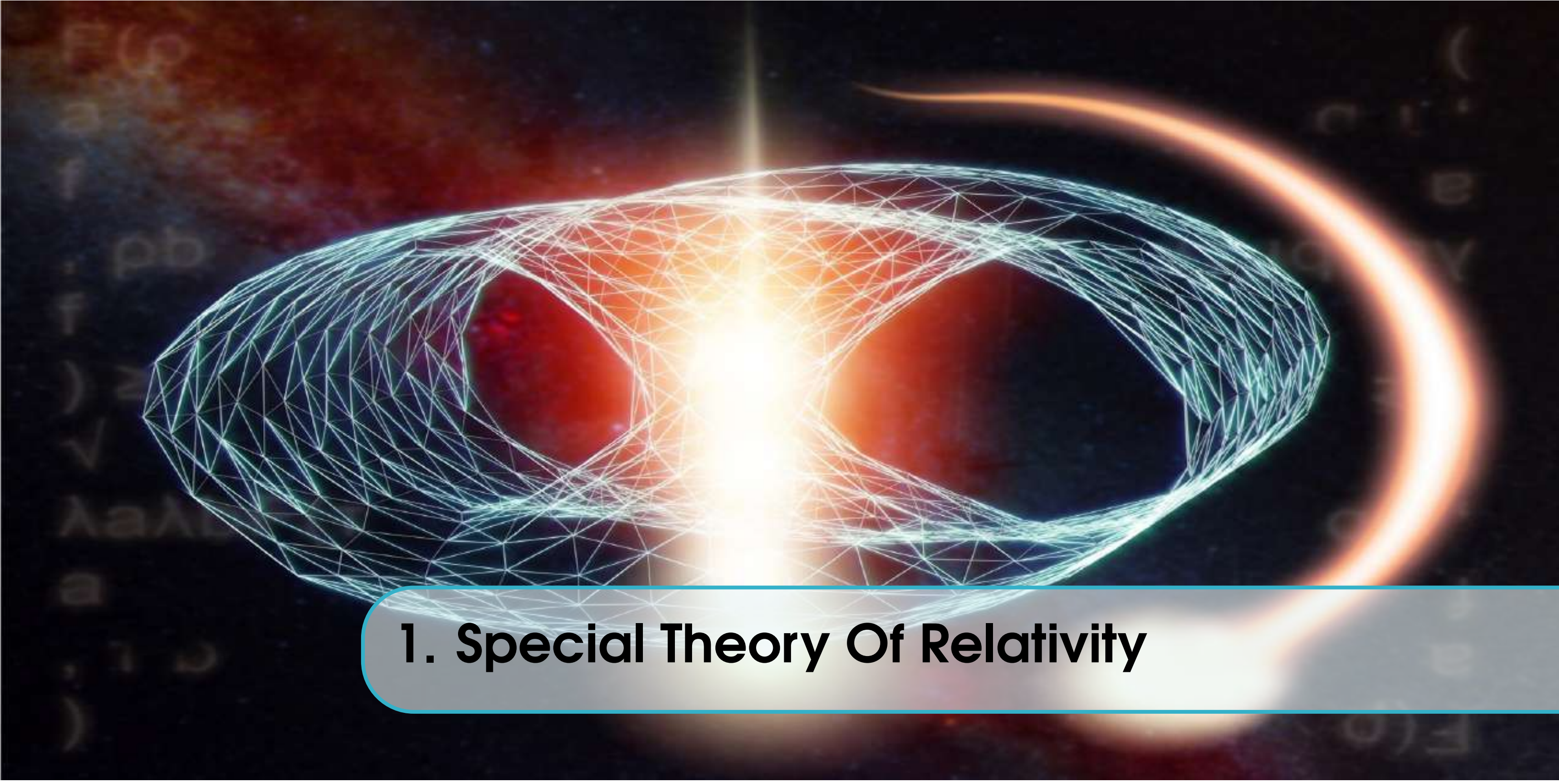


1. Special Theory Of Relativity

[3][9]

1.1 Introduction

In a macroscopic world, the speed of moving objects and mechanical waves with respect to any observer is quite low compared to the speed of light. But in a microscopic world, speed of these particles can reach quite close to that of light. But for these particles with very high speed, Newtonian mechanics fails to explain their motion.

In 1905, Albert Einstein published the special theory of relativity that predicts the result of mechanical experiments for the entire range of speed from very less to nearly equal to that of light's speed.

1.2 Ether Frame

Newtonian Relativity is based on Galilean Transformations. Galilean Transformation states that time-interval and space-interval measurements and absolute. Newtonian Relativity states that we can only speak of the relativity motion of one frame with respect to another and there is no absolute frame of reference. But when this idea was extended to electromagnetism, it inferred that the Maxwell's equations were different for different frames of reference. For a long time, it was considered that the relativity principle exists for mechanics but in case of electrodynamics, there is a preferred inertial frame, called the ether frame. This also meant that we should be able to locate this frame experimentally. Various experiments were performed over the years and it turned out that the ether hypothesis was unsupportable. They all concluded that the speed of light is same in all inertial frames and is independent of the relative motion of source and observer. So we needed a new relativity principle that was applicable to both mechanics and electromagnetism.

Einstein made two assumptions the relativity principle and the principle of constancy of the speed of light. The principle of relativity states that the laws of physics are same in all inertial frames. The second principle is obvious from the name that the speed of light is same in all inertial system and has the value c as calculated from electromagnetism.

$$c = \frac{1}{\sqrt{\mu_o \varepsilon_o}} = 2.99792458 \times 10^8 \, ms^{-1}$$

The special theory of relativity says that *the time and length interval measurements are relative and depend on the observer's frame of reference.* This explains why the measured speed of light is same for all. As Newton's equation of motion are invariant under a Galilean transformation, Einstein's laws are invariant under a Lorentz transformation.

1.3 Lorentz Transformations

Suppose we have two observers in two different inertial frames, say S and S', where S' is moving with constant velocity v with respect to S in x-direction. The coordinates of an event $\mathscr{A}$ are (ct, x, y, z) in frame S and (ct', x', y', z') in frame S'.Here I multiplied time by the speed of light to match it's units with other coordinates. So the Lorentz transformation of coordinates from S to S' are,

$$\begin{aligned}
ct' &= \frac{ct - (\frac{v}{c})x}{\sqrt{1 - \frac{v^2}{c^2}}} \\
x' &= \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}} \\
y' &= y \\
z' &= z
\end{aligned}$$

We then substitute $\beta = \frac{v}{c}$ and $\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$. Let's write it in matrix form.

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

$$\Lambda(v) = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

To get transformations from S' to S, we just reverse the velocity v. In matrix form, we take the inverse of $\Lambda(v)$.

$$\Lambda^{-1}(v) = \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$\vec{V} \to (ct, x, y, z)$ is a four dimensional vector, also called as *four-vector*.

Let $(\Delta t, \Delta x, \Delta y, \Delta z)$ be the difference between coordinates of $\mathscr{A}$ and $\mathscr{B}$ in some frame $\mathscr{O}$. The interval between these two events is defined as,

$$\Delta s^2 = -c^2(\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$$

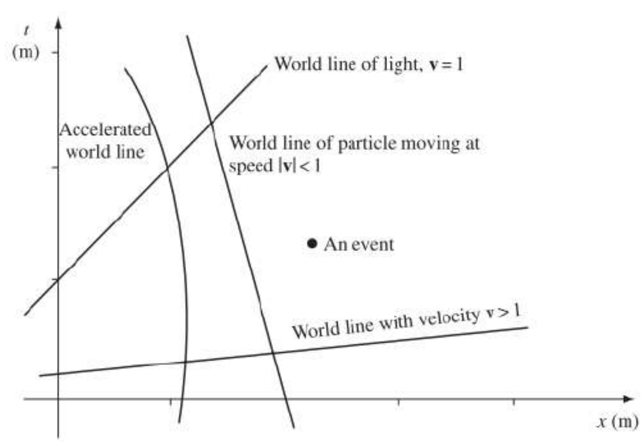
This quantity is Lorentz invariant or Lorentz scalar, i.e. it is same for any observer in any other frame, say $\mathscr{O}'$.

$$\Delta s^2 = (\Delta s')^2$$

Because Δs^2 is a property only of the two events and not of the observer, it can be used to classify the relation between the events. If Δs^2 is positive (so that the spatial increments dominate Δt), the events are said to be *spacelike* separated.If Δs^2 is negative, the events are said to be *timelike* separated.If Δs^2 is zero (so the events are on the same light path), the events are said to be *lightlike* or *null* separated.

