A direct bijective proof of the hook-length formula

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Theorem (Hook-length formula)

Fix a partition $\lambda := (\lambda_1, \ldots, \lambda_l)$ *with* $\lambda_1 \geq \cdots \geq \lambda_l > 0$ *and* $\lambda \vdash n$. *Let* f^λ *denote the number of standard* λ*-tableaux. Then*

$$f^\lambda = \frac{n!}{\prod_{c \in \lambda} h_c}$$

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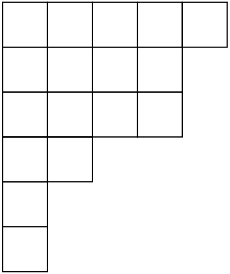
$(5, 4, 4, 2, 1, 1) \qquad \longleftrightarrow$



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1	2	4	7
3	5	8	
6			

1	3	5	7
2	4	6	
8			

1	2	5	8
3	6	7	
4			

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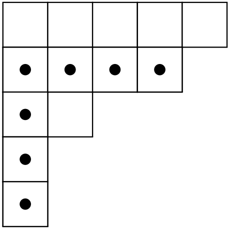
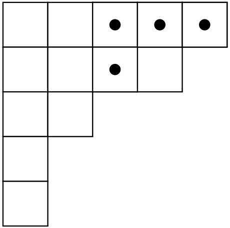


$h_{(1,1)} = 7$



$h_{(0,2)} = 4$

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- There are $n!$ bijective maps from λ to $\{1, \dots, n\}$. Such a bijection can be viewed as a filling of λ with numbers $1, \dots, n$ such that every number occurs exactly once.

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- For a cell $(i, j) \in \lambda$, define $L_{i,j} := \{(i', j) \in \lambda \mid i' > i\}$ and $A_{i,j} := \{(i, j') \in \lambda \mid j' > j\}$. Let $l_{i,j} := |L_{i,j}|$ and $a_{i,j} := |A_{i,j}|$. Then $h_{i,j} = l_{i,j} + a_{i,j} + 1$ for all $(i, j) \in \lambda$.
- A *hook tableau* on λ is a map $J : \lambda \to \mathbb{Z}$ such that $-l_{i,j} \leq J_{i,j} \leq a_{i,j}$ for all $(i, j) \in \lambda$.

We define the total order $\leq$ on λ by

$$(i,j) \leq (i',j') \iff j > j' \text{ or } j = j' \text{ and } i \geq i'$$

For a fixed cell $c \in \lambda$, let $T_{\leq c}$ (resp. $T_{<c}$) denote the restriction of T to the cells in λ that are less than or equal to c (resp. less than).

Algorithm I: $T \to (P, J)$

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Example. If $\lambda = \ydiagram{4,3,3,2}$ and the cells of λ are labelled $c_1 < \cdots < c_{12}$ then we get

c_{12}	c_8	c_4	c_1
c_{11}	c_7	c_3	
c_{10}	c_6	c_2	
c_9	c_5		

Algorithm I: $T \to (P, J)$

We will make use of a procedure called a *forward slide*:

procedure NPS
Input: $c = (i, j) \in \lambda$ such that $T_{<c}$ is a standard tableau
  while $T_{\leq c}$ is not standard **do**
    Let c' be the cell of $\min\{T_{i+1,j}, T_{i,j+1}\}$
    Exchange T_c and $T_{c'}$ and let $c := c'$
  end while
$\triangleright$ The cells through which c passes we will call the *exchange path*
end procedure

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5	$\underline{1}$
$\underline{2}$	3
4	

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Algorithm I: $T \to (P, J)$

(Label the cells of λ in the given order $c_1 < c_2 < \cdots < c_n$.)

Goal: given T obtain (P, J)

- We start with the pair (T_0, J_0) where $T_0 := T$ and J_0 is the hook tableau with all zeros.
- For $k = 0, \ldots, n-1$, we obtain T_{k+1} by applying a forward slide to T_k starting from $c_{k+1} = (i_0, j_0)$.
- If (i', j') is the cell at the end of this exchange path, then J_{k+1} is J_k except that $(J_{k+1})_{h,j_0} = \begin{cases} (J_k)_{h+1,j_0} - 1 \text{ for } i_0 \leq h < i' \\ j' - j_0 \text{ for } h = i' \end{cases}$. (Note that if J_k is a hook tableau, then so is J_{k+1}.)
- (T_n, J_n) is a pair of a standard tableau and a hook tableau.

