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PH807 Advanced General Relativity

3D GRAVITY AND BTZ BLACKHOLE

Course Project

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Abstract

Lower-dimensional models find widespread utility in various branches of physics, notably in quantum mechanics, where constant potentials in a single spatial dimension offer the most straightforward instances of energy quantization and tunneling. These simplified models hold significance by serving as catalysts for innovative ideas and facilitating fresh perspectives on their higher-dimensional counterparts. Additionally, they furnish a convenient backdrop for demonstrating fundamental physical phenomena, circumventing the intricate mathematics often associated with four dimensions.

Initially, Einstein's general theory of relativity may seem inconsequential when examined in fewer than four dimensions. In three-dimensional spacetime, Einstein's gravity lacks dynamic degrees of freedom, implying the absence of gravitons in its quantum rendition. Nevertheless, the absence of genuine dynamical content in a theory does not preclude intriguing applications. Three-dimensional gravity shares notable characteristics with its four-dimensional counterpart, such as non-trivial global and topological effects that can manifest within the theory. Moreover, three-dimensional gravity's gauge group is the diffeomorphism group, and the absence of dynamics may render the role of diffeomorphisms in quantum theory more transparent and comprehensible.

In this comprehensive review, I will present an overview of solutions in three-dimensional gravity as discussed in the work of Deser, Jackiw, and Hooft [6] and delve into the asymptotic structure of three-dimensional gravity, as explored in Brown's book [1]. In the later part, we will survey BTZ blackholes, the blackhole solution in 3D gravity from the work of Carlip [2]. And we will end with conditions for gravitational collapse in 3D gravity.

Contents

Chapter 1

Introduction to Three-dimensional Gravity

Einstein-Hilbert action for gravity coupled to matter in $2+1$ dimensions is given by

$$I = \frac{1}{16\pi G}\int_M d^3x\sqrt{-g}(R-2\Lambda) + I_{matter} \tag{1.1}$$

1.1 Dimensional Analysis

This section is referred from [9].

We will adopt natural units, where $\hbar$ and c are set to 1. In this system, the dimensions of different quantities are related as follows: $[dx^\mu] = [ds] = L$. Since we have the metric relation $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$, it follows that $[g_{\mu\nu}] = 1$, and consequently, $[g] = 1$.

As the Riemann curvature tensor involves second derivatives with respect to the metric, its dimensions are $[R] = L^{-2}$. For the action I to be dimensionless, we must have $[G] = L = M^{-1}$, indicating that the gravitational constant G has mass dimension -1. This classification makes this theory "power-counting non-renormalizable", which means it contains terms with coupling constants of negative mass dimension. Such terms lead to divergences that cannot be eliminated through a finite number of counterterms.

The cosmological constant Λ has dimensions $[\Lambda] = L^{-2}$, providing a dimensionless quantity in the theory. This allows us to define a characteristic length scale, denoted as "l", as follows:

$$l \sim \frac{1}{G|\Lambda|^{\frac{1}{2}}} \tag{1.2}$$

This characteristic length scale is determined by the interplay between the gravitational constant and the cosmological constant and has important implications for the theory.

1.2 Equations of motion

This section is referred from [9] & [1].

Einstein gravity in three spacetime dimensions exhibits some unusual features, which can be deduced from the properties of the Einstein field equations and the curvature tensor. Einstein's field equation for general relativity is given as:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu} \tag{1.3}$$

Taking the trace and resubstituting back R, this equation can also be expressed as:

$$R_{\mu\nu} = 2\Lambda g_{\mu\nu} + \kappa (T_{\mu\nu} - g_{\mu\nu} T) \tag{1.4}$$

Here, Λ is the cosmological constant, and the coupling constant κ has dimensions of M^{-1} in 3D. In D dimensions, the number of independent components of the Riemann tensor is $D^2(D^2-1)/12$ while that of the Ricci tensor is $D(D+1)/2$. Riemann tensor can be split into two tensors, the Ricci tensor and the Weyl tensor defined as:

$$C_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} - \frac{2}{D-2}(g_{\mu[\rho} R_{\sigma]\nu} - g_{\nu[\rho} R_{\sigma]\mu}) + \frac{2}{(D-1)(D-2)} R g_{\mu[\rho} g_{\sigma]\nu} \tag{1.5}$$

in $D \geq 3$ dimensions. In a hand wavy argument, we can say that, in 2+1 dimensions, it is interesting to note that both these tensors have six independent components. Hence, the Riemann tensor can be written completely in terms of the Ricci tensor and vice versa. Thus the Weyl tensor vanishes and we get

$$R_{\mu\nu\lambda\rho} = g_{\mu\lambda} R_{\nu\rho} + g_{\nu\rho} R_{\mu\lambda} - g_{\mu\rho} R_{\nu\lambda} - g_{\nu\lambda} R_{\mu\rho} - \frac{1}{2} R (g_{\mu\lambda} g_{\nu\rho} - g_{\mu\rho} g_{\nu\lambda}) \tag{1.6}$$

As for a formal definition, the Weyl tensor vanishes in any spacetime which is conformally flat. Therefore, in any conformally flat spacetime with a metric of the form $g_{\mu\nu} = f^2(x)\eta_{\mu\nu}$, the curvature tensor can be expressed entirely in terms of $R_{\mu\nu}$, $g_{\rho\sigma}$ and R. The converse is also true for all spacetimes with $D > 3$; that is, in four and higher dimensions, the vanishing of the Weyl tensor ensures that the spacetime is conformally flat. The situation is special for $D = 2, 3$. All $D = 2$ spaces are conformally flat and the Weyl tensor is undefined for $D = 2$. In $D = 3$ the Weyl tensor vanishes identically but all three-dimensional spaces are not conformally flat. If a space is indeed conformally flat with $g_{\mu\nu} = f^2(x)\eta_{\mu\nu}$, then one should be able to make a conformal transformation from $g_{\mu\nu}$ to $\eta_{\mu\nu}$. Since we can determine how the curvature tensor changes under the conformal transformation, demanding that the new curvature tensor should vanish will lead to a differential equation for $f(x)$. The integrability condition for this equation to possess solutions turns out to be identically satisfied if the Weyl tensor vanishes in $D > 3$. But in $D = 3$, one term in the relevant equation (which has a coefficient $(D-3)$) vanishes and the integrability requires an additional condition to be satisfied:

$$C^{\alpha\beta} \equiv 2\epsilon^{\alpha\gamma\delta} \nabla_\delta [R^\beta_\gamma - \frac{1}{4}\delta^\beta_\gamma R] = 0. \tag{1.7}$$

This tensor $C^{\alpha\beta}$ is known as the Cotton-York tensor, which plays a role similar to the Weyl tensor in higher dimensions. The tensor vanishes if and only if the three-dimensional spacetime is conformally flat. It is also symmetric, traceless, and covariantly conserved.

Using (1.4) in the right-hand side of (1.6) shows that the full curvature tensor is completely determined by the local matter distribution $T^{\mu\nu}$ and the cosmological constant Λ. In particular, the source-free regions of spacetime, with $T^{\mu\nu} = 0$, are regions of constant curvature with a curvature tensor:

$$R_{\mu\nu\lambda\rho} = \Lambda(g_{\mu\lambda}g_{\nu\rho} - g_{\mu\rho}g_{\nu\lambda}) \tag{1.8}$$

and the curvature scalar $R = 6\Lambda$. This means that any curvature effects produced by matter do not propagate through spacetime using gravitational waves; there are no dynamical degrees of freedom. In regions that are free of matter, spacetime is locally flat ($\Lambda = 0$), de Sitter ($\Lambda > 0$), or anti-de Sitter ($\Lambda < 0$), depending on the value of the cosmological constant.

The lack of dynamics in three-dimensional Einstein gravity can also be seen from the canonical point of view by counting degrees of freedom. The two-dimensional spatial metric and its conjugate each contain three algebraically independent components. Out of these six components, one is needed to specify the choice of space-like hypersurfaces, while two more are needed to specify coordinates on these two-dimensional hypersurfaces. Finally, there are three initial value constraints that completely determine the remaining components.

As $2 + 1$ gravity has no propagating degrees of freedom, it has an interesting Newtonian limit. In this limit, we can show that the geodesic equation reduces to

$$\frac{d^2x^i}{dt^2} + 2\left(\frac{d-3}{d-2}\right)\partial_i\Phi = 0$$

i.e. in $D = 3$ gravity, static masses do not experience Newtonian gravitational force.

