Problem 1

For each red ball, a portion p_1 came from Bag 1 and the other $p_2 = 1 - p_1$ from Bag 2.

So, $\hat{\mu}_1 = \frac{p_1}{1+p_1}$.

$\hat{\mu}_2 = \frac{p_2}{2+p_2}$.

Each red ball is either from Bag 1 or Bag 2.

Likelihood for each is :

$\frac{1}{2}(1-\mu_1) \times \left(\frac{1}{2}\mu_1\right)^3 \times \frac{1}{2}(1-\mu_2) \times \frac{1}{2}(1-\mu_2) \rightarrow$ Bag 1

$\frac{1}{2}(1-\mu_1) \times \left(\frac{1}{2}\mu_2\right)^3 \times \frac{1}{2}(1-\mu_2) \times \frac{1}{2}(1-\mu_2) \rightarrow$ Bag 2.

The log-likelihood for these 2 cases :

$LLH1 = \ln(1-\mu_1) + 3\ln\mu_1 + 2\ln(1-\mu_2) - 6\ln 2$ (Bag 1)

$LLH2 = \ln(1-\mu_1) + 3\ln\mu_2 + 2\ln(1-\mu_2) - 6\ln 2$ (Bag 2)

Expected Log-Likelihood is :

$p_1 \times LLH1 + p_2\, LLH_2$.

$= \ln(1-\mu_1) + 3p_1\ln(\mu_1) + 3p_2\ln(\mu_2) + 2\ln(1-\mu_2) - 6\ln 2$.

Maximization Step:

$$\frac{\partial LLH}{\partial \mu_1} = \frac{-1}{1-\mu_1} + \frac{3p_1}{\mu_1} = 0.$$

$$\Rightarrow \quad -\mu_1 + 3p_1 - 3p_1\mu_1 = 0.$$

$$\mu_1(1+3p_1) = 3p_1$$

$$\mu_1 = \frac{3p_1}{1+3p_1}. \longrightarrow (1)$$

$$\frac{\partial LLH}{\partial \mu_2} = \frac{3p_2}{\mu_2} - \frac{2}{1-\mu_2} = 0.$$

$$\Rightarrow \quad 3p_2 - 3p_2\mu_2 - 2\mu_2 = 0.$$

$$\mu_2 = \frac{3p_2}{2+3p_2}. \longrightarrow (2).$$

$$p_1 = \frac{\mu_1}{\mu_1+\mu_2} \longrightarrow (3) \quad ; \quad p_2 = \frac{\mu_2}{\mu_1+\mu_2} \longrightarrow (4).$$

Combining the above 4 equations.

$$\mu_1 \leftarrow \frac{3\mu_1}{4\mu_1+\mu_2} \qquad \mu_2 \leftarrow \frac{3\mu_2}{2\mu_1+5\mu_2}.$$

Intutively,
since we have 6 balls and probability of picking each bag is $\frac{1}{2}$, we can say three balls from each were picked.

Since green ball can be picked only from bag 1 and blue only from bag 2, we can say that the three balls from Bag 1 are 1 Green & 2 Reds, while the three balls from Bag 2 are 2 Blue & 1 Red.

Green Red Red | Red Blue Blue.

So, the estimates: $\hat{\mu}_1 = \frac{2}{3}$ & $\hat{\mu}_2 = 1/3$.

Initial Mu1 =0.9, Mu2 = 0.1	**Mu1**	**Mu2**
Iteration 1	0.730	0.130
Iteration 2	0.718	0.185
Iteration 3	0.705	0.235
Iteration 4	0.692	0.273
Iteration 5	0.683	0.298
Iteration 6	0.676	0.313
Iteration 7	0.672	0.322
Iteration 8	0.670	0.327
Iteration 9	0.668	0.330
Iteration 10	0.668	0.331
Iteration 11	0.667	0.332
Iteration 12	0.667	0.333

Initial Mu1 =0.7, Mu2 = 0.3	**Mu1**	**Mu2**
Iteration 1	0.677	0.310
Iteration 2	0.673	0.320
Iteration 3	0.670	0.326
Iteration 4	0.669	0.329
Iteration 5	0.668	0.331
Iteration 6	0.667	0.332
Iteration 7	0.667	0.333
Iteration 8	0.667	0.333

Initial Mu1 =0.5, Mu2 = 0.5	**Mu1**	**Mu2**
Iteration 1	0.600	0.429
Iteration 2	0.636	0.385
Iteration 3	0.652	0.361

Iteration 4	0.659	0.348
Iteration 5	0.662	0.342
Iteration 6	0.664	0.338
Iteration 7	0.665	0.336
Iteration 8	0.666	0.335
Iteration 9	0.666	0.334
Iteration 10	0.666	0.334
Iteration 11	0.667	0.334
Iteration 12	0.667	0.333
Iteration 13	0.667	0.333
Iteration 14	0.667	0.333
Iteration 15	0.667	0.333
Iteration 16	0.667	0.333
Iteration 17	0.667	0.333

