RADIATION OF THE ELECTROMAGNETIC FIELD BEYOND THE DIPOLE APPROXIMATION

Group 9 : PH444 : Electromagnetic theory project

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Abstract

Multipole expansion is a well-known technique in the calculation of electromagnetic, and gravitational fields at large distances, but we need a more elaborative approach beyond the lowest order approximation in the case of the electromagnetic radiation field. The goal of this project is to derive an expression for the radiated power in the approximation order beyond the dipole approximation, using the basic equations of electrodynamics and the techniques from vector and tensor calculus. The derivation only requires knowledge of the fundamental relations from an introductory course in classical electrodynamics and involves straightforward mathematical transformations.

1. Poynting Vector And Radiated Power

We know the energy density in Gaussian units is

$$w = \frac{\mathbf{E}^2 + \mathbf{B}^2}{8\pi} \tag{1}$$

Let's look at the partial time derivative of the energy density,

$$\frac{\partial w}{\partial t} = \frac{1}{4\pi}\left(\mathbf{E}\cdot\frac{\partial \mathbf{E}}{\partial t} + \mathbf{B}\cdot\frac{\partial \mathbf{B}}{\partial t}\right) \tag{2}$$

In a region with an absence of charges, we have the Maxwell equations as

$$\frac{\partial \mathbf{E}}{\partial t} = c\boldsymbol{\nabla}\times\mathbf{B}, \qquad \frac{\partial \mathbf{B}}{\partial t} = -c\boldsymbol{\nabla}\times\mathbf{E} \tag{3}$$

Substituting the Maxwell equations in (2), we get

$$\frac{\partial w}{\partial t} = \frac{c}{4\pi}\left(\mathbf{E}\cdot(\boldsymbol{\nabla}\times\mathbf{B}) - \mathbf{B}\cdot(\boldsymbol{\nabla}\times\mathbf{E})\right) \tag{4}$$

Using the identity that, $\boldsymbol{\nabla}\cdot(\mathbf{A}\times\mathbf{B}) = (\boldsymbol{\nabla}\times\mathbf{A})\cdot\mathbf{B} - (\boldsymbol{\nabla}\times\mathbf{A})\cdot\mathbf{B}$, we get

$$\frac{\partial w}{\partial t} = -\frac{c}{4\pi}(\boldsymbol{\nabla}\cdot(\mathbf{E}\times\mathbf{B})) \tag{5}$$

Substituting the value of the Poynting vector in the above equation

$$\frac{\partial w}{\partial t} = -\frac{c}{4\pi}(\boldsymbol{\nabla}\cdot\mathbf{S}) \tag{6}$$

here, $\mathbf{S}$ is the energy flux density (the Poynting Vector)

$$\mathbf{S} = \frac{c}{4\pi}\mathbf{E}\times\mathbf{B}. \tag{7}$$

The Energy Conservation law for the electromagnetic field in the absence of charges or currents is

$$\frac{\partial w}{\partial t} + \boldsymbol{\nabla}\cdot\mathbf{S} = 0, \tag{8}$$

Using Gauss's theorem to equation (8), the radiated Power

$$I = -\frac{dW}{dt} \equiv \frac{d}{dt}\int w\,dV$$

in terms of the surface integral of Poynting Vector

$$I = \oint_{\Sigma}\mathbf{S}\cdot d\Sigma = \int_{\Omega=4\pi}|\mathbf{S}|r^2\,d\Omega, \tag{9}$$