In a source-free region, we can observe that local curvature remains unaffected by matter, but matter has global effects. If $\Lambda = 0$, according to equation (1.8), the Riemann curvature is zero in a source-free region, and the metric becomes Minkowski, denoted as $g_{\mu\nu} = \eta_{\mu\nu}$. However, this transformation is not generally well-defined everywhere, and the region outside sources as a whole may not be identical to a region of Minkowski spacetime. In simpler scenarios, such as a point mass source, we can observe nontrivial global geometrical effects in 3D gravity. When $\Lambda = 0$, spacetime is flat everywhere except along the world line of the particle. If spacetime is static, we can choose coordinates in which each of the $t =$ constant spatial sections are identical. These spatial sections will be flat everywhere except at a single point, which is the location of the particle. A cone is a unique two-dimensional spatial geometry that fits this description. It can be demonstrated that the three-dimensional spacetime for a point source is obtained by removing a "wedge" from Minkowski space with an angle κm, where κ is the gravitational constant. This applies to any arbitrary number of static point masses as well. Just as when $\Lambda = 0$, these solutions correspond to deleting a wedge from de Sitter or anti-de Sitter space and identifying points across the wedge [6].

In three-dimensional Einstein gravity, matter cannot affect the local curvature in source-free regions. Therefore, the conical-helical geometry that applies to a point source characterizes the spacetime outside more general compact matter distributions. Similar to four-dimensional general relativity, the total energy and angular momentum associated with a compact distribution of matter are reflected in the asymptotic behavior of spacetime geometry and can be expressed as surface integrals at spatial infinity. In three-dimensional gravity with $\Lambda = 0$, energy and angular momentum are the only "global charges" that can be defined. This is because global charges are intimately associated with asymptotic symmetries, and in the generic conical-helical geometry, the only asymptotic symmetries present are time translations (associated with energy) and spatial rotations (angular momentum).

An alternative to Einstein gravity in three dimensions is the "topologically massive gravity." This theory includes the Chern-Simons secondary characteristic class in the gravitational action:

$$S_{CS} = \frac{1}{2}\int d^3x \epsilon^{\mu\nu\sigma}\left[R^{\alpha}_{\beta\mu\nu}\Gamma^{\beta}_{\alpha\sigma} - \frac{2}{3}\Gamma^{\alpha}_{\beta\mu}\Gamma^{\beta}_{\gamma\nu}\Gamma^{\gamma}_{\alpha\sigma}\right] \tag{1.9}$$

The variation of the Chern-Simons class yields the Cotton-York tensor:

$$C^{\mu\nu} \equiv \frac{1}{2\sqrt{-g}}\frac{\delta S_{CS}}{\delta g_{\mu\nu}} = \frac{\epsilon^{\alpha\beta\mu}}{\sqrt{-g}}\nabla_{\beta}(R^{\nu}_{\alpha} - \frac{1}{4}\delta^{\nu}_{\alpha}R) \tag{1.10}$$

The action for topologically massive gravity is obtained by adding $(4\kappa\mu)^{-1}S_{CS}$ to the Einstein-Hilbert action, where μ is the coupling constant with units of L^{-1}. The resulting equations of motion are:

$$G^{\mu\nu} + \Lambda g^{\mu\nu} + \frac{1}{\mu}C^{\mu\nu} = 0 \tag{1.11}$$

It's important to note that this equation of motion is third order in the derivatives of the metric. This equation can also be split into trace and trace-free parts as follows:

$$R = 6\Lambda \tag{1.12}$$

$$C^{\mu\nu} = -\mu\left(R^{\mu\nu} - \frac{1}{3}g^{\mu\nu}R\right) \tag{1.13}$$

Similar to Einstein gravity, these equations hold for a spacetime of constant curvature, i.e., de Sitter ($\Lambda > 0$), anti-de Sitter ($\Lambda < 0$), or flat ($\Lambda = 0$) spacetime. However, unlike three-dimensional Einstein gravity, topologically massive gravity allows for dynamics, resulting in curvature with massive excitations characterized by the mass parameter μ. This is evident when recognizing that any metric $g_{\mu\nu}$ satisfying (1.13) will also satisfy an equation obtained by applying the tensor operator $\sqrt{-g}\epsilon_{\nu\alpha\sigma}\nabla^{\alpha}$ to (1.13), including contraction on the index ν. The resulting equation can be expressed more simply as:

$$(-\nabla^{\alpha}\nabla_{\alpha} + \mu^2)(R_{\mu\nu} - \frac{1}{3}g_{\mu\nu}R) = g_{\mu\nu}R^{\alpha\beta}R_{\alpha\beta} - 3R_{\mu\alpha}R^{\alpha}_{\nu} \tag{1.14}$$

This demonstrates that curvature excitations are indeed massive, and their propagation is causal. In fact, for $\Lambda = 0$, (1.14) implies that in the linearized limit, the trace-free part of the Ricci tensor satisfies the free Klein-Gordon equation with mass μ.

1.3 Static N-body solutions

This section is referred from [6].

In the context of a static scenario, let's break down the metric as follows:

$$g_{00} = -N^2(\mathbf{r}), \quad g_{0i} = 0, \quad g_{ij} = \gamma_{ij}(\mathbf{r}), \quad \sqrt{-g} = N\sqrt{\gamma} \tag{1.15}$$

Here, γ_{ij} is the induced metric, N is represented as a scalar function of space, and g and γ are the determinants of $g_{\mu\nu}$ and γ_{ij}. The Einstein tensor takes the form:

$$-\sqrt{\gamma}G^0_0 = \frac{1}{2}\sqrt{\gamma}R, \quad G^0_i = 0, \quad G_{ij} = -\frac{1}{2N}(D_iD_j - \gamma_{ij}D^2)N \tag{1.16}$$

Here, D_i denotes the covariant derivative with respect to the spatial metric γ_{ij}. The spatial components of the Einstein tensor are simplified due to the vanishing of the two-dimensional Einstein tensor. It's worth noting that $\sqrt{\gamma}G^0_0$ represents the Euler invariant density, which is a total 2-divergence. In regions without sources, R becomes zero, resulting in flat spatial 2-surfaces.

Now, let's consider the source as a collection of point particles at rest with masses m_n located at positions r_n, and the stress-energy tensor density is given by:

$$T^{00} = \sum_n m_n\delta^2(\mathbf{r} - \mathbf{r}_n), \quad T^{0i} = 0 = T^{ij} \tag{1.17}$$

For initially static objects, the geodesic equation simplifies to:

$$\ddot{x}^i = -\frac{1}{2}g^{ij}\partial_j g_{00} \tag{1.18}$$

Regardless of the spatial gauge choice available, the $G_{ij} = 0$ equations clearly imply that $D^2N = 0$ and $D_iD_jN = 0$, indicating that N is indeed a constant. Setting $N = 1$ represents a time calibration. Therefore, $g_{00} = -1$, meaning that acceleration consistently vanishes for our solution.

We will solve the G_{00} equation using isotropic coordinates $\gamma_{ij} = \phi\delta_{ij}$ (a valid choice in two dimensions) and then transform to curvature (Schwarzschild) coordinates to reveal the global aspects. In this frame, $\frac{1}{2}\gamma R$ simplifies to $-\frac{1}{2}\nabla^2 \ln\phi$, where ∇^2 is the flat Laplacian. Given that its Green's function is $\ln r$, with $\nabla^2 \ln r = 2\pi\delta^2(\mathbf{r})$, our solution to the time-time component of Einstein's equation is:

$$\ln\phi = -8\pi G\sum_n m_n \ln|r - r_n| + \ln C \tag{1.19}$$

This results in the metric becoming:

$$g_{00} = -1, \quad g_{0i} = 0, \quad g_{ij} = C\delta_{ij}\prod_n |r - r_n| - 8Gm_n \tag{1.20}$$

The constant C can be eliminated through a rescaling of r, except in the singular case where $\sum m_n = 1/4G$. The metric (1.20) represents the general static N-body solution. Notably, there are no particle interactions or hidden "rods" holding the particles fixed, as there is no curvature between them.

1.4 Angular Momentum: Rotating Sources

This section is referred from [6].