1.4 Spacetime Diagram

Figure below shows a two dimensional slice of spacetime, the $t - x$ plane.



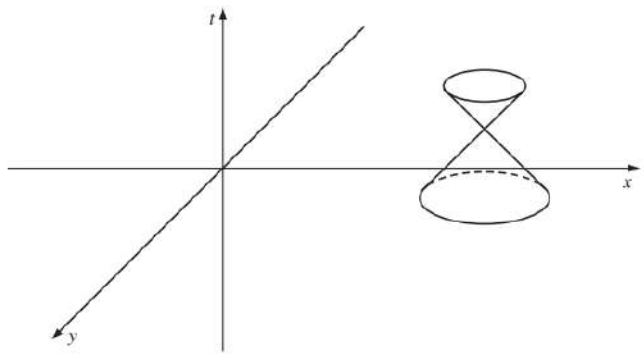
A spacetime diagram in natural units.

A point with fixed x and t is an *event*. A line shows the relation between the particle position and time. It is called a *worldline*. Its slope is given by,

$$\text{slope} = \frac{dt}{dx}$$

The worldline of a photon always has slope = 1.

The events that are lightlike separated from any particular event $\mathscr{A}$, lie on a cone whose apex is $\mathscr{A}$. This cone is illustrated in the figure below. This is called the light cone of $\mathscr{A}$. All events within the light cone are timelike separated from $\mathscr{A}$; all events outside it are spacelike separated. So only the events that are inside the cone can affect $\mathscr{A}$ and vice versa. As no signal can travel faster than light, no event outside the cone can affect $\mathscr{A}$ directly by a physical object. All events in the above part of the apex will happen in future and those in the below part of the apex must have happened in the past.



The light cone of an event. The z-dimension is suppressed.

1.5 Four Velocity

Suppose a point in S' frame, which is moving along x-axis with constant velocity v, is moving with velocity u', then the transformations are,

$$u_x = \frac{u'_x + v}{1 + \frac{u'_x v}{c^2}}$$

$$u_y = \frac{1}{\gamma} \frac{u'_y}{\left(1 + \frac{u'_y v}{c^2}\right)}$$

This transformations are very complicated. So, we break the velocity u into its components. This way the transformations become quite simple. We define a 4-dimensional velocity vector or *four velocity vector* as,

$$u = \begin{pmatrix} \frac{1}{\sqrt{1-\frac{u^2}{c^2}}} \\ \frac{u_x}{\sqrt{1-\frac{u^2}{c^2}}} \\ \frac{u_y}{\sqrt{1-\frac{u^2}{c^2}}} \\ \frac{u_z}{\sqrt{1-\frac{u^2}{c^2}}} \end{pmatrix}$$

where u_x, u_y and u_z are components of u such that $u = \sqrt{(u_x)^2 + (u_y)^2 + (u_z)^2}$. Now the transformations become,

$$u' = \Lambda(v)u$$

In special relativity, there is a relativity in simultaneity. Two events that are simultaneous to one observer will not to simultaneous to another observer with different speed. The measurement of lengths is more complicated in the theory of relativity than in classical mechanics. In classical mechanics, lengths are measured based on the assumption that the locations of all points involved are measured simultaneously. But in the theory of relativity, the notion of simultaneity is dependent on the observer. The term proper distance provides an invariant measure whose value is the same for all observers The *proper length* (L_o) of an object is the length of the object measured by an observer which is at rest relative to it, by applying standard measuring rods on the object. The measurement of the object's endpoints doesn't have to be simultaneous, since the endpoints are constantly at rest at the same positions in the object's rest frame, so it is independent of Δt

$$\begin{aligned} &\Delta t = 0 \\ &\Delta s^2 = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 = (\Delta\sigma)^2 \\ &L_o = \Delta\sigma = \sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2} \end{aligned}$$

.

In relativity, *proper time* (τ) along a timelike world line is defined as the time measured by a clock following that line. It is thus independent of coordinates, and is a Lorentz scalar.

$$\begin{aligned} \Delta x =& 0 \\ \Delta y =& 0 \\ \Delta z =& 0 \\ \Delta s^2 =& -c^2(\Delta t)^2 = -c^2(\Delta\tau)^2 \end{aligned}$$

2. General Theory Of Relativity

[1][6][4]

2.1 Introduction

Being one of the most amazing theory in the history of physics, Einsten published it in 1915. Special theory of Relativity lacks the explanation for gravity and that's where General Relativity comes in. To understand it we first need to learn what tensors and one-forms are and their mathematics.

2.2 Vector Analysis

Just to make everything easier, from now, we will consider the speed of light to be 1 and change the units accordingly. Here onwards we are going to study in terms of vectors. The *displacement vector* points from one event to another and has components equal to the coordinate differences.

$$\Delta\vec{x} \underset{\mathscr{O}}{\longrightarrow} (\Delta t, \Delta x, \Delta y, \Delta z)$$

The $\mathscr{O}$ underneath it means 'in the frame $\mathscr{O}$'. the components will always be in the order t, x, y, z (their indices will be in the order 0, 1, 2, 3). In index form, we write the vector as,

$$\Delta\vec{x} \underset{\mathscr{O}}{\longrightarrow} \{\Delta x^{\alpha}\}$$

where α takes the values 0, 1, 2, 3. Now the transformations from one frame to another goes as,

$$\Delta x^{\bar{\alpha}} = \sum_{\beta=0}^{3} \Lambda^{\bar{\alpha}}_{\beta} \Delta x^{\beta}$$

The *Einstein summation convention* says that whenever an expression contains one index as a superscript and the same index as a subscript, a summation is implied over all values that index can take. So finally, we write the Lorentz transformation as,

$$\Delta x^{\bar{\alpha}} = \Lambda^{\bar{\alpha}}_{\beta} \Delta x^{\beta}$$

In any frame $\mathscr{O}$ there are four special vectors, defined by giving their components,

$$\begin{aligned}
\vec{e}_0 &\underset{\mathscr{O}}{\longrightarrow} (1,0,0,0) \\
\vec{e}_1 &\underset{\mathscr{O}}{\longrightarrow} (0,1,0,0) \\
\vec{e}_2 &\underset{\mathscr{O}}{\longrightarrow} (0,0,1,0) \\
\vec{e}_3 &\underset{\mathscr{O}}{\longrightarrow} (0,0,0,1)
\end{aligned}$$

This vectors are defined in each frame. So, they, generally, are not same for different frames.

The β component of $\vec{e}_\alpha$ is the Kronocker delta, i.e. it is 1 when $\alpha = \beta$ and 0 otherwise.

$$((\vec{e}_\alpha)^{\beta}) = (\delta_\alpha)^{\beta})$$

Now, if any vector A has its components A^0, A^1, A^2, A^3 then it can be represented as,

$$\begin{aligned}
\vec{A} &= A^0\vec{e}_0 + A^1\vec{e}_1 + A^2\vec{e}_2 + A^3\vec{e}_3 \\
\vec{A} &= A^{\alpha}\vec{e}_\alpha \\
A^{\bar{\alpha}}\vec{e}_{\bar{\alpha}} &= A^{\alpha}\vec{e}_\alpha
\end{aligned}$$

The Lorentz transformations are,

$$\begin{aligned}
\vec{e}_\alpha &= \Lambda^{\bar{\beta}}_{\alpha}\vec{e}_{\bar{\beta}} \\
\vec{A}_\alpha &= \Lambda^{\bar{\beta}}_{\alpha}\vec{A}_{\bar{\beta}}
\end{aligned}$$

For inverse Lorentz transformations we just reverse the velocity,

$$\begin{aligned}
\vec{e}_\alpha &= \Lambda^{\bar{\beta}}_{\alpha}(v)\vec{e}_{\bar{\beta}} \\
\vec{e}_{\bar{\mu}} &= \Lambda^{\nu}_{\bar{\mu}}(-v)\vec{e}_{\nu}
\end{aligned}$$