Since the R.H.S of the integral is over the complete solid angle, only those fields contribute to the radiation, which has a dependence

$$|\mathbf{S}| \propto \frac{1}{r^2}, \quad r\to\infty$$

The magnetic and electric field in the far zone can be written as

$$\mathbf{E} = \mathbf{E_0} + \mathbf{E_1}, \quad B = \mathbf{B_0} + \mathbf{B_1}, \tag{10}$$

where $\mathbf{E_1}$, and $\mathbf{B_1} \propto \frac{1}{r}$. The expression for $\mathbf{E_1}$ and $\mathbf{B_1}$ can be seen through fields for the Liénard–Wiechert potential. The Lienard-Wiechert potentials ϕ scalar potential field and $\mathbf{A}$ vector potential field can be written as

$$\phi(r,t) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{(1-\mathbf{n_s}\cdot\beta_s)|\mathbf{r}-\mathbf{r_s}|}\right)_t \tag{11}$$

and

$$\mathbf{A}(r,t) = \frac{\mu_0 c}{4\pi}\left(\frac{q\beta_s}{(1-\mathbf{n_s}\cdot\beta_s)|\mathbf{r}-\mathbf{r_s}|}\right)_t = \frac{\beta_s(t_r)}{c}\phi(r,t) \tag{12}$$

where t_r means retarded time and

$$\beta_s(t) = \frac{\mathbf{v_s}(t)}{c} \tag{13}$$

$$\mathbf{n_s} = \frac{r-r_s}{|r-r_s|} \tag{14}$$

After calculation and substitution of the above equation in the expression we obtain the following expression for $\mathbf{E_1}$ and $\mathbf{B_1}$

$$\mathbf{E}(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0}\Bigg(\frac{q(\mathbf{n}_s-\beta_s)}{\gamma^2(1-\mathbf{n}_s\cdot\beta_s)^3|\mathbf{r}-\mathbf{r}_s|^2} \tag{15}$$

$$+\frac{q\mathbf{n}_s\times((\mathbf{n}_s-\beta_s)\times\dot{\boldsymbol{\beta}}_s}{c(1-\mathbf{n}_s\cdot\beta_s)^3|\mathbf{r}-\mathbf{r}_s|}\Bigg)_{t_r} \tag{16}$$

$$\mathbf{B}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\Bigg(\frac{qc(\beta_s\times\mathbf{n}_s)}{\gamma^2(1-\mathbf{n}_s\cdot\beta_s)^3|\mathbf{r}-\mathbf{r}_s|^2} \tag{17}$$

$$+\frac{q\mathbf{n}_s\times(\mathbf{n}_s\times((\mathbf{n}_s-\beta_s)\times\dot{\boldsymbol{\beta}}_s}{(1-\mathbf{n}_s\cdot\beta_s)^3|\mathbf{r}-\mathbf{r}_s|}\Bigg)_{t_r} \tag{18}$$

$$= \frac{\mathbf{n}_s(t_r)}{c}\times\mathbf{E}(\mathbf{r},t) \tag{19}$$

Since the vector $\mathbf{E_1}$ and $\mathbf{B_1}$, both are equal in magnitude and orthogonal, the Poynting vector for the radiation can be written as

$$|\mathbf{S_1}| = \frac{c}{4\pi}|\mathbf{E_1}\times\mathbf{B_1}| = \frac{c}{4\pi}|\mathbf{B_1}|^2. \tag{20}$$

Radiated power as

$$I = \frac{c}{4\pi}\int_{\Omega=4\pi}|\mathbf{B_1}|^2 r^2\, d\Omega. \tag{21}$$

The magnetic field is

$$\mathbf{B}(r,t) = \nabla\times\mathbf{A}(r,t), \tag{22}$$

The calculation for vector potential $\mathbf{A}$(r,t) we will see in the next section.

2. Vector Potential

Consider a system of charges in volume V, and observation point $r \gg V^{1/3}$

$$A(r,t) = \frac{1}{c}\int dV'\frac{1}{|\mathbf{r}-\mathbf{r}'|}\mathbf{j}\left(\mathbf{r}',t-\frac{1}{c}|\mathbf{r}-\mathbf{r}'|\right) \tag{23}$$

For the approximation, $r' \ll r$

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} \simeq \frac{1}{r} \tag{24}$$

$$|\mathbf{r}-\mathbf{r}'| \simeq r - \mathbf{r}'.\nabla r = r - \frac{\mathbf{r}'.\mathbf{r}}{r} \tag{25}$$

or, using the unit vector $\mathbf{n} = \mathbf{r}/r$.