Example.

$$\begin{array}{|c|c|c|}\hline 10 & 11 & 5 \\ \hline 7 & 9 & 6 \\ \hline 2 & 1 & 4 \\ \hline 8 & 3 \\ \cline{1-2}\end{array}\,,\ \begin{array}{|c|c|c|}\hline 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 \\ \cline{1-2}\end{array}$$

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Algorithm I: $T \to (P, J)$

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$$\begin{array}{|c|c|c|}\hline 10&11&5\\\hline 7&9&6\\\hline 2&1&4\\\hline 8&3\\\cline{1-2}\end{array}, \begin{array}{|c|c|c|}\hline 0&0&0\\\hline 0&0&0\\\hline 0&0&0\\\hline 0&0\\\cline{1-2}\end{array}$$

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Algorithm I: $T \to (P, J)$

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$$\xrightarrow{9}\ \begin{array}{|c|c|c|}\hline 10&11&4\\\hline 7&1&5\\\hline 2&3&6\\\hline 8&9\\\cline{1-2}\end{array}\ ,\ \begin{array}{|c|c|c|}\hline 0&0&-2\\\hline 0&-1&0\\\hline 0&-1&0\\\hline 0&0\\\cline{1-2}\end{array}$$

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Algorithm I: $T \to (P, J)$

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$$\xrightarrow{10} \begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2&5&10\\\hline 6&7&11\\\hline 8&9\\\cline{1-2}\end{array}, \begin{array}{|c|c|c|}\hline -2&-2&-2\\\hline 2&-2&0\\\hline 1&1&0\\\hline 0&0\\\cline{1-2}\end{array}$$

- Start with a pair $(T_n, J_n) := (P, J)$ where P is a standard tableau and J is a hook tableau.
- Assume (T_k, J_k) has been constructed. Which cells could have been the end of the path for the forward slide $\mathrm{NPS}(c_k)$?

Algorithm II: $(P, J) \to T$

- Start with a pair $(T_n, J_n) := (P, J)$ where P is a standard tableau and J is a hook tableau.
- Assume (T_k, J_k) has been constructed. Which cells could have been the end of the path for the forward slide NPS(c_k)?
- The set of *candidate cells* for $c_k = (i_0, j_0)$ in T_k is

$$\mathcal{C}_k := \{(i,j) \in \lambda \mid i \geq i_0,\ j = j_0 + (J_k)_{i,j_0},\ (J_k)_{i,j_0} \geq 0\}$$

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For example, if

$$(T_{10}, J_{10}) = \begin{array}{|c|c|c|c|c|} \hline 13 & 18 & 10 & 1 & 4 \\ \hline 17 & 15 & 2 & 3 & 5 \\ \hline 14 & 16 & 6 & 7 & 11 \\ \hline 19 & 12 & 8 & 9 & \\ \cline{1-4} \end{array}\ , \ \begin{array}{|c|c|c|c|c|} \hline 0 & 0 & 0 & -2 & -2 \\ \hline 0 & 0 & -1 & -2 & 0 \\ \hline 0 & 0 & 1 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & \\ \cline{1-4} \end{array}$$

then $\mathcal{C}_k = \{(2,3), (3,2)\}$

Algorithm II: $(P, J) \to T$

For $k = n, \dots, 1$ we obtain T_{k-1} from T_k by performing a *backward slide*:

---

procedure SPN

Pick $c = (i, j) \in \mathcal{C}_k$, assume $c_k = (i_0, j_0)$

while $c \neq c_k$ **do**

Let c' be the cell of $\begin{cases} \max\left\{(T_k)_{i-1,j}, (T_k)_{i,j-1}\right\} & \text{if } j > j_0 \text{ and } i > 0 \\ (T_k)_{i,j-1} \text{ if } j > j_0 \text{ and } i = 0 \\ (T_k)_{i-1,j} \text{ if } j = j_0 \text{ and } i > 0 \end{cases}$

Exchange T_c and $T_{c'}$ and let $c := c'$

end while

$\triangleright$ The cells through which c passes we will call its *reverse path*

end procedure

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Which $c \in \mathcal{C}_k$ should we pick?

Algorithm II: $(P, J) \to T$

Which $c \in \mathcal{C}_k$ should we pick?