We define the interval as,

$$\vec{A}\cdot\vec{A} = -(A^0)^2 + (A^1)^2 + (A^2)^2 + (A^3)^2$$

This must be a Lorentz scalar. If the product $\vec{A}\cdot\vec{A}$ is negative, $\vec{A}$ is said to be "timelike", "spacelike" if product is positive and "lightlike" or "null" if product is 0. For more general notation, we define a scalar product. This is a Lorentz scalar as well.

$$\begin{aligned}\vec{A}\cdot\vec{B} &= -A^0B^0 + A^1B^1 + A^2B^2 + A^3B^3\\ &= (A^\alpha\vec{e}_\alpha)\cdot(B^\beta\vec{e}_\beta)\\ &= A^\alpha B^\beta(\vec{e}_\alpha\cdot\vec{e}_\beta)\\ &= A^\alpha B^\beta\eta_{\alpha\beta}\end{aligned}$$

$$\eta_{\alpha\beta} = \begin{bmatrix}-1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

$\eta_{\alpha\beta}$ is called the *metric* tensor. For small element,

$$ds^2 = d\vec{x}\cdot d\vec{x} = \eta_{\alpha\beta}dx^\alpha dx^\beta$$

The *4-momentum vector* is defined as $\vec{p} = m\vec{u}$, where m is the rest mass and u is the 4-velocity vector. The time component of momentum is Energy, i.e. $p^0 = E$.

$$\begin{aligned}\vec{u} &= (\gamma, \gamma v)\\ \vec{u}\cdot\vec{u} &= -\gamma^2 + \gamma^2 v^2 = -1\\ \vec{p}\cdot\vec{p} &= m^2\vec{u}\cdot\vec{u} = -m^2 = -E^2 + |p|^2\\ E^2 - |p|^2 &= m^2\end{aligned}$$

Just like Newton's laws of motion, here, 4-momentum of a system of particles is conserved.

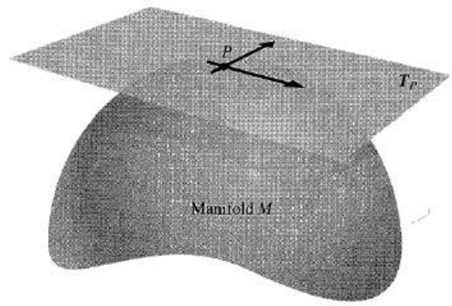


FIGURE 1.8 A suggestive drawing of the tangent space T_p, the space of all vectors at the point p.

Four vectors in spacetime are quite different than normal vectors that we know of. To each point in spacetime, we define a set of all possible at that point and this set is known as *tangent space* at p or T_p. Based on this vector space T_p, we define another vector space known as *dual vector space* T_p^*. The dual vector space is a space of all linear maps from the original vector space to a real number, i.e. if $\omega\,\varepsilon\,T_p^*$ and $V\,\varepsilon\,T_p$ then $\omega(V)\,\varepsilon\,\mathbb{R}$. We can express this in terms of basis vectors and components, which we label with lower indices, as,

$$\omega = \omega_\mu \hat{\theta}^{(\mu)}$$

Elements of T_p (vectors) are also called *contravariant* vectors and elements of T_p *(dual vectors)* are called *covariant* vectors. Another name for dual vectors is *one-forms*. The Lorentz transformations are, $\omega_{\mu'} = (\Lambda^\nu)_{\mu'}\omega_\nu$.

In spacetime the gradient of a scalar function is a dual vector.

$$\begin{aligned}
d\phi &= \frac{\partial\phi}{\partial x^\mu}\hat{\theta}^{(\mu)} \\
\frac{\partial\phi}{\partial x^\mu} &= \partial_\mu\phi \\
\frac{\partial\phi}{\partial x^{\mu'}} &= \Lambda^\mu{}_{\mu'}\frac{\partial\phi}{\partial x^\mu} \\
\partial_\mu\phi\frac{\partial\phi}{\partial\lambda} &= \frac{d\phi}{d\lambda}
\end{aligned}$$

2.3 Tensor analysis

A *tensor* is just a generalization of vectors and dual vectors. A tensor of type (m,n) is a linear function of m one forms and n vectors into a real number. A *tensor product* of a (k,l) tensor T and a (m,n) tensor S is defined as a $(k+m,l+n)$ as $T\otimes S$. In components notation, we write a tensor as,

$$T = T^{\mu_1\cdots\mu_k}{}_{\nu_1\cdots\nu_l} e_{(\hat{\mu}_1)} \otimes \cdots \otimes e_{(\hat{\mu}_k)} \otimes \theta^{(\hat{\nu}_1)} \otimes \cdots \otimes \theta^{(\hat{\nu}_l)}$$
$$T^{\mu_1'\cdots\mu_k'}{}_{\nu_1'\cdots\nu_l'} = \Lambda^{\mu_1'}{}_{\mu_1} \cdots \Lambda^{\mu_k'}{}_{\mu_k} \Lambda^{\nu_1'}{}_{\nu_1} \cdots \Lambda^{\nu_l'}{}_{\nu_l} T^{\mu_1\cdots\mu_k}{}_{\nu_1\cdots\nu_l}$$

The metric tensor mentioned above is a type $(0,2)$. The inverse metric tensor $\eta^{\nu\mu}$ is a $(2,0)$ type tensor such that,

$$\eta^{\mu\nu}\eta_{\nu\rho} = \eta_{\rho\nu}\eta^{\nu\mu} = \delta^{\nu}_{\rho}$$

The metric and inverse metric can be used to turn vectors into dual-vectors, i.e. raise or lower indices on a tensor.

$$V_\mu = \eta_{\mu\nu}V^\nu$$
$$\omega^\mu = \eta^{\mu\nu}\omega_\nu$$
$$T^{\alpha\beta\mu}{}_\delta = \eta^{\mu\gamma}T^{\alpha\beta}{}_{\gamma\mu}$$
$$T_\mu{}^\beta{}_{\gamma\delta} = \eta_{\mu\alpha}T^{\alpha\beta}{}_{\gamma\mu}$$
$$T^{\sigma}_{\mu\nu} = \eta_{\mu\alpha}\eta_{\nu\beta}\eta^{\rho\gamma}\eta^{\sigma\delta}T^{\alpha\beta}{}_{\gamma\mu}$$

2.4 Stress Energy Tensor

We think of the path of a particle as a parameterized curve $x^\mu(\lambda)$. It is quite convenient to use proper time as a parameter as it is measured by a clock on that path. So we can think of the path as $x^\mu(\tau)$. We define the velocity four vector and momentum four vector as,

$$U^\mu = \frac{dx^\mu}{d\tau}$$
$$p^\mu = mU^\mu$$

Rather than defining the momentum vector for individual vector, we define a system called *fluid*, a continuum characterized by quantities like density, pressure, etc. We then define the energy and momentum of the fluid with a tensor named *Energy-Momentum Tensor* or also called *Stress Energy Tensor*.

$$T^{\mu\nu} = \rho U^\mu U^\nu$$

where ρ is defined as the energy density in the rest frame. For a *perfect fluid*, one that can be specified by two quantities the rest frame energy density ρ and isotropic rest frame pressure p, the Stress Energy Tensor is,

$$T^{\mu\nu} = (\rho + p)U^{\mu}U^{\nu} + p\eta^{\mu\nu}$$

The Stress Energy Tensor is divergence free, i.e. $\partial_{\mu}T^{\mu\nu} = 0$.

2.5 Principles Of Equivalence

The principles of Equivalence relates Special relativity with General relativity. One of the principle of equivalence is the *Weak Equivalence Principle*. It states, *the motion of freely-falling particles are the same in gravitational field and a uniformly accelerated frame in a small region of spacetime.* There is an another principle called as *Einstein Equivalence Principle*e which states, *in small enough region of spacetime, the laws of physics reduce to that of special relativity; it is impossible to detect the existence of a gravitational field by means of local experiments*. Non-uniformities in gravitational fields are called *tidal forces*. These tidal forces prevent the construction of global inertial frames.