Problem 2. The probability to get a green ball is $P(g) = \frac{1}{2}(1-\mu_1)$, to get a blue ball $P(b) = \frac{1}{2}(1-\mu_2)$ and to get a red ball $P(r) = \frac{1}{2}\mu_1 + \frac{1}{2}\mu_2$, thus the log-likelihood of the incomplete data is,

$$LLH = \Sigma \ln p = N_g \ln P(g) + N_r \ln P(r) + N_b \ln P(b).$$

$$= N_g \ln \frac{1}{2}(1-\mu_1) + N_r \ln \left(\frac{1}{2}\mu_1 + \frac{1}{2}\mu_2\right) + N_b \ln \frac{1}{2}(1-\mu_2).$$

$$= N_g \ln (1-\mu_1) + N_r \ln(\mu_1+\mu_2) + N_b \ln(1-\mu_2) - N \ln 2.$$

Taking derivative wrt μ_1, μ_2 and set them to zero, we have,

$$\frac{\partial L}{\partial \mu_1} = -\frac{N_g}{1-\mu_1} + \frac{N_r}{\mu_1+\mu_2} = 0.$$

$$\frac{\partial L}{\partial \mu_2} = -\frac{N_b}{1-\mu_2} + \frac{N_r}{\mu_1+\mu_2} = 0.$$

Solving this we have $\mu_2 = 1 - \frac{N_b}{N_g} + \frac{N_b}{N_g}\mu_1$,

taking this and putting into above eqns we have

$\mu_1 = 1 - 2\frac{N_g}{N}$ and $\mu_2 = 1 - 2\frac{N_b}{N}$

The likelihood $\mathcal{L}_1 = N_g \ln \frac{N_g}{N} + N_b \ln \frac{N_b}{N} + N_r \ln \frac{N_r}{N}$. — ①

But in this, μ_1 & μ_2 are less than 0 when $N_g > \frac{N}{2}$ or $N_b > \frac{N}{2}$. So we need likelihood with $\mu_1 \geq 0$ $\mu_2 \geq 0$.

μ_1 and μ_2 are implicitly less than 1 from the log-likelihood formula.

Set $\mu_1 = 0$ and solve for optimal $\mu_2 = \frac{N_r}{N_r + N_b}$,

which corresponds to a likelihood of

$$\mathcal{L}_2 = N_b \ln \frac{N_b}{N_r + N_b} + N_r \ln \frac{N_r}{N_r + N_b} - N \ln 2. \quad ②$$

Similarly, setting $\mu_2 = 0$, optimal $\mu_1 = \frac{N_r}{N_r + N_g}$

and $$\mathcal{L}_3 = N_g \ln \frac{N_g}{N_r + N_g} + N_r \ln \frac{N_r}{N_r + N_g} - N \ln 2. \quad ③$$

We need to select the maximum likelihood among ①, ② & ③,

we compute

$$① - ② = N_g \ln \frac{N_g}{N} + N_b \ln \frac{N_r + N_b}{N} + N_r \ln \frac{N_r + N_b}{N} + N \ln 2.$$

$$= N\left[\frac{N_g}{N}\ln\frac{N_g}{N} + \left(1-\frac{N_g}{N}\right)\ln\left(1-\frac{N_g}{N}\right) + \ln 2\right].$$

Let $x = \frac{N_g}{N}$ we have $L_1 - L_2 = N\left[x\ln x + (1-x)\ln(1-x) + \ln 2\right]$.

Take derivative wrt x & set it to 0, we can see that the derivative is positive when $x > \frac{1}{2}$ and negative when $x < \frac{1}{2}$. ~~At~~ The difference $L_1 - L_2 = 0$ when $x = \frac{1}{2}$, so we see that $L_1 \geq L_2$ for all values of $\frac{N_g}{N}$. Similarly, we can see that $L_1 \geq L_3$ for all values of $\frac{N_g}{N}$.

So the maximum likelihood estimates for μ_1, μ_2 are

① $\mu_1 = 1 - 2\frac{N_g}{N}$ and $\mu_2 = 1 - 2\frac{N_b}{N}$ when $N_g \leq \frac{N}{2}$ and $N_b \leq \frac{N}{2}$.

② $\mu_1 = 0$, $\mu_2 = \frac{N_r}{N_b + N_r}$ when $N_g > \frac{N}{2}$.

③ $\mu_1 = \frac{N_r}{N_r + N_g}$, $\mu_2 = 0$ when $N_b > \frac{N}{2}$.

Intuitively this suggests than μ_1 decreases as N_g increases relative to N. That is, as the proportion of green balls in the sample increases, the estimate for μ_1 decreases.

This makes sense because if there are more green balls in the sample, there is less room for uncertainty (represented by μ_1) about the probability of drawing a green ball. Similarly, as Ng decreases relative to N, μ_1 increases. This reflects the increased uncertainity when there are fewer green balls in the sample. Similarly, for μ_2. As N_b increases relative to N, μ_2 decreases, indicating less uncertainty about the probability of drawing a blue ball when there are more blue balls in the sample. Conversely, as N_b decreases relative to N, μ_2 increases, reflecting increased uncertainity when there are fewer blue balls in the sample.

N	Ng	Nb	Initial Mu1	Initial Mu2	Final Mu1	Final Mu2
6	1	2	0.9	0.1	0.667	0.333
6	1	2	0.7	0.3	0.667	0.333
6	1	2	0.5	0.5	0.667	0.333
6	1	2	0.2	0.8	0.667	0.333
100	10	20	0.9	0.1	0.8	0.6
100	10	20	0.7	0.3	0.8	0.6
100	10	20	0.5	0.5	0.8	0.6
100	10	20	0.2	0.8	0.8	0.6
100	30	10	0.9	0.1	0.4	0.8
100	30	10	0.7	0.3	0.4	0.8
100	30	10	0.5	0.5	0.4	0.8
100	30	10	0.2	0.8	0.4	0.8
1000	250	100	0.1	0.8	0.5	0.8
1000	250	100	0.4	0.8	0.5	0.8
1000	250	100	0.7	0.8	0.5	0.8
1000	250	100	0.9	0.1	0.5	0.8