$$|\mathbf{r}-\mathbf{r}'| \simeq r - \mathbf{n}.\mathbf{r}' \tag{26}$$

with the previous approximations, we get

$$A(r,t) = \frac{1}{cr}\int_V dV'\mathbf{j}\left(\mathbf{r}',t-\frac{r}{c}+\frac{1}{c}\mathbf{n}.\mathbf{r}'\right) \tag{27}$$

Expanding as a taylor series in $(\mathbf{n}\cdot\mathbf{r}')/c$

$$\begin{aligned} A(r,t) =& \frac{1}{cr}\int_V dV'\mathbf{j}\left(\mathbf{r}',t-\frac{r}{c}\right) \\ &+ \frac{d}{dt}\frac{1}{c^2 r}\int_V dV'(\mathbf{n}\cdot\mathbf{r}')\mathbf{j}\left(\mathbf{r}',t-\frac{r}{c}\right) \\ &+ \frac{d^2}{dt^2}\frac{1}{2c^3 r}\int_V dV'(\mathbf{n}\cdot\mathbf{r}')^2\mathbf{j}\left(\mathbf{r}',t-\frac{r}{c}\right)+\cdots \end{aligned} \tag{28}$$

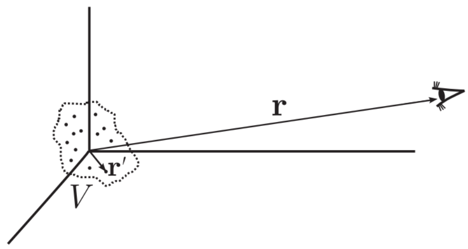

Figure 1: The system of charges is located near the origin and a distant observer sits at point $\mathbf{r}$.

Now, the current density can be written as

$$\mathbf{j}(\mathbf{r},\tau) = \sum_i e_i\mathbf{v}_i(\tau)\delta(\mathbf{r}-\mathbf{r}_i(\tau)) \tag{29}$$

The first term in the expansion of 28

$$\begin{aligned} \int_V dV'\mathbf{j}\left(\mathbf{r}',t-\frac{r}{c}\right) = \sum_i e_i\mathbf{v}_i(\tau) = \frac{d}{d\tau}\sum_i e_i\mathbf{r}_i(\tau) \\ = \frac{d}{d\tau}\mathbf{d}(\tau) = \dot{\mathbf{d}}(\tau) \end{aligned} \tag{30}$$

Thus, in the leading order,

$$\mathbf{A}(\mathbf{r},t) = \frac{\dot{\mathbf{d}}(t-r/c)}{cr}+\cdots \tag{31}$$

We now obtain the next order correction by making the following transformation

$$\begin{aligned} \int_V dV'(\mathbf{n}\cdot\mathbf{r}')\mathbf{j}(\mathbf{r}',t-\frac{r}{c}) &= \sum_i e_i v_i(\tau)\int_V dV'\mathbf{n}\cdot\mathbf{r}'\delta(\mathbf{r}'-\mathbf{r}_i(\tau)) \\ &= \sum_i e_i v_i(\mathbf{n}\cdot\mathbf{r}') = \frac{1}{2}\sum_i e_i\frac{d\mathbf{r}_i}{d\tau}(\mathbf{n}\cdot\mathbf{r}') \\ &= \frac{1}{2}\frac{d\mathbf{r}_i}{d\tau}\sum_i e_i\mathbf{r}_i(\mathbf{n}\cdot\mathbf{r}')+\frac{1}{2}\sum_i e_i\mathbf{n}\times(\mathbf{v}_i\times\mathbf{r}_i) \end{aligned} \tag{32}$$

The vector potential appears only as a cross product with the unit vector $\mathbf{n}$. Thus, we can add to $\mathbf{A}$ an arbitrary vector proportional to $\mathbf{n}$. We thus change the first term of equation 32 to

$$\frac{1}{2}\frac{d}{d\tau}\sum_i e_i\{\mathbf{r}_i\left(\mathbf{n}\cdot\mathbf{r}_i-\mathbf{n}\frac{r_i^2}{3}\right)\}$$