2.6 Tensor Derivatives

In plane polar coordinates, we define the metric tensor as,

$$\vec{e}_{\alpha} \cdot \vec{e}_{\beta} = g_{\alpha\beta} = diag(-1, 1, r^2, 1)$$
$$ds^2 = -dt^2 + dr^2 + r^2 d\phi^2 + dz^2$$

A *covariant derivative* of a vector, one-form and tensor is defined as,

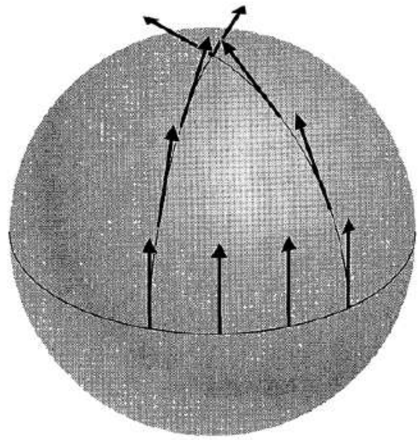
$$\nabla_{\beta}V^{\alpha} = \partial_{\beta}V^{\alpha} + V^{\mu}\Gamma^{\alpha}{}_{\beta\mu}$$
$$\nabla_{\beta}P_{\alpha} = \partial_{\beta}P_{\alpha} - P_{\mu}\Gamma^{\mu}{}_{\beta\alpha}$$
$$\nabla_{\alpha}T^{\beta}{}_{\gamma} = \partial_{\alpha}T^{\beta}{}_{\gamma} + \Gamma^{\beta}{}_{\alpha\mu}T^{\mu}{}_{\gamma} - \Gamma^{\mu}{}_{\alpha\gamma}T^{\beta}{}_{\mu}$$

$\Gamma^{\mu}{}_{\alpha\beta}$ is called *Christoffel* symbol.

$$\Gamma^{\mu}{}_{\alpha\beta} = \frac{1}{2}g^{\mu\gamma}(\partial_{\alpha}g_{\beta\gamma} + \partial_{\beta}g_{\gamma\alpha} - \partial_{\gamma}g_{\alpha\beta})$$
$$\nabla_{\gamma}g_{\alpha\beta} = 0$$

2.7 Parallel Transport and Geodesics

The concept of moving a vector along a path keeping it constant all the while is called *Parallel transport*. The crucial difference between flat and curved spaces is that in curved space the result of parallel transporting a vector from one point to another will depend on the path taken between the points.



Parallel transportation of a vector along a sphere

For a given curve $x^\mu(\lambda)$, the *directional covariant derivative* is defined as,

$$\frac{D}{d\lambda} = \frac{dx^\mu}{d\lambda}\nabla_\mu$$

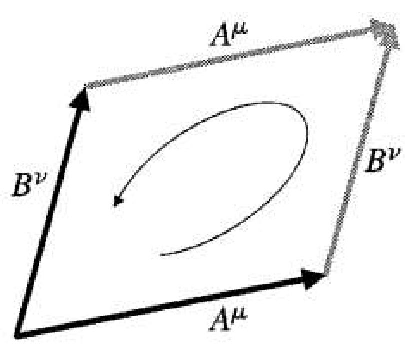
For a Tensor to be parallel transported along a path, its directional covariant derivative must vanish. For a vector, it takes the form,

$$\frac{d}{d\lambda}V^\mu + \Gamma^\mu{}_{\sigma\rho}\frac{dx^\sigma}{d\lambda}V^\rho = 0$$

A *geodesic* is a curve along which the tangent vector is parallel transported. The *geodesic equation* is,

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu{}_{\rho\sigma}\frac{dx^\rho}{d\lambda}\frac{dx^\sigma}{d\lambda} = 0$$

The Reimann Curvature Tensor



The Reimann curvature tensor quantifies curvature. If we parallel transport a vector V^ρ by moving it along vector A^μ then along B^ν and then backwards along A^μ and B^ν to return to its starting point, than the Reimann tensor gives us the change δV^ρ,

$$\delta V^\rho = R^\rho{}_{\sigma\mu\nu} V^\sigma A^\mu B^\nu$$
$$R^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho{}_{\nu\sigma} - \partial_\nu \Gamma^\rho{}_{\mu\sigma} + \Gamma^\rho{}_{\mu\lambda}\Gamma^\lambda{}_{\nu\sigma} - \Gamma^\rho{}_{\nu\lambda}\Gamma^\lambda{}_{\mu\sigma}$$

When we contract the Reimann tensor, we get the *Ricci* tensor.

$$R_{\mu\nu} = R^\lambda{}_{\mu\lambda\nu} = R_{\nu\mu}$$

The trace of Ricci tensor is the Ricci scalar or the curvature scalar.

$$R = R^\mu{}_\mu = g^{\mu\nu} R_{\mu\nu}$$

To sum it all, we define the *Einstein* Tensor as,

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$$

This tensor, by its name, is going to be very important. It is a divergence free tensor, i.e. $\nabla^\mu G_{\nu\mu} = 0$.

2.8 Einstein Field Equation

The Poisson equation for Newtonian Gravitational potential is,

$$\nabla^2 \phi = 4\pi G \rho$$

where $\nabla^2 = \delta^{ij}\partial_i\partial_j$ is the Laplacian operator and ρ is the mass density. If we look at the right hand side of this equation, ρ is a component of the Stress Energy tensor. A law is not quite good if it picks out only a component of a tensor. So the tensor equivalent of ρ is the stress energy tensor $T_{\mu\nu}$.

The left hand side of this equation is a double derivative of a potential. The Reimann tensor has a product of two Christoffel symbols in it, i.e. two derivatives. As the Reimann tensor has 4 indices and Stress Energy tensor has only two, we look at the next best choice, i.e. the Ricci tensor $R_{\mu\nu}$. Stress Energy tensor is divergence free but Ricci tensor isn't. Thus we consider the Einstein tensor which has two indices and is divergence free. The units doesn't match. So we put in a constant. Thus the Einstein Field Equation is,

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu}$$
$$or$$
$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \;\; for\; c \neq 1$$

Einstein equation relates curvature with mass ,i.e. energy. As you can see, G is quite small while c^4 is a very large number. So huge amount of energy can curve very small amount of spacetime. Thus it's very hard to bend spacetime.

Now one might think that the metric tensor also has two indices and is divergence free. So can we take this into account in the field equation? And the answer is YES! So now the field equation is written as,

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}$$

where Λ is called the cosmological constant.

3. Cosmology

3.1 Mathematical models of different universes

[1][2][6]

Cosmological models are based on the Copernican principle − the universe is same everywhere. I know this can be a bit weird thing to say as how can we be the center of the entire universe. But we just consider this at very very large scale. Mathematically, the Copernican principle says that the manifold has to be Isotropic and Homogeneous. Isotropy says that the manifold looks the same in all directions along any point. Homogeneity states that the manifold has same properties everywhere. Think of isotropy as invariance under rotations and boosts and homogeneity as invariance under translation. These together states that the space is Maximally Symmetric i.e. it has its maximum possible number of Killing vectors.

For a maximally symmetric n-dimensional manifold, the Reimann tensor and the Ricci tensor can be written as,

$$R_{\rho\sigma\mu\nu} = \frac{R}{n(n-1)}(g_{\rho\mu}g_{\sigma\nu} - g_{\rho\nu}g_{\sigma\mu})R_{\mu\nu} = \frac{R}{n}g_{\mu\nu}$$

where R is the Ricci scalar.

Our universe is spatially symmetric on large scales but not timely. What does ‘large’ means? We need to probe the largest scales in our universe. For that we need to go back nearly 13.7 billion years and we see the Cosmic Microwave Background (CMB). The CMB deviates from regularity about 10^{-5} so its an adequate basis for approximation of spacetime for large scales.

