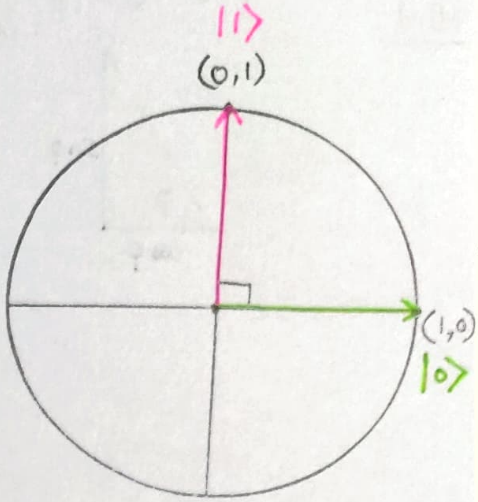
Single Qubit

A vector in complex 2D Hilbert Space

$$|\psi\rangle = a|0\rangle + \beta|1\rangle$$

a, β are called amplitudes

$a, \beta \in C$ (complex numbers)



Probability that the qubit collapses to $|0\rangle$ on measurement

$$\begin{aligned} P(|\psi\rangle = |0\rangle) &= |\langle 0|\psi\rangle|^2 \\ &= |\langle 0|(a|0\rangle + \beta|1\rangle)|^2 \\ &= |(a\langle 0|a\rangle + \beta\langle 0|1\rangle)|^2 \\ &= |a(1) + \beta(0)|^2 \\ &= |a|^2 \end{aligned}$$

likewise

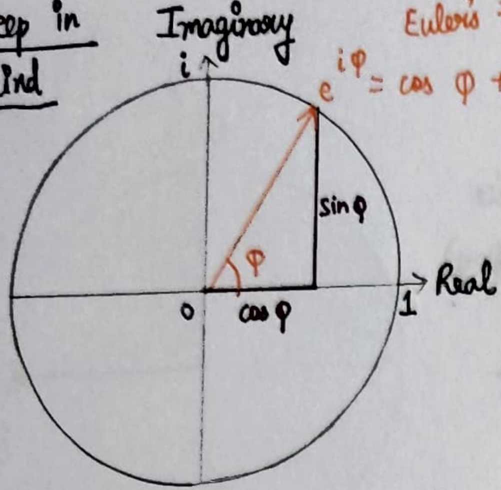
Probability that the qubit collapses to $|1\rangle$ on measurement

$$\begin{aligned} P(|\psi\rangle = |1\rangle) &= |\langle 1|\psi\rangle|^2 \\ &= |(a\langle 1|0\rangle + \beta\langle 1|1\rangle)|^2 \\ &= |(a(0) + \beta(1))|^2 \\ &= |\beta|^2 \end{aligned}$$

So probability = $|\text{amplitude}|^2$

$|a|^2 + |\beta|^2 = 1 = \langle\psi|\psi\rangle \rightarrow$ normalised state

Keep in Mind



Euler's Formula

$$e^{i\varphi} = \cos\varphi + i\sin\varphi$$

Complex number $z = x + iy$

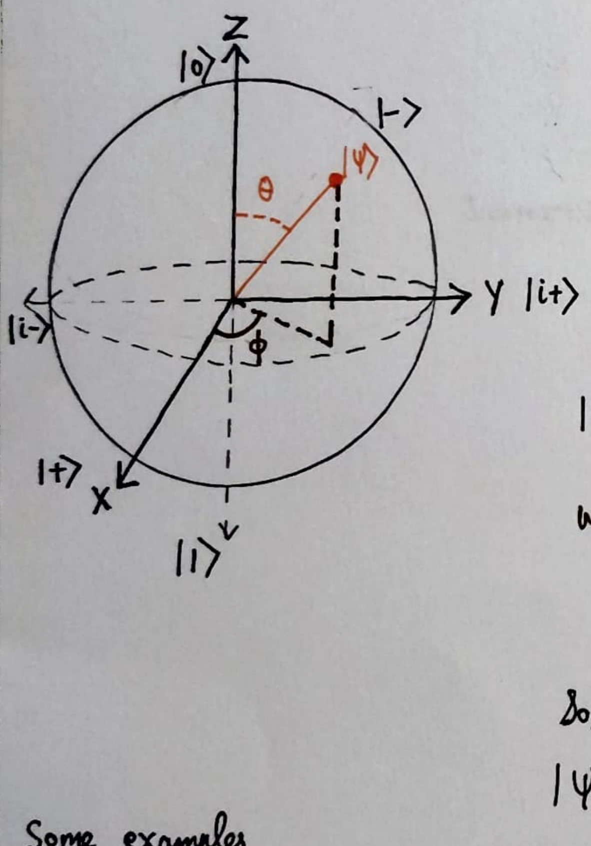
$$z = |z|(\cos\varphi + i\sin\varphi)$$

$$z = r e^{i\varphi}$$

$$r = |z| = \sqrt{x^2+y^2} \qquad \varphi = \tan^{-1}\left(\frac{y}{x}\right)$$

BLOCH SPHERE

- The state of a qubit can be represented as a point on the Bloch Sphere
- It is a unit sphere, $r = 1$



α, β are complex

$\alpha = |\alpha|e^{i\phi_\alpha}$

$\beta = |\beta|e^{i\phi_\beta}$

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

$$|\psi\rangle = |\alpha|e^{i\phi_\alpha}|0\rangle + |\beta|e^{i\phi_\beta}|1\rangle$$

$$|\psi\rangle = \underbrace{e^{i\phi_\alpha}}_{\text{global phase}}\left[|\alpha||0\rangle + |\beta|\underbrace{e^{i(\phi_\beta - \phi_\alpha)}}_{\text{Relative phase}}|1\rangle\right]$$

global phase is insignificant. therefore, ignorable

Relative phase $\phi = \phi_\beta - \phi_\alpha$

$$|\psi\rangle = |\alpha||0\rangle + |\beta|e^{i\phi}|1\rangle$$

we know $|\alpha|^2 + |\beta|^2 = 1$

$$|\alpha| = \cos(\theta/2) \qquad |\beta| = \sin(\theta/2)$$

$$\cos^2(\theta/2) + \sin^2(\theta/2) = 1$$

So,

$$|\psi\rangle = \cos(\theta/2)|0\rangle + \sin(\theta/2)e^{i\phi}|1\rangle$$

Some examples

→ $\theta = 0$ $\quad |\psi\rangle = \cos(0)|0\rangle + \sin(0)e^{i\phi}|1\rangle = |0\rangle$

→ $\theta = \pi$ $\quad |\psi\rangle = \cos(\pi/2)|0\rangle + \sin(\pi/2)e^{i\phi}|1\rangle = e^{i\phi}|1\rangle = |1\rangle$ global phase ignorable

→ $\theta = \frac{\pi}{2}, \phi = 0$ $\quad |\psi\rangle = \cos\left(\frac{\pi}{4}\right)|0\rangle + \sin\left(\frac{\pi}{4}\right)e^{i0}|1\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} = |+\rangle$

→ $\theta = \frac{\pi}{2}, \phi = \pi$ $\quad |\psi\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}} = |-\rangle$

→ $\theta = \frac{\pi}{2}, \phi = \frac{\pi}{2}$ $\quad |\psi\rangle = \cos(\pi/4)|0\rangle + \sin(\pi/4)e^{i\pi/2}|1\rangle = \frac{|0\rangle}{\sqrt{2}} + \frac{e^{i\pi/2}|1\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle) = |i+\rangle$

→ $\theta = \frac{\pi}{2}, \phi = -\frac{\pi}{2}$ $\quad |\psi\rangle = \frac{|0\rangle + e^{-i\pi/2}|1\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle) = |i-\rangle$