The sum over i is a contraction of the electric quadrupole moment tensor with the unit vector $\mathbf{n}$ yielding some vector $\mathcal{D}$,

$$Q_{jk} = \sum_i e_i x_j(i)x_k(i) - \frac{r_i^2}{3}\delta_{jk} \qquad \mathcal{D} = \sum_k Q_{jk}n_k$$

, The remaining part of this correction involves the cross product of $\mathbf{m} = \frac{1}{2c}\sum_i e_i\mathbf{r_i}\times v_i$ with the unit vector $\mathbf{n}$

The last term in the expansion of 28 can be transformed as:

$$\frac{d^2}{dt^2}\frac{1}{2c^3 r}\int_V dV'(\mathbf{n}\cdot\mathbf{r}')^2\mathbf{j}\left(\mathbf{r}', t-\frac{r}{c}\right)$$
$$= \frac{1}{2c^3 r}\frac{d^2}{dt^2}\int_V dV'\mathbf{e}_i j_i\left(\mathbf{r}', t-\frac{r}{c}\right)n_k x_k' n_l x_l' \tag{33}$$
$$= \frac{1}{2c^2 r}\ddot{\mathcal{M}}\left(\mathbf{r}', t-\frac{r}{c}\right)n_k n_l \mathbf{e}_i$$

Here $\mathbf{e}_i$ are the unit Cartesian coordinate vectors and $\ddot{\mathcal{M}}$ is the third rank current quadrupole tensor. Summation over repeated indices is implied. $\ddot{\mathcal{M}} = \frac{1}{c}\int_V d_V' j_i x_k' x_l'$

Combining all the contributions from equation 28, we arrive at the vector potential

$$\mathbf{A}(\mathbf{r}, t) = \frac{\dot{\mathbf{d}}}{cr} + \frac{\dot{\mathbf{m}}\times\mathbf{n}}{cr} + \frac{1}{2c^2 r}\ddot{\mathcal{D}} + \frac{1}{2c^2 r}\ddot{\mathcal{M}}n_k n_l \mathbf{e}_i \tag{34}$$

Here all the quantities are evaluated at the retarded time $\tau = t - r/c$.

3. Bringing it all together

Now let's calculate the Magnetic field $\mathbf{B}$ using vector potential eq (34).

$$\mathbf{B} = \nabla\times\mathbf{A} = \nabla\times\left[\frac{1}{cr}\dot{\boldsymbol{d}}\left(t-\frac{r}{c}\right) + \frac{1}{cr}\dot{\boldsymbol{m}}\left(t-\frac{r}{c}\right)\times\boldsymbol{n}\right. \tag{35}$$
$$\left. + \frac{1}{2c^2 r}\ddot{\mathcal{D}}\left(t-\frac{r}{c}\right) + \frac{1}{2c^2 r}\ddot{\mathcal{M}}_{ikl}\left(t-\frac{r}{c}\right)\right] \tag{36}$$
$$= \frac{\ddot{\boldsymbol{d}}\times\boldsymbol{n}}{c^2 r} + \frac{(\ddot{\boldsymbol{m}}\times\boldsymbol{n})\times\boldsymbol{n}}{c^2 r} + \frac{\dddot{\mathcal{D}}\times\boldsymbol{n}}{2c^3 r} + \frac{1}{2c^3 r}\dddot{\mathcal{M}}_{ikl}n_k n_l \epsilon_{irs} n_r \mathbf{e_s} \tag{37}$$

here we used the following identity

$$\nabla\times\boldsymbol{f}\left(t-\frac{r}{c}\right) = \nabla\left(t-\frac{r}{c}\right)\times\dot{\boldsymbol{f}}\left(t-\frac{r}{c}\right) = \frac{1}{c}\dot{\boldsymbol{f}}\left(t-\frac{r}{c}\right)\times\boldsymbol{n} \tag{38}$$

