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Annual intrinsic value of a company in a competitive insurance market

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ABSTRACT

In this paper we analyze a measure of the insurance company's value in an extended Lundberg model which includes the effect of competition on pricing. The extended model is designed to be an integral part of a multi-year controlled risk model of a company operating on both competitive insurance and financial markets, when insureds migrate in seeking for better rates and investors migrate in seeking for higher return on investments.

1. Introduction and rationale

Filip Lundberg introduced his model in 1903. He sought to investigate the property and liability insurance business of a firm during a period of, say, one year regarding the total risk business as an economic system developing in time, and in every instant subject to random fluctuations.[1] With Lundberg's approach,[2] "it is no longer necessary to consider each contract in the portfolio in order to determine the probability distribution of the total amount of claim payments. Instead, this distribution is built up from … two distributions which it should be possible to estimate from the records of the company".

On the one hand,[3] "this approximation of the reality … led to very remarkable results. A great part of these results, particularly with regard to the ruin theory, should have been very difficult to reach, if more realistic models had been introduced from the beginning". On the other hand,[4] "from a practical point of view, the theory of collective risk, as initiated by Filip Lundberg, has missed the point, because the underlying model is unrealistic, too simplified. For one thing, a stationary business should give stationary reserves, as predicted by the control theory". Significant shortcomings of the standard ruin theory in terms of its practical application were noted by many experts.[5]

In Malinovskii (2007, 2008a,b, 2009, 2012), the standard Lundberg model was upgraded to a dynamic multi-year model with annual controls and annual probability mechanisms of insurance. Not necessarily the same in separate years, these mechanisms can be chosen in many different ways. When they are standard Lundberg, the multi-year model's conclusions about the long-term reserves and ruin probabilities are no more preposterous but reasonable and depend on the control strategies.

The real insurance market is in a permanent disequilibrium. It comes from an imminent duality of participants' objectives, as some firms seek stable current income and other firms ignore short term income and seek growth. It is due to the very essence of this market, where a better price attracts customers, a higher return on investments attracts investors, and profits influence insurers' business strategies. The disequilibrium manifests itself in the form of underwriting cycles.

In Malinovskii (2010, 2013b), the cycles were modeled in the framework of an extended dynamic multi-year controlled

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[1] In fact, we have quoted Cramér (1976) who digested Lundberg's contribution as follows: "The net result of the risk business of an insurance company during a period of, say, one year, could be regarded as the sum of the net results of all the individual insurances. If these were supposed to be mutually independent, the connection with the classical "central limit theorem" in probability theory was evident. But there was also the possibility of regarding the total risk business as an economic system developing in time, and in every instant subject to random fluctuations. Systems of this kind had been considered in some pioneering works which today appear as forerunners of the modern theory of stochastic processes".

[2] We quote further in this sentence Borch (1967, p. 439).

[3] We quote further in this sentence Philipson (1968, p. 49).

[4] Further on, we quote H. Bohman' farewell interview as retiring Chief Editor of the Scandinavian Actuarial Journal; see Bohman (1987, p. 2). It was given at the end of his distinguished career as a member of the Swedish Probability School and, which makes it more valuable, as an experienced person from the industry.

[5] Discussing Borch (1967), C.-O. Segerdhal used the following grotesque paradox: "I should think that if a manager of an insurance company came to his board or to his policyholders and said something like this: "Gentlemen, I am running this company along lines proposed by modern economics. This means that the company will certainly go broke. The probability of ruin is equal to one. It will go broke, but I will try to postpone as long as possible the deplorable but inevitable moment when you lose your money. Or, alternatively, before that happens, I will try to make as much money as possible to distribute. I do not care what happens then", I think such a managing director would not need any *deus ex machina* to be relieved of the burden of his duties. His board would see to that immediately".

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model allowing for price competition and migration of insureds. In Malinovskii (2013b), an emphasis was put on reflexivity.[6] In brief, reflexivity means a connection between the market participants' thinking and market participants' acting. It sheds light on the concept of market price and reveals the internal causes of the competition-originated cycles.

Besides managers, policyholders, supervisors or regulators, the major party of the insurance business is shareholders owing this business and supplying the investment capital. In the long run, they are concerned with good prospects of their investments and wish business solvency and growth, but in the short run they wish profit. The insurance and financial markets are closely related,[7] and the multi-year model extended further to include in consideration migration of shareholders is even more applicable.[8]

Managers are a hired personnel whose primary job is to earn a high level of return on the investments of shareholders, i.e. their employers. To be successful, managers have to satisfy the policyholders, or business consumers, who wish to have a reliable insurance protection at a reasonable price. Managers must comply with the regulators' formal requirements which aim at the overall market stability, i.e. at prevention of insolvencies and destructive competition among the companies on the market.

Seeking to maintain harmony within the company, insurance managers cannot opt out of competition on both insurance and financial markets. On the former, they have to compete for customers' money. On the latter, they have to withstand the competition for investors' money. The primary tool to attract new and to retain old investors is regularly reporting the company's per-share book value.[9]

The book value is an oversimplified indicator of a limited worth. More elaborated measures are needed to assess the real company's position. An all-important concept is the company's intrinsic value which is defined as the discounted value of the cash that can be taken out of a business during its remaining life.[10] Its assessment is not simple. It is a subjective estimate[11] rather than a precise figure which varies over time and under different scenario assumptions. Bearing in mind the reflexivity insight into the causes of cycles, the publication of the company's intrinsic value exercises influence on the market as well as the company's price.

The idea that the firm's intrinsic value must depend on the profitability of its annual underwriting operations and on its future lifetime related to solvency is plain but very difficult to implement in a model which integrates all cash flows over successive account years.[12] Suffice is to say that for a long time the firm's investment operations may be much more profitable than just underwriting. However, it is unlikely that any dividends will be paid out in a particular year if the firm's total annual earnings fall below the value of compulsory reserves needed to have an adequate level of solvency over the next year.

Aiming to translate these considerations into a quantitative analysis, we have to explore the multi-year models with emphasis on rational competitive behavior. In this paper, which is a step in this direction, we concentrate on Lundberg annual probability mechanisms of insurance which are the integral part of the multi-year models. We assume that throughout a single insurance year the market is homogeneous, i.e., claim sizes are identically distributed over the market and over time. Modeling the company's annual intrinsic value, we focus on the ratio of expected profit to capital needed to maintain a prescribed level of solvency. The main technical difficulty is that this capital is a complex function of the company's price. We overcome this difficulty by applying the complex analytical results of Malinovskii (2013a, in preparation).

The rest of the paper is arranged as follows.

In Section 2, following Malinovskii (2010, 2013b,c), we extend the standard Lundberg model to the model with migration of insureds. We pick up a family of simple but rational migration rate functions and discuss its merits. We introduce the ratio called the annual intrinsic value.

In Section 3, we address the asymptotic bounds[13] for α-level initial capital, called for brevity ruin capital. It is derived in Malinovskii (2013a, in preparation) in the framework of the standard Lundberg model, and we extend it to the model with migration of insureds.

In Section 4, we obtain the asymptotic lower bounds for the company's annual intrinsic value. We focus on the case of a firm with a steady position on the profitable market. It means that the interval of policyholders' migration sensitivity that contains the market price lies entirely to the right of the marginal cost of insurance. It follows that the most aggressive company's pricing behavior, i.e. the choice of the lowest price within the interval of migration sensitivity, still provides profitability of the company's business. Planning to apply in the future these results to different years within profitable phase of the insurance cycle, we consider wider and narrower intervals of migration sensitivity.

In Section 5, we compare the company's annual intrinsic value with a simplified measure of this type. The simplification consists in using a constant baseline capital rather than the ruin capital which is a function of varying insurer's price.

Section 6 contains the main technicalities.

2. Migration of insureds and annual risk model

Migration rate functions were introduced in Malinovskii (2010) and discussed further on in Malinovskii (2013b,c), where they are defined as follows.

6 Reflexivity is put forth by Soros (1994). He called life a fertile fallacy and accentuated connections between the participants' thinking and the events in which they participate. Emphasizing links between ideas and material conditions, he claimed that "neither markets nor elections constitute an objective criterion, only an expression of the prevailing bias; but that is the best available in an imperfect world".

7 On the insurance market, the risks are pooled and premiums are collected. On the financial stock market, additional funds are raised through the sale of shares to investors. Both sources are paramount. Note only that the company is legally obliged to have sufficient funds to be admitted to the market. Being business owners, the investors bear all the risk of loss of their investments in the case of bankruptcy.

8 Being a fertile idealization, an extended Lundberg model may be called applicable. Seldom used by managing directors in their daily practice, it is indispensable for training and for assessing complex but intuitive considerations. To be particular, it was used for a rational insight into managers' self-inflicting pricing conduct (see Malinovskii, 2013c), into the difference between rate cuts of large and small firms (see Malinovskii, submitted for publication-a), into distinctions between perils which face large and small firms at or near underwriting cycle's nadir (see Malinovskii, submitted for publication-b).

9 Quoting a message from Warren E. Buffett dated 1996, "we regularly report our per-share book value, an easily calculable number, though one of limited use".

10 We have quoted Buffett further on: "intrinsic value is an all-important concept that offers the only logical approach to evaluating the relative attractiveness of investments and businesses. Intrinsic value can be defined simply: it is the discounted value of the cash that can be taken out of a business during its remaining life".

11 Quoting Buffett once more, "that is one reason we never give you our estimates of intrinsic value. What our annual reports do supply, though, are the facts that we ourselves use to calculate this value".

12 See, e.g., Eq. (1.1.1) in Daykin et al. (1996) which relies on another extension of the Lundberg model.

13 By asymptotic bounds we mean the relations like $x_t \leqslant y_t\,(1+\bar{o}(1))$, as $t \to \infty$. It means that for any $\epsilon > 1$ there exists $t(\epsilon)$ so large that $x_t \leqslant \epsilon y_t$ for all $t > t(\epsilon)$.

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Definition 1 (*Migration Rate Functions*). Consider the family of functions $r_s(d, \mathfrak{z})$ of three non-negative variables: time s, migration factor $\mathfrak{z}$ and ratio d of insurer's price P to market price P_M. We call them migration rate functions if they satisfy the following. They must be identically unit in any of these three cases: $s = 0$, or $d = 1$, or $\mathfrak{z} = 0$. Moreover, for $s > 0$ and $d > 0$ fixed, the function $r_s(d, \mathfrak{z})$ of the variable $\mathfrak{z} > 0$

- being unit, as $\mathfrak{z} = 0$, is monotone increasing, as $\mathfrak{z}$ increases, and is bounded from above by c_{U}, if $0 < d < 1$,
- being unit, as $\mathfrak{z} = 0$, is monotone decreasing, as $\mathfrak{z}$ increases, and is bounded from below by c_{L}, if $d > 1$.

For $s > 0$ and $\mathfrak{z} > 0$ fixed, the function $r_s(d, \mathfrak{z})$ of the variable $d > 0$

- is monotone decreasing, as d decreases, being bounded from above by c_{U} and bounded from below by c_{L}, and passes through one, as $d = 1$.

For $\mathfrak{z} > 0$ and $d > 0$ fixed, the function $r_s(d, \mathfrak{z})$ of the variable $s > 0$

- being unit, as $s = 0$, is monotone increasing, as s increases, and is bounded from above by $r(d, \mathfrak{z}) = \lim_{s\to+\infty} r_s(d, \mathfrak{z}) \leqslant \mathsf{c}_{\mathsf{U}}$, if $0 < d < 1$,
- being unit, as $s = 0$, is monotone decreasing, as s increases, and is bounded from below by $r(d, \mathfrak{z}) = \lim_{s\to+\infty} r_s(d, \mathfrak{z}) \geqslant \mathsf{c}_{\mathsf{L}}$, if $d > 1$.

We call ultimate migration rate functions the family $r(d, \mathfrak{z}) = \lim_{s\to+\infty} r_s(d, \mathfrak{z})$.

Without a significant loss of generality, we assume also that

$$\lim_{\mathfrak{z}\to\infty} r(d, \mathfrak{z}) = \mathsf{U}(d), \quad d > 0,$$

where $\mathsf{U}(d)$ is a step function equal to c_{U}, as $0 < d < 1$, to 1, as $d = 1$, and to c_{L}, as $d > 1$.

Migration rate functions are an idealization of the actual behavior of policyholders immigrating in the firms with better rates and emigration from the firms with inferior rates. This idealization is based on some rational considerations. In particular, these include the surveys of policyholders.[14]

Rationally chosen migration rate functions differ for small and large firms (see Malinovskii, submitted for publication-a) and for different stages of the underwriting cycles (see Malinovskii, submitted for publication-b). Some observations which underlie the rational choice of migration rate functions are given in Section 2.1 of Malinovskii (2013c).

Seeking for mathematical convenience, migration rate functions may be defined through the standard distributions of mathematical statistics. The well-known Beta and Gamma distributions are used for that in Sections 2.2 and 2.3 of Malinovskii (2013c).

Since in this paper we confine ourselves to the probability mechanisms of insurance within a single year, we assume that migration factor $\mathfrak{z}$ is fixed. So, seeking for brevity, we omit $\mathfrak{z}$ in our further notation[15] and switch from the argument $d = P/P_M$ to P. This is fair because we focus on migration rate functions as functions of price P, assuming the market price P_M known and fixed throughout the whole insurance year.

In the same way as in Section 2.3 of Malinovskii (2013c), we focus on the annual migration rate functions of the form

$$r_t(P) = r(P) + (1 - r(P))\,\varsigma_t = r(P)(1 - \varsigma_t) + \varsigma_t, \quad t \geqslant 0, \quad (2.1)$$

where ς_t with $\varsigma_0 = 1$ refers to the speed of migration, as time t goes on. We assume that ς_t is monotone decreasing to zero, as $t \to \infty$, and $\int_0^t \varsigma_s ds = \bar{o}(t)$, as $t \to \infty$. When $\varsigma_t = e^{-kt}$ for some $k > 0$, we speak about exponential speed of migration. When $\varsigma_t = (1 + t)^{-k}$ for some $k > 0$, we speak about power speed of migration.

Let the ultimate migration rate functions $r(P)$ in Eq. (2.1) be defined as follows. Having the upper and lower limits $0 < \mathsf{c}_{\mathsf{L}} < 1 < \mathsf{c}_{\mathsf{U}} < \infty$ called upper and lower migration capacities, we focus on the finite interval $[\mathsf{p}_{\mathsf{U}}, \mathsf{p}_{\mathsf{L}}]$ of policyholders' migration sensitivity, or clients' readiness to migration, taking $0 < \mathsf{p}_{\mathsf{U}} < P_M < \mathsf{p}_{\mathsf{L}} < \infty$. The ultimate migration rate functions are

$$r(P) = \begin{cases} \mathsf{c}_{\mathsf{U}}, & 0 \leqslant P \leqslant \mathsf{p}_{\mathsf{U}}, \\ 1 + (\mathsf{c}_{\mathsf{U}} - 1)\, F_l\left(\dfrac{P_M - P}{P_M - \mathsf{p}_{\mathsf{U}}}\right), & \mathsf{p}_{\mathsf{U}} \leqslant P \leqslant P_M, \\ 1 - (1 - \mathsf{c}_{\mathsf{L}})\, F_{l'}\left(\dfrac{P - P_M}{\mathsf{p}_{\mathsf{L}} - P_M}\right), & P_M \leqslant P \leqslant \mathsf{p}_{\mathsf{L}}, \\ \mathsf{c}_{\mathsf{L}}, & P \geqslant \mathsf{p}_{\mathsf{L}}, \end{cases} \quad (2.2)$$

where the functions $F_l(x)$, $l > 0$, define the shape of $r(P)$. Suggestive is to take them c.d.f. on $[0, 1]$, so that $F_l(0) = 0$, $F_l(1) = 1$ and $F_{l_1}(x) \leqslant F_{l_2}(x)$ for all $x \in [0, 1]$, as $l_1 < l_2$. The particular case of this choice is

$$F_l(x) = \begin{cases} 0, & x < 0, \\ x^l, & 0 \leqslant x \leqslant 1, \\ 1, & x > 1, \end{cases} \quad (2.3)$$

which yields[16]

$$r(P) = \begin{cases} \mathsf{c}_{\mathsf{U}}, & 0 \leqslant P \leqslant \mathsf{p}_{\mathsf{U}}, \\ 1 + (\mathsf{c}_{\mathsf{U}} - 1)\left(\dfrac{P_M - P}{P_M - \mathsf{p}_{\mathsf{U}}}\right)^{l}, & \mathsf{p}_{\mathsf{U}} \leqslant P \leqslant P_M, \\ 1 - (1 - \mathsf{c}_{\mathsf{L}})\left(\dfrac{P - P_M}{\mathsf{p}_{\mathsf{L}} - P_M}\right)^{l'}, & P_M \leqslant P \leqslant \mathsf{p}_{\mathsf{L}}, \\ \mathsf{c}_{\mathsf{L}}, & P \geqslant \mathsf{p}_{\mathsf{L}}. \end{cases} \quad (2.4)$$

By cumulative migration rate function we mean

$$\Upsilon_t(P) = \int_0^t r_s(P)ds = \mathrm{t}r(P) + (1 - r(P)) \int_0^t \varsigma_s ds. \quad (2.5)$$

It is noteworthy that, being simple enough, migration rate functions (2.1)–(2.2) embrace all of the major components of a rational choice: migration capacity, migration sensitivity and shapes diversity within the interval of policyholders' migration sensitivity. Plainly, migration rate functions (2.2) may be asymmetric with respect to the point P_M, where all of them are one. More discussion of the choice of migration rate functions contains in Section 2.2 of Malinovskii (2013c).

[14] Quote from Subramanian (1998, p. 39): "Surveys of policyholders have consistently demonstrated some reluctance to switch insurers. In a survey of 2462 policyholders by Cummins et al. (1974), 54% of respondents confessed never to have shopped around for auto insurance prices. To the question "Which is the most important factor in your decision to buy insurance?", 40% responded the company, 29% the agent, and only 27% the premium. A similar survey of 2004 Germans (see Schlesinger and von der Schulenburg, 1993) indicated that, despite the fact that 67% of those responding knew that considerable price differences exist between automobile insurers, only 35% chose their carrier on the basis of their favorable premium. Therefore, we will assume that, given the opportunity to switch for a reduced premium, one-third of the policyholders will do so".

[15] Dependence of migration rate functions on migration factor $\mathfrak{z}$ is essential in multi-year models. A variation from year to year of the migration factor is named among the main causes of the underwriting cycles in Malinovskii (2013b), because it influences the prevailing bias of individual insurers.

[16] See illustrations for $l = l' = 1$, $l = l' = 0.3$, and $l = l' = 3.3$ in Figs. 5–8.

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Assume that the annual probability mechanism of insurance is modeled by the annual risk reserve process[17]

$$\mathcal{R}_s(u,\lambda,P) = u + P\lambda\Upsilon_s(P) - \mathcal{V}_s,\quad 0 < s \leqslant t, \tag{2.6}$$

where $t > 0$ is the length of the insurance year, $u > 0$ is the capital at time 0 and $\lambda > 0$ is the initial portfolio size, the cumulative premium amount at time t is $P\lambda\Upsilon_t(P)$, the claims payout process is

$$\mathcal{V}_s = \sum_{i=1}^{\mathcal{N}_s} Y_i,\quad 0 < s \leqslant t, \tag{2.7}$$

or 0, if $\mathcal{N}_s = 0$, where Y_i, $i = 1, 2, \dots$, are claim amounts i.i.d. and exponential with parameter $1/\mathsf{E}Y$, and

$$\mathcal{N}_s,\quad 0 < s \leqslant t, \tag{2.8}$$

is the inhomogeneous Poisson claims arrival process with intensity $\lambda\Upsilon_s(P)$, independent of the claims Y_i, $i = 1, 2, \dots$.

The intensity of the Poisson process is varying with time $0 < s \leqslant t$ and depends on price P through the cumulative migration rate function $\Upsilon_s(P)$ only. For P fixed, it models a growth or decrease of a unit of portfolio's volume with time, while λ stays for the number of such units at the beginning of the year, as time is zero. Greater or lesser price intensity P means selling policies more expensive or cheaper throughout the year. Recall that it is selected at the beginning of the year and after that it remains fixed all the year long. Insured's migration, which is their reaction on selection of this price, affects the portfolio size and, indirectly, both annual premiums and claims arrival processes.

In the standard Lundberg model, by time we mean operational[18] rather than calendar time. In our case, calling t the length of the insurance year, we mean a ball-park figure of financial transactions rather than, e.g., the end of December. Assuming t large, we focus on the firms with large annual volume of transactions, or large portfolio size.

The role of the size parameter[19] λ becomes visible when we look at the difference between larger and smaller companies,[20] as in Malinovskii (submitted for publication-b), or when we investigate dynamic multi-year models, concentrating on the variation of portfolio size from year to year. In this paper, dealing with one-year model, we treat λ merely as a dummy parameter which will be useful for future research.

By the probability of ruin within time t we call

$$\boldsymbol{\psi}_t(u,\lambda,P) = \mathsf{P}\Big\{\inf_{0<s\leqslant t}\mathcal{R}_s(u,\lambda,P) < 0\Big\}. \tag{2.9}$$

We highlight $u_{\alpha,t}(P)$, the solution (w.r.t. u) of the equation

$$\boldsymbol{\psi}_t(u,\lambda,P) = \alpha.$$

We call it α-level ($0 < \alpha < \frac{1}{2}$) initial capital, or ruin capital. We call the ratio

$$\begin{aligned}\mathcal{E}_{\alpha,t}(P) &= \max\left\{0,\ \frac{\mathsf{E}\mathcal{R}_t(u,\lambda,P)|_{u=u_{\alpha,t}(P)}}{u_{\alpha,t}(P)}\right\}\\ &= \max\left\{0,\ 1 + \frac{P\lambda\Upsilon_t(P) - \mathsf{E}\mathcal{V}_t}{u_{\alpha,t}(P)}\right\}\end{aligned} \tag{2.10}$$

company's intrinsic value. Hereinafter, for the level $0 < \alpha < \frac{1}{2}$ and for the standard normal c.d.f. $\Phi_{(0,1)}(x)$, we use the notation $\kappa_{\alpha/2} = \Phi^{-1}_{(0,1)}(1-\alpha/2)$ and $\kappa_\alpha = \Phi^{-1}_{(0,1)}(1-\alpha)$.

3. Asymptotic bounds for α-level initial capital

For $P > 0$, we put $u^*_{\alpha,t}(P) = \mathsf{E}Y\sqrt{2\lambda\Upsilon_t(P)}\,\kappa_{\alpha/2}(1+\bar{o}(1))$, $t \to \infty$, and for $P \geqslant \mathsf{E}Y$ introduce

$$P^*_{\alpha,t}(P) = \mathsf{E}Y\left(\frac{u^*_{\alpha,t}(P) - \mathsf{E}Y\ln\alpha}{u^*_{\alpha,t}(P) + \mathsf{E}Y\ln\alpha}\right).$$

We note[21] that $P^*_{\alpha,t}(P) > \mathsf{E}Y$.

Theorem 3.1. *In the assumptions of the model (2.6)–(2.8) with migration rate functions (2.1)–(2.2), for t sufficiently large, we have $u_{\alpha,t}(P) \leqslant \overline{u}_{\alpha,t}(P)$ for $P > 0$, where*

$$\overline{u}_{\alpha,t}(P) = \begin{cases} \lambda\Upsilon_t(P)(\mathsf{E}Y - P) + u^*_{\alpha,t}(P), & 0 < P \leqslant \mathsf{E}Y,\\ \dfrac{(u^*_{\alpha,t}(P) + \mathsf{E}Y\ln\alpha)^2}{4(\mathsf{E}Y)^2\ln\alpha}P \\ \quad - \dfrac{(u^*_{\alpha,t}(P) - \mathsf{E}Y\ln\alpha)^2}{4(\mathsf{E}Y)^2\ln\alpha}\mathsf{E}Y, \\ \mathsf{E}Y < P \leqslant P^*_{\alpha,t}(P), \\ -\dfrac{P\,\mathsf{E}Y\ln\alpha}{P - \mathsf{E}Y}, & P \geqslant P^*_{\alpha,t}(P), \end{cases} \tag{3.1}$$

and $u_{\alpha,t}(P) \geqslant \underline{u}_{\alpha,t}(P)$ for $0 < P \leqslant \mathsf{E}Y$, where

$$\underline{u}_{\alpha,t}(P) = \lambda\Upsilon_t(P)(\mathsf{E}Y - P) + \frac{\kappa_\alpha}{\kappa_{\alpha/2}}u^*_{\alpha,t}(P). \tag{3.2}$$

It is noteworthy that up to the terms of a smaller order, as $t \to \infty$, the upper bound considered as a function of increasing P is

- linear with negative slope $-\mathsf{c}_\mathsf{U}t\lambda$ for $0 < P \leqslant \min\{\mathsf{p}_\mathsf{U}, \mathsf{E}Y\}$,
- monotone decreasing for $\min\{\mathsf{p}_\mathsf{U}, \mathsf{E}Y\} < P \leqslant \min\{\mathsf{p}_\mathsf{L}, \mathsf{E}Y\}$,
- linear with negative slope $-\mathsf{c}_\mathsf{L}t\lambda$ for $\min\{\mathsf{p}_\mathsf{L}, \mathsf{E}Y\} < P \leqslant \mathsf{E}Y$,
- monotone decreasing for $\mathsf{E}Y < P < \max\{P > \mathsf{E}Y : P \geqslant P^*_{\alpha,t}(P)\}$,
- monotone decreasing[22] and *independent* on t and λ, i.e. on the portfolio size, for $P > \max\{P > \mathsf{E}Y : P \geqslant P^*_{\alpha,t}(P)\}$.

Up to the terms of a smaller order, as $t \to \infty$, the lower bound considered as a function of increasing P is

- linear with negative slope $-\mathsf{c}_\mathsf{U}t\lambda$ for $0 < P \leqslant \min\{\mathsf{p}_\mathsf{U}, \mathsf{E}Y\}$,
- monotone decreasing for $\min\{\mathsf{p}_\mathsf{U}, \mathsf{E}Y\} < P \leqslant \min\{\mathsf{p}_\mathsf{L}, \mathsf{E}Y\}$,
- linear with negative slope $-\mathsf{c}_\mathsf{L}t\lambda$ for $\min\{\mathsf{p}_\mathsf{L}, \mathsf{E}Y\} < P \leqslant \mathsf{E}Y$.

In Figs. 1 and 2, the graph of $u_{\alpha,t}(P)$ calculated numerically using Theorem 6.1 and the graph of its upper bound $\overline{u}_{\alpha,t}(P)$ defined in Theorem 3.1 are shown as functions of P, for $P > \mathsf{E}Y$.

In Figs. 3 and 4, the graphs of $P^*_{\alpha,t}(P)$ and of the linear function P used in Eq. (3.1) for definition of $\overline{u}_{\alpha,t}(P)$ are shown as functions of P, for $P > \mathsf{E}Y$. Here $r_t(P)$ is defined in (2.1) and (2.4), with $\mathsf{E}Y < \mathsf{p}_\mathsf{U}$ and $l = l' = 1$ and with wider and narrower intervals $[\mathsf{p}_\mathsf{U}, \mathsf{p}_\mathsf{L}]$ of

17 Having Eq. (2.6) written, we see that P is the premium income per unit of time and per unit of portfolio volume. Though it would be more correct to call it premium rate or price rate, we continue to call it insurer's price.

18 The unit of measurement of this variable is money.

19 This value is dimensionless. Roughly speaking, it counts the number of units which consist of the portfolio at the beginning of the year.

20 Throughout this paper, we focus on the asymptotics, as $t \to \infty$. Doing that, we assume that the company under consideration is large. Speaking further about large and small companies and using the asymptotics, we mean actually that both are large, but one is, e.g., ten times larger than another.

21 To check that $P^*_{\alpha,t}(P) > \mathsf{E}Y$, note that $0 < \alpha < 1/2$.

22 Being defined as $-P\,\mathsf{E}Y\ln\alpha/(P - \mathsf{E}Y)$, this function is decreasing to $-\mathsf{E}Y\ln\alpha > 0$, as $P \to \infty$. The non-zero limit, small for small $\alpha > 0$, arises from using (see Section 5) fundamental auxiliary result which is a coarsened, but explicitly written bound of Theorem 2.5 in Malinovskii (in preparation) rather than more accurate, but using a solution of a non-linear equation, bound of Theorem 2.4 in Malinovskii (in preparation).

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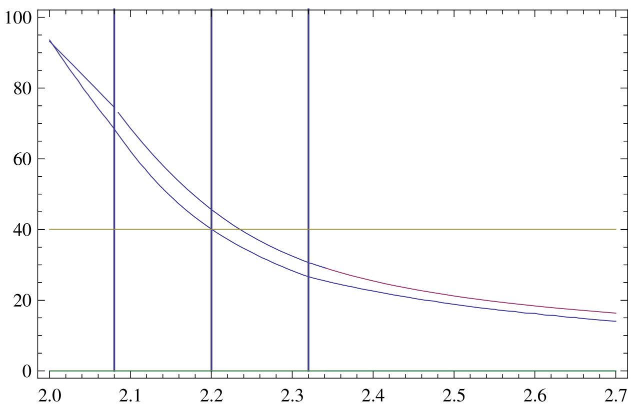

Fig. 1. Graphs of $u_{\alpha,t}(P)$ calculated numerically, using Theorem 6.1, and its upper bound $\overline{u}_{\alpha,t}(P)$, as functions of P (X-axis) with $\alpha = 0.12$, $\lambda = 1$, $\mathsf{E}Y = 2.0$, $\mathsf{p_U} = 2.08$, $P_M = 2.2$, $\mathsf{p_L} = 2.32$, $\mathsf{c_U} = 1.5$, $\mathsf{c_L} = 0.5$, $t = 300$. Horizontal line: $u_{\alpha,t}(P_M) = 40.1134$. Note that $\overline{u}_{\alpha,t}(P_M) = 45.6673$.

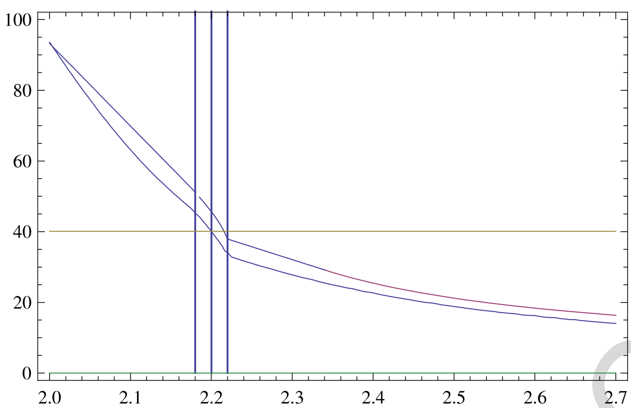

Fig. 2. Graphs as in Fig. 1, but with narrower interval of policyholders' migration sensitivity: $\mathsf{p_U} = 2.18$, $\mathsf{p_L} = 2.22$.

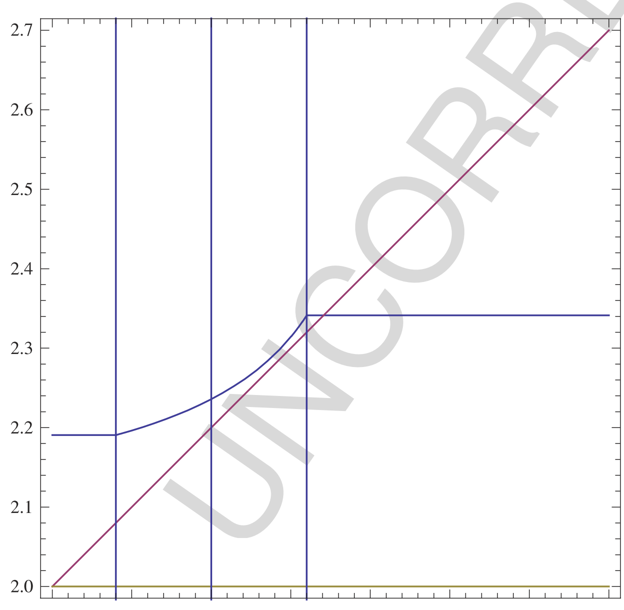

Fig. 3. Graphs of $P^*_{\alpha,t}(P)$ and P, as functions of P (X-axis). Parameters are set as in Fig. 1.

policyholders' migration sensitivity. The speed of migration over time in $r_t(P)$ is taken exponential, i.e. $\varsigma_t = e^{-t}$.

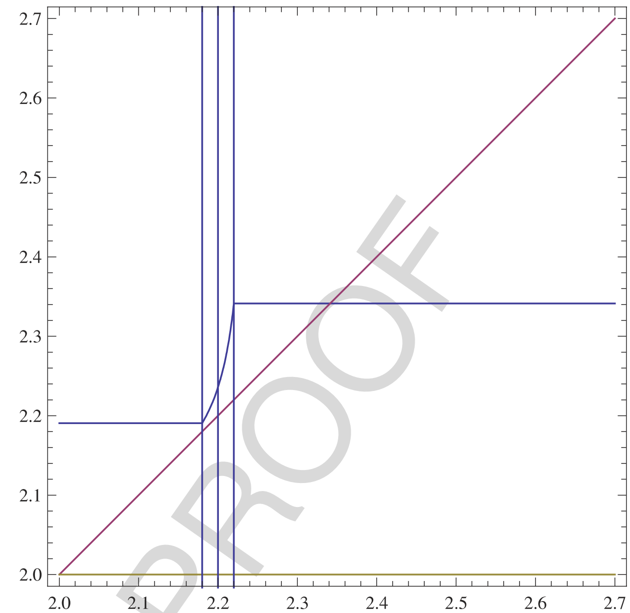

Fig. 4. Graphs as in Fig. 3, but with narrower interval of policyholders' migration sensitivity: $\mathsf{p_U} = 2.18$, $\mathsf{p_L} = 2.22$.

4. Bounds for a firm holding a steady position on profitable market

In this section, we address lower, cautious bounds for the company's annual intrinsic value. We focus on the profitable market (i.e., $\mathsf{E}Y > P_M$) and on the company which makes a decision about price P within the interval $[\mathsf{p_U}, \mathsf{p_L}]$ of policyholders' migration sensitivity, holding a steady market position. This means that $\mathsf{E}Y < \mathsf{p_U}$.

For $t \to \infty$, we introduce

$$\underline{\mathcal{E}}_{\alpha,t}(P) = \begin{cases} \max\left\{0,\, 1 + \dfrac{(P-\mathsf{E}Y)\lambda\varUpsilon_t(P)}{\lambda\varUpsilon_t(P)(\mathsf{E}Y-P) + \frac{\kappa_\alpha}{\kappa_{\alpha/2}}u^*_{\alpha,t}(P)}\right\}, \\ \quad 0 < P \leqslant \mathsf{E}Y, \\ 1 + \dfrac{4(\mathsf{E}Y)^2 \ln\alpha\,(P-\mathsf{E}Y)\lambda\varUpsilon_t(P)}{(u^*_{\alpha,t}(P) + \mathsf{E}Y\ln\alpha)^2 P - (u^*_{\alpha,t}(P) - \mathsf{E}Y\ln\alpha)^2\mathsf{E}Y}, \\ \quad \mathsf{E}Y < P \leqslant P^*_{\alpha,t}(P), \\ 1 - \dfrac{(P-\mathsf{E}Y)^2\lambda\varUpsilon_t(P)}{P\mathsf{E}Y\ln\alpha}, \quad P > P^*_{\alpha,t}(P). \end{cases}$$

Theorem 4.1. *Assume that $\mathsf{E}Y < \mathsf{p_U} < P_M$ in the model (2.6)–(2.8) with migration rate functions (2.1)–(2.2). Then $\underline{\mathcal{E}}_{\alpha,t}(P) \leqslant \mathcal{E}_{\alpha,t}(P)$ for $P > 0$, as t is sufficiently large.*

Proof of Theorem 4.1. Bearing in mind that $\mathsf{E}\mathcal{V}_t = \mathsf{E}Y\lambda\varUpsilon_t(P)$, rewrite (2.10) as

$$\mathcal{E}_{\alpha,t}(P) = \max\left\{0,\, 1 + \frac{(P-\mathsf{E}Y)\lambda\varUpsilon_t(P)}{u_{\alpha,t}(P)}\right\}.$$

Since $\varUpsilon_t(P)$ is positive for all $P > 0$ and the difference $P - \mathsf{E}Y$ changes the sign in the point $P = \mathsf{E}Y$, we have from Theorem 3.1 that $\underline{\mathcal{E}}_{\alpha,t}(P) \leqslant \mathcal{E}_{\alpha,t}(P)$ for $P > 0$, where

$$\underline{\mathcal{E}}_{\alpha,t}(P) = \begin{cases} \max\left\{0,\, 1 + \dfrac{(P-\mathsf{E}Y)\lambda\varUpsilon_t(P)}{\underline{u}_{\alpha,t}(P)}\right\}, & 0 < P \leqslant \mathsf{E}Y, \\ 1 + \dfrac{(P-\mathsf{E}Y)\lambda\varUpsilon_t(P)}{\overline{u}_{\alpha,t}(P)}, & P > \mathsf{E}Y. \end{cases}$$

The proof is easily completed by using (3.1) and (3.2). $\square$

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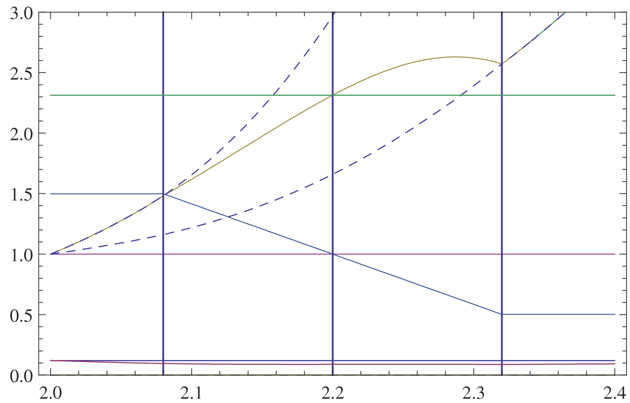

Fig. 5. Graphs of $\underline{\mathcal{E}}_{\alpha,t}(P)$ (upper solid line), $r_t(P)$ (middle solid line) defined in (2.1) and (2.4) with $l = l' = 1$, and $\boldsymbol{\psi}_t(u, \lambda, P)$ with $u = \overline{u}_{\alpha,t}(P)$ which lies below $\alpha = 0.12$ uniformly in P. Dashed lines: $\underline{\mathcal{E}}^{\mathsf{U}}_{\alpha,t}(P)$ and $\underline{\mathcal{E}}^{\mathsf{L}}_{\alpha,t}(P)$. Parameters are set as in Fig. 1, i.e. $\lambda = 1$, $\mathsf{E}Y = 2.0$, $\mathsf{p_U} = 2.08$, $P_M = 2.2$, $\mathsf{p_L} = 2.32$, $\mathsf{c_U} = 1.5$, $\mathsf{c_L} = 0.5$, $t = 300$. Upper horizontal line: $\underline{\mathcal{E}}_{\alpha,t}(P_M) = 2.3139$.

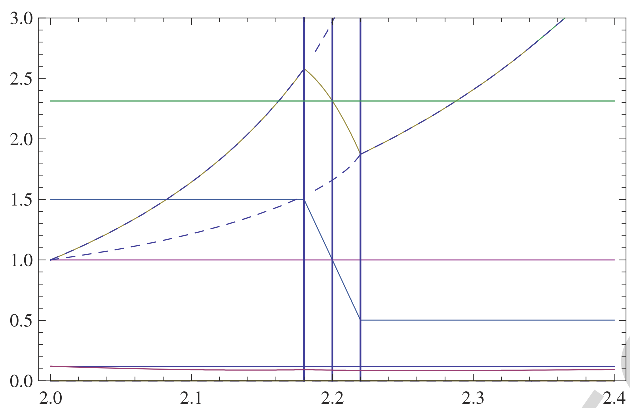

Fig. 6. Graphs as in Fig. 5, but with narrower interval of migration sensitivity: $\mathsf{p_U} = 2.18$, $\mathsf{p_L} = 2.22$.

Introduce

$$\underline{\mathcal{E}}^{\mathsf{U}}_{\alpha,t}(P) = \begin{cases} 1 + \dfrac{4(\mathsf{E}Y)^2 \ln\alpha\,(P - \mathsf{E}Y)\lambda \mathsf{c_U} t}{(u^*_{\alpha,t}(P) + \mathsf{E}Y \ln\alpha)^2 P - (u^*_{\alpha,t}(P) - \mathsf{E}Y\ln\alpha)^2 \mathsf{E}Y}, \\ \quad \mathsf{E}Y < P \leqslant P^*_{\alpha,t}(P), \\ 1 - \dfrac{(P - \mathsf{E}Y)^2 \lambda \mathsf{c_U} t}{P\mathsf{E}Y \ln\alpha}, \quad P > P^*_{\alpha,t}(P), \end{cases}$$

and

$$\underline{\mathcal{E}}^{\mathsf{L}}_{\alpha,t}(P) = \begin{cases} 1 + \dfrac{4(\mathsf{E}Y)^2 \ln\alpha\,(P - \mathsf{E}Y)\lambda \mathsf{c_L} t}{(u^*_{\alpha,t}(P) + \mathsf{E}Y \ln\alpha)^2 P - (u^*_{\alpha,t}(P) - \mathsf{E}Y\ln\alpha)^2 \mathsf{E}Y}, \\ \quad \mathsf{E}Y < P \leqslant P^*_{\alpha,t}(P), \\ 1 - \dfrac{(P - \mathsf{E}Y)^2 \lambda \mathsf{c_L} t}{P\mathsf{E}Y \ln\alpha}, \quad P > P^*_{\alpha,t}(P). \end{cases}$$

It is noteworthy that for $P > \mathsf{E}Y$ and for t sufficiently large

$$\underline{\mathcal{E}}^{\mathsf{L}}_{\alpha,t}(P) \leqslant \underline{\mathcal{E}}_{\alpha,t}(P) \leqslant \underline{\mathcal{E}}^{\mathsf{U}}_{\alpha,t}(P).$$

Introduce additionally

$$\underline{\mathcal{E}}^{\mathsf{M}}_{\alpha,t}(P) = \begin{cases} 1 + \dfrac{4(\mathsf{E}Y)^2 \ln\alpha\,(P - \mathsf{E}Y)\lambda t}{(u^*_{\alpha,t}(P) + \mathsf{E}Y \ln\alpha)^2 P - (u^*_{\alpha,t}(P) - \mathsf{E}Y\ln\alpha)^2 \mathsf{E}Y}, \\ \quad \mathsf{E}Y < P \leqslant P^*_{\alpha,t}(P), \\ 1 - \dfrac{(P - \mathsf{E}Y)^2 \lambda t}{P\mathsf{E}Y \ln\alpha}, \quad P > P^*_{\alpha,t}(P). \end{cases}$$

It is the middle line, for which $\underline{\mathcal{E}}^{\mathsf{L}}_{\alpha,t}(P) \leqslant \underline{\mathcal{E}}^{\mathsf{M}}_{\alpha,t}(P) \leqslant \underline{\mathcal{E}}^{\mathsf{U}}_{\alpha,t}(P)$ for $P > \mathsf{E}Y$ and for t sufficiently large. It is easily seen that $\underline{\mathcal{E}}_{\alpha,t}(P_M) = \underline{\mathcal{E}}^{\mathsf{M}}_{\alpha,t}(P_M)$. Moreover, for t large

$$\underline{\mathcal{E}}_{\alpha,t}(P) = \begin{cases} \underline{\mathcal{E}}^{\mathsf{U}}_{\alpha,t}(P), & 0 < P < \mathsf{p_U}, \\ \underline{\mathcal{E}}^{\mathsf{L}}_{\alpha,t}(P), & P > \mathsf{p_L} \end{cases}$$

and $\mathcal{E}_{\alpha,t}(\mathsf{E}Y) = \underline{\mathcal{E}}_{\alpha,t}(\mathsf{E}Y) = \underline{\mathcal{E}}^{\mathsf{M}}_{\alpha,t}(\mathsf{E}Y) = \underline{\mathcal{E}}^{\mathsf{L}}_{\alpha,t}(\mathsf{E}Y) = \underline{\mathcal{E}}^{\mathsf{U}}_{\alpha,t}(\mathsf{E}Y) = 1$. The marginal cost of insurance $\mathsf{E}Y$ is an equilibrium point, where the annual revenues are equal to the annual reserves needed to ensure the company's non-ruin with a given probability.

In Figs. 5 and 6, the lower bounds $\underline{\mathcal{E}}_{\alpha,t}(P)$ for the annual intrinsic value $\mathcal{E}_{\alpha,t}(P)$ are shown as functions of P, for $P > \mathsf{E}Y$. The graph of $\underline{\mathcal{E}}_{\alpha,t}(P)$ lies between the dashed lines which are the graphs of $\underline{\mathcal{E}}^{\mathsf{U}}_{\alpha,t}(P)$ and $\underline{\mathcal{E}}^{\mathsf{L}}_{\alpha,t}(P)$. The lower solid lines show the level $\alpha = 0.12$ and the probability of ruin $\boldsymbol{\psi}_t(u, \lambda, P)$ where $\overline{u}_{\alpha,t}(P)$ is put instead of u. The graph of this probability lies below the level α, but close to it, uniformly in P. For convenience, we draw also the graph of migration rate function $r_t(P)$.

The case of a narrow interval $[\mathsf{p_U}, \mathsf{p_L}]$ which illustrates high policyholders' price sensitivity is shown in Fig. 5. In this case, immigration or emigration up to the maximal or minimal firm's capacity is triggered by rather insignificant moves of its prices. It may happen when the company is incumbent, and its actions are quickly noticed, or when the market is thirsty for migration.

The case of a wide interval $[\mathsf{p_U}, \mathsf{p_L}]$ which implies low policyholders' price sensitivity is shown in Fig. 6. It may be the case of a small newcomer who cuts prices aggressively for several years in a row, but its actions may not be noticed, responses of rivals may be delayed. Such newcomers may increase their market share rapidly, and their revenues climb, but it all ends with ruin if the annual incomes lag behind the growth of the annual initial capitals needed to maintain solvency. This study is continued in Malinovskii (submitted for publication-a,b), where more interpretation in terms of dynamics of the competitive insurance process is given.

Graphs in Figs. 7 and 8 correspond to $r_t(P)$ defined in (2.1) and (2.4) with a non-linear shape within the interval $[\mathsf{p_U}, \mathsf{p_L}]$. Fig. 7 corresponds to the case $l = l' = 0.3$. Fig. 8 corresponds to the case $l = l' = 3.3$. The speed of migration over time in $r_t(P)$ is always taken exponential, i.e. $\varsigma_t = e^{-t}$.

5. Deficiencies of a simplified measure of the company's value

A simplified measure of the company's value[23] which naturally comes to mind is

$$\begin{aligned} E_{\alpha,t}(P) &= \max\left\{0,\ \frac{\mathsf{E}\mathcal{R}_t(u,\lambda,P)|_{u=u_{\alpha,t}(P_M)}}{u_{\alpha,t}(P_M)}\right\} \\ &= \max\left\{0,\ +\frac{P\lambda\Upsilon_t(P) - \mathsf{E}\mathcal{V}_t}{u_{\alpha,t}(P_M)}\right\}. \end{aligned}$$

It is the ratio of expected profit to ruin capital which guarantees a given probability of non-ruin at the price equal to P_M. The calculation of $u_{\alpha,t}(P_M)$, which is a constant rather than a function of P, seems much easier.

We are focused on the case $\mathsf{E}Y < \mathsf{p_U} < P_M$. We write

$$\underline{E}_{\alpha,t}(P) = 1 + \frac{(P - \mathsf{E}Y)\lambda\Upsilon_t(P)}{\overline{u}_{\alpha,t}(P_M)}.$$

The following result is straightforward.

[23] The use of simplified measures is common in the actuarial risk theory. For example (see Daykin et al., 1996), the value at risk is a simplified measure of solvency. It is often used instead of the probability of ruin.

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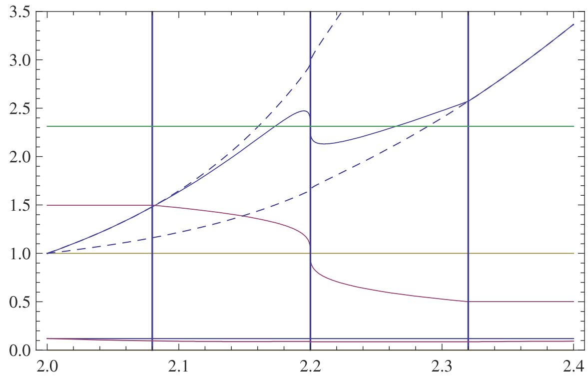

Fig. 7. Graphs as in Fig. 5, but with $l = l' = 0.3$.

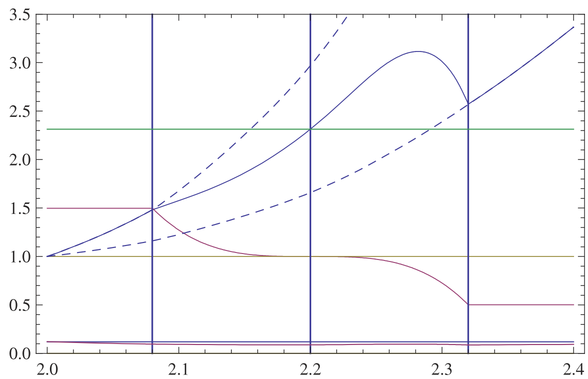

Fig. 8. Graphs as in Fig. 5, but with $l = l' = 3.3$.

Theorem 5.1. *Assume that* $\mathsf{E}Y < \mathsf{p}_{\mathsf{U}} < P_M$ *in the model* (2.6)–(2.8) *with migration rate functions* (2.1)–(2.2). *Then* $\underline{E}_{\alpha,t}(P) \leqslant E_{\alpha,t}(P)$ *for* $P > \mathsf{E}Y$, *as* t *is sufficiently large.*

Introduce

$$\underline{E}^{\mathsf{U}}_{\alpha,t}(P) = 1 + \frac{\mathsf{c}_{\mathsf{U}} t\lambda(P - \mathsf{E}Y)}{\overline{u}_{\alpha,t}(P_M)}$$

and

$$\underline{E}^{\mathsf{L}}_{\alpha,t}(P) = 1 + \frac{\mathsf{c}_{\mathsf{L}} t\lambda(P - \mathsf{E}Y)}{\overline{u}_{\alpha,t}(P_M)}.$$

In Figs. 9 and 10, shown is the graph of the function $\underline{E}_{\alpha,t}(P)$. It lies between the graphs of the functions $\underline{E}^{\mathsf{U}}_{\alpha,t}(P)$ and $\underline{E}^{\mathsf{L}}_{\alpha,t}(P)$ drawn with dashed lines. The lower solid lines are the constant level $\alpha = 0.12$ and the graph of the probability of ruin $\boldsymbol{\psi}_t(u, \lambda, P)$, where $\overline{u}_{\alpha,t}(P_M)$ is put instead of u.

Comparing Figs. 5 and 9 at first, and Figs. 6 and 10 at second, we see that the function $\boldsymbol{\psi}_t(u, \lambda, P)\,|_{u=\overline{u}_{\alpha,t}(P)}$ lies below but very close to the level α uniformly in $P > \mathsf{E}Y$, unlike the function $\boldsymbol{\psi}_t(u, \lambda, P)\,|_{u=\overline{u}_{\alpha,t}(P_M)}$ which is dramatically larger than α for $\mathsf{E}Y < P < P_M$ and dramatically smaller than α for $P > P_M$.

Comparing Figs. 5 and 9 further on, we see that $\mathcal{E}_{\alpha,t}(P)$ is substantially less than $\underline{E}_{\alpha,t}(P)$ for $\mathsf{E}Y < P < P_M$ and significantly larger than $\underline{E}_{\alpha,t}(P)$ for $P > P_M$. In addition, the shapes of these functions are quite different, which may be misleading for the development of a good control. This discrepancy is as smoother, as narrower is the interval $[\mathsf{p}_{\mathsf{U}}, \mathsf{p}_{\mathsf{L}}]$ of policyholders' migration sensitivity (see Figs. 6 and 10).

6. Auxiliary results and proof of Theorem 3.1

The following result is used for numerical calculations in Figs. 5–10. Denote by $I_n(z)$ the modified Bessel function of n-th order and put for brevity $g(P) = P/\mathsf{E}Y$, $\mu = 1/\mathsf{E}Y$. For brevity, we omit below the argument of the ratio $g(P)$.

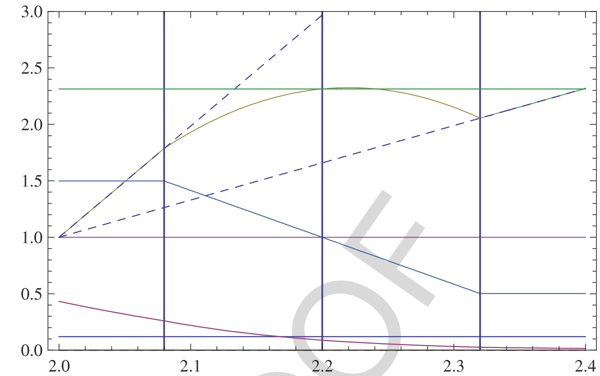

Fig. 9. Graphs of $\underline{E}_{\alpha,t}(P)$ (upper solid line), $r_t(P)$ (middle solid line) defined in (2.1) and (2.4) with $l = l' = 1$, $\boldsymbol{\psi}_t(u, \lambda, P)$ with $u = \overline{u}_{\alpha,t}(P_M)$ (here $\overline{u}_{\alpha,t}(P_M) = 45.6673$), and level $\alpha = 0.12$. Dashed lines: $\underline{E}^{\mathsf{U}}_{\alpha,t}(P)$ and $\underline{E}^{\mathsf{L}}_{\alpha,t}(P)$. Parameters are set as in Fig. 1, i.e. $\lambda = 1$, $\mathsf{E}Y = 2.0$, $\mathsf{p}_{\mathsf{U}} = 2.08$, $P_M = 2.2$, $\mathsf{p}_{\mathsf{L}} = 2.32$, $\mathsf{c}_{\mathsf{U}} = 1.5$, $\mathsf{c}_{\mathsf{L}} = 0.5$, $t = 300$. Upper horizontal line: $\mathcal{E}_{\alpha,t}(P_M) = 2.3139$.

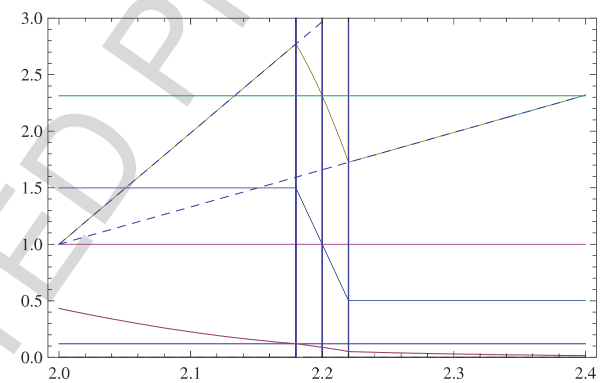

Fig. 10. Graphs as in Fig. 9, but with narrower interval of migration sensitivity: $\mathsf{p}_{\mathsf{U}} = 2.18$, $\mathsf{p}_{\mathsf{L}} = 2.22$.

Theorem 6.1 (*Theorem 3.3 in* Malinovskii, 2010). *In the assumptions of the model* (2.6)–(2.8) *with migration rate functions* (2.1)–(2.2), *we have*

$$\boldsymbol{\psi}_t(u, \lambda, P) = e^{-u\mu} \sum_{n\geqslant 0} \frac{(u\mu)^n}{n\,!} g^{-(n+1)/2} \times \int_0^{\lambda\Upsilon_t(P)} \frac{n+1}{x} e^{-(1+g)x} I_{n+1}(2x\sqrt{g})\, dx, \qquad (6.1)$$

or, alternatively,

$$\boldsymbol{\psi}_t(u, \lambda, P) = \boldsymbol{\psi}_\infty(u, \lambda, P) - \frac{1}{\pi}\int_0^\pi f(x \mid u, \lambda, P)\, dx, \qquad (6.2)$$

where[24]

$$\boldsymbol{\psi}_\infty(u, \lambda, P) = \begin{cases} g^{-1}\exp\{-u\mu(1-g^{-1})\}, & g > 1, \\ 1, & g \leqslant 1 \end{cases} \qquad (6.3)$$

and

$$f(x \mid u, \lambda, P) = g^{-1}(1 + g^{-1} - 2g^{-1/2}\cos x)^{-1} \times \exp\{u\mu(g^{-1/2}\cos x - 1)$$

[24] Bear in mind that the right hand side of Eq. (6.3) does not depend on λ.

$$
\begin{aligned}
&- \lambda \Upsilon_t(P) g \big(1 + g^{-1} - 2g^{-1/2} \cos x\big)\big\} \\
&\times \big[\cos\big(u\mu g^{-1/2} \sin x\big) \\
&- \cos\big(u\mu g^{-1/2} \sin x + 2x\big)\big].
\end{aligned}
$$

By the standard classical Lundberg model we mean the homogeneous risk model, where the risk reserve process is[25]

$$R_s = u + cs - V_s, \quad s \geqslant 0, \tag{6.4}$$

and $V_s = \sum_{i=1}^{N_s} Y_i$ or 0, if $N_s = 0$ (or $T_1 > s$), $N_s = \max\big\{n > 0 : \sum_{i=1}^{n} T_i \leqslant s\big\}$ or 0, if $T_1 > s$, $u > 0$, $c > 0$, and T_i, $i = 1, 2, \ldots$, Y_i, $i = 1, 2, \ldots$, are mutually independent sequences of i.i.d. exponentially distributed random variables with intensities $\lambda > 0$ and $\mu > 0$ respectively. So, the time arrival process is a homogeneous Poisson process with rate λ. By $u_{\alpha,t}(c \mid \lambda, \mu)$ we mean the corresponding α-level initial capital, or ruin capital.

Theorem 6.2 (*Theorem 3.1 in Malinovskii, 2012*). *In the classical Lundberg model, we have*

$$u_{\alpha,t}(\lambda/\mu \mid \lambda, \mu) = \frac{\sqrt{2t\lambda}}{\mu} \kappa_{\alpha/2}(1 + \bar{o}(1)), \quad t \to \infty. \tag{6.5}$$

The following result is a combination of Theorem 4.2 in Malinovskii (2013a) and Theorem 2.5 in Malinovskii (in preparation).

Theorem 6.3. *In the classical Lundberg model, we have for t sufficiently large*

$$
\begin{aligned}
&\left(\frac{\lambda}{\mu} - c\right) t + \frac{\kappa_\alpha}{\kappa_{\alpha/2}} u_{\alpha,t}(\lambda/\mu \mid \lambda, \mu) \leqslant u_{\alpha,t}(c \mid \lambda, \mu) \\
&\leqslant \left(\frac{\lambda}{\mu} - c\right) t + u_{\alpha,t}(\lambda/\mu \mid \lambda, \mu)
\end{aligned}
\tag{6.6}
$$

for $c \leqslant \lambda/\mu$ and

$$
u_{\alpha,t}(c \mid \lambda, \mu) \leqslant
\begin{cases}
\dfrac{(\mu u_{\alpha,t}(\lambda/\mu \mid \lambda, \mu) + \ln\alpha)^2}{4\lambda \ln\alpha} \\
\quad c - \dfrac{(\mu u_{\alpha,t}(\lambda/\mu \mid \lambda, \mu) - \ln\alpha)^2}{4\mu \ln\alpha}, \\
\dfrac{\lambda}{\mu} < c \leqslant \dfrac{\lambda}{\mu} \left(\dfrac{\mu u_{\alpha,t}(\lambda/\mu \mid \lambda, \mu) - \ln\alpha}{\mu u_{\alpha,t}(\lambda/\mu \mid \lambda, \mu) + \ln\alpha}\right), \\
-\dfrac{\ln\alpha}{\mu - \lambda/c}, \\
\quad c \geqslant \dfrac{\lambda}{\mu} \left(\dfrac{\mu u_{\alpha,t}(\lambda/\mu \mid \lambda, \mu) - \ln\alpha}{\mu u_{\alpha,t}(\lambda/\mu \mid \lambda, \mu) + \ln\alpha}\right)
\end{cases}
\tag{6.7}
$$

for $c > \lambda/\mu$.

Proof of Theorem 3.1. We have to prove that in the assumptions of the model (2.6)–(2.8) with migration rate functions (2.1)–(2.2), for t sufficiently large

$$
\begin{aligned}
&\lambda \Upsilon_t(P)(\mathsf{E}Y - P) + \frac{\kappa_\alpha}{\kappa_{\alpha/2}} u^*_{\alpha,t}(P) \leqslant u_{\alpha,t}(P) \\
&\qquad \leqslant \lambda \Upsilon_t(P)(\mathsf{E}Y - P) + u^*_{\alpha,t}(P)
\end{aligned}
\tag{6.8}
$$

for $P \leqslant \mathsf{E}Y$ and

$$
u_{\alpha,t}(P) \leqslant
\begin{cases}
\dfrac{(u^*_{\alpha,t}(P) + \mathsf{E}Y \ln\alpha)^2}{4(\mathsf{E}Y)^2 \ln\alpha} P - \dfrac{(u^*_{\alpha,t}(P) - \mathsf{E}Y \ln\alpha)^2}{4(\mathsf{E}Y)^2 \ln\alpha} \mathsf{E}Y, \\
\mathsf{E}Y < P \leqslant \mathsf{E}Y \left(\dfrac{u^*_{\alpha,t}(P) - \mathsf{E}Y \ln\alpha}{u^*_{\alpha,t}(P) + \mathsf{E}Y \ln\alpha}\right), \\
-\dfrac{P\mathsf{E}Y \ln\alpha}{P - \mathsf{E}Y}, \quad P \geqslant \mathsf{E}Y \left(\dfrac{u^*_{\alpha,t}(P) - \mathsf{E}Y \ln\alpha}{u^*_{\alpha,t}(P) + \mathsf{E}Y \ln\alpha}\right)
\end{cases}
\tag{6.9}
$$

for $P > \mathsf{E}Y$, where

$$u^*_{\alpha,t}(P) = \mathsf{E}Y \sqrt{2\lambda \Upsilon_t(P)}\, \kappa_{\alpha/2}(1 + \bar{o}(1)), \quad t \to \infty. \tag{6.10}$$

This proof is based on two main observations. First, uniformly in $P > 0$ we have $\Upsilon_t(P) = \mathrm{t}r(P) + (1 - r(P)) \int_0^t \varsigma_s ds > \mathrm{c_L} t$, where $\mathrm{c_L} > 0$. So, $\Upsilon_t(P) \to \infty$ uniformly in $P > 0$, as $t \to \infty$. Second, we can switch from the model (2.6)–(2.8) to the standard classical Lundberg model with homogeneous claims arrival process. For that, we use the standard[26] time-change theorem. For the probability of ruin defined in (2.9), it yields

$$\boldsymbol{\psi}_t(u, \lambda, P) = \mathsf{P}\Big\{\inf_{0 < s \leqslant \Upsilon_t(P)} R_s(u, \lambda P) < 0\Big\}, \tag{6.11}$$

where $R_s = R_s(u, \lambda P)$ is defined in Eq. (6.4) with $c = \lambda P$. For each $P > 0$, we apply[27] the bounds of Theorem 6.3 to the right hand side of Eq. (6.11). This completes the proof. □

Uncited references

Feldblum, 2001 and Feldblum, 2007.

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[25] Below in this section, we apply notation used in Malinovskii (2012) and in Malinovskii (2013c).

[26] See Bühlmann (1970, pp. 38–39 and Section 2.2.3), or Grandell (1991, Section 2.1 on p. 33), or Asmussen (2000, Remark 1.6 on p. 60). It was used in the proof of Theorem 3.4 in Malinovskii (2013c).

[27] Applying Theorem 6.3, we set $c = \lambda P$ and recall that $\mu = 1/\mathsf{E}Y$.

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