Indian Institute of Technology Bombay

Department of Physics

QUANTIZATION IN $D \to 4$ EINSTEIN-GAUSS-BONNET GRAVITY

Supervised Learning Project Report

Presenter:
Apurv Keer
210260007

Guide:
Prof. S. Shankaranarayanan

Abstract

One of the key aspects of Lovelock gravity models is the inclusion of the Gauss-Bonnet term in the action. This is particularly important for gaining insight into modified gravity theories. Our discussion begins with a review of Glavin and Lin's approach to rescaling the coupling constant to account for the dynamic effects of this term. We also examine various attempts to create well-defined and self-consistent theories that incorporate it.

In the next section, we delve into T. Padmanabhan's research paper on the Plank length as the lower limit. We discuss his method of approaching quantum gravity through the quantization of conformal fluctuations in flat spacetime and his conclusion regarding the Plank length as the ultimate limit for all physical scales.

Finally, we turn to my research collaboration with Professor Shankaranarayanan, which revolves around developing a quantization theory for $D \to 4$ Einstein-Gauss-Bonnet, using a methodology similar to Padmanabhan's.

Contents

Chapter 1

Einstein-Gauss-Bonnet Theory in 4D

Ever since its inception, there have been attempts to provide alternatives to Einstein's theory of General Relativity. These alternatives to GR are introduced for a variety of mathematical, philosophical, and observational reasons, but almost all have the common function of generalizing the theory that Einstein initially proposed.

Holding a special place amongst this zoo of possibilities is the Einstein-Gauss-Bonnet theory, initially proposed by Lanczos and subsequently generalized by Lovelock. These theories are unique in requiring no extra fundamental fields beyond those that go into GR while maintaining the property that the field equations of the theory can be written with no higher than second derivatives of the metric (a sufficient condition to prevent Ostrogradsky instability). They are therefore particularly well motivated and hold a uniquely privileged position among the pantheon of alternatives to GR.

Although there is an obvious interest in studying gravity in four dimensions, the Einstein-Gauss-Bonnet extension of GR was for a long time thought to be trivial in this case. This changed in 2020 when Glavan & Lin proposed a re-scaling of the coupling constant of the theory that potentially allowed for the consequences of Einstein-Gauss-Bonnet to be noticed even in the four-dimensional case [10]. The theories that resulted from this idea have come to be known as "4D Einstein-Gauss-Bonnet" (4D EGB) gravity, and have a number of interesting properties.

Throughout this report, we will use a Lorentzian metric with signature $(-,+,+,+,....)$. Greek letters will be used to represent the space-time components of vectors and tensors in a coordinate basis, and Latin letters for spatial components. All expressions will be presented in units in which $\hbar = c = 1$, and we will use the notation $\Box = \nabla_\mu \nabla^\mu$, $(\nabla\phi)^2 = \nabla_\mu\phi\nabla^\mu\phi$ and $\nabla^2\phi = \nabla_\mu\nabla^\mu\phi$ where ∇_μ represents the covariant derivative with respect to the metric $g_{\mu\nu}$.

1.1 The Gauss-Bonnet Term

This is closely based on the review [4]. We will consider the following Lagrangian density:

$$\mathcal{L} = \sqrt{-g}[\alpha_0 + \alpha_1 R + \alpha_2 \mathcal{G}] \tag{1.1}$$

where

$$\mathcal{G} = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \tag{1.2}$$

is known as the *Gauss-Bonnet* term. Extremization of the action associated with this Lagrangian gives the Lanczos tensor

$$A^{\mu}_{\nu} = -\frac{1}{2}\alpha_0\delta^{\mu}_{\nu} + \alpha_1\left(R^{\mu}_{\nu} - \frac{1}{2}\delta^{\mu}_{\nu}R\right) + \alpha_2\left(2R^{\mu\alpha\rho\sigma}R_{\nu\alpha\rho\sigma} - 4R^{\rho\sigma}R^{\mu}_{\rho\nu\sigma} - 4R^{\mu\rho}R_{\nu\rho} + 2RR^{\mu}_{\nu} - \frac{1}{2}\delta^{\mu}_{\nu}\mathcal{G}\right) \tag{1.3}$$

The contribution term from the Gauss-Bonnet term, the term in the brackets after α_2, vanishes identically due to dimensionally-dependent identities of A^{μ}_{ν}[7]. It is for this reason that the Gauss-Bonnet term in $D = 4$ is often referred to as a "topological term", and neglected. This is despite the fact that generically $\mathcal{G} \neq 0$ in $D = 4$.

In an attempt to introduce the Gauss-Bonnet term in 4D gravity directly, Glavan & Lin proposed rescaling the coupling constant α_2 such that

$$\alpha_2 \to \frac{\alpha_2}{(D-4)} \tag{1.4}$$

This quantity is clearly divergent in the limit $D \to 4$ but Glavan & Lin made the non-trivial suggestion that if this re-scaling were introduced into the Lanczos tensor then the terms that contain this quantity as a factor might remain finite and non-zero. That is, they postulated that the divergence they introduced into α_2 might be sufficient to cancel out the fact that additional terms in Eq. (1.3) tend to zero as $D \to 4$. If this were the case, then the Gauss-Bonnet term would be allowed to have a direct effect on the 4D theory of gravity.

The trace of the Lanczos tensor(1.3) in D dimensions is

$$A^{\mu}_{\mu} = -\frac{1}{2}D\alpha_0 - \frac{1}{2}(D-2)\alpha_1 R - \frac{1}{2}(D-4)\alpha_2\mathcal{G} \tag{1.5}$$

The vanishing of the term from the Einstein tensor in $D = 2$ and the vanishing of the Gauss-Bonnet term in $D = 4$ are both made explicit here, and both can be seen to be due to a pre-factor of the form $(D - n)$ (recall that R and $\mathcal{G}$ can be non-zero only if $D > 1$ and $D > 3$, respectively). Using the re-scaling in Eq. (1.4) can then be seen to entirely remove the factor that usually results in the contribution from the Gauss-Bonnet term vanishing, and leaves a term that can, in general, be non-zero in the limit $D \to 4$. The additional term that results in the trace of the field equations (1.5), after the re-scaling given in Eq. (1.4), is strongly motivated from studying quantum corrections to the stress-energy tensor in the presence of gravity. In this case the renormalized vacuum expectation value for the trace of $T^{\mu\nu}$ includes terms that are proportional to $\mathcal{G}$, in just the same way that they are found in the trace of the left-hand side of the field equations in Eq. (1.5). This is known as the "conformal" or "trace" anomaly in the quantum field theory literature, and a natural interpretation of the Glavan & Lin re-scaling is that it is a way of accounting for the conformal anomaly in the gravitational sector of the theory.

1.2 Einstein-Gauss-Bonnet Theory in 4D

Let us start by considering the typical Einstein-Gauss-Bonnet action in D dimensions, where for the moment we neglect any contributions from matter fields, focusing on the purely gravitational sector

$$S = \int d^D x\sqrt{-g}(R - 2\Lambda + \frac{\alpha}{(D-4)}\mathcal{G}) \tag{1.6}$$

Varying and extremizing the action with respect to the metric results in the field equations of the theory, which read

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{\alpha}{(D-4)} H_{\mu\nu} \tag{1.7}$$

where

$$H_{\mu\nu} = -2\left(RR_{\mu\nu} - 2R_{\mu\alpha\nu\beta}R^{\alpha\beta} + R_{\mu\alpha\beta\sigma}R_{\nu}^{\alpha\beta\sigma} - 2R_{\mu\alpha}R_{\nu}^{\alpha} - \frac{1}{4}g_{\mu\nu}\mathcal{G}\right) \tag{1.8}$$

The right-hand side of this equation is anti-symmetrized over five indices, and so must vanish in dimensions $D < 5$. The variation of the Gauss-Bonnet term, $H_{\mu\nu}$ given in Eq. (1.8), can be written in D dimensions in terms of the Weyl tensor as

$$H_{\mu\nu} = 2(H_{\mu\nu}^{(1)} + H_{\mu\nu}^{(2)}) \tag{1.9}$$

where

$$H_{\mu\nu}^{(1)} = C_{\mu\alpha\beta\sigma}C_{\nu}^{\alpha\beta\sigma} - \frac{1}{4}g_{\mu\nu}C_{\alpha\beta\sigma\rho}C^{\alpha\beta\sigma\rho} \tag{1.10}$$

and

$$\begin{aligned} H_{\mu\nu}^{(2)} = & \frac{(D-4)(D-3)}{(D-2)(D-1)}\Bigg[-\frac{2(D-1)}{(D-3)}C_{\mu\rho\nu\sigma}R^{\rho\sigma} - \frac{2(D-1)}{(D-2)}R_{\mu\rho}R_{\nu}^{\rho} \\ & + \frac{D}{(D-2)}R_{\mu\nu}R + \frac{1}{(D-2)}g_{\mu\nu}\left((D-1)R_{\rho\sigma}R^{\rho\sigma} - \frac{D+2}{4}R^2\right)\Bigg] \end{aligned} \tag{1.11}$$

where the D-dimensional Weyl tensor is given as,

$$C_{\mu\alpha\nu\beta} = R_{\mu\alpha\nu\beta} - \frac{2}{D-2}\left(g_{\mu[\nu}R_{\beta]\alpha} - g_{\alpha[\nu}R_{\beta]\beta} + \frac{2}{(D-1)(D-2)}Rg_{\mu[\nu}g_{\beta]\alpha}\right) \tag{1.12}$$

After rescaling the coupling constant $\hat{\alpha}$ as per Eq. (1.4), the term $\hat{\alpha}H_{\mu\nu}^{(2)}$ is well defined. The same is not true about $\hat{\alpha}H_{\mu\nu}^{(1)}$. But this term identically vanishes in 4D. The poor behavior of this term in the 4-dimensional limit is problematic, but if one were to simply ignore the above contribution to the field equations, the finite part resulting from the Gauss-Bonnet term would not be covariantly conserved, which would clearly be unacceptable. Rescaling of the constant has such similar shortcomings. Given such concerns, alternative regularizations have been sought for a well-defined version of the Einstein-Gauss-Bonnet theory in four dimensions.

1.2.1 Counter-term regularization

Due to the rescaling of the coupling constant, the 4D limit of the equation of motion is not well-defined. However, the trace of the field equations is well-defined. One can attempt to solve this indeterminacy of the field equations by adding a counter-term to the action, in order to cancel the resulting ill-defined terms. This can be done by adding to the action (1.6) the term

$$-\frac{\alpha}{(D-4)}\int d^Dx\sqrt{-\tilde{g}}\tilde{\mathcal{G}} \tag{1.13}$$

where the tilde denotes a quantity constructed from the conformal geometry defined by

$$\tilde{g}_{\mu\nu} = e^{2\phi}g_{\mu\nu} \tag{1.14}$$

We can write the Gauss-Bonnet term of the conformal metric in terms of the original one, expand the exponential around $D = 4$, neglecting terms of order $(D-4)^2$ or higher and taking the four-dimensional limit, we get our action as,

$$S = \int_{\mathcal{M}} d^4x\sqrt{-g}\left[R + \alpha\left(4G^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - \phi\mathcal{G} + 4(\nabla\phi)^2 + 2(\nabla\phi)^4\right)\right] + S_m \tag{1.15}$$

which can be seen to be a four-dimensional action free of divergences.

1.2.2 Regularized Kaluza-Klein reduction

We perform a Kaluza-Klein compactification of D-dimensional Einstein-Gauss-Bonnet gravity on a maximally symmetric space-time of $(D-4)$ dimensions. We start the Kaluza-Klein regularization process by parametrizing the D-dimensional metric as

$$ds_D^2 = ds_4^2 + e^{-2\phi} d\Sigma^2_{D-4,\lambda} \tag{1.16}$$

Here, ds_D^2 is the 4-dimensional line element. $d\Sigma^2_{D-4,\lambda}$ corresponds to the $D-4$ dimensional internal maximally symmetric spacetime whose curvature tensor is given by

$$R_{\mu\nu\rho\sigma} = \lambda(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}) \tag{1.17}$$

with λ as a constant representing the curvature of the internal space. We assume that the scalar field ϕ only depends on the 4D coordinates. This is because we choose a cylindrical gauge setting $g^D_{ab} = 0$ where a and b run from $D-4$ to D[1].Under these assumptions, the Einstein-Gauss-Bonnet action ((1.6)) reduces to

$$\begin{aligned} S = \frac{1}{16\pi G}\int d^4x\sqrt{-g}e^{-(D-4)\phi}\Big\{ & R + (D-4)(D-5)((\nabla\phi)^2 + \lambda e^{2\phi}) \\ & + \alpha\big(\mathcal{G} - (D-4)(D-5)\{2G^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - \lambda R e^{2\phi}\} \\ & - (D-4)(D-5)(D-6)\{-2(\nabla\phi)^2\Box\phi + (D-5)(\nabla\phi)^4\} \\ & + (D-4)(D-5)(D-6)(D-7)\{2\lambda(\nabla\phi)^2 e^{2\phi} + \lambda^2 e^{4\phi}\}\big)\Big\} \end{aligned} \tag{1.18}$$

As we are interested in the $D \to 4$ limit, we expand the exponent around $D = 4$ and discard terms of order $(D-4)^2$. There is also a bare $\mathcal{G}$ term which can be removed by introducing a counter-term,

$$-\frac{\alpha}{D-4}\int d^4x\sqrt{-\tilde{g}}\tilde{\mathcal{G}} \tag{1.19}$$

where the tilded quantities are a result of the conformal scaling, $\tilde{g}_{\mu\nu} = e^{2\phi}g_{\mu\nu}$.The above processes result in the following action,

$$\begin{aligned} S = \int d^4x\sqrt{-g}\Big[& R + \alpha\big(4G^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - \phi\mathcal{G} + 4\Box\phi(\nabla\phi)^2 \\ & + 2(\nabla\phi)^4 - 2\lambda e^{2\phi}\big[R + 6(\nabla\phi)^2 + 3\lambda e^{2\phi}\big]\big)\Big] \end{aligned} \tag{1.20}$$

1.3 Gravity with a generalized conformal scalar field

The field equation that is produced by the aforementioned regularisation procedures fully decouples from the scalar field and is solely geometric in nature. Consequently, we attempt to investigate the reason behind the complete decoupling of a particular combination of the field equations from the scalar field.

It can also be verified that the scalar field equations resulting from these theories are conformally invariant. We define a conformal transformation as,

$$g_{\mu\nu} \to g_{\mu\nu}e^{2\sigma}, \quad \text{and} \quad \phi \to \phi - \sigma \tag{1.21}$$

Conformal symmetry of the matter fields is well-known to be associated with simplifications of the field equations. In the above equations, the full field equations are not conformally invariant but only the matter field equations are. This suggests that the previously mentioned simplification of the equations of motion might in fact be related to the conformal invariance of the matter field equations, and not of the action.

We start by considering the transformation of Eq. (1.21) in its infinitesimal form, such that $\delta_\sigma g_{\mu\nu} = 2\sigma g_{\mu\nu}$ and $\delta_\sigma \phi = -\sigma$, where $\delta_s igma$ denotes the change under an infinitesimal conformal transformation. Assuming an action principle that describes a theory that depends solely on the metric $g_{\mu\nu}$ and a scalar field ϕ, $S[\phi, g]$, we find that the transformation induces the variation

$$\begin{aligned}\delta_\sigma S &= \int d^4x \left(\frac{\delta S[\phi,g]}{\delta g_{\mu\nu}} \delta_\sigma g_{\mu\nu} + \frac{\delta S[\phi,g]}{\delta\phi} \delta_\sigma \phi \right) \\ &= -\int d^4x \left(-2g_{\mu\nu} \frac{\delta S[\phi,g]}{\delta g_{\mu\nu}} + \frac{\delta S[\phi,g]}{\delta\phi} \right) \sigma \end{aligned} \tag{1.22}$$

where the first and second terms in brackets can be identified with the trace and the scalar field equations, respectively. We want this theory to have a conformally invariant scalar field equation and thus $\delta_\sigma S$ should be independent of ϕ. Thus the transformed action contains exactly the same scalar field dependence as the original one, resulting in the same scalar field equation.

Ref. [5] has shown that the most general scalar-tensor theory with second-order equations of motion and a conformally invariant scalar field equation is given by the action

$$\begin{aligned} S = \int d^4x\sqrt{-g}\Big[&R - 2\Lambda - \beta e^{2\phi}(R + 6(\nabla\phi)^2) - 2\gamma e^{4\phi} \\ &+ \alpha \left(4G^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - \phi\mathcal{G} + 4\Box\phi(\nabla\phi)^2 + 2(\nabla\phi)^4 \right) \Big] + S_m \end{aligned} \tag{1.23}$$

where α, β and γ are constants.

The most general scalar-tensor theory having second-order field equations in 4D is called a Horndeski theory[6]. In four dimensions, the Lagrangian for these theories is given by

$$\begin{aligned} \mathcal{L} = {} & G_2(\phi, X) - G_3(\phi, X) + G_4(\phi, X)R + G_{4X}[(\Box\phi)^2 - \nabla^\mu\nabla^\nu\phi\nabla_\mu\nabla_\nu\phi] \qquad (1.24) \\ & + G_5(\phi, X)G^{\mu\nu}\nabla_\mu\nabla_\nu\phi - \frac{G_{5X}}{6}[(\Box\phi)^3 - 3\Box\phi\nabla^\mu\nabla^\nu\phi^{\mu\nu}\nabla_\mu\nabla_\nu\phi + 2\nabla_\mu\nabla_\nu\phi\nabla^\nu\nabla^\lambda\phi\nabla^\mu\nabla_\lambda\phi] \end{aligned}$$

where $X = -(\nabla\phi)^2/2$, G_2, G_3, G_4 and G_5 are arbitrary functions of ϕ and X and the notation used is $f_X = \partial f/\partial X$ for a function f of ϕ and X. It can be seen that our theory belongs to the Horndeski class with

$$\begin{gathered} G_2 = -2\Lambda - 2\gamma e^{4\phi} + 12\beta e^{2\phi}X + 8\alpha X^2; \qquad (1.25) \\ G_3 = 8\alpha X; \qquad G_4 = 1 - \beta e^{2\phi} + 4\alpha X; \qquad G_5 = 4\alpha \log X \end{gathered}$$

The non-minimal coupling of the form $G^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi$ can be written either as $G_4 = X$ or $G_5 = -\phi$ by performing by parts integration. The coupling of the Gauss-Bonnet term in this is non-trivial. It can be shown that terms $\phi\mathcal{G}$ and $-4\log X$ are equivalent[6].

Chapter 2

Lower Bound To All Physical Length Scales

This is the review of the article **Planck Length as the Lower Bound to all Physical Length Scales**[8] by **T. Padmanabhan**.

In classical general relativity, the proper distance between two events x^μ and $x^\mu + dx^\mu$ is given by the line element

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu \tag{2.1}$$

In the absence of a gravitational field, our proper distance becomes

$$ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu = -dt^2 + dx^2 + dy^2 + dz^2 \tag{2.2}$$

Classically, we say that we can measure $g_{\mu\nu}$ at a single event x^μ can be measured to an arbitrary accuracy, and thus the proper length can be calculated as accurately as we need i.e.

$$\lim_{x^\mu \to y^\mu} ds^2 = 0 \tag{2.3}$$

for $y^\mu = x^\mu + dx^\mu$. Thus this implies the proper interval goes to zero as two events approach each other.

Quantum gravity, on the other hand, is not like classical gravity. It would be more accurate to refer to the flat space-time as the vacuum state of quantum gravity. The quantum fluctuations cause the metric tensor $g_{\mu\nu}$ to turn into a quantum field. The vacuum fluctuations of gravity will exist even in the flat space-time. Consequently, measuring the value of $g_{\mu\nu}$ at a single event x^μ and obtaining a unique value for the space-time interval is no longer feasible.

2.1 Quantum Conformal Fluctuations

Quantum gravity can be approached through the path integral,

$$K = \int \mathfrak{D}g_{\mu\nu} \exp\{iS[g_{\mu\nu}]\} = \int \mathfrak{D}g_{\mu\nu} \; \exp\left\{ i\frac{1}{12L_p^2}\int d^4x\sqrt{-g}R \right\} \tag{2.4}$$

Most of the contributions to the path integral are expected to arise from the classical solution $g_{\mu\nu} = \bar{g}_{\mu\nu}$. In considering the quantum conformal fluctuations, one evaluates the path integral in equation 2.4 over a class of metrics which are conformal to $g_{\mu\nu}$ i.e., we take

$$g_{\mu\nu} = [1+\phi(x)]^2\bar{g}_{\mu\nu} \tag{2.5}$$

Thus our path integral becomes

$$K = \int \mathfrak{D}\phi \, exp\Big\{ - \frac{i}{2L_p^2} \int \Big[\phi^i \phi_i - \frac{1}{6}\bar{R}(1+\phi)^2 \Big] \sqrt{-\bar{g}} d^4x \Big\} \tag{2.6}$$

We want the probability that the conformal fluctuation has a given value $\phi(x)$ in the gravitational vacuum (flat space). It is given by

$$\mathcal{P}[\phi(\mathbf{x})] = N \ \exp\left[- \int \frac{d^3\mathbf{k}}{(2\pi)^3} |\mathbf{k}| |q_{\mathbf{k}}|^2 \right] \tag{2.7}$$

$$= N \ \exp\left[- \frac{1}{4\pi^2 L_P^2} \int d^3\mathbf{x} \int d^3\mathbf{y} \left\{ \frac{\nabla_{\mathbf{x}}\phi \nabla_{\mathbf{y}}\phi}{|\mathbf{x}-\mathbf{y}|^2} \right\} \right] \tag{2.8}$$

where

$$\phi(\mathbf{r}) = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \, q_{\mathbf{k}} \, e^{i\mathbf{k}\cdot\mathbf{r}} \tag{2.9}$$

The expression (2.8) denotes the square of the "ground state wave functional" for gravity and is derived in Appendix A. The time independence of P reflects the fact that the ground state is a stationary state.

2.2 Quantum Fluctuations and Length Measurements

Consider an experiment that achieves the measurement of proper length between points in space $\mathbf{x}$and $\mathbf{y}$(at time t) in flat space with a spatial resolution of L (say). If the sensitivity profile of the set-up is denoted by $f(\mathbf{r})$ then we will actually be measuring the field $\phi(\mathbf{x})$ "coarse-grained" over the scale L:

$$\phi_f(\mathbf{x}) \equiv \int \phi(\mathbf{x}+\mathbf{r}) f(\mathbf{r}) d^3\mathbf{r} \tag{2.10}$$

The function f is taken to be zero for $|\mathbf{r}| > L$ and is of the order of unity for $|\mathbf{r}| < L$. The probability that ϕ_f has a particular value η is given by

$$\mathfrak{G}[\phi_f = \eta] = \int \mathfrak{D}\phi(\mathbf{x}) \, \delta(\phi_f - \eta) \, \mathcal{P}[\phi(\mathbf{x})] \tag{2.11}$$

This expression has been evaluated in the Fourier space in Appendix A. We get

$$\mathfrak{G}[\phi_f = \eta] = \left(\frac{1}{2\pi\Delta^2} \right)^{1/4} \exp\left(- \frac{\eta^2}{4\Delta^2} \right) \tag{2.12}$$

where

$$\Delta^2 = L_P^2 \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{|f_{\mathbf{k}}|^2}{2|\mathbf{k}|} \tag{2.13}$$

Suppose we take $f(\mathbf{r})$ to be the Gaussian sensitivity profile,

$$f(\mathbf{r}) = \left(\frac{1}{2\pi L^2} \right)^{3/2} \exp\left(- \frac{|\mathbf{r}|^2}{2L^2} \right) \tag{2.14}$$

then, from 2.13 we get

$$\Delta^2 = L_P^2 \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{|f_{\mathbf{k}}|^2}{2|\mathbf{k}|} = \frac{L_P^2}{4\pi^2 L^2} \tag{2.15}$$

Thus L essentially measures the width of $f(\mathrm{r})$ for any distribution. As long as one considers length measurements averaged over many Planck lengths (i.e., for $L >> L_P$), Δ is almost zero, and the probability in 2.12 is sharply peaked at $\eta = 0$. In this case, quantum fluctuations hardly affect the length measurements. However, as the resolution of the apparatus L goes to zero the fluctuations in η go on increasing. When the conformal factor has a value η, the proper length – between $(t, \mathbf{x})$ and $(t, \mathbf{y})$ – is given by

$$R^2 = (1+\eta)^2 R_o^2; \qquad R_o^2 = |\mathbf{x} - \mathbf{y}|^2 \tag{2.16}$$

Therefore, the probability for the events $(t, \mathbf{x})$ and $(t, \mathbf{y})$ to be separated by a proper length R is given by

$$\mathcal{P}(R)dR = \mathcal{P}(\eta(R))\frac{d\eta}{dR}dR \tag{2.17}$$

Therefore,

$$\mathcal{P}(R) = \left(\frac{2}{\pi\sigma^2}\right)^{1/2} \exp\left\{-\frac{(R-R_0)^2}{2\sigma^2}\right\} \tag{2.18}$$

with

$$R \geq 0, \qquad R_0 = |\mathbf{x} - \mathbf{y}|, \qquad \sigma^2 = R_0^2 \frac{L_P^2}{L^2}\frac{1}{4\pi^2} \tag{2.19}$$

To have a well-defined concept of length between two points, one must have $\sigma^2 \ll R_0^2$ implying $L \gg L_P$. As the measurement becomes more and more accurate, we can only talk about the probability of a particular value for the length. The concept of definite proper length breaks down at $L \sim L_P$. Equation (2.18) is the main result of this analysis.

2.3 Lower Bound to Proper Length

Classically, two events can be arbitrarily close and the proper distance tends to be zero. However, it is physically meaningless to talk about distances below the resolution limit L. If this resolution limit is taken to zero, then equation (2.18) predicts infinite uncertainty in the proper length.

Instead of considering the fluctuations in the conformal factor, one may look at the expectation value of the line interval

$$\langle 0|ds^2|0\rangle \equiv \langle g_{\mu\nu}(x)\rangle dx^\mu dx^\nu = (1 + \langle\phi^2(x)\rangle)dx^\mu dx^\nu \bar{g}_{\mu\nu} \tag{2.20}$$

Now, $\langle\phi^2\rangle$ diverges for quantum fields. Also definition of ds^2 involves x^μ and $y^\mu = x^\mu + dx^\mu$. So we consider $\langle\phi^2\rangle$ as the limit

$$\langle\phi^2(x)\rangle = \lim_{x\to y}\langle\phi(x)\phi(y)\rangle \tag{2.21}$$

In flat space, the limiting distance between two space points $\mathbf{x}$and $\mathbf{y}$(at some time t, say) is given by

$$\lim_{\mathbf{x}\to\mathbf{y}} l^2(\mathbf{x}, \mathbf{y}) = \lim_{\mathbf{x}\to\mathbf{y}} |\mathbf{x} - \mathbf{y}|^2 = 0 \tag{2.22}$$

in the classical limit. When quantum fluctuations are included, this is replaced by

$$\lim_{\mathbf{x}\to\mathbf{y}}\langle l^2(\mathbf{x}, \mathbf{y})\rangle = \lim_{\mathbf{x}\to\mathbf{y}}(1 + \langle\phi(\mathbf{x}, t)\phi(\mathbf{y}, t)\rangle)|\mathbf{x} - \mathbf{y}|^2 \tag{2.23}$$

We can calculate this expectation value as

$$\langle\phi(\mathbf{x}, t)\phi(\mathbf{y}, t)\rangle = -\frac{(4\pi^2)^{-1}L_P^2}{|\mathbf{x} - \mathbf{y}|^2} \tag{2.24}$$

Therefore

$$\lim_{\mathbf{x}\to\mathbf{y}} l^2(\mathbf{x},\mathbf{y}) = \lim_{\mathbf{x}\to\mathbf{y}} \left\{ \left[1 - \frac{(4\pi^2)L_P^2}{|\mathbf{x}-\mathbf{y}|^2} \right] |\mathbf{x}-\mathbf{y}|^2 \right\}$$
$$= -\frac{L_P^2}{4\pi^2} \tag{2.25}$$

In other words, the expectation value of the proper length between two events is bounded at Planck length.

This result is far more general than indicated by the derivation above. First of all, for any two events x^μ and $x^\mu + \epsilon^\mu$, in flat space, the expectation value has the form

$$\langle 0|\phi(x+\epsilon)\phi(x)|0\rangle = -\frac{(4\pi^2)^{-1}L_P^2}{(\epsilon^\mu \epsilon_\mu)} \tag{2.26}$$

Thus,

$$\lim_{\epsilon\to 0}\langle l^2(x, x+\epsilon)\rangle = \lim_{\epsilon\to 0} \left[1 - \frac{(4\pi^2)^{-1}L_P^2}{(\epsilon^\mu \epsilon_\mu)} \right] \epsilon^\mu \epsilon_\mu = -(4\pi^2)^{-1}L_P^2 \tag{2.27}$$

(The minus sign shows that the lower bound arises from the limiting value of time-like separations.) The result can also be generalized to arbitrary curved spacetime because of the following fact: In any space-time,

$$\lim_{x\to y} = \langle \phi(x)\phi(y)\rangle = -\frac{(L_P/2\pi)^2}{s^2} \tag{2.28}$$

where s^2 is proper interval between x and y.

Chapter 3

Quantization in $D \to 4$ EGB gravity

In Chapter 1, we derived the action for a theory of $D \to 4$ Einstein-Gauss-Bonnet (EGB) gravity using various regularization techniques. The resulting action is as follows:

$$S = S_e + S_l + S_a, \tag{3.1}$$

where the components of the action are defined as:

$$\begin{aligned}
S_e &= \int d^4x\sqrt{-g}\left[(1-\beta e^{2\phi})R - 6\beta(\nabla\phi)^2 - 2\gamma e^{4\phi}\right], \\
S_l &= \int d^4x\sqrt{-g}\left[-\alpha\phi\mathcal{G}\right], \\
S_a &= \int d^4x\sqrt{-g}\left[\alpha(4G^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi + 4\Box\phi(\nabla\phi)^2 + 2(\nabla\phi)^4)\right].
\end{aligned}$$

To facilitate further analysis, we perform a conformal transformation to convert this action from the Jordan frame to the Einstein frame. We choose a conformal factor as follows:

$$\bar{g}_{\mu\nu} = 2(1-\beta e^{2\phi})g_{\mu\nu}, \tag{3.2}$$

and consider the limit where $\beta \ll 1$, ignoring terms of second and higher order in β. This transformation results in the following Einstein frame action:

$$S_e = \int d^4x\sqrt{-g}\left\{\frac{1}{2}(R-\gamma e^{4\phi}) - \beta\left[e^{2\phi}(\gamma e^{4\phi} - 3\nabla^\mu\phi\nabla_\mu\phi)\right]\right\} \tag{3.3}$$

$$S_l = -\int d^4x\sqrt{-g}\alpha\left\{\phi\mathcal{G} + \beta\left[32e^{2\phi}G^{\mu\nu}\phi\nabla_\mu\phi\nabla_\nu\phi + 8e^{2\phi}G^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi\right]\right\} \tag{3.4}$$

$$S_a = \int d^4x\sqrt{-g}\alpha\left\{4G^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi + 4\nabla_\mu\phi\nabla^\mu\phi\nabla_\nu\nabla^\nu\phi + 2\nabla_\mu\phi\nabla^\mu\phi\nabla_\nu\phi\nabla^\nu\phi\right\} \tag{3.5}$$

3.1 Perturbation around a Maximally symmetric spacetime

In our initial analysis, we conducted a perturbation around flat spacetime with the metric parameterized as $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$. We also perturbed the scalar field as $\phi = \phi_c + \phi'$. For flat spacetime to serve as a solution to our theory, it is necessary for ϕ_c to be zero. Considering terms up to the second order in both h and ϕ', we arrive at the Lagrangian density $\mathcal{L}_e$ as follows:

$$\begin{aligned}\mathcal{L}e =&\frac{1}{8}(\partial\rho h_{\mu\nu}\partial^\rho h^{\mu\nu} - 2\partial_\rho h_{\mu\nu}\partial^\nu h^{\mu\rho} + 2\partial_\mu h^{\mu\nu}\partial_\nu h - \partial_\mu h\partial^\mu h)\\ &-\frac{\gamma}{2}(8\phi'^2 + 2\phi' h^\mu_\mu + \frac{1}{8}(h^\mu_\mu)^2 - \frac{1}{4}h^{\mu\nu}h_{\mu\nu})\\ &-\beta\gamma(18\phi'^2 + 3\phi' h^\mu_\mu + \frac{1}{8}(h^\mu_\mu)^2 - \frac{1}{4}h^{\mu\nu}h_{\mu\nu})\\ &-3\beta\partial_\mu\phi'\partial^\mu\phi'.\end{aligned} \tag{3.6}$$

Due to our perturbation parameterization, Lagrangian densities $\mathcal{L}_l$ and $\mathcal{L}_a$ would yield terms of higher order, which are negligible for our analysis. The absence of a significant contribution from the Gauss-Bonnet term is primarily due to the vanishing of ϕ_c in this context.

Subsequently, we extended our analysis by performing perturbations around a maximally symmetric spacetime. The metric was parameterized as follows:

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}, \tag{3.7}$$

where $\bar{g}$ represents a 4D maximally symmetric spacetime with a constant curvature parameter κ. The Riemann tensor for $\bar{g}$ is given by:

$$\bar{R}_{\mu\nu\rho\sigma} = \kappa(\bar{g}_{\mu\rho}\bar{g}_{\nu\sigma} - \bar{g}_{\mu\sigma}\bar{g_{\nu\rho}}). \tag{3.8}$$

Also, $\phi = \phi_c + \phi'$, where ϕ_c is a constant and ϕ' is the perturbation. Thus, equations (3.3), (3.4), and (3.5) result in

$$\begin{aligned}\mathcal{L}_e = \sqrt{-\bar{g}}\Bigg[&\frac{1}{8}\big(-4\kappa(4h_{\mu\nu}h^{\mu\nu} - h^2) + 4\nabla_\nu h^{\mu\nu}\nabla_\rho h^\rho_\mu + \nabla_\rho h\nabla^\rho h\\ &-2\nabla_\nu h\nabla_\mu h^{\mu\nu} - 2\nabla_\nu h_{\mu\rho}\nabla^\rho h^{\mu\nu} - \nabla_\rho h^{\mu\nu}\nabla^\rho h_{\mu\nu}\big) + 3\beta e^{2\phi_c}\nabla_\mu\phi'\nabla^\mu\phi'\\ &-\frac{1}{2}\gamma\left(8e^{4\phi_c}(\phi' - \phi_c)^2 + 2e^{4\phi_c}(\phi' - \phi_c)h + \frac{1}{8}e^{4\phi_c}(h^2 - 2h^{\mu\nu}h_{\mu\nu})\right)\\ &-\gamma\beta\left(18e^{6\phi_c}(\phi' - \phi_c)^2 + 3e^{6\phi_c}(\phi' - \phi_c)h + \frac{1}{8}e^{6\phi_c}(h^2 - 2h^{\mu\nu}h_{\mu\nu})\right)\Bigg]\end{aligned} \tag{3.9}$$

$$\begin{aligned}\mathcal{L}_l =& -\alpha\sqrt{-\bar{g}}\Big\{\phi_c\Big[-19\kappa^2(4h_{\mu\nu}h^{\mu\nu} - h^2)\\ &+\kappa\big(-96\beta e^{2\phi_c}\phi_c\nabla^\mu\phi'\nabla_\mu\phi' - 24\beta e^{2\phi_c}\nabla^\mu\phi'\nabla_\mu\phi' + \phi'\left[4(\nabla_\nu\nabla_\mu h^{\mu\nu} - \nabla_\mu\nabla^\mu h)\right]\\ &-8\nabla_\nu h^{\mu\nu}\nabla_\mu h + 30\nabla_\nu h^{\mu\nu}\nabla_\rho h^\rho_\nu + 3\nabla_\mu h\nabla^\mu h - 16\nabla_\rho h^{\mu\nu}\nabla_\nu h^\rho_\mu - 9\nabla_\rho h_{\mu\nu}\nabla^\rho h^{\mu\nu}\big)\\ &-2\nabla_\nu\nabla_\rho h^\lambda_\gamma\nabla^\gamma\nabla_\mu h^{\mu\nu} - 2\nabla_\gamma\nabla_\rho h^\rho_\nu\nabla^\gamma\nabla_\mu h^{\mu\nu} + 2\nabla_\nu\nabla_\rho h^\rho_\gamma\nabla^\gamma\nabla^\nu h - \nabla_\gamma\nabla_\nu h\nabla^\gamma\nabla^\nu h\\ &+2\nabla_\gamma\nabla_\rho h^\rho_\nu\nabla^\gamma\nabla^\nu h + \nabla_\nu\nabla_\mu h^{\mu\nu}\nabla_\rho\nabla_\gamma h^{\rho\gamma} - 2\nabla_\nu\nabla^\nu h\nabla_\rho\nabla_\gamma h^{\gamma\rho} - \nabla_\gamma\nabla^\gamma h^{\mu\nu}\nabla_\rho\nabla^\rho h_{\mu\nu}\\ &+4\nabla^\gamma\nabla_\mu h^{\mu\nu}\nabla_\rho\nabla^\rho h_{\nu\gamma} - 2\nabla^\gamma\nabla^\nu h\nabla_\rho\nabla^\rho h^{\gamma\nu} + \nabla_\nu\nabla^\nu h\nabla_\mu\nabla^\mu h + \nabla_\nu\nabla_\mu h_{\gamma\rho}\nabla^\rho\nabla^\gamma h^{\mu\nu}\\ &-2\nabla_\rho\nabla_\nu h_{\mu\gamma}\nabla^\rho\nabla^\gamma h^{\mu\nu} + \nabla_\rho\nabla_\gamma h_{\mu\nu}\nabla^\rho\nabla^\gamma h^{\mu\nu}\Big]\Big\}\end{aligned} \tag{3.10}$$

$$\mathcal{L}_a = -12\alpha\sqrt{-\bar{g}}\kappa\nabla^\mu\phi'\nabla_\mu\phi' \tag{3.11}$$

This is the final action that we got on which we will be working on from here on.

Appendix A

Lower Bound related calculations

A.1 Ground state wave functional

We know that the Ricci scalar for a conformally transformed spacetime is given by[2]

$$g_{ik} = \Omega^2(x)\bar{g}_{ik}$$
$$\sqrt{-g} = \Omega^D\sqrt{-\bar{g}}$$
$$R = \frac{\bar{R}}{\Omega^2} + 2(D-1)\bar{g}^{ef}\frac{\bar{\nabla}_e\bar{\nabla}_f\Omega}{\Omega^3} + (D-1)(D-4)\bar{g}^{ef}\frac{(\bar{\nabla}_e\Omega)(\bar{\nabla}_f\Omega)}{\Omega^4} \tag{A.1}$$

Using equations 2.5 and A.1 and applying by parts integration on $\phi\Box\phi$, we get

$$K = \int \mathfrak{D}\phi \; exp\bigg\{-\frac{i}{2L_p^2}\int d^4x\sqrt{-\bar{g}}\Big[\phi^i\phi_i - \frac{1}{6}\bar{R}(1+\phi)^2\Big]\bigg\} \tag{A.2}$$

We want the probability that the conformal fluctuations have a given value $\phi(x)$ in the gravitational vacuum (flat space). We consider a collection of harmonic oscillators each labeled by a vector k. The ground state of the quantum field can be obtained by multiplying together the ground state wave functions of all oscillators.

$$\Psi[\phi(x)] = \prod_k (\frac{\omega_k}{\pi})^{1/4}\, exp\Big(-\frac{1}{2}\omega_k|q_k|^2\Big) = N\; exp\Big[-\frac{1}{2}\int\frac{d^3k}{(2\pi)^3}\omega_k|q_k|^2\Big] \tag{A.3}$$

where $\bar{N}$ is a normalization factor independent of q_k which we will ignore for the moment. The argument of the exponent can be expressed in a more useful form in real space in terms of $\nabla\phi$.

Note that (at $t = 0$, say),

$$\phi(r) = \int \frac{d^3k}{(2\pi)^3}\, q_k\, e^{ik\cdot r}$$
$$\nabla\phi(r) = \int \frac{d^3k}{(2\pi)^3}\, ikq_k\, e^{ik\cdot r}$$
$$ikq_k = \int d^3x\, \nabla\phi\, e^{-ik\cdot r}$$

For a massless field, $\omega_k = |k|$ and

$$\int\frac{d^3k}{(2\pi)^3}\omega_k|q_k|^2 = \int\frac{d^3k}{(2\pi)^3}\frac{|k|^2|q_k|^2}{|k|}$$
$$= -\int\frac{d^3k}{(2\pi)^3}\int d^3x\int d^3y\frac{1}{|k|}\nabla_x\phi\nabla_y\phi e^{-ik(x-y)} \tag{A.4}$$

Let's look at the integral of k in polar coordinates,

$$\begin{aligned}
\int \frac{d^3k}{(2\pi)^3}\frac{1}{|k|}e^{-ik(x-y)} &= \int \frac{d^3k}{(2\pi)^3}\frac{1}{|k|}e^{-ik\cdot r} \\
&= -\frac{1}{4\pi^2}\int_0^\infty\int_0^\pi \frac{dk\,k^2}{k}e^{ikrcos\theta}d(cos\theta) \\
&= \frac{1}{2\pi^2}\int_0^\infty dk\,k\,cos(kr) \\
&= \frac{1}{2\pi^2}\left[\frac{1}{2}\int_0^\infty dk\,k\,e^{ikr} + \frac{1}{2}\int_0^\infty dk\,k\,e^{-ikr}\right]
\end{aligned}$$

To evaluate these integrals, we take $r \to r + i\epsilon$ for the first integral and $r \to r - i\epsilon$ for the second. Take $\alpha = \epsilon - ir$ and $\beta = \epsilon + ir$,

$$\begin{aligned}
&= \frac{1}{2\pi^2}\left[\frac{1}{2}\int_0^\infty dk\,k\,e^{-\alpha k} + \frac{1}{2}\int_0^\infty dk\,k\,e^{-\beta k}\right] \\
&= \frac{1}{2\pi^2}\left[\frac{1}{2}\left(-\frac{e^{-\alpha k}(1+\alpha k)}{\alpha^2}\right)\Bigg|_0^\infty + \frac{1}{2}\left(-\frac{e^{-\beta k}(1+\beta k)}{\beta^2}\right)\Bigg|_0^\infty\right] \\
&= \frac{1}{2\pi^2}\left[\frac{1}{2}\left(\frac{1}{\alpha^2}\right) + \frac{1}{2}\left(\frac{1}{\beta^2}\right)\right] \\
&= -\frac{1}{2\pi^2}\frac{1}{r^2} \quad \text{for limit } \epsilon \to 0.
\end{aligned}$$

Thus the integral in equation A.4 becomes,

$$\int \frac{d^3k}{(2\pi)^3}\omega_k|q_k|^2 = \frac{1}{2\pi^2}\int d^3x \int d^3y \left\{\frac{\nabla_x\phi\nabla_y\phi}{|x-y|^2}\right\} \tag{A.5}$$

Therefore, the probability amplitude to observe a field configuration $\phi(r)$ in the vacuum state is

$$\begin{aligned}
\mathcal{P}[\phi(x)] &= |\Psi[\phi(x)]|^2 \\
&= N\,exp\left[-\frac{1}{4\pi^2L_p^2}\frac{\nabla_x\phi\nabla_y\phi}{|x-y|^2}\right]
\end{aligned} \tag{A.6}$$

A.2 Probability amplitude for measurement of ϕ_f

Note that

$$\begin{aligned}
\delta(\phi_f - \eta) &= \int_{-\infty}^{\infty} \exp\left[i\lambda(\phi_f - \eta)\right]\frac{d\lambda}{2\pi} = \int_{-\infty}^{\infty} e^{-i\lambda\eta}\exp\left[i\lambda\int \phi(\mathbf{r}+\mathbf{y})f(\mathbf{y})d^3\mathbf{y}\right]\frac{d\lambda}{2\pi} \\
&= \int_{-\infty}^{\infty}\left\{e^{-i\lambda\eta}\exp\left[i\lambda\int\frac{d^3\mathbf{k}}{(2\pi)^3}\int\frac{d^2\mathbf{p}}{(2\pi)^3}q_{\mathbf{k}}e^{i\mathbf{k}\cdot(\mathbf{r}+\mathbf{y})}f_{\mathbf{p}}e^{-\mathbf{p}\cdot\mathbf{y}}d^3\mathbf{y}\right]\right\}\frac{d\lambda}{2\pi} \\
&= \int_{-\infty}^{\infty}\left\{e^{-i\lambda\eta}\exp\left[i\lambda\int\frac{d^3\mathbf{k}}{(2\pi)^3}q_{\mathbf{k}}f_{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{r}}\right]\right\}\frac{d\lambda}{2\pi}
\end{aligned} \tag{A.7}$$

Substituting this Fourier expansion in (2.11),

$$
\begin{aligned}
\mathfrak{G}[\phi_f = \eta] &= \int \mathfrak{D}\phi(\mathbf{x})\, \delta(\phi_f - \eta)\, \mathcal{P}[\phi(\mathbf{x})] \\
&= \prod_{\mathbf{p}} \int dq_{\mathbf{p}} \int_{-\infty}^{\infty} \left\{ e^{-i\lambda\eta} \exp\left\{ - \int \frac{d^3\mathbf{k}}{(2\pi)^3} \left(\frac{1}{2}\omega_{\mathbf{k}} |q_{\mathbf{k}}|^2 + i\lambda q_{\mathbf{k}} f_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}} \right) \right\} \right\} \frac{d\lambda}{2\pi} \\
&= \prod_{\mathbf{p}} \int_{-\infty}^{\infty} \frac{d\lambda}{2\pi} e^{i\lambda\eta} \int_{-\infty}^{\infty} \exp\left\{ - \sum_{\mathbf{k}} \left(\frac{1}{2}\omega_{\mathbf{k}} |q_{\mathbf{k}}|^2 + i\lambda q_{\mathbf{k}} f_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}} \right) \right\} dq_{\mathbf{p}}
\end{aligned} \tag{A.8}
$$

Since $q_{\mathbf{p}}$ is a complex number, the symbol $\int q_{\mathbf{p}}$ actually represents integration over both real and imaginary parts of $q_{\mathbf{p}}$. Performing the Gaussian integrations for each $\mathbf{k}$. we get

$$
\begin{aligned}
\mathfrak{G}(\phi_f = \eta) &= (\text{const}) \int_{-\infty}^{\infty} e^{-\lambda\eta} \exp\left\{ -\lambda^2 \left(\sum_{\mathbf{k}} \frac{|f(\mathbf{k})|^2}{2|\mathbf{k}|} \right) \right\} \frac{d\lambda}{2\pi} \\
&= (\text{const}) \exp\left\{ - \frac{\eta^2}{2 \sum_{\mathbf{k}} |f_{\mathbf{k}}|^2 / |\mathbf{k}|} \right\}
\end{aligned} \tag{A.9}
$$

Replacing $\sum_{\mathbf{k}}$ by $\int d^3\mathbf{k}/(2\pi)^3$ and normalizing $\mathfrak{G}(\phi_f = \eta)$, we get eq. (2.12)

Appendix B

Conformal transformations

We consider the conformal transformation of the form

$$\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu} \tag{B.1}$$

This results in the transformations of other quantities as follows,[2].

$$\nabla_\mu \phi \nabla_\mu \phi = \Omega^2 \tilde{\nabla}_\mu \phi \tilde{\nabla}_\mu \phi \tag{B.2}$$

$$\sqrt{-g} = \Omega^{-4} \sqrt{-\tilde{g}} \tag{B.3}$$

$$R = \Omega^2 \left[\tilde{R} + 6 \frac{\tilde{\Box}\Omega}{\Omega} - 12 \frac{\tilde{\nabla}_\mu \Omega \tilde{\nabla}^\mu \Omega}{\Omega^2} \right] \tag{B.4}$$

$$\begin{aligned} \mathcal{G} = \Omega^4 \Bigg\{ & \tilde{\mathcal{G}} - 4\Omega^{-1} \left[2\tilde{R}_{\mu\nu} \tilde{\nabla}^\mu \tilde{\nabla}^\nu \Omega - \tilde{R} \tilde{\Box} \Omega \right] + 4\Omega^{-2} \Big[2(\tilde{\Box}\Omega)^2 - 2\tilde{\nabla}_\mu \tilde{\nabla}_\nu \Omega \tilde{\nabla}^\mu \tilde{\nabla}^\nu \Omega \\ & - \tilde{R} \tilde{\nabla}_\mu \Omega \tilde{\nabla}^\mu \Omega \Big] - 6\Omega^{-3} \left[4(\tilde{\Box}\Omega) \tilde{\nabla}_\mu \Omega \tilde{\nabla}^\mu \Omega - 4\Omega^{-1} (\tilde{\nabla}_\mu \Omega \tilde{\nabla}^\mu \Omega)^2 \right] \Bigg\} \end{aligned} \tag{B.5}$$

Now we want to move to the Einstein frame. For $f(R)$ theories, there is a specific transformation that allows one to move from Jordon frame to Einstein frame[3]. For action of the form,

$$S = \int d^4x \sqrt{-g} \left[\frac{1}{2} \Omega^2 R - \frac{1}{2} \omega(\varphi) (\nabla \varphi)^2 - U(\varphi) \right] + S_M(g_{\mu\nu}, \Psi_M) \tag{B.6}$$

we perform the conformal transformation to get the action in Einstein frame as,

$$S_E = \int d^4x \sqrt{-\tilde{g}} \left[\frac{1}{2} \tilde{R} - \frac{1}{2} (\tilde{\nabla} \phi)^2 - V(\phi) \right] + S_M(F^{-1} \tilde{g}_{\mu\nu}, \Psi_M) \tag{B.7}$$

where $V = U/F^2$. Here we have introduced a new scalar field ϕ to make the kinetic term canonical.

$$\phi \equiv \sqrt{6 \left(\frac{1}{\Omega} \frac{d\Omega}{d\varphi} \right)^2 + \frac{\omega}{\Omega^2}} \tag{B.8}$$

However, in our work, we did not define such a new scalar field.

Bibliography

[1] Roberto Balbinot and Antonio Barletta. Five-dimensional quantum gravity and the residual length. *Physical Review D*, 34(12):3666, 1986.

[2] Diego F Carneiro, Emanuel A Freiras, Bruno Gonçalves, Andre G de Lima, and Ilya L Shapiro. On useful conformal tranformations in general relativity. *arXiv preprint gr-qc/0412113*, 2004.

[3] Antonio De Felice and Shinji Tsujikawa. f (r) theories. *Living Reviews in Relativity*, 13(1):1–161, 2010.

[4] Pedro G S Fernandes, Pedro Carrilho, Timothy Clifton, and David J Mulryne. The 4d einstein–gauss–bonnet theory of gravity: a review. *Classical and Quantum Gravity*, 39(6):063001, feb 2022.

[5] Pedro G. S. Fernandes. Gravity with a generalized conformal scalar field: Theory and solutions. *Physical Review D*, 103(10), may 2021.

[6] Tsutomu Kobayashi. Horndeski theory and beyond: a review. *Reports on Progress in Physics*, 82(8):086901, jul 2019.

[7] David Lovelock. Dimensionally dependent identities. *Mathematical Proceedings of the Cambridge Philosophical Society*, 68(2):345–350, 1970.

[8] T Padmanabhan. Planck length as the lower bound to all physical length scales. *General relativity and gravitation*, 17:215–221, 1985.