- Define total order on the reverse paths $\mathcal{R}_k$ of cells in $\mathcal{C}_k$.
- Any such path can be written as a string of Ns and Ws encoding the sequence of north/west steps taken. This sequence is read backward and then padded with $\emptyset$.
- This allows us to define a lexicographical order on the paths by stipulating that $N < \emptyset < W$.

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When performing SPN we pick the candidate cell (i', j') corresponding to the largest reverse path.

- J_{k-1} remains the same as J_k except that

$$(J_{k-1})_{h,j_0} = \begin{cases} (J_k)_{h-1,j_0} + 1 \text{ if } i_0 < h \leq i' \\ 0 \text{ if } h = i_0 \end{cases}.$$

(It's unclear that J_{k-1} is a hook tableau given that J_k is one, as $(J_k)_{h-1,j_0} + 1$ may exceed a_{h,j_0} for some $i_0 < h \leq i'$.)

Example.

$$
\begin{array}{|c|c|c|}
\hline
\underline{1} & \mathbf{2} & 7 \\
\hline
3 & 4 & \mathbf{8} \\
\hline
5 & 9 & 10 \\
\hline
6 & \mathbf{11} \\
\cline{1-2}
\end{array}
,\quad
\begin{array}{|c|c|c|}
\hline
1 & 1 & -2 \\
\hline
2 & -1 & 0 \\
\hline
-1 & 1 & 0 \\
\hline
1 & 0 \\
\cline{1-2}
\end{array}
$$

Algorithm II: $(P, J) \to T$

Example.

$$\begin{array}{|c|c|c|}\hline \underline{1} & \mathbf{2} & 7 \\ \hline 3 & 4 & \mathbf{8} \\ \hline 5 & 9 & 10 \\ \hline 6 & \mathbf{11} \\ \cline{1-2} \end{array}, \begin{array}{|c|c|c|}\hline 1 & 1 & -2 \\ \hline 2 & -1 & 0 \\ \hline -1 & 1 & 0 \\ \hline 1 & 0 \\ \cline{1-2} \end{array}$$

$$\xrightarrow{8} \begin{array}{|c|c|c|}\hline 8 & 1 & 2 \\ \hline \underline{3} & 4 & \mathbf{7} \\ \hline 5 & 9 & 10 \\ \hline 6 & \mathbf{11} \\ \cline{1-2} \end{array}, \begin{array}{|c|c|c|}\hline 0 & 1 & -2 \\ \hline 2 & -1 & 0 \\ \hline -1 & 1 & 0 \\ \hline 1 & 0 \\ \cline{1-2} \end{array}$$

Algorithm II: $(P, J) \to T$

Example.

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\begin{array}{|c|c|c|}\hline \underline{1} & \mathbf{2} & 7 \\ \hline 3 & 4 & \mathbf{8} \\ \hline 5 & 9 & 10 \\ \hline 6 & \mathbf{11} \\ \cline{1-2} \end{array}\,,\;
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$$

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\xrightarrow{8}
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\quad \xrightarrow{11} \quad
\begin{array}{|c|c|c|}\hline 8 & 1 & 2 \\ \hline 7 & 3 & 4 \\ \hline 11 & 5 & 10 \\ \hline \underline{\mathbf{6}} & 9 \\ \cline{1-2} \end{array}\,,\;
\begin{array}{|c|c|c|}\hline 0 & 1 & -2 \\ \hline 0 & -1 & 0 \\ \hline 0 & 1 & 0 \\ \hline 0 & 0 \\ \cline{1-2} \end{array}
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$$\xrightarrow{6} \begin{array}{|c|c|c|}\hline 8 & \underline{1} & \mathbf{2} \\ \hline 7 & 3 & 4 \\ \hline 11 & 5 & \mathbf{10} \\ \hline 6 & \mathbf{9} \\ \cline{1-2}\end{array}\,,\ \begin{array}{|c|c|c|}\hline 0 & 1 & -2 \\ \hline 0 & -1 & 0 \\ \hline 0 & 1 & 0 \\ \hline 0 & 0 \\ \cline{1-2}\end{array}$$

Algorithm II: $(P, J) \to T$

Example.

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\begin{array}{|c|c|c|}\hline \underline{1} & \mathbf{2} & 7 \\ \hline 3 & 4 & \mathbf{8} \\ \hline 5 & 9 & 10 \\ \hline 6 & \mathbf{11} \\ \cline{1-2} \end{array}\,,\;
\begin{array}{|c|c|c|}\hline 1 & 1 & -2 \\ \hline 2 & -1 & 0 \\ \hline -1 & 1 & 0 \\ \hline 1 & 0 \\ \cline{1-2} \end{array}
$$

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\xrightarrow{8}\;
\begin{array}{|c|c|c|}\hline 8 & 1 & 2 \\ \hline \underline{3} & 4 & \mathbf{7} \\ \hline 5 & 9 & 10 \\ \hline 6 & \mathbf{11} \\ \cline{1-2} \end{array}\,,\;
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$$
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\begin{array}{|c|c|c|}\hline 8 & \underline{1} & \mathbf{2} \\ \hline 7 & 3 & 4 \\ \hline 11 & 5 & \mathbf{10} \\ \hline 6 & \mathbf{9} \\ \cline{1-2} \end{array}\,,\;
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$$

$$
\xrightarrow{2}\;
\begin{array}{|c|c|c|}\hline 8 & 2 & 1 \\ \hline 7 & \underline{3} & 4 \\ \hline 11 & 5 & \mathbf{10} \\ \hline 6 & \mathbf{9} \\ \cline{1-2} \end{array}\,,\;
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$$

Algorithm II: $(P, J) \to T$

Example...

$$\xrightarrow{2} \begin{array}{|c|c|c|}\hline 8 & 2 & 1 \\ \hline 7 & \underline{3} & 4 \\ \hline 11 & 5 & \mathbf{10} \\ \hline 6 & \mathbf{9} \\ \cline{1-2} \end{array}, \begin{array}{|c|c|c|}\hline 0 & 0 & -2 \\ \hline 0 & -1 & 0 \\ \hline 0 & 1 & 0 \\ \hline 0 & 0 \\ \cline{1-2} \end{array}$$

Example...

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Algorithm II: $(P, J) \to T$

Example...

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$$\xrightarrow{4} \begin{array}{|c|c|c|}\hline 8 & 2 & 4 \\ \hline 7 & 10 & \underline{1} \\ \hline 11 & 3 & \mathbf{5} \\ \hline 6 & 9 \\ \cline{1-2}\end{array}, \begin{array}{|c|c|c|}\hline 0 & 0 & 0 \\ \hline 0 & 0 & -1 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 \\ \cline{1-2}\end{array}$$

Algorithm II: $(P, J) \to T$

Example...

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$$\xrightarrow{5} \begin{array}{|c|c|c|}\hline 8 & 2 & 4 \\ \hline 7 & 10 & 5 \\ \hline 11 & 3 & \underline{\mathbf{1}} \\ \hline 6 & 9 \\ \cline{1-2} \end{array}, \begin{array}{|c|c|c|}\hline 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 \\ \cline{1-2} \end{array}$$

Algorithm II is well-defined

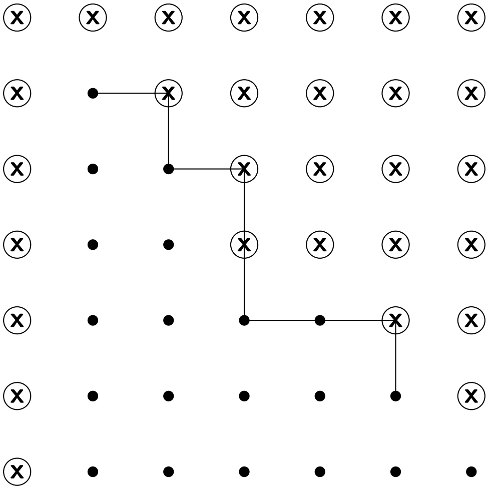
Lemma (Characterizing Reverse Paths)

Suppose all paths in $\mathcal{R}_k$ go through $c_k = (i_0, j_0)$, and $r' \in \mathcal{R}_k$ is a reverse path that starts at (i', j'). Then r' is the largest reverse path if and only if any initial cell (i, j) of $r \in \mathcal{R}_k$, $r \neq r'$, satisfies:

(1) $i_0 \leq i \leq i'$ *and* (i, j) *is west and weakly south of* r', *or*

(2) $i' < i$ *and* r *enters row* i' *weakly west of* r'.

Algorithm II is well-defined



Algorithm II is well-defined

Lemma (Bounding Reverse Paths)

Suppose some cell of a reverse path $r'' \in \mathcal{R}_{k-1}$ is south of a reverse path $r' \in \mathcal{R}_k$ which goes through c_k. Then r'' goes through c_{k-1}.

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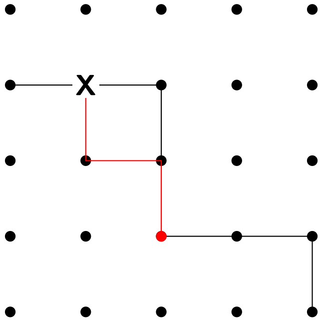

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Theorem (SPN is well-defined)

For all k, the hook tableau J_k is well defined and all reverse paths go through $c_k = (i_0, j_0)$ in T_k.

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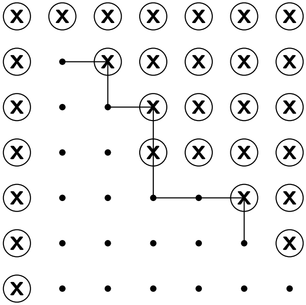

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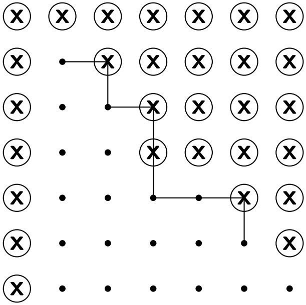
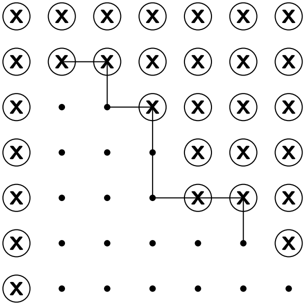
For all k, the hook tableau J_k is well defined and all reverse paths go through $c_k = (i_0, j_0)$ in T_k.





Algorithm II inverts Algorithm I

If (T_k, J_k) is obtained from (T_{k-1}, J_{k-1}) by an application of NPS, with a forward slide p' from (i_0, j_0) to (i', j'), then we have the following lemma:

Lemma (Characterizing Reverse Paths II)

Suppose all paths in $\mathcal{R}_{k-1}$ go through c_{k-1}, and $r'' \in \mathcal{R}_{k-1}$ is a reverse path with initial cell (i'', j''). If p' does not consist solely of E (east) steps then

(1) *If $i_0 + 1 \leq i'' \leq i'$ then (i'', j'') is south and weakly west of p'.*

(2) *If $i'' > i'$ then r'' enters row i' weakly west of p'.*

Algorithm II inverts Algorithm I

Theorem (SPN inverts NPS)

Suppose (T_k, J_k) is obtained from $(T, 0)$ with k applications of NPS. Then all reverse paths in $\mathcal{R}_k$ go through $c_k = (i_0, j_0)$ in T_k, and furthermore applying SPN to (T_k, J_k) we get (T_{k-1}, J_{k-1}).

Algorithm II inverts Algorithm I

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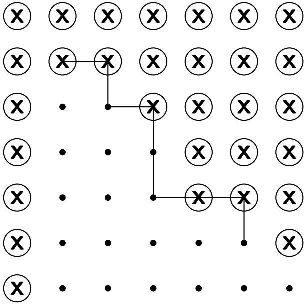

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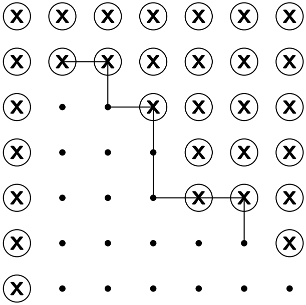

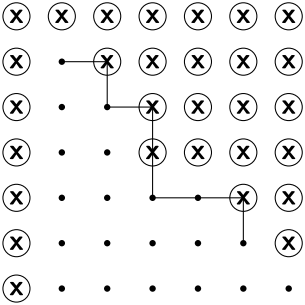

Algorithm II inverts Algorithm I

(1) If the pair (T_{k-1}, J_{k-1}) is derived from $k-1$ applications of NPS to $(T, 0)$, then applying NPS followed by SPN is the identity map.

(2) If the pair (T_k, J_k) is derived from $n-k$ applications of SPN to (P, J), then applying SPN followed by NPS is the identity map.

The first statement is the content of the second theorem. The second statement is true because forward and backward slides are step-by-step inverses, and there is no question about which cell to use to undo a backward slide.

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