We take the spacetime metric to be,

$$ds^2 = -dt^2 + R^2(t)\gamma_{ij}dx^i dx^j$$

$R(t)$ is called the scale factor. We assume dx to be dimensionless and all the length scales are considered in the scale factor. We assume the observer, at rest in the frame, to be co-moving with the spacetime i.e. it just tracks the spacetime. We write the four velocity to be $\vec{U} = diag(1,0,0,0)$. So for a spatial metric,

$$\begin{aligned} R_{ijkl} &= k(\gamma_{ik}\gamma_{jl} - \gamma_{il}\gamma_{jk}) \\ R_{jl} &= 2k\gamma_{jl} \\ R &= 6k \end{aligned}$$

Here k can only take the values {-1, 0, 1}. As γ_{ij} is isotropic, it has to be spherically symmetric. So

$$\begin{aligned} \gamma_{ij}dx^i dx^j &= f(\bar{r})d\bar{r} + \bar{r}^2 d\Omega \\ d\Omega &= d\theta^2 + sin^2\theta d\phi^2 \end{aligned}$$

where $f(\bar{r})$ is some function of radius. $\bar{r}$ just says that its dimensionless. By doing some manipulations on the Ricci tensor and setting $f = 1$ at $\bar{r} = 0$, we get the line element as,

$$ds^2 = -dt^2 + R^2(t)\left[\frac{d\bar{r}^2}{1-k\bar{r}^2} - \bar{r}^2 d\Omega\right]$$

This is called the Robertson-Walker (RW) metric. Now let us look at some notations and terminology. We define a radial coordinate as χ where,

$$d\chi = \frac{d\bar{r}}{\sqrt{1-k\bar{r}^2}}$$

By integrating this, we get,

$$\bar{r} = \begin{cases} sin\chi & \text{if } k = 1 \\ \chi & \text{if } k = 0 \\ sin\chi & \text{if } k = -1 \end{cases}$$

The universe with $k = 1$ is called ‘closed’, the one with $k = 0$ is called ‘flat’ and the one with $k = -1$ is called ‘open’. Now we choose a special value of the scale factor for present time as R_o. So we define new parameters $a(t)$ and κ as,

$$\begin{aligned} a(t) &= \frac{R(t)}{R_o} \\ r &= R_o\bar{r} \\ \kappa &= \frac{k}{{R_o}^2} \\ ds^2 = -dt^2 &+ a^2(t)\left[\frac{dr^2}{1-\kappa r^2} + r^2 d\Omega\right] \end{aligned}$$

Until now, we were just looking at the geometry of the spacetime. Now let's put this in the Einstein Field Equation. We consider our source to be a perfect fluid. If the fluid is at rest, it satisfies homogeneity and isotropy. Our Stress Energy tensor is,

$$T_{\mu\nu} = (\rho + p)U_\mu U_\nu + pg_{\mu\nu}$$

The Einstein equations can also be written as,

$$R_{\mu\nu} = 8\pi G(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T)$$

where T is the trace of stress energy tensor. Putting everything in, we get two equations. First one from considering the sum of $R_{00} + R_{ii}$ and second one from just R_{00}.

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{\kappa}{a^2} \tag{3.1}$$

$$\left(\frac{\ddot{a}}{a}\right) = -\frac{4\pi G}{3}(\rho + 3P) \tag{3.2}$$

These are called the two Friedmann Equations. When we solve these to find different line elements, we get Friedmann-Robertson-Walker metrics.

Let's define some more terminologies. We define,

$$H = \left(\frac{\dot{a}}{a}\right)$$

H is called the Hubble parameter. H_o is the value of the Hubble parameter at present time. It's value is somewhat between 72 to 73 km/s/Mpc. The change in scale factor tells us how to particles appear to move away from us with time while the Hubble parameter tells how much they have seperated after small time. We then define,

$$\rho_{crit} = \frac{3H^2}{8\pi G}$$
$$\Omega = \frac{\rho}{\rho_{crit}}$$

Using this the first Friedmann equation can be written as,

$$\Omega - 1 = \frac{\kappa}{H^2 a^2}$$

From this we can say that, if $\Omega < 1$ we have an open universe, spatially flat for $\Omega = 0$ and closed for $\Omega > 1$. Also let's say,

$$\rho_{curv} = -\frac{3H^2}{8\pi G a^2}$$
$$\Omega_{curv} = \frac{\rho_{curv}}{\rho_{crit}}$$

Now the Friedmann equation just becomes,

$$\Omega + \Omega_{curv} = 1$$

To make progress, we need to define an equation of state i.e. pressure as a function of density. We assume it to be,

$$P = w\rho$$

We start by imagining a universe dominated by a certain species ρ_i. For this, $P = w_i\rho_i$. From conservation of energy equation, we can find that,

$$\left(\frac{\dot{\rho}_i}{\rho_i}\right) = -3(1+w_i)\left(\frac{\dot{a}}{a}\right)$$
$$\frac{\rho}{\rho_o} = \left(\frac{a}{a_o}\right)^{-3(1+w)}$$

Now let's look at our first species, matter. By matter we mean a pressureless species i.e. $w_M = 0$. So putting it in the above equation we get,

$$\rho = \rho_o a^{-3}$$

Second species is radiation. Here $w_R = \frac{1}{3}$. So,

$$\rho = \rho_o a^{-4}$$

For third species, we have the cosmological constant. For this $w_c = -1$ and density is constant.

Putting this in the Friedmann equation, we get $a \;\alpha\; t^{\frac{2}{3(1+w)}}$. A matter dominated universe expands as $t^{\frac{2}{3}}$ and a radiation dominated universe expands as $t^{\frac{1}{2}}$. It gets a little tricky for the cosmological constant. But fear not! We can find this from the original Friedmann equation. We get,

$$a \;\alpha\; e^{\pm\sqrt{\frac{\Lambda}{3}}t}$$

where the cosmological constant $\Lambda = 8\pi G\rho$.

Our universe was radiation dominated at early times and matter dominated from from $a \sim 1/3000$ to $a \sim 1/2$. At $a = 0$, we have a singularity known as the Big Bang. It is the creation of the universe from a single state and not an explosion of matter into a pre-existing spacetime.

As the radiation travels across the universe, product of scale factor and the energy measured by a comoving observer is constant. As $E = \omega$, we get,

$$\frac{\omega_{obs}}{\omega_{emit}} = \frac{a_{emit}}{a_{obs}}$$

We define red shift (z_{em}) as,

$$z_{em} = \frac{\lambda_{obs} - \lambda_{em}}{\lambda_{em}}$$
$$a_{em} = \frac{1}{1 + z_{em}}$$

Now we need to understand the distance measured in cosmology. We define the instantaneous physical distance as $D_p = a(t)R_o\chi$. The observed velocity simply becomes,

$$v_{apparent} = \dot{D}_p = \dot{a}R_o\chi = \frac{\dot{a}}{a}D_p$$
$$v_{apparent} = H_o D_p$$

This is the very famous Hubble Expansion law. The observed velocity is directly proportional to the distance, for galaxies not too far.

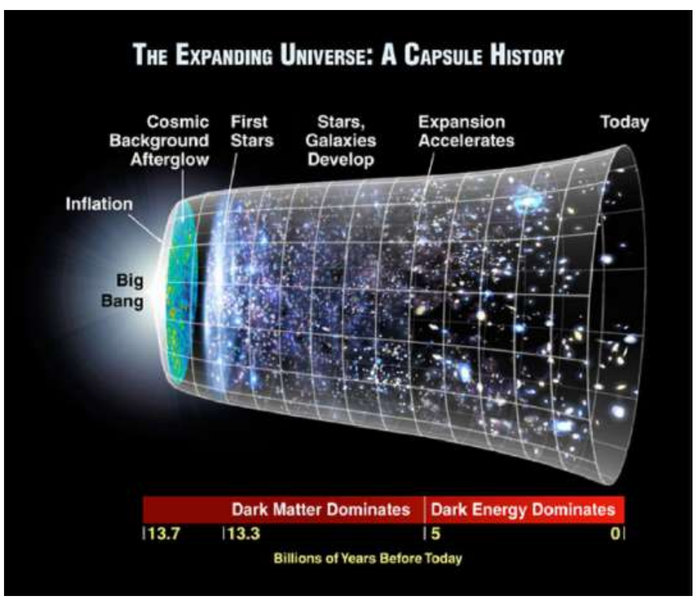
There are few other types of distances namely, Luminosity distance d_L, the proper distance motion d_M and the angular diameter distance d_A. They are defined as,

$$d_L^2 = \frac{L}{4\pi F}$$
$$d_M = \frac{u}{\dot{\theta}}$$
$$d_A = \frac{R}{\theta}$$

where L is the absolute luminosity of the source, F is the flux measured by the observer, u is the proper transverse velocity, $\dot{theta}$ is the observed angular velocity, R is the proper size of the object and θ is the observed angular diameter.