The four terms of $\mathbf{B}$ correspond to the electric dipole, magnetic dipole, electric quadrupole, and current quadrupole term, respectively.

$$\mathbf{B} = \mathbf{B}_d + \mathbf{B}_m + \mathbf{B}_Q + \mathbf{B}_{\mathcal{M}}$$

$$\mathbf{B}_d = \frac{\ddot{\boldsymbol{d}}\times\boldsymbol{n}}{c^2 r} \propto \frac{1}{c^2} \tag{39}$$

$$\mathbf{B}_m = \frac{\boldsymbol{n}\times(\ddot{\boldsymbol{m}}\times\boldsymbol{n})}{c^2 r} \propto \frac{1}{c^3} \tag{40}$$

$$\mathbf{B}_Q = \frac{\dddot{\mathcal{D}}\times\boldsymbol{n}}{2c^3 r} \propto \frac{1}{c^3} \tag{41}$$

$$\mathbf{B}_{\mathcal{M}} = \frac{1}{2c^3 r}\dddot{\mathcal{M}}_{ikl}n_k n_l \epsilon_{irs} n_r \mathbf{e_s} \propto \frac{1}{c^4} \tag{42}$$

The magnitude of the pointing vectors contains the following terms

$$|\mathbf{S}_1| = \frac{1}{4\pi}\Bigg(\underbrace{c|\mathbf{B}_d|^2}_{\propto 1/c^3} + \underbrace{2c\mathbf{B}_d.\mathbf{B}_m}_{\propto 1/c^4} + \underbrace{2c\mathbf{B}_d.\mathbf{B}_Q}_{\propto 1/c^4} \tag{43}$$
$$+ \underbrace{c|\mathbf{B}_m|^2}_{\propto 1/c^5} + \underbrace{c|\mathbf{B}_Q|^2}_{\propto 1/c^5} + \underbrace{2c\mathbf{B}_m.\mathbf{B}_Q}_{\propto 1/c^5} + \underbrace{2c\mathbf{B}_d.\mathbf{B}_{\mathcal{M}}}_{\propto 1/c^5}$$
$$+ O(c^{-6})\Bigg)$$

3.1. Electric and magnetic dipole radiation

Looking at the above expression leading contribution to the radiated power is given by the electric dipole term

$$I_d = \frac{c}{4\pi}\int_{\Omega=4\pi}|\mathbf{B}_d|^2 r^2\, d\Omega = \frac{1}{4\pi c^3}\int_{\Omega=4\pi}|\ddot{\mathbf{d}}\times\mathbf{n}|^2\, d\Omega \tag{44}$$

Assume that $\ddot{\mathbf{d}}$ is directed towards the z-axis and θ is the angle between $\ddot{\mathbf{d}}$ & $\mathbf{n}$

$$I_d = \frac{\ddot{\mathbf{d}}}{4\pi c^3}\int_{\Omega=4\pi}\sin^2\theta\, d\Omega = \frac{\ddot{\mathbf{d}}}{4\pi c^3}\int_0^{2\pi}d\phi\int_0^{\pi}\sin^2\theta\, d\theta$$
$$= \frac{2\ddot{\mathbf{d}}^2}{3c^3} \propto \frac{1}{c^3} \tag{45}$$

similarly, magnetic dipole contribution is

$$I_m = \frac{c}{4\pi}\int_{\Omega=4\pi}|\mathbf{B}_m|^2 r^2\, d\Omega = \frac{1}{4\pi c^3}\int_{\Omega=4\pi}|(\ddot{\mathbf{m}}\times\mathbf{n})\times\mathbf{n}|^2\, d\Omega$$
$$= \frac{2\ddot{\mathbf{m}}^2}{3c^3} \propto \frac{1}{c^5} \tag{46}$$