When there are two or more moving particles that are not in a straight line, the system possesses both orbital angular momentum and energy. In this scenario, we derive a solution known as the "Kerr" solution, which corresponds to a spatially localized spinning source with no energy density but only angular momentum density. Naturally, the metric becomes a transformation of the Minkowski metric, denoted as $\eta_{\mu\nu}$, since the exterior spacetime remains flat. The angular momentum in this case is represented by a single quantity, denoted as J:

$$J = \frac{1}{2}\epsilon_{ij} J^{ij} = \frac{1}{2}\epsilon_{ij} \int d^2x (x^i T^{0i} - x^j T^{0i}) \tag{1.21}$$

In 4D, the Kerr metric in Boyer–Lindquist coordinates is given as:

$$ds^2 = -\left(1 - \frac{r_s r}{\Sigma}\right) dt^2 + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2 + \left(r^2 + a^2 + \frac{r_s r a^2}{\Sigma}\sin^2\theta\right)\sin^2\theta d\phi^2 - \frac{2r_s r a \sin^2\theta}{\Sigma} dt d\phi \tag{1.22}$$

where the coordinates r, θ, ϕ are standard oblate spheroidal coordinates, which are equivalent to the cartesian coordinates:

$$\begin{aligned} x &= \sqrt{r^2 + a^2}\sin\theta\cos\phi \\ y &= \sqrt{r^2 + a^2}\sin\theta\sin\phi \\ x &= r\cos\theta \end{aligned} \tag{1.23}$$

and

$$\begin{aligned} r_s &= 2GM \\ a &= \frac{J}{M} \\ \Sigma &= r^2 + a^2\cos^2\theta \\ \Delta &= r^2 - r_s r + a^2 \end{aligned} \tag{1.24}$$

Because there are no gravitational field contributions, our solution is significantly simpler than this 4-dimensional Kerr metric.

We consider a stationary (time-independent but not static) axially symmetric scenario using circular coordinates:

$$ds^2 = dr^2 + f^2(r)d\theta^2 + 2g_{02}(r)dtd\theta - g_{00}(r)dt^2 \tag{1.25}$$

For later convenience, we have chosen the spatial gauge $g_{11} = 1$, but changing the radial coordinates would yield different forms if needed. The coefficient of g_{02}, when expressed in Cartesian coordinates ($x \equiv r\cos\theta, y \equiv r\sin\theta$), can be written as:

$$dtd\theta = dtr^{-2}(xdy - ydx) = dt(dy\partial_x - dx\partial_y)\ln r \tag{1.26}$$

Since $\ln r$ serves as the Green's function, this part of the metric is expected to result in a singularity in the field equations, appropriate for a localized spinning source.

This is the initial indication of the presence of angular momentum. Additionally, notice that g_{02} has dimensions of length, similar to angular momentum itself: $GJ \propto (Gm)vr \propto r$ in gravitational units.

The component $\sqrt{-g}G^{0i}$ is precisely $-2D_j\pi^{ij}$, where in the stationary case:

$$\pi_{ij} = -\frac{\sqrt{\gamma}}{2N}[D_i N_j + D_j N_i - 2\gamma_{ij} D_l N^l] \tag{1.27}$$

with $N_i \equiv g_{0i}$, $N^2 \equiv \gamma^{ij} N_i N_j - g_{00}$; all operations are performed with respect to the spatial metric γ_{ij}. The exterior G^0_i equation can be expressed as:

$$D_j \pi^{ij} = 0 \tag{1.28}$$

In our metric (1.25), we find that:

$$\begin{aligned} &\pi_{11} = \pi_{22} = \pi^i_i = 0 \\ &\pi_{12} = [2(\frac{\rho'}{\rho})N_2 - N_2']\frac{\rho}{2N} = -(\rho^{-2}N_2)'\frac{\rho^2}{2N} \end{aligned} \tag{1.29}$$

where a prime denotes differentiation with respect to r. Consequently, in this frame, $D_j \pi^j_i$ becomes $\partial_j \pi^j_i$, and (1.28) simplifies into an ordinary divergence equation:

$$\partial_j \pi^j_i \equiv 0, \quad \partial_j \pi^j_2 = \pi_{12}' = 0 \implies \pi_{12} = A \tag{1.30}$$

The absence of covariant differentiation in (1.30) indicates the lack of a gravitational contribution to angular momentum. To determine the spatial metric, we next calculate $\rho(r)$ from the G^0_0 equation, which is:

$$\sqrt{\gamma}R = \frac{1}{\sqrt{\gamma}}\pi_{ij}\pi^{ij} = 2(\pi_{12})^2\rho^{-3} \tag{1.31}$$

Even though there appears to be a "gravitational energy density" due to π^2_{ij}, we will see that the angular momentum (i.e., the energy) actually vanishes for our spinning model with $T^{00} = 0$. The spatial curvature does not vanish, of course; this is a consequence of our choice of spacetime slicing using curved 2-surfaces in flat spacetime. Equation (1.31) leads to an equation for ρ:

$$-2\rho'' = 2A^3\rho^{-3} \tag{1.32}$$

and its solution is:

$$\rho^2 = (Br + C)^2 - \left(\frac{A^2}{B^2}\right) \tag{1.33}$$

We can set the integration constant C to zero and absorb the scale B into unity, as any other value of B would introduce a conical singularity unrelated to our source. This leads to:

$$\rho^2 = r^2 - A^2 \tag{1.34}$$

Finally, we must determine N from the $G_{ij} = 0$ equations. The 1,1-component of this equation is particularly crucial:

$$\frac{\rho' N'}{\rho} + \frac{A^2 N}{\rho^4} = 0 \tag{1.35}$$

Equation (1.35) allows us to find N as:

$$N^2 = \frac{\tau^2(\rho^2 + A^2)}{\rho^2} = \tau^2 r^2 (r^2 - A^2)^{-1} \tag{1.36}$$

and the constant τ can be absorbed through a time rescaling. Finally, we determine N_2 from (1.27) or (1.29) to be:

$$N_2 = A + F\rho^2 \tag{1.37}$$

Choosing $F = 0$, we find that N_2 is a constant, which, as we shall see, is proportional to the angular momentum J. Furthermore:

$$-g_{00} \equiv N^2 - n_i N^i = \rho^{-2}(A^2 + \rho^2) - \rho^{-2}A^2 = 1 \tag{1.38}$$

Summarizing our solution, the line element becomes:

$$ds^2 = (dr^2 + r^2 d\theta^2) - (dt^2 - 2A\, dt\, d\theta + A^2 d\theta^2) \tag{1.39}$$

Now, we can examine the singularity structure of π^{ij} and calculate the angular momentum using (1.21). The angular momentum is given in terms of gravitational variables by the standard formula:

$$J^{ik} = -(8\pi G)^{-1} \int d^2x [x^i \partial_j \pi^{kj} - x^k \partial_j \pi^{ij}] \tag{1.40}$$

$$= -(8\pi G)^{-1} \oint dS_j (x^i \pi^{kj} - x^k \pi^{ij}) \tag{1.41}$$

Here, we've used $8\pi G\sqrt{-g}T^{0i} = \sqrt{-g}G^{0i} = -2\partial_j \pi^{ij}$. From $\pi^{r\theta} = g^{rr}g^{\theta\theta}\pi_{r\theta} = A(r^2 - A^2)$, we can obtain the Cartesian components π^{ij} by converting from circular to Cartesian coordinates, resulting in:

$$\begin{aligned} \pi^{ij} =& - A(r^2 - A^2)^{-1} r^{-2} (\varepsilon^{im} x^m x^j + \varepsilon^{jm} x^m x^i) \\ &\xrightarrow{r \to \infty} - \frac{A}{r^4} (\varepsilon^{im} x^m x^j + \varepsilon^{jm} x^m x^i) \\ =& \frac{1}{2} A (\varepsilon^{im} \partial_m \partial_j + \varepsilon^{jm} \partial_m \partial_i) \ln r \end{aligned} \tag{1.42}$$

The second line of (1.42) represents the asymptotic value of π^{ij} that we are interested in for the surface integral of (1.40). Asymptotically, $\partial_j \pi^{kj}$ (or $\partial_j \pi^j_k$) is $\frac{1}{2} A \varepsilon^{km} \partial_m (\nabla^2 \ln r)$, which indeed represents an effective localized spin source. The value of J^{ik} is:

$$J^{ik} = -(4G)^{-1} A \varepsilon^{ik} = J \varepsilon^{ik} \tag{1.43}$$

This determines the constant A to be $-4GJ$. The interval (1.39) can be transformed into Minkowski form:

$$ds^2 = dr^2 + r^2 d\theta^2 dT^2 \tag{1.44}$$

by changing the time coordinate according to:

$$T = t + 4GJ\theta \tag{1.45}$$

However, a singularity exists, analogous to the mass defect in the static case. Specifically, at constant t, as θ reaches 2π, which is identified with $\theta = 0$, T experiences a jump of $8\pi GJ$ because times that differ by $8\pi GJ$ must be identified to maintain single-valuedness. This "time-helical" structure may have interesting consequences in a quantized theory.