3.2 Epochs Of Our Universe

[10][5]



The Standard Model of cosmology attempts to explain how the universe physically developed after the ‘Big Bang’. This model is based on the Friedmann-Lemaître-Robertson-Walker metric. A singularity arises from this metric so it cannot explain what actually happened at the Big bang. We think quantum gravity might explain this properly.

Planck epoch

till 10^{-43} seconds (1 Planck Time)

This is the closest that current physics can get to the absolute beginning of time, and very little can be known about this period. General relativity proposes a gravitational singularity before this time (although even that may break down due to quantum effects), and it is hypothesized that the four fundamental forces (electromagnetism, weak nuclear force, strong nuclear force and gravity) all have the same strength, and are possibly even unified into one fundamental force, held together by a perfect symmetry which some have likened to a sharpened pencil standing on its point (i.e. too symmetrical to last). At this point, the universe spans a region of only 10^{-35} meters (1 Planck Length), and has a temperature of over $10^{32^o}C$ (the Planck Temperature).

Grand Unification Epoch

from 10^{-43} to 10^{-36} seconds

The force of gravity separates from the other fundamental forces (which remain unified), and the earliest elementary particles (and antiparticles) begin to be created.

Inflationary Epoch

from 10^{-36} to 10^{-32} seconds

Triggered by the separation of the strong nuclear force, the universe undergoes an extremely rapid exponential expansion, known as cosmic inflation. The linear dimensions of the early universe increases during this period of a tiny fraction of a second by a factor of at least 10^{26} to around 10 centimeters (about the size of a grapefruit). The elementary particles remaining from the Grand Unification Epoch (a hot, dense quark-gluon plasma, sometimes known as “quark soup”) become distributed very thinly across the universe.

Electroweak Epoch

from 10^{-36} to 10^{-12} seconds

As the strong nuclear force separates from the other two, particle interactions create large numbers of exotic particles, including W and Z bosons and Higgs bosons (the Higgs field slows particles down and confers mass on them, allowing a universe made entirely out of radiation to support things that have mass).

Quark Epoch

from 10^{-12} to 10^{-6} seconds

Quarks, electrons and neutrinos form in large numbers as the universe cools off to below 10 quadrillion degrees, and the four fundamental forces assume their present forms. Quarks and antiquarks annihilate each other upon contact, but, in a process known as baryogenesis, a surplus of quarks (about one for every billion pairs) survives, which will ultimately combine to form matter.

Hadron Epoch

Between 10^{-6} to 1 seconds

The temperature of the universe cools to about a trillion degrees, cool enough to allow quarks to combine to form hadrons (like protons and neutrons). Electrons colliding with protons in the extreme conditions of the Hadron Epoch fuse to form neutrons and give off massless neutrinos, which continue to travel freely through space today, at or near to the speed of light. Some neutrons and neutrinos re-combine into new proton-electron pairs. The only rules governing all this apparently random combining and re-combining are that the overall charge and energy (including mass-energy) be conserved.

Lepton Epoch

Between 1 second to 3 minutes

After the majority (but not all) of hadrons and antihadrons annihilate each other at the end of the Hadron Epoch, leptons (such as electrons) and antileptons (such as positrons) dominate the mass of the universe. As electrons and positrons collide and annihilate each other, energy in the form of photons is freed up, and colliding photons in turn create more electron-positron pairs.

Nucleosynthesis

Between 3 minutes to 20 minutes

The temperature of the universe falls to the point (about a billion degrees) where atomic nuclei can begin to form as protons and neutrons combine through nuclear fusion to form the nuclei of the simple elements of hydrogen, helium and lithium. After about 20 minutes, the temperature and density of the universe has fallen to the point where nuclear fusion cannot continue.

Photon Epoch (or Radiation Domination)

from 3 minutes to 240,000 years

During this long period of gradual cooling, the universe is filled with plasma, a hot, opaque soup of atomic nuclei and electrons. After most of the leptons and antileptons had annihilated each other at the end of the Lepton Epoch, the energy of the universe is dominated by photons, which continue to interact frequently with the charged protons, electrons and nuclei.

Recombination/Decoupling

from 240,000 to 300,000 years

As the temperature of the universe falls to around 3,000 degrees (about the same heat as the surface of the Sun) and its density also continues to fall, ionized hydrogen and helium atoms capture electrons (known as "recombination"), thus neutralizing their electric charge. With the electrons now bound to atoms, the universe finally becomes transparent to light, making this the earliest epoch observable today. It also releases the photons in the universe which have up till this time been interacting with electrons and protons in an opaque photon-baryon fluid (known as "decoupling"), and these photons (the same ones we see in today's cosmic background radiation) can now travel freely. By the end of this period, the universe consists of a fog of about 75% hydrogen and 25% helium, with just traces of lithium.

Dark Age (or Dark Era)

from 300,000 to 150 million years

The period after the formation of the first atoms and before the first stars is sometimes referred to as the Dark Age. Although photons exist, the universe at this time is literally dark, with no stars having formed to give off light. With only very diffuse matter remaining, activity in the universe has tailed off dramatically, with very low energy levels and very large time scales. Little of note happens during this period, and the universe is dominated by mysterious “dark matter”.

Reionization

150 million to 1 billion years

The first quasars form from gravitational collapse, and the intense radiation they emit reionizes the surrounding universe, the second of two major phase changes of hydrogen gas in the universe (the first being the Recombination period). From this point on, most of the universe goes from being neutral back to being composed of ionized plasma.

Star and Galaxy Formation

300 - 500 million years onwards

Gravity amplifies slight irregularities in the density of the primordial gas and pockets of gas become more and more dense, even as the universe continues to expand rapidly. These small, dense clouds of cosmic gas start to collapse under their own gravity, becoming hot enough to trigger nuclear fusion reactions between hydrogen atoms, creating the very first stars. The first stars are short-lived supermassive stars, a hundred or so times the mass of our Sun, known as Population III (or “metal-free”) stars. Eventually Population II and then Population I stars also begin to form from the material from previous rounds of star-making. Larger stars burn out quickly and explode in massive supernova events, their ashes going to form subsequent generations of stars. Large volumes of matter collapse to form galaxies and gravitational attraction pulls galaxies towards each other to form groups, clusters and superclusters.

Solar System Formation

8.5 - 9 billion years

Our Sun is a late-generation star, incorporating the debris from many generations of earlier stars, and it and the Solar System around it form roughly 4.5 to 5 billion years ago (8.5 to 9 billion years after the Big Bang

Today

13.7 billion years

The expansion of the universe and recycling of star materials into new stars continues.

4. Dark Matter

[8][7][2]

4.1 Introduction

According to the standard model of cosmology, our universe consists of 4% ordinary baryonic matter, $\sim 23\%$ dark matter, and $\sim 73\%$ dark energy, with a tiny abundance of relic neutrinos. We know the baryonic content both from element abundances produced in primordial nucleosynthesis roughly 100 seconds after the Big Bang, and from measurements of anisotropies in the cosmic microwave background (CMB). The evidence for the existence of dark matter is overwhelming, and comes from a wide variety of astrophysical measurements.

Dark matter cannot consist of baryons. There are two reasons for this. First, if baryons made up all the dark matter, the cosmic microwave background and cosmic web of structure would look radically different. Second, the abundance of light elements created during big-bang nucleosynthesis depends strongly on the baryon density (more precisely, on the baryon-to-photon ratio) of the Universe. Observed abundances of deuterium and ^{4}He constrain give similar constraints on the baryon density in the Universe as those coming from cosmic microwave background observations.

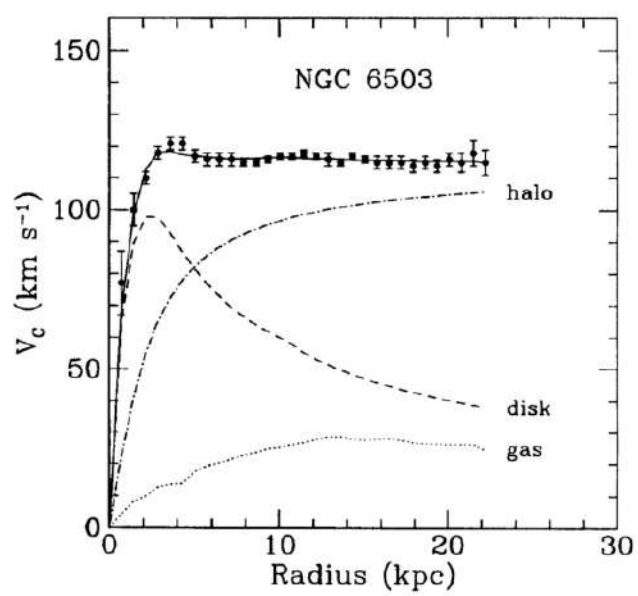
Dark matter cannot consist of light (sub-keV-mass) particles unless they were created via a phase transition in the early Universe (like QCD axions). This is because light particles are relativistic at early times, and thus fly out of small-scale density perturbations.

There are strong constraints on the electromagnetics of dark matter. If dark matter had either a small charge or a small electric or magnetic dipole moment, it would couple to the photon-baryon fluid before recombination, thus altering the sub-degree-scale features of the cosmic microwave background as well as the matter power spectrum.

4.2 Evidence

The evidence that 95% of the mass of galaxies and clusters is made of some unknown component of Dark matter (DM) comes from (i) rotation curves (out to tens of kpc), (ii) gravitational lensing (out to 200 kpc), and (iii) hot gas in clusters and cosmic abundances.

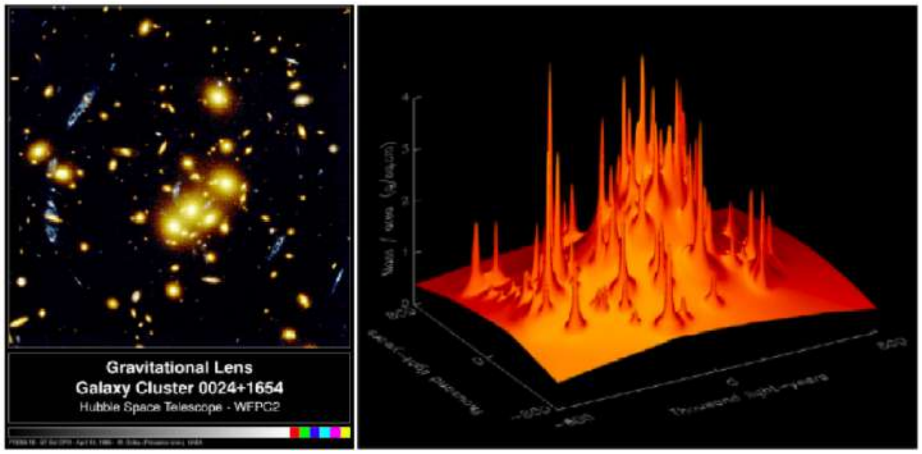
Rotatoion Curves



Galactic rotation curve for NGC 6503 showing disk and gas contribution plus the dark matter halo contribution needed to match the data

It was discovered that the velocities of objects (stars or gas) orbiting the centers of galaxies, rather than decreasing as a function of the distance from the galactic centers as had been expected, remain constant out to very large radii. The simplest explanation is that galaxies contain far more mass than can be explained by the bright stellar objects residing in galactic disks. This mass provides the force to speed up the orbits. To explain the data, galaxies must have enormous dark halos made of unknown dark matter. Indeed, more than 95% of the mass of galaxies consists of dark matter.

Lensing

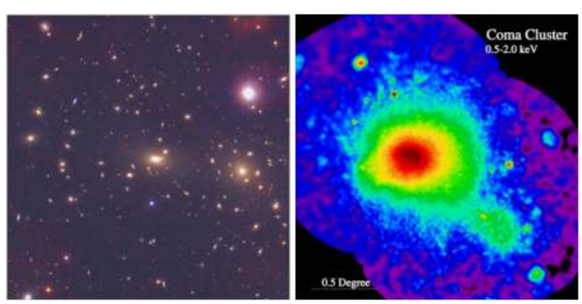


Left: the foreground cluster of galaxies gravitationally lenses the blue background galaxy into multiple images. Right: a computer reconstruction of the lens shows a smooth background component not accounted for by the mass of the luminous objects.

Einstein's theory of General Relativity predicts that mass bends, or lenses, light. This effect can be used to gravitationally ascertain the existence of mass even when it emits no light. Lensing measurements confirm the existence of enormous quantities of dark matter both in galaxies and in clusters of galaxies. Observations are made of distant bright objects such as galaxies or quasars. As the result of intervening matter, the light from these distant objects is bent towards the regions of large mass. Hence there may be multiple images of the distant objects, or, if these images cannot be individually resolved, the background object may appear brighter. Some of these images may be distorted or sheared.

The key success of the lensing of DM to date is the evidence that DM is seen out to much larger distances than could be probed by rotation curves: the DM is seen in galaxies out to 200 kpc from the centers of galaxies, in agreement with N-body simulations. On even larger Mpc scales, there is evidence for DM in filaments (the cosmic web).

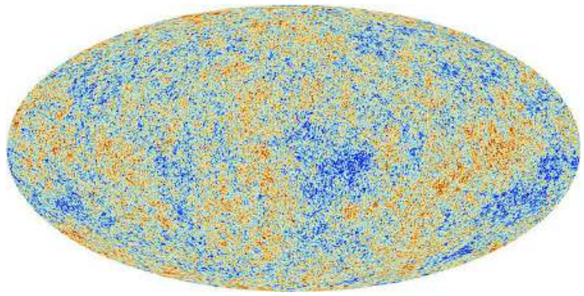
Hot Gas in Clusters



COMA Cluster: Left panel: optical image. Right panel: X-ray image.

Another piece of gravitational evidence for dark matter is the hot gas in clusters. The X-ray image indicates the presence of hot gas. The existence of this gas in the cluster can only be explained by a large dark matter component that provides the potential well to hold on to the gas.

The Cosmic Microwave Background



The Cosmic Microwave Background

The CMB is the remnant radiation from the hot early days of the universe. The photons underwent oscillations that froze in just before decoupling from the baryonic matter at a redshift of 1000. The angular scale and height of the peaks (and troughs) of these oscillations are powerful probes of cosmological parameters, including the total energy density, the baryonic fraction, and the dark matter component.

Primordial Nucleosynthesis

When the universe was a few hundred seconds old, at a temperature of ten billion degrees, deuterium became stable: $p+n \longrightarrow D+\gamma$. Once deuterium forms, helium and lithium form as well.The predictions from the Big Bang are 25% Helium-4, 10^{-5} deuterium, and 10^{-10} Li-7 abundance by mass. These predictions exactly match the data as long as atoms are only 4% of the total constituents of the universe.

4.3 Dark Matter Candidates

The only major non-particle candidate for dark matter is the primordial black hole, which would have collapsed directly from highly overdense regions of the early Universe. The good news is that cosmologists don't need to "invent" new particles. Candidates already exist in particle physics for other reasons.

Weakly-interacting massive particles (WIMPs)

This class of candidate was originally introduced by Steigman and Turner. The key features of this particle class are exactly as described: interactions around or near typical weak-force interactions (the fine-structure constant α near the weak-scale coupling $\sim 10^{-2}$), particle masses near the weak scale (m $\sim$ 100 GeV in particle-physics units , similar to the mass of a silver atom).

These particles, if present in thermal abundances in the early universe, annihilate with one another so that a predictable number of them remain today. The relic density of these particles comes out to be the right value:

$$\Omega_\chi h^2 = (3\times 10^{-26} cm^3/sec)/\langle\sigma v\rangle_{ann}$$

where the annihilation cross section $\langle \sigma v \rangle_{ann}$ of weak interaction strength automatically gives the right answer. This coincidence is known as “the WIMP miracle”

Axions

Axions with masses in the range $10^{-(3-6)}$ eV arise in the Peccei-Quinn solution to the strong-CP problem in the theory of strong interactions. Axions are in some ways less natural than WIMPs because it is tricky to get their comoving number density to match the observed dark-matter density

Gravitinos

While supersymmetric neutralinos are the dark-matter candidate of choice in some swaths of the MSSM, the gravitino, the supersymmetric partner of the graviton, may be dark matter in other swaths. Depending on exactly how supersymmetry is broken, the gravitino could be anywhere in the mass range of $\sim$eV to TeV, although masses $\lesssim$ keV are disfavored because they wash out too much small-scale structure. In order for lighter gravitinos to be dark matter, one typically must introduce some non-standard cosmology. Because these massive gravitinos are born out of decays at relatively high momentum, they can smear out primordial density perturbations on small scales. Gravitinos are not nearly as beloved as WIMPs as dark-matter candidates because of the difficulty of getting the abundance just right and because they are much harder to detect using conventional methods.

There are other dark-matter candidates that are plausible and solve some other problems in physics, although they to not provide quite the same bargain-hunting thrill of the previously discussed candidates.

4.4 Dark Energy

We know that there is an additional ingredient in the universe’s energy budget, dark energy, a substance whose equation of state ω is close to -1 and does not participate in gravitational collapse. We have strong evidence that the universe is Euclidean, with total density parameter close to 1. Since $\Omega_m = 0.3$ is very far from 1 (and radiation is totally negligible today), something that does not clump as does matter has to make up this budgetary shortfall. Second, the expansion of the universe is accelerating, as measured by standard candles and rulers. Accelerated expansion ($\ddot{a} > 0$) occurs only if the dominant constituent in the universe has a negative equation of state, i.e. negative pressure.

The two groups in 1998 measured the apparent magnitudes of dozens of Type Ia supernovae, which are known to be standardizable candles, i.e., they have absolute magnitude that can be determined from other observables, in particular the characteristic time it takes for the luminosity to decay after the peak.By carefully accounting for statistical and systematic errors in the distance estimation, one can then obtain the best-fit parameters describing the expansion history of the universe. The result is based on the assumption that dark energy is a cosmological constant, but not restricted to a Euclidean universe. The two free cosmological parameters are then the matter density parameter Ω_m and the corresponding parameter Ω_Λ for the cosmological constant. A universe with $\Lambda = 0$ (and hence $\Omega_\Lambda = 0$) is not compatible with observations. Instead, supernovae point to the concordance value of $\Omega_\Lambda \simeq 0.7$.

We now have another piece of independent evidence for dark energy: the Baryon Acoustic Oscillation (BAO) standard ruler provides both a measurement of the angular diameter distance to a given redshift and the distance interval corresponding to a certain redshift interval. This is the derivative of the comoving distance with respect to redshift,

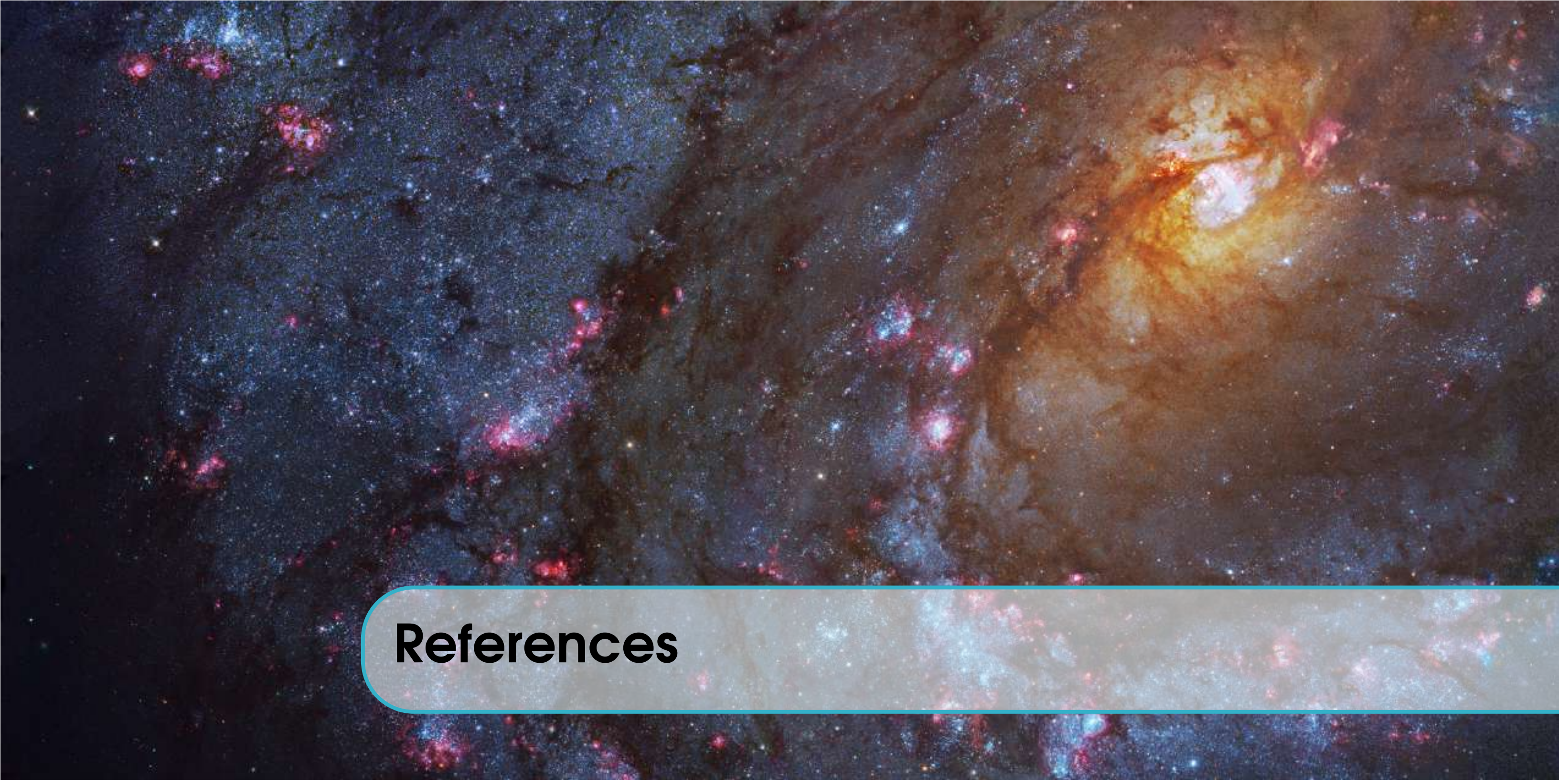
$$\frac{d\chi}{dz} = \frac{1}{H(z)}$$

This measurement not allows for independent constraints on dark energy but it also beautifully shows us directly that the expansion of the universe is accelerating. In a universe with matter and radiation (or any constituent with vanishing or positive pressure), the quantity $H(z)/(1+z) = aH = \dot{a}$ is monotonically decreasing. However, we see that aH has to increase in order to meet up with the locally measured Hubble rate. So, standard candles and rulers now allow us to see the presence of dark energy directly.

The existence of dark energy can be inferred not only using probes that measure the expansion history directly (sometimes called geometric probes). The accelerated expansion also directly affects the evolution of structure in the universe. Growth of structure probes independently support the Euclidean concordance cosmology with $\Omega_\Lambda \simeq 0.7$. A compelling argument for the existence of dark energy is that both geometric (background) and dynamic (structure) probes agree on the same cosmological model.

So far, we have always talked about the cosmological constant Λ, with the one free parameter being the energy density associated with it, Ω_Λ. However, this is only the simplest possibility for what dark energy could be, and introducing a constant carries its own set of problems. This is why we use "dark energy" rather than cosmological constant as a moniker. One generalization is to make the constant dynamical, turning the energy density associated with Λ into the potential energy of a scalar field $V(\phi)$. This possibility is often referred to as *quintessence*. Yet another possibility is to modify general relativity itself, so that the acceleration is due to a modified behavior of gravity

We know that the energy-momentum tensor is completely general and is dictated by the symmetries of the FLRW spacetime. Hence, defining pressure via the equation of state $\omega_{DE}(a)$, the effect of a general dark energy on the expansion history is completely determined by the function $\omega_{DE}(a)$. The cosmological constant simply adds a term $\Lambda\delta^\mu{}_\nu$ to the Einstein equations (when written with one upper index). Comparing this with the energy momentum tensor shows that the cosmological constant effectively has an energy-momentum tensor that is of perfect fluid form, with $\mathscr{P} = -\rho\alpha\Lambda$ which implies an equation of state of $\omega_\Lambda = 1$. For a dynamical dark energy (e.g. quintessence), $\omega_{DE} \geq -1$ (but still significantly below 0). Measuring the dark energy density as a function of cosmic time (i.e. at different redshifts) then allows us to constrain ω_{DE} and hence distinguish between different dark energy scenarios.



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