3.2. Terms with zero contributions

$$\mathbf{B_d}\cdot\mathbf{B_m} = (\ddot{\mathbf{d}}\times\mathbf{n})\cdot[(\ddot{\mathbf{m}}\times\mathbf{n})\times\mathbf{n}]$$
$$= (\ddot{\mathbf{d}}\times\mathbf{n})\cdot[\mathbf{n}(\mathbf{n}\cdot\ddot{\mathbf{m}}) - \ddot{\mathbf{m}}] \tag{47}$$
$$= (\ddot{\mathbf{d}}.\underbrace{(\mathbf{n}\times\mathbf{n})}_{=0})(\mathbf{n}\cdot\ddot{\mathbf{m}}) - (\ddot{\mathbf{d}}\times\mathbf{n})\cdot\ddot{\mathbf{m}} = \mathbf{n}\cdot(\ddot{\mathbf{d}}\times\ddot{\mathbf{m}})$$

Integration of the respective term in Eq.(9) using spherical coordinates with θ corresponding to the angle between $\mathbf{n}$ and $(\ddot{\mathbf{d}}\times\ddot{\mathbf{m}})$ gives zero due to integration of $\sin\theta\cos\theta$ term from $0\rightarrow\pi$
We can do similar calculations for $\mathbf{B_d}\cdot\mathbf{B_Q}$

$$\mathbf{B_d}\cdot\mathbf{B_Q} = (\ddot{\mathbf{d}}\times\mathbf{n})\cdot(\dddot{\mathcal{D}}\times\mathbf{n}) = \ddot{\mathbf{d}}\cdot[\mathbf{n}\times(\dddot{\mathcal{D}}\times\mathbf{n})]$$
$$= \ddot{\mathbf{d}}\cdot\dddot{\mathcal{D}} - (\mathbf{n}\cdot\ddot{\mathbf{d}})(\mathbf{n}\cdot\dddot{\mathcal{D}}) \tag{48}$$

Let's look at the terms individually

$$\ddot{\mathbf{d}}\cdot\dddot{\mathcal{D}} \equiv \ddot{d}_i.\dddot{\mathcal{D}}_i = \ddot{d}_i.\dddot{Q}_{ij}n_j \tag{49}$$

$$(\mathbf{n}\cdot\ddot{\mathbf{d}})(\mathbf{n}\cdot\dddot{\mathcal{D}}) = \ddot{d}_i\dddot{Q}_{jk}n_i n_j n_k \tag{50}$$

It is easy to show that the integration of the n_i components over the complete solid angle gives zero, also same goes for triple product The above two contributions contain products of an odd number of n_i components They yield zero upon integration over the complete solid angle as a consequence of the independence of choice of axes orientation The product of an odd number of n_i factors is an odd-rank symmetric tensor and the only tensor of this type invariant under axes rotation is zero.int (WInter 2018) The same rationale would apply to $\mathbf{B_m}\cdot\mathbf{B}_{\mathcal{M}}$ & $\mathbf{B_Q}\cdot\mathbf{B}_{\mathcal{M}}$.
Let's look at the only vector terms

$$\mathbf{B_m}\cdot\mathbf{B}_{\mathcal{M}} \propto (\boldsymbol{n}\times(\ddot{\boldsymbol{m}}\times\boldsymbol{n}))\cdot\boldsymbol{e}_s = \ddot{\mathbf{m}}\cdot\boldsymbol{e}_s - (\mathbf{n}\cdot\boldsymbol{e}_s)(\mathbf{n}\cdot\ddot{\boldsymbol{m}}) \tag{51}$$

3.3. Electric quadrupole radiation

We now look at the $|\mathbf{B}_Q|^2$ term i.e. $|\dddot{\mathcal{D}} \times \mathbf{n}|^2$.