In conclusion, angular momentum results in a flat spacetime, but the coordinate time T exhibits a jump property. Alternatively, in the original form (1.39), the metric is singular at the spatial origin and at $r = A$. This singularity structure has been replaced by the jump in the new time coordinate. Thus, these calculations reveal anomalies in the Minkowski coordinates and relate them to source strengths.

Chapter 2

The Asymptotic Structure of Three-Dimensional Gravity

This section is referred in [1]

Asymptotic symmetries play a crucial role in any gauge theory formulated in an open space. These symmetries encompass transformations that leave the asymptotic form of the field unchanged, leading to conserved global charges that characterize the physical system.

In the Hamiltonian formalism, these global charges emerge as the canonical generators of the asymptotic symmetries inherent in the theory. For each infinitesimal symmetry denoted as ξ, there corresponds a phase space function $H[\xi]$. This function generates the corresponding transformation of the canonical variables. It's generally assumed that the Poisson bracket algebra of these generators $H[\xi]$ is isomorphic to the Lie algebra of infinitesimal asymptotic symmetries, represented as:

$$\{H[\xi], H[\eta]\} = H[[\xi, \eta]] \tag{2.1}$$

However, it turns out that while this equation holds true in many important cases, it's not universally valid. In the generic scenario, global charges may only yield a "projective" representation of the asymptotic symmetry group, resulting in:

$$\{H[\xi], H[\eta]\} = H[[\xi, \eta]] + K[\xi, \eta] \tag{2.2}$$

In this context, the "central charges" $K[\xi, \eta]$ remain constant and do not depend on the canonical variables. Typically, these central charges are nontrivial and cannot be eliminated by adding constants C_ξ to the generators $H[\xi]$. The existence of classical central charges is not unique to general relativity and gauge theories; it naturally arises in Hamiltonian classical mechanics. This phenomenon stems from the non-uniqueness of the canonical generator associated with a given phase space vector field. Such a generator is only determined up to the addition of a constant that commutes with all variables.

This issue also arises in asymptotic symmetries in gauge theories. In this case, the Hamiltonian generator $H[\xi]$ for a given asymptotic symmetry ξ^A consists of a linear combination of constraints $\varphi_A(x)$ from the canonical formalism, along with a surface term $J[\xi]$:

$$H[\xi] = \int d^n x . \xi^A(x) \varphi_A(x) + J[\xi] \tag{2.3}$$

The surface term $J[\xi]$ is constructed to ensure that the generator $H[\xi]$ has well-defined functional derivatives concerning the canonical variables. However, this construction leaves $J[\xi]$, and therefore $H[\xi]$, indeterminate up to the addition of an arbitrary constant. The presence of these constants in the canonical generators signifies the potential for central charges.

For three-dimensional Einstein gravity with a negative cosmological constant, the asymptotic symmetry group can be either $R \times SO(2)$ or the conformal group in two dimensions, depending on the chosen boundary conditions at spatial infinity. In the latter case, a nontrivial central charge emerges in the Poisson bracket algebra of the canonical generators. This example illustrates the possibility of central charge terms in the algebra of global charges, as described in (2.2). Additionally, it underscores the significance of the boundary conditions imposed at infinity, which determine the asymptotic structure of the theory but are not solely dictated by the theory itself.

It's worth noting that the presence of a true central charge can be ruled out when asymptotic symmetries can be realized as exact symmetries of some background field configuration. In this scenario, the charges evaluated for that background remain invariant under an asymptotic symmetry transformation since the background itself remains unchanged. Adjusting the arbitrary constant in $H[\xi]$ such that $H[\xi](\text{background}) = 0$ leads to $K[\xi, \eta]$ vanishing. However, this analysis doesn't cover all applications of the asymptotic symmetry group concept, particularly when dealing with the conformal group of asymptotic symmetries in $D = 3$ gravity with $\Lambda < 0$ or the B.M.S. group of asymptotic symmetries in $D = 4$ gravity with $\Lambda = 0$, which cannot be realized as the group of isometries in some spacetime.

2.1 Solutions To Three Dimensional Gravity With negative Cosmological constant

This subsection delves into a solution within Einstein gravity in the context of 2+1 dimensions with a negative cosmological constant. This particular solution serves as motivation for selecting suitable boundary conditions to apply to the metric in general.

In three dimensions, the gravitational field lacks dynamical degrees of freedom. Consequently, in regions devoid of sources, the spacetime locally resembles the solution to Einstein's equations for empty space, specifically anti-de Sitter space when the cosmological constant Λ is negative. This equivalence arises because the entire curvature tensor can be expressed in terms of the Einstein tensor, and where the empty space Einstein equations are valid, the curvature tensor simplifies to that of anti-de Sitter space. Matter, when localized, does not influence the local geometry of source-free regions, only affecting the global spacetime geometry. Therefore, the primary solution to consider is locally anti-de Sitter space with a curvature radius denoted as $R = (-1/\Lambda)^{1/2}$:

$$dS^2 = -\left(\frac{\bar{r}^2}{R^2} + 1\right) d\bar{t}^2 + \left(\frac{\bar{r}^2}{R^2} + 1\right)^{-1} d\bar{r}^2 + \bar{r}^2 d\bar{\phi}^2 \tag{2.4}$$

However, this solution features an unusual identification of points that modifies the global geometry. By identifying points $(\bar{t} = t', \bar{r} = r', \bar{\phi} = \phi')$ with points

($\bar{t} = t' - 2\pi A, \bar{r} = r', \bar{\phi} = \phi' + 2\pi\alpha$) for all t', r', and ϕ', a "wedge" of coordinate angle $2\pi(1-\alpha)$ is removed, and a "jump" of $2\pi A$ is introduced in coordinate time. The unusual identification only affects the origin at $\bar{r} = 0$, and the vacuum Einstein equations remain satisfied everywhere else.

The reason for considering this spacetime is its analogy to conic geometry in 2+1 gravity with $\Lambda = 0$, where the wedge parameter $\alpha \neq 1$ corresponds to total energy, and the jump parameter $A \neq 0$ relates to total angular momentum. Additionally, as in the de Sitter case, a wedge cut from anti-de Sitter space serves as a solution to Einstein's equations with the energy-momentum tensor of a point mass. The metric (2.4) can also be applied to the empty region exterior to a more general compact source distribution. The geometrically invariant nature of the wedge and the jump can be demonstrated as follows, independent of the details within the spacetime containing the source. Despite the spacetime being locally maximally symmetric, the only Killing vector fields consistent with the unusual identification of points are linear combinations of $d/d\bar{t}$ and $d/d\bar{\phi}$. Notably, $d/d\bar{t}$ and $d/d\bar{\phi}$ can be uniquely singled out (up to normalization) as the only two Killing vector fields orthogonal to each other throughout the spacetime. Similarly, $d/d\bar{r}$ is the unique vector field everywhere orthogonal to all Killing vector fields (up to normalization). As such, the curves serving as the $\bar{t}, \bar{r}, \bar{\phi}$ coordinate lines for the metric (2.4) can always be distinguished. Furthermore, the proper length L of a trajectory curve of $d/d\bar{\phi}$ between intersections with a trajectory curve of $d/d\bar{t}$ can be analyzed. The change dL as the curve moves a proper distance dS along the direction $d/d\bar{r}$ is given by:

$$\frac{dL}{dS} = \left[\frac{L^2}{R^2} + (2\pi\alpha)^2\right]^{1/2} \tag{2.5}$$

For $\alpha < 1$, the length L increases at a slower rate with proper distance compared to a globally anti-de Sitter space. Finally, the jump A is proportional to the proper time distance between the intersection points of the aforementioned trajectories. Henceforth, it is more convenient to express the metric (2.4) with a continuous time variable. A coordinate transformation is applied: $t = \bar{t} + (A/\alpha)\bar{\phi}, r = \bar{r}, \phi = (1/\alpha)\bar{\phi}$, yielding:

$$dS^2 = -\left(\frac{r^2}{R^2} + 1\right)(dt - Ad\phi)^2 + \left(\frac{r^2}{R^2} + 1\right)^{-1} dr^2 + \alpha^2 r^2 d\phi^2 \tag{2.6}$$

In this coordinate system, the Killing vector fields are linear combinations of d/dt and $d/d\phi$. It's worth noting that the trajectories of $d/d\phi$ form closed timelike curves unless $|A| = \alpha|R|$ and:

$$r^2 > \frac{A^2 R^2}{\alpha^2 R^2 - A^2} \tag{2.7}$$

Consequently, this spacetime serves as a valid solution to Einstein gravity only for $|A| = \alpha|R|$ and large values of r, particularly in the asymptotic limit as $r \to \infty$.

2.2 Global Charges and the $R \times SO(2)$ Asymptotic Symmetries

The process of determining the global charges within the Hamiltonian framework of a gauge theory has been firmly established. The initial step involves establishing the boundary conditions at spatial infinity that generic fields must adhere to. Subsequently, one identifies the asymptotic symmetries that preserve these asymptotic conditions. In the context of gravitational theories, it is essential to translate the boundary conditions imposed on the spacetime metric into equivalent conditions on the canonical variables, namely, g_{ij} and π^{ij}. Additionally, the asymptotic symmetries of the spacetime dictate the permissible surface deformation vectors, denoted as ξ^μ (with μ representing both time and spatial components), for the spacelike hypersurfaces under consideration.

For these boundary conditions and asymptotic symmetries to be acceptable in a gravitational theory, it must be feasible to express the Hamiltonian as the standard linear combination of constraints, along with an appropriate surface term denoted as $J[\xi]$. This surface term, referred to as the charge hereafter, must exhibit a variation that cancels out the undesired surface terms in the variation of the Hamiltonian. Consequently, the Hamiltonian takes the form:

$$H[\xi] = \int d^n x \xi^\mu(x) \mathcal{H}_\mu(x) + J[\xi] \tag{2.8}$$

This ensures that the Hamiltonian possesses well-defined variational derivatives and can be employed as the generator for the permissible surface deformations. In practical applications, the charges $J[\xi]$ are often determined by examining the surface terms originating from the variation of the "volume piece" of the Hamiltonian, specifically:

$$-\lim_{r\to\infty} \oint d^{n-1} S_l \left\{ G^{ijkl} \left[\xi^\perp \delta g_{ij;k} - \xi^\perp_{,k} \delta g_{ij} \right] + 2\xi_i \delta\pi^{il} + (2\xi^i \pi^{kl} - \xi^l \pi^{ik}) \delta g_{ik} \right\} \tag{2.9}$$

Here, $G^{ijkl} = \frac{1}{2}\sqrt{g}(g^{ik}g^{jl} + g^{il}g^{jk} - 2g^{ij}g^{kl})$, and the semicolon indicates covariant differentiation within a spacelike hypersurface. By leveraging the assumed asymptotic behavior of the fields g_{ij}, π^{ij}, and vectors ξ^μ, this expression is reformulated as the total variation of a surface integral. The negative of this surface integral, up to a constant, represents the charge $J[\xi]$. Parameters α and A place constraints on the metric outside the source, defining the boundary conditions for the metric. In this scenario, the asymptotic symmetries align with the Killing vector fields d/dt and $d/d\phi$. Consequently, the associated asymptotic symmetry group with these boundary conditions is $R \times SO(2)$. Notably, there is a departure from asymptotic anti-de Sitter invariance, as the asymptotic symmetry group does not correspond to the full anti-de Sitter group $SO(2, 2)$. This departure likely arises from the specific boundary conditions imposed by the chosen family of metrics characterized by parameters α and A.

Next, we proceed to compute the charges associated with the vectors d/dt and $d/d\phi$. Let's assume ξ as some linear combination of our Killing vector fields, with components ${}^{(3)}\xi^\alpha$ where α takes values of t, r, ϕ in the spacetime coordinate system.

The $\perp, r, \phi$ components ξ^ν of this vector describe an allowable deformation of the surface outside the source. They are linked to the spacetime components as follows:

$$\xi^\perp = N\ {}^{(3)}\xi^t \tag{2.10a}$$
$$\xi^r =^{(3)} \xi^r + N^r\ {}^{(3)}\xi^t \tag{2.10b}$$
$$\xi^\phi =^{(3)} \xi^\phi + N^\phi\ {}^{(3)}\xi^t \tag{2.10c}$$

Here, N represents the lapse, while N^r and N^ϕ correspond to the shifts in the spacetime coordinate system. We can calculate these values by comparing our line element in (2.6) with the metric employed in the ADM formalism[4]. The functions are determined as follows:

$$N = \alpha\left[\frac{r^2+R^2}{\alpha^2R^2-A^2}\right]^{1/2}\left[1-\frac{A^2R^2}{(\alpha^2R^2-A^2)r^2}\right]^{-1/2}$$
$$N^r = 0$$
$$N^\phi = \frac{A(r^2+R^2)}{r^2(\alpha^2R^2-A^2)-A^2R^2}$$

Given that ${}^{(3)}\xi^r = 0$, the component $\xi^r = 0$ remains constant. The only nonzero components of the canonical variables needed for computing the expression in (2.9) are as follows:

$$g_{rr} = \left(\frac{r^2}{R^2}+1\right)^{-1} \tag{2.11a}$$
$$g_{\phi\phi} = \alpha^2r^2 - A^2\left(\frac{r^2}{R^2}+1\right) \tag{2.11b}$$
$$\pi^r_\phi = \alpha A \tag{2.11c}$$

This results in:

$$-\delta J[\xi] = 4\pi\left[{}^{(3)}\xi^t\delta(\alpha) -^{(3)} \xi^\phi\delta(\alpha A)\right] \tag{2.12}$$

Consequently, the charges associated with the symmetries d/dt and $d/d\phi$ are, up to constants:

$$J[d/dt] = 4\pi(1-\alpha) \tag{2.13a}$$
$$J[d/d\phi] = 4\pi\alpha A \tag{2.13b}$$

Remarkably, these charges correspond precisely to the energy and angular momentum of locally flat $2+1$ gravity, thereby ensuring the validity of these charges as Λ tends to 0.

2.3 The Conformal Group Of Asymptotic Symmetries

The boundary conditions can be relaxed slightly, leading to an expansion of the group of asymptotic symmetries to include the anti-de Sitter group in $2+1$ dimensions, denoted as $O(2,2)$. The inspiration for these modified boundary conditions arises from the reparametrization of the metric (2.6) through the following substitutions:

$$\begin{aligned}
t &\to \frac{t}{\alpha}\left(\alpha^2 - \frac{A^2}{R^2}\right) \\
r &\to r\left(\alpha^2 - \frac{A^2}{R^2}\right)^{-1/2} \\
\phi &\to \phi - \frac{A}{\alpha R^2} t
\end{aligned} \tag{2.14}$$

This transformation results in the metric (2.15), which now takes the form:

$$ds^2 = -\left(\frac{r^2}{R^2} + \alpha^2\right) dt^2 + 2A\alpha dr d\phi + \left(\frac{r^2 - A^2}{R^2} + \alpha^2\right)^{-1} dr^2 + (r^2 - A^2) d\phi^2 \tag{2.15}$$

It's worth noting that when $A = 0$, the leading contributions in this metric and those of a globally anti-de Sitter space are in agreement, yielding:

$$ds^2 \to -\left(\frac{r^2}{R^2}\right) dt^2 + \left(\frac{R^2}{r^2}\right) dr^2 + r^2 d\phi^2 \tag{2.16}$$

In this context, it appears reasonable to consider the metric (2.15), particularly when $A = 0$, as "asymptotically anti-de Sitter." However, to precisely define the concept of "asymptotically anti-de Sitter," specific boundary conditions for the metric must be established. For the anti-de Sitter group to be part of the asymptotic symmetries that preserve these conditions, the metric resulting from an anti-de Sitter transformation acting on (2.15), whether with or without $A = 0$, must also be "asymptotically anti-de Sitter."

By applying all possible anti-de Sitter group transformations to (2.15), the following boundary conditions are generated:

$$g_{tt} = -\frac{r^2}{R^2} + \mathcal{O}(1) \tag{2.17a}$$

$$g_{tr} = \mathcal{O}(1/r^3) \tag{2.17b}$$

$$g_{t\phi} = \mathcal{O}(1) \tag{2.17c}$$

$$g_{rr} = \frac{R^2}{r^2} + \mathcal{O}(1/r^4) \tag{2.17d}$$

$$g_{r\phi} = \mathcal{O}(1/r^3) \tag{2.17e}$$

$$g_{\phi\phi} = r^2 + \mathcal{O}(1) \tag{2.17f}$$

It's intriguing to compare these boundary conditions (2.17) with the boundary conditions on the metric in the context of gravity in $3 + 1$ dimensions with $\Lambda < 0$. When restricting the spatial sections in the $3 + 1$ scenario to two dimensions (e.g., by setting $\theta = \pi/2$), it becomes evident that the permitted metrics, as opposed to anti-de Sitter space, must decay faster by one power of l/r in $3 + 1$ dimensions than in $2 + 1$ dimensions.

Chapter 3

First Order Formalism

This section is referred mainly from [9]. Other references were [8] and Appendix A.2 of [7].

The fundamental variables in the first-order formalism are known as vielbeins, denoted as e^a_μ. Vielbeins act as transformation matrices that relate the tangent space to the coordinate frame. In the context of $2+1$ dimensions, these vielbeins are often referred to as triads. Assuming that the basis vectors in the local tangent space to spacetime are orthonormal with respect to the Minkowski metric, the vielbeins satisfy the following relations:

$$g^{\mu\nu} e^a_\mu e^b_\nu = \eta^{ab} \tag{3.1}$$

$$\eta_{ab} e^a_\mu e^b_\nu = g_{\mu\nu} \tag{3.2}$$

In the above expressions, Greek indices represent spacetime indices, while Latin indices denote vielbein (or triad in the case of $2+1$ dimensions) indices. The collection of all possible vielbeins at each point on the spacetime manifold M is collectively known as a frame or vielbein bundle. Utilizing these vielbeins, represented as V^a_μ, we can introduce a set of basis 1-forms in the tangent Minkowski space-time, denoted as $V^a = V^a_\mu dx^\mu$. The covariant derivative of V^a is given by:

$$D_\mu V^a = \partial_\mu V^a + \omega^a_{\mu b} V^b \tag{3.3}$$

Here, $\omega^a_{\mu b}$ is a connection in the vielbein basis and is commonly referred to as the spin connection. To determine the specific form of $\omega^a_{\mu b}$, one can demand that the net parallel transport of e^a_μ results in a vanishing covariant derivative:

$$D_\mu e^a_\nu = \partial_\mu e^a_\nu - \Gamma^\rho_{\mu\nu} e^a_\rho + \epsilon^{abc} \omega_{\mu b} e_{\nu c} = 0 \tag{3.4}$$

When the connection $\Gamma^\rho_{\mu\nu}$ is torsion-free, the following equation holds:

$$T^a = D_\omega e^a = de^a + \omega^a_b \wedge e^b = 0 \tag{3.5}$$

This relationship is referred to as Cartan's first structure equation. The curvature tensor can be defined as:

$$[D_\mu, D_\nu] V^a = R^a_{\mu\nu b} V^b \tag{3.6}$$

Expanding the covariant derivatives, the Riemann tensor can be expressed as:

$$dx^\mu \wedge dx^\nu R^a_{\mu\nu b} = (\partial_{[\mu} \omega^b_{\nu]a} - \omega^c_{[\mu|a]} \omega^b_{|\nu]c}) dx^\mu \wedge dx^\nu \tag{3.7}$$

$$= d\omega^b_a + \omega^b_c \wedge \omega^c_a \tag{3.8}$$

This form is analogous to the field strength in Yang-Mills theory:

$$F = dA + A \wedge A \tag{3.9}$$

With the metric and spin connection, the Einstein action can be expressed in the first-order formalism as:

$$I = k \int \left[\epsilon_{a_1 a_2 \ldots a_D} R^{a_1 a_2} \wedge e^{a_3} \wedge \ldots e^{a_D} + \frac{\Lambda}{D!} \epsilon_{a_1 a_2 \ldots a_D} e^{a_1} \wedge e^{a_2} \wedge \ldots e^{a_D} \right] \tag{3.10}$$

Here, $R^{a_1 a_2}$ represents a curvature two-form, specifically $R^{a_1 a_2} \equiv R^{a_1 a_2 a_3 a_4} e_{a_3} \wedge e_{a_4}$.

In the context of $2+1$ dimensions, vielbeins, and spin connections can be written as one-forms:

$$e^a = e^a_\mu dx^\mu, \quad \omega^a = \frac{1}{2} \epsilon^{abc} \omega_{\mu bc} dx^\mu$$

Here, the $3D$ Levi-Civita tensor ϵ^{abc} is an invariant tensor of $SO(3,1)$. It follows the convention $\epsilon^{012} = 1$.

The Einstein-Hilbert action in three dimensions can be expressed as:

$$I = \frac{1}{8\pi G} \int \left[e^a \wedge (d\omega_a + \frac{1}{2} \epsilon_{abc} \omega^b \omega^c) + \frac{\Lambda}{6} \epsilon_{abc} e^a \wedge e^b \wedge e^c \right] \tag{3.11}$$

One of the equations of motion can be obtained by varying ω_a:

$$T_a = de_a + \epsilon_{abc} \omega^b \wedge e^c = 0 \tag{3.12}$$

Varying the action with respect to e^a yields:

$$d\omega_a + \frac{1}{2} \epsilon_{abc} \omega^b \wedge \omega^c + \frac{\Lambda}{2} \epsilon_{abc} e^b \wedge e^c = 0 \tag{3.13}$$

This equation can be written as:

$$R_a = d\omega_a + \frac{1}{2} \epsilon_{abc} \omega^b \wedge \omega^c = -\frac{\Lambda}{2} \epsilon_{abc} e^b \wedge e^c \tag{3.14}$$

which corresponds to Einstein's equation in the language of vielbeins and the spin connection.

The action, up to boundary terms, remains invariant under two sets of gauge symmetries:

(a) Local Lorentz Transformations (LLT):

$$\begin{aligned} \delta_l e^a &= \epsilon^{abc} e_b \tau_c \\ \delta_l \omega^a &= d\tau^a + \epsilon^{abc} \omega_b \tau_c \end{aligned} \tag{3.15}$$

Here, τ_a represents a local function.

(b) Local Translations (LT):

$$\begin{aligned} \delta_t e^a &= d\rho^a + \epsilon^{abc} \omega_b \rho_c \\ \delta_t \omega^a &= -\Lambda \epsilon^{abc} e_b \rho_c \end{aligned} \tag{3.16}$$

The subscripts t and l on δ are used for labeling purposes. These transformations are referred to as local Lorentz transformations and local translations because the number of components in τ and ρ corresponds precisely to the number of parameters for Lorentz transformations ($\frac{d(d-1)}{2}$ in d dimensions) and translations (d in d dimensions), respectively.

Chapter 4

BTZ Blackholes

This section is referred from [2]. Although (2+1)-dimensional gravity has been widely recognized as a useful laboratory for studying conceptual issues—the nature of observables, for example, and the "problem of time"—it has been widely believed that the model is too physically unrealistic to give much insight into real gravitating systems in 3+1 dimensions. In particular, general relativity in 2+1 dimensions has no Newtonian limit and no propagating degrees of freedom. It, therefore, came as a considerable surprise when Bañados, Teitelboim, and Zanelli (BTZ) showed in 1992 that (2+1)-dimensional gravity has a black hole solution. The BTZ black hole differs from the Schwarzschild and Kerr solutions in some important respects: it is asymptotically anti-de Sitter rather than asymptotically flat and has no curvature singularity at the origin. Nonetheless, it is clearly a black hole: it has an event horizon and (in the rotating case) an inner horizon, it appears as the final state of collapsing matter, and it has thermodynamic properties much like those of a (3+1)-dimensional black hole.

The BTZ black hole in "Schwarzschild" coordinates is described by the metric

$$ds^2 = -(N^\perp)^2 dt^2 + f^{-2} dr^2 + r^2 (d\phi + N^\phi dt)^2 \tag{4.1}$$

with lapse and shift functions and radial metric

$$\begin{aligned} N^\perp = f &= \left(-M + \frac{r^2}{l^2} + \frac{J^2}{4r^2}\right) \\ N^\phi &= -\frac{J}{2r^2} \end{aligned} \tag{4.2}$$

and $|J| \leq Ml$. This is the solution of the (2+1)-dimensional Einstein's equation

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{1}{l^2} g_{\mu\nu} \tag{4.3}$$

with a cosmological constant $\Lambda = -1/l^2$. The metric (4.1) is stationary and axially symmetric, with Killing vectors ∂_t and ∂_ϕ, and generically has no other symmetries.

Parameters M and J are the standard ADM mass and angular momentum which determine the asymptotic behavior of the solution. To see this, one can write the Einstein action in ADM form,

$$I = \frac{1}{16\pi G} \int_0^T dt \int d^2x \left[\pi^{ij} \dot{g}_{ij} - N^\perp \mathcal{H} - N^i \mathcal{H}_i\right] + B \tag{4.4}$$

where the boundary term B is required to cancel surface integrals in the variation of I. If one now considers variations of the spatial metric that preserves the asymptotic form of the BTZ solution, one finds that

$$\delta B = T\left[-\delta M + N^{\phi}\delta J\right] \tag{4.5}$$

As in 3+1 dimensions, the conserved charges can be read off from δB: J is the angular momentum as measured at infinity, while M is the mass associated with asymptotic translations in the "Killing time" t. It may also be checked that M and J are the conserved charges associated with the asymptotic Killing vectors ∂_t and ∂_ϕ for asymptotically anti-de Sitter spacetimes.

The metric (4.1) is singular when $r = r_\pm$ where

$$r_\pm^2 = \frac{Ml^2}{2}\left\{1 \pm \left[1 - \left(\frac{J}{ML}\right)^2\right]^{1/2}\right\} \tag{4.6}$$

i.e.,

$$\begin{aligned} M =& \frac{r_+^2 + r_-^2}{l^2} \\ J =& \frac{2r_+ r_-}{l} \end{aligned} \tag{4.7}$$

As we shall see below, these are merely coordinate singularities, closely analogous to the singularity at $r = 2m$ of the ordinary Schwarzschild metric. The time-time component g_{00} of the metric vanishes at $r = r_{erg}$, where

$$r_{erg} = M^{\frac{1}{2}} l = \left(r_+^2 + r_-^2\right)^{\frac{1}{2}} \tag{4.8}$$

As in the Kerr solution in $3+1$ dimensions, $r < r_{erg}$ determines an ergosphere: timelike curves in this region necessarily have $d\phi/d\tau > 0$ (when $J > 0$), so all observers are dragged along by the rotation of the black hole. Note that the $r_\pm$ becomes complex if $|J| > Ml$, and the horizons disappear, leaving a metric that has a naked conical singularity at $r = 0$. The $M = -1, J = 0$ metric may be recognized as that of ordinary anti-de Sitter space; it is separated by a mass gap from the $M = 0, J = 0$ "massless black hole".

The BTZ metric is a genuine black hole can be seen most easily by transforming to Eddington-Finkelstein-like coordinates

$$\begin{aligned} dv =& dt + \frac{dr}{(N^\perp)^2} \\ d\tilde{\phi} =& d\phi - \frac{N^\phi}{(N^\perp)^2} dr \end{aligned} \tag{4.9}$$

in which the metric becomes

$$ds^2 = -(N^\perp)^2 dv^2 = 2dvdr + r^2(d\phi + N^\phi dv)^2 \tag{4.10}$$

It is now easy to see that the horizon $r = r_+$, where $N^\perp$ vanishes, is a null surface, generated by geodesics

$$r(\lambda) = r_+ \quad , \quad \frac{d\tilde{\phi}}{d\lambda} + N^\phi(r_+)\frac{dv}{d\lambda} = 0 \tag{4.11}$$

Moreover, this surface is evidently a marginally trapped surface: at $r = r_+$, any null geodesic satisfies

$$\frac{dv}{d\lambda}\frac{dr}{d\lambda} = -\frac{r_+^2}{2}\left(\frac{d\tilde{\phi}}{d\lambda} + N^\phi(r_+)\frac{dv}{d\lambda}\right)^2 \leq 0 \tag{4.12}$$

so r decreases or remains constant as v increases.

Like the outer horizon of the Kerr metric, the surface $r = r+$ is also a Killing horizon. The Killing vector normal to this surface is

$$\chi = \partial_v - N^\phi(r_+)\partial_{\tilde{\phi}} \tag{4.13}$$

from which the surface gravity κ, defined by

$$\kappa^2 = -\frac{1}{2}\nabla^a\chi^b\nabla_a\chi_b \tag{4.14}$$

may be computed to be

$$\kappa = \frac{r_+^2 - r_-^2}{l^2 r_+} \tag{4.15}$$

For a more complete description of the BTZ solution, we can transform it instead to Kruskal-like coordinates. To do so, let us define new null coordinates

$$\begin{aligned} u &= \rho(r)e^{-at} \\ v &= \rho(r)e^{at} \end{aligned} \tag{4.16}$$

with

$$\frac{d\rho}{dr} = \frac{a\rho}{(N^\perp)^2}$$

As in the case of the Kerr metric, we need two patches, $r_- < r < \infty$ and $0 < r < r_+$, to cover the BTZ spacetime. In each patch, the metric (4.1) takes the form

$$ds^2 = \Omega^2 dudv + r^2(d\tilde{\phi} + N^\phi dt)^2 \tag{4.17}$$

where for $r_- < r < \infty$

$$\begin{aligned} \Omega_+^2 &= \frac{(r^2 - r_-^2)(r + r_+)^2}{a_+^2 r^2 l^2}\left(\frac{r - r_-}{r + r_-}\right)^{r_-/r_+} \\ \tilde{\phi}_+ &= \phi + N^\phi(r_+)t \qquad a_+ = \frac{r_+^2 - r_-^2}{l^2 r_+} \end{aligned} \tag{4.18}$$

and for $0 < r < r_+$

$$\begin{aligned} \Omega_-^2 &= \frac{(r_+^2 - r^2)(r + r_-)^2}{a_-^2 r^2 l^2}\left(\frac{r_+ - r}{r_+ + r}\right)^{r_+/r_-} \\ \tilde{\phi}_- &= \phi + N^\phi(r_-)t \qquad a_- = \frac{r_-^2 - r_+^2}{l^2 r_-} \end{aligned} \tag{4.19}$$

with r and t viewed as implicit functions of u and v. As in the case of the Kerr black hole, an infinite number of such Kruskal patches may be joined together to form a maximal solution, whose Penrose diagram is shown in Figure **??**. This diagram differs from that of the Kerr metric at $r = \infty$, reflecting the fact that the BTZ black hole is asymptotically anti-de Sitter rather than asymptotically flat, but the overall structure is similar. In particular, it is evident that $r = r_+$ is an event horizon, while the inner horizon $r = r_-$ is a Cauchy horizon for the region I. When $J = 0$, the Penrose diagram collapses to that of Figure **??**, which is similar in structure—except for its asymptotic behavior—to the diagram for the ordinary Schwarzschild solution, while for the extreme case, $J = \pm Ml$, the Penrose diagram is that of Figure 4.2.

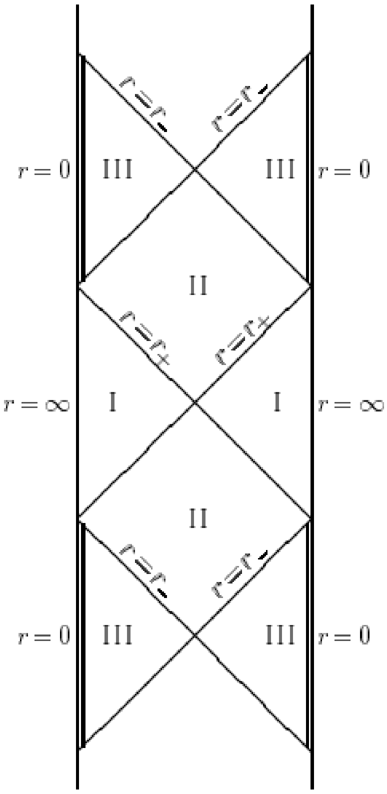

Figure 4.1: Krushkal patches for Kerr BH

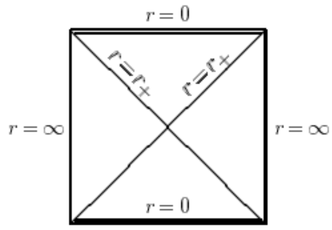

Figure 4.2: Kerr BH with $J = 0$

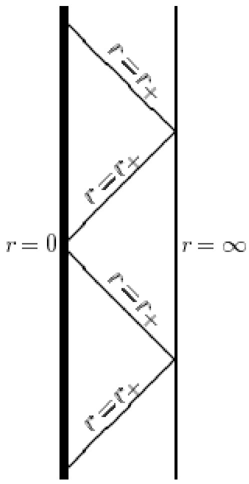

Figure 4.3: Kerr BH with $J = \pm Ml$

Let's look at geodesics behavior[5]. The Killing vectors associated with the BTZ metric are two ∂_t and ∂_ϕ. Thus, the constants of motion along the geodesic are

$$E = -g_{ab}\xi^a u^b = \left[-M + \frac{r^2}{l^2}\right]\left(\frac{dt}{d\lambda}\right) + \frac{J}{2}\left(\frac{d\phi}{d\lambda}\right) \tag{4.20}$$

where $\xi^a = (\partial/\partial t)^a$ denotes the static Killing vector, and

$$L = g_{ab}\Phi^a u^b = r^2\left(\frac{d\phi}{d\lambda}\right) - \frac{J}{2}\left(\frac{dt}{d\lambda}\right) \tag{4.21}$$

where $\Phi^a = (\partial/\partial\phi)^a$ is the rotational Killing vector and $u^a = dx^a/d\lambda$ is the tangent to a curve parameterized by λ, which is normalized by the condition

$$u^a u_a = -m^2 \tag{4.22}$$

where m is 1 for timelike geodesic and 0 for null geodesics. The constant E cannot be interpreted as the local energy of the particle at infinity since the black-hole field is not asymptotically flat.

Making the following re-scalings,

$$r \to \frac{r}{l\sqrt{M}}, \quad \phi \to \phi\sqrt{M}, \quad t \to t\frac{\sqrt{M}}{l},$$

$$\lambda \to \frac{\lambda}{l}, \quad E \to \frac{E}{\sqrt{M}}, \quad L \to \frac{L}{l\sqrt{M}},$$

$$J \to \frac{J}{lM}$$

and using (4.20), (4.21), (4.22), we obtain the geodesic equations

$$r^2\dot{r}^2 = -m^2(r^4 - r^2 + \frac{J}{4}) + (E^2 - L^2)r^2 + L^2 - JEL \tag{4.23}$$

$$\dot{\phi} = \frac{(r^2 - 1)L + \frac{1}{2}JE}{(r^2 - r_+^2)(r^2 - r_-^2)} \tag{4.24}$$

$$\dot{t} = \frac{Er^2 - \frac{1}{2}JL}{(r^2 - r_+^2)(r^2 - r_-^2)} \tag{4.25}$$

where dot means $d/d\lambda$. Equations (4.23) to (4.25) describe the motion of test particles in a 2+1 black-hole background. From these equations, we will obtain the orbits for massless and massive particles and the effective radial potential.

Chapter 5

Black Holes and Gravitational Collapse

The physical importance of the (3+1)-dimensional black hole comes from its role as the final state of gravitational collapse. Since (2+1)-dimensional gravity has no Newtonian limit, one might fear that no such interpretation exists for the BTZ black hole. In fact, however, it was shown shortly after the discovery of the BTZ solution that this black hole arises naturally from collapsing matter.

Consider a (2+1)-dimensional spacetime containing a spherical cloud of dust surrounded by empty space. For the exterior, we take the metric to be of the BTZ form (4.1), with $J = 0$ for simplicity; for the interior, we may choose comoving coordinates, in which the geometry is given by a Robertson-Walker metric,

$$ds^2 = -dt^2 + a^2(t)\left(\frac{d\tilde{r}^2}{1-k\tilde{r}^2} + \tilde{r}^2 d\phi^2\right) \tag{5.1}$$

The stress-energy tensor for pressureless dust is

$$T_{\mu\nu} = \rho(t) u_\mu u_\nu \tag{5.2}$$

with $u_\mu = (1,0,0)$ in comoving coordinates; as usual, conservation implies that $\rho a^2 = \rho_0 a_0^2$. It is then easy to show that the field equations in the interior are solved by

$$\begin{aligned} a(t) =& a_0 \cos\frac{t}{l} + l\dot{a}_0 \sin\frac{t}{l} \\ \dot{a}_0^2 =& 8\pi G\rho_0 a_0^2 - k - \frac{a_0^2}{l^2} \end{aligned} \tag{5.3}$$

Note that for arbitrary initial values, $a(t)$ always reaches zero in a finite proper time.

We must now join the interior and exterior solutions, using the standard matching conditions that the spatial metric g_{ij} and the extrinsic curvature K_{ij} be continuous at the boundary. According to [10], this requires

$$M = 8\pi G\rho_0 a_0^2 r_0^2 - 1 \tag{5.4}$$

where $\tilde{r} = r_0$ is the position of the surface of the collapsing dust in the interior (comoving) coordinate, equivalent to the exterior radial coordinate $r = r_0 a(t)$. The collapse closely resembles the Oppenheimer-Snyder solution in 3+1 dimensions. In particular, the surface of the collapsing dust crosses the horizon in a finite amount of comoving time, but light emitted from the surface is infinitely red-shifted at the horizon, and the collapse appears to take infinitely long to a static exterior observer. The mass M of the final black hole depends on three parameters, ρ_0, r_0, and a_0, or equivalently ρ_0, r_0, and v_0, where $v_0 = \dot{r}|_{t=0} = r_0\dot{a}_0$ is the initial velocity. If these parameters are such that $M < 0$ in eqn. (5.4), the final state is not a black hole, but rather a naked conical singularity in an asymptotically anti-de Sitter spacetime. While pressureless dust in 2+1 dimensions necessarily collapses (provided that $\Lambda < 0$), a (2+1)-dimensional ball of fluid can be stabilized by internal pressure.

A BTZ black hole can also form from a collapsing pulse of radiation. The most useful coordinates to describe this process are the Eddington-Finkelstein coordinates of (4.10), in which a suitable metric ansatz — analogous to the Vaidya metric in 3+1 dimensions — is

$$ds^2 = \left[\frac{r^2}{l^2} + m(v)\right] dv^2 + 2dvdr - j(v)dvd\tilde{\phi} + r^2 d\tilde{\phi}^2 \tag{5.5}$$

For the stress-energy tensor, we take that of a rotating null fluid,

$$\begin{aligned} T_{vv} =& \frac{\rho(v)}{r} + \frac{j(v)\omega(v)}{2r} \\ T_{v\tilde{\phi}} =& -\frac{\omega(v)}{r} \end{aligned} \tag{5.6}$$

where $\rho(v)$ and $\omega(v)$ are arbitrary functions and the form of the r dependence follows from the conservation law for the stress-energy tensor. The Einstein field equations then reduce to

$$\begin{aligned} \frac{d\,m(v)}{dv} =& 2\pi\rho(v) \\ \frac{d\,j(v)}{dv} =& 2\pi\omega(v) \end{aligned} \tag{5.7}$$

(in BTZ units, 8G=1). In particular, any distribution of radiation for which $m(v) \sim M$ and $j(v) \sim J$, as v goes to infinity, will approach a BTZ black hole. The pulse of radiation of the form,

$$\rho(v) = A sech^2\frac{v}{b} \tag{5.8}$$

leads asymptotically to a black hole with mass $2\pi A$.

One of the important open questions in black hole physics is that of the stability of the inner horizon. It is apparent from the Penrose diagram of Figure **??**, **??**, and 4.2 that in the case of a rotating black hole, an infalling observer need not hit the singularity at $r = 0$, but can escape through the inner horizon $r = r_-$ to a new exterior region. On the other hand, infalling radiation is infinitely blue-shifted at $r = r_-$, suggesting that the inner horizon is not stable; and indeed, simple models in 3+1 dimensions indicate that this horizon may be destroyed by the back reaction of ingoing and back-scattered outgoing radiation. This phenomenon has been an important focus of research in (3+1)-dimensional general relativity, where it has been shown that the internal mass function of a Reissner-Nordstrom black hole diverges at the Cauchy horizon ("mass inflation"), but that tidal forces may nevertheless remain weak enough that physical objects can survive passage through the horizon.

Let us model a thin shell of outgoing radiation by joining an "exterior" metric of the form (5.5), with mass function $m_1(v_1)$, to an "interior" metric, also of the form (5.5), with a mass function $m_2(v_2)$. By choosing appropriate matching conditions for the interior and exterior regions, we can model the interaction of infalling radiation — $\rho(v)$ in eqn. (5.5) — with this shell of outgoing radiation. A careful analysis of these matching conditions then shows that $m2$ necessarily diverges at the Cauchy horizon. (For the spinning case, it is actually the quantity

$$E_2(v) = m_2(v) - \frac{j_2^2(v)}{4r^2} \tag{5.9}$$

that diverges). The (2+1)-dimensional black hole thus exhibits mass inflation. On the other hand, as in 3+1 dimensions, tidal forces lead to only a finite distortion at the Cauchy horizon, so it is not clear that passage through the horizon is forbidden. Moreover, in contrast to the (3+1)-dimensional case, the Kretschmann scalar $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ remains finite. Further investigation of more complicated solutions — in particular, solutions with realistic outgoing radiation — seems feasible, and could provide valuable information on the question of stability.

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