$$|\dddot{\mathcal{D}} \times \mathbf{n}|^2 = (\dddot{\mathcal{D}} \times \mathbf{n}) \cdot (\dddot{\mathcal{D}} \times \mathbf{n}) = \dddot{\mathcal{D}} \cdot \left(\mathbf{n} \times (\dddot{\mathcal{D}} \times \mathbf{n})\right)$$

where we got the second expression by using the cycle permutation property of the scalar-vector product. We use the property of three vector product, $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$

$$\begin{aligned} |\dddot{\mathcal{D}} \times \mathbf{n}|^2 &= \dddot{\mathcal{D}} \cdot \left((\mathbf{n} \cdot \mathbf{n})\dddot{\mathcal{D}} - (\mathbf{n} \cdot \dddot{\mathcal{D}})\mathbf{n}\right) \\ &= \dddot{\mathcal{D}} \cdot \dddot{\mathcal{D}} - (\dddot{\mathcal{D}} \cdot \mathbf{n})(\dddot{\mathcal{D}} \cdot \mathbf{n}) \end{aligned}$$

as $\mathbf{n}$ is a unit vector. Writing this in index notation,

$$\begin{aligned} |\dddot{\mathcal{D}} \times \boldsymbol{n}|^2 &= \dddot{\mathcal{D}}_i \dddot{\mathcal{D}}_i - n_i \dddot{\mathcal{D}}_i n_j \dddot{\mathcal{D}}_j \\ &= \dddot{Q}_{ij} n_j \dddot{Q}_{ik} n_k - n_i \dddot{Q}_{ik} n_k n_j \dddot{Q}_{jl} n_l \end{aligned} \qquad (52)$$

where we substitute the form of $\mathcal{D}_i$ from equation (**??**). Thus, the power of the electric quadrupole radiation is equal to

$$\begin{aligned} I_Q &= \frac{c}{4\pi} \frac{1}{4c^6} \int_{\Omega=4\pi} \left\{ \dddot{Q}_{ij} \dddot{Q}_{ik} n_j n_k - \dddot{Q}_{ik} \dddot{Q}_{jl} n_i n_k n_j n_l \right\} d\Omega \\ &= \frac{1}{20c^5} \dddot{Q}_{ij} \dddot{Q}_{ij} \end{aligned} \qquad (53)$$

where the integral of two and four unit vectors is in Jackson's book eq. (9.47).

$$\begin{aligned} \int_{\Omega=4\pi} n_i n_j d\Omega &= \frac{4\pi}{3} \delta_{ij} \\ \int_{\Omega=4\pi} n_i n_j n_k n_l d\Omega &= \frac{4\pi}{15} \left(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right) \end{aligned} \qquad (54)$$

and the fact that the electric quadrupole moment tensor is traceless, $Q_{ii} = 0$.

3.4. Anapole radiation

The next term

$$\begin{aligned} 2\mathbf{B}_d \cdot \mathbf{B}_{\mathcal{M}} &= 2 \frac{1}{c^2 r} \varepsilon_{pqt} \ddot{d}_p n_q \mathbf{e}_t \cdot \frac{1}{2c^3 r} \dddot{\mathcal{M}}_{ikl} n_k n_l \varepsilon_{irs} n_r \mathbf{e}_s \\ &= \frac{1}{c^5 r^2} \underbrace{\mathbf{e}_t \cdot \mathbf{e}_s}_{=\delta_{ts}} \ddot{d}_p \dddot{\mathcal{M}}_{ikl} \varepsilon_{pqt} \varepsilon_{irs} n_k n_l n_q n_r \\ &= \frac{1}{c^5 r^2} \ddot{d}_p \dddot{\mathcal{M}}_{ikl} \varepsilon_{pqt} \varepsilon_{irt} n_k n_l n_q n_r \\ &= \frac{1}{c^5 r^2} \ddot{d}_p \dddot{\mathcal{M}}_{ikl} (\delta_{pi}\delta_{qr} - \delta_{pr}\delta_{qi}) n_k n_l n_q n_r \\ 2\mathbf{B}_d \cdot \mathbf{B}_{\mathcal{M}} &= \frac{1}{c^5 r^2} \left(\ddot{d}_i \dddot{\mathcal{M}}_{ikl} n_k n_l - \ddot{d}_q \dddot{\mathcal{M}}_{ikl} n_k n_l n_q n_i \right) \end{aligned} \qquad (55)$$

Thus the power radiated from this term will be

$$\begin{aligned} I_A &= \frac{c}{4\pi} \int_{\Omega=4\pi} 2\mathbf{B}_d \cdot \mathbf{B}_{\mathcal{M}} d\Omega \\ &= -\frac{2}{15c^4} \left(\dddot{\mathcal{M}}_{kki} - 2\dddot{\mathcal{M}}_{ikk} \right) \ddot{d}_i \end{aligned} \qquad (56)$$

We define a quantity, the anapole moment or toroidicity $\mathbf{T}$ as

$$\mathbf{T}(\tau) = \frac{1}{10c} \int_V dV' \left\{ (\mathbf{j}(\mathbf{r}', \tau) \cdot \mathbf{r}')\, \mathbf{r} - 2r'^2 \mathbf{j}(\mathbf{r}', \tau) \right\} \qquad (57)$$

Thus the power of the anapole radiation simplifies to,

$$I_A = -\frac{4}{3c^4} \dddot{\mathbf{T}} \cdot \ddot{\mathbf{d}} \qquad (58)$$

4. Discussion

In summary, the radiated power up to $1/c^5$ is given by

$$\begin{aligned} I &= I_d + I_m + I_Q + I_A \\ &= \underbrace{\frac{2\ddot{\mathbf{d}}^2}{3c^3}}_{1/c^3} + \underbrace{\frac{2\ddot{\mathbf{m}}^2}{3c^3} + \frac{1}{20c^5} \dddot{Q}_{ij} \dddot{Q}_{ij} - \frac{4}{3c^4} \dddot{\mathbf{T}} \cdot \ddot{\mathbf{d}}}_{1/c^5} \end{aligned} \qquad (59)$$

The second and fourth terms are proportional to $1/c^5$ as a $1/c$ factor appears in both magnetic moment and toroidicity. Any subsequent terms would yield corrections of the order $1/c^7$ and higher. We would only get terms with odd powers of $1/c$. The even power terms will vanish as it did in the case of $(\mathbf{B}_d \cdot \mathbf{B}_m)$. So the terms proportional to $(\mathbf{B}_m \cdot \mathbf{B}_{\mathcal{M}})$ and $(\mathbf{B}_Q \cdot \mathbf{B}_{\mathcal{M}})$ will also vanish. The reason being that they contain products of an odd number of n_i components. Such expressions always yield zero upon integration over the complete solid angle.

Generally, the anapole term is often neglected when considering the radiated power. This is because the electric dipole radiated power I_d is a sufficient approximation and the calculation of the corrections is required only if $I_d = 0$. But in this case $\ddot{\mathbf{d}} = 0$ and hence $I_A = 0$ as well. This means that the anapole term becomes relevant only if the radiated power requires a higher precision than the dipole term alone.

In the static case, the electromagnetic field of the torus is zero outside the system but for time-dependent sources, such a moment is known as toroidicity, toroidal dipole or anapole moment. In higher approximations, other toroidal multipoles appear as well.

Using a similar analysis from section 3.1, we can show that the radiation of the torus is given by

$$I_{\text{torus}} = \frac{2\dddot{\mathbf{T}}^2}{3c^5} \propto \frac{1}{c^7} \qquad (60)$$

This contribution, together with the electric dipole (45) and anapole (56) radiation, completes the square

$$I_{d+t} = \frac{2\ddot{\mathbf{d}}}{3c^3} - \frac{4}{3c^4} \dddot{\mathbf{T}} \cdot \ddot{\mathbf{d}} + \frac{2\dddot{\mathbf{T}}}{3c^5} = \frac{2}{3c^3} \left(\ddot{\mathbf{d}} - \frac{1}{c} \dddot{\mathbf{T}} \right)^2 \qquad (61)$$

In summary, we have obtained an expression for the power of electromagnetic radiation in the approximation beyond the dipole approximation, i.e., with terms proportional to $1/c^5$, which is an accuracy sufficient for comparisons with most present-day measurements.

References

, WInter 2018. Evaluation of someintegrals over solid angles. URL: