Polyominoes

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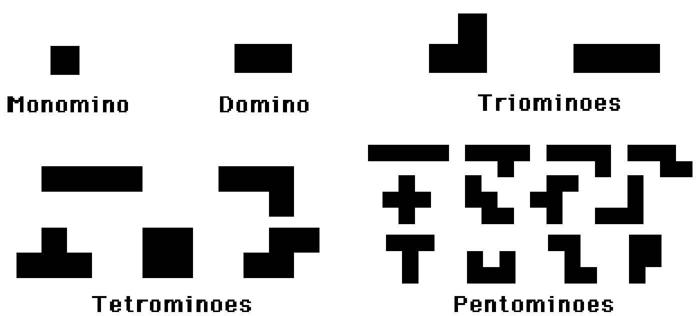
1 Introduction

Define a polyomino to be a plane geometric figure formed by joining one or more unit squares edge to edge. We will use different polyominoes to tile various boards and develop identities involving so-called “tiling numbers”.

The Fibonacci numbers ($F_1 = 1$, $F_2 = 1$, $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$) are an example of such numbers [1]. Indeed, if we let f_n count the number of ways to tile a 1 x n board with monominoes (squares) and dominoes (defining $f_0 = 1$ for convenience), we easily get the recursion $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$. It follows that $f_n = F_{n+1}$ for $n \geq 0$.



2 Tiling 2 x N Boards

Definition 1. Let ℓ_n count the number of ways to tile a 2 x n board with L-trominoes and dominoes. Note that $\{\ell_n\}_{n\geq 0}$: $1, 1, 2, 5, ..$ (in particular, $\ell_0 = 1$ and $\ell_i = 0$ for $i < 0$).

Definition 2. Let ℓ'_n count the number of ways to tile (with L-trominoes and dominoes) a mutilated 2 x n board , i.e. a 2 x n board with a unit square attached to the side. Also $\ell_0 = 0$.

[1] Some of the techniques employed in this article to generate identities were inspired by Arthur T. Benjamin and Jennifer J. Quinn’s wonderful book [1]

Identity 1. $\ell'_n = \sum_{i=0}^{n-1} \ell_i$ for $n \geq 1$.

Proof. By conditioning on the first figure to be placed in the tiling (either a tromino or a horizontal domino), we get $\ell'_n = \ell'_{n-1} + \ell_{n-1}$. The claim follows by induction.

Identity 2. $\ell_n = \ell_{n-1} + \ell_{n-2} + 2\sum_{i=0}^{n-3} \ell_i$ for $n \geq 3$.

Proof. There are three cases. We begin tiling the board from left to right. If we first place a vertical domino, there are ℓ_{n-1} to finish the tiling. If we place two horizontal dominoes, we have ℓ_{n-2} ways to finish tiling. Now suppose the tiling begins with a tromino (which can be oriented in two ways). There are $2\ell'_{n-2} = 2\sum_{i=0}^{n-3} \ell_i$ to finish tiling.

Identity 3. $\ell'_n + \ell'_{n-1} = \ell_n + \ell_{n-2}$ for $n \geq 2$.

Proof. Let N denote the set of 2 x n and 2 x $(n-2)$ tilings and M denote the set of mutilated 2 x n and 2 x $(n-1)$ tilings. We establish a bijection $L : N \mapsto M$ between the set N and the set M.

I) Map every regular 2 x n tiling according to the following rule:

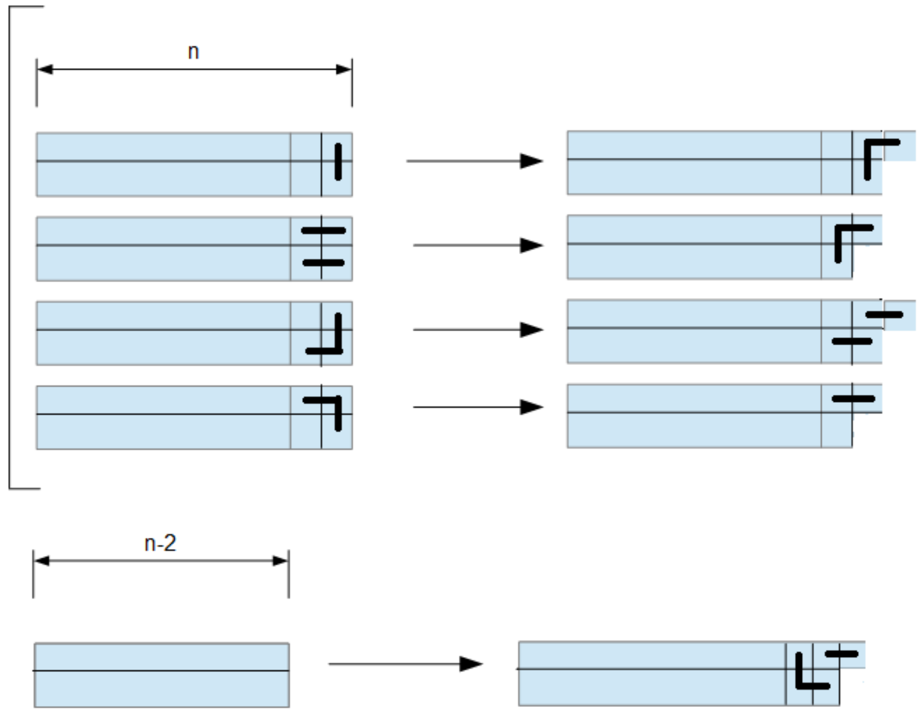
1) If the tiling ends in a vertical domino, attach a square to the top right side of the board, creating a mutilated 2 x n tiling which ends in a tromino.

2) If the tiling ends in two horizontal dominoes, remove them and add a tromino in shape of a vertically reflected "L" to create a mutilated 2 x $(n-1)$ tiling.

3) If the tiling ends in a tromino, it can be oriented in two ways – a) shaped like a horizontally reflected "L" and b) horizontally and vertically reflected "L". For case a), remove the tromino and attach two horizontal dominoes to create a mutilated 2 x n tiling. For b), delete the bottom right unit square to create a mutilated 2 x $(n-1)$ tiling.

II) Map every regular 2 x $(n-2)$ tiling to a mutilated 2 x n tiling by attaching an "L" shaped tromino and a horizontal domino to the right side of the 2 x $(n-2)$ tiling (see below).

We have therefore mapped every element in N to exactly one element in M. The mapping is bijective because the "inverse" procedure can be applied to an element in M to get a unique element in N.



The mapping L: N --> M

Identity 4. $2(\ell'_{n-1} + \ell_{n-1} + \ell_{n-2}) + 3\ell_n = \ell_{n+2} + \ell_{n-2}$ for $n \geq 2$.

Proof. Let A denote the set of 2 x n boards, 2 x $(n-1)$ boards, 2 x $(n-2)$ boards, and mutilated 2 x $(n-1)$ boards. Let B denote the set of 2 x $(n-2)$ boards and 2 x $(n+2)$ boards. We define the mapping Q between A and B:

I) Map every 2 x n board in A to three boards in B:

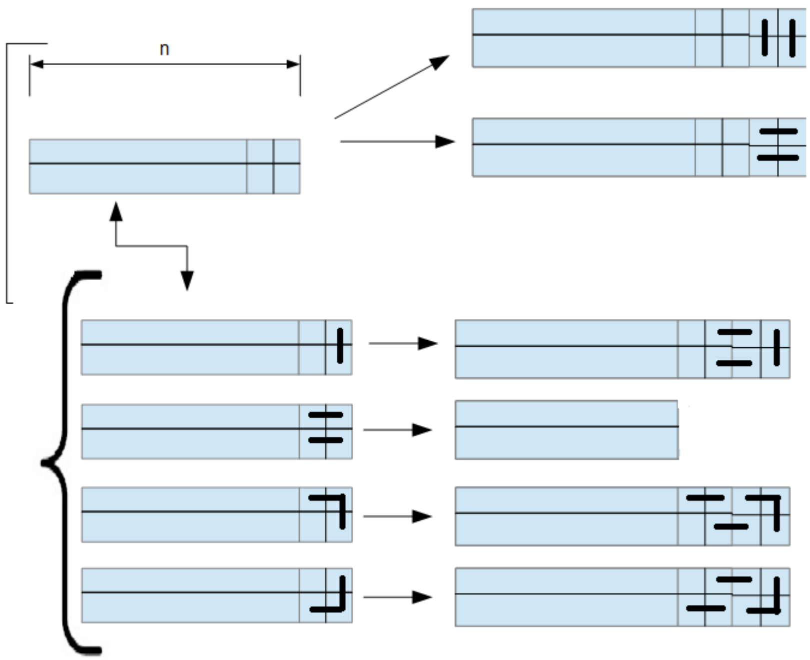
i) Add two vertical dominoes to the end of the board to create a 2 x $(n+2)$ board.

ii) Add two horizontal dominoes to the end of the board to create a 2 x $(n+2)$ board.

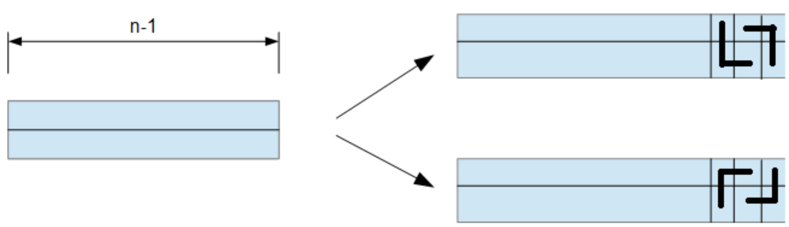
iii) (a) If the tiling ends in a vertical domino, add two horizontal dominoes before that vertical domino to create a 2 x $(n+2)$ board.

(b) If the tiling ends in two horizontal dominoes, remove them to create a 2 x $(n-2)$ tiling.

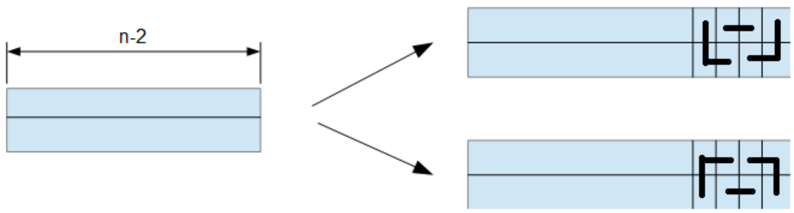
(c) If the tiling ends in a tromino, remove it and add two horizontal dominoes followed by a tromino to create a 2 x $(n+2)$ tiling.



II) Map every 2 x $(n-1)$ board in A to two 2 x $(n+2)$ boards in B – simply attach a 2 x 3 board (tiled with two trominoes) to the 2 x $(n-1)$ board. The two distinct orientations of these trominoes give rise to two mappings for each board.



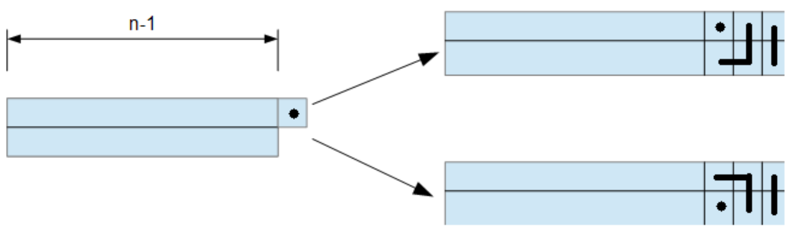
III) Map every 2 x $(n-2)$ board in A to two 2 x $(n+2)$ boards in B – simply attach a 2 x 4 board which is tiled with two trominoes and a horizontal domino in this order – tromino $\rightarrow$ horizontal domino $\rightarrow$ tromino. The two orientations give rise to two mappings.



IV) Map every mutilated 2 x $(n-1)$ board to two 2 x $(n+2)$ boards.

i) Attach a tromino to the end of the tiling to create a regular 2 x $(n+1)$ tiling and then attach a vertical domino to complete the mapping.

ii) Take the same mutilated board and reflect it vertically. Again attach a tromino to the end of the tiling to create a regular 2 x $(n+1)$ tiling and then attach a vertical domino.



Every element in A is mapped to some number of elements in B (depending on its type). Furthermore, it is easy to see that for every element $b \in B$ there is exactly one element $a \in A$ such that $b \in Q(a)$. The identity follows.

Identity 5. $\ell_n = \sum_{k=0}^{\lfloor \frac{n-1}{3} \rfloor} 2^k \ell_{n-3k-1} + \sum_{k=0}^{\lfloor \frac{n-2}{3} \rfloor} 2^k \ell_{n-3k-2} + 2 \sum_{k=0}^{\lfloor \frac{n-4}{3} \rfloor} 2^k \ell'_{n-3k-3}$ for $n \geq 3$.

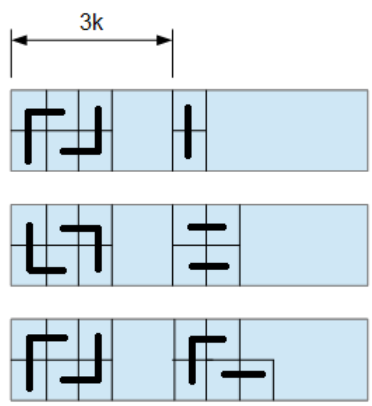
Proof. Condition on the location of the first domino. There are three disjoint cases.

1) The first domino is a vertical domino. Notice that this domino must oc-

cur at column $3k+1$ for some $0 \leq k \leq \lfloor \frac{n-1}{3} \rfloor$. This is because the trominoes preceding this domino come in groups of two, each group covering a 2 x 3 board. The 2 different orientations for each of these k groups gives rise to a total of 2^k orientations for a 2 x $3k$ board covered completely in trominoes. The remaining portion of the board can be tiled in ℓ_{n-3k-1} ways.

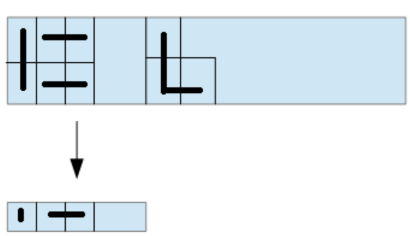
2) The first (two) dominoes are horizontal and come in a pair covering column $3k+1$ and $3k+2$ for $0 \leq k \leq \lfloor \frac{n-2}{3} \rfloor$. This case is analogous to case (1).

3) One horizontal domino (two orientations) occurring at column $3k+2$ and $3k+3$ for $0 \leq k \leq \lfloor \frac{n-4}{3} \rfloor$. One portion of the board can be tiled in 2^k ways, the other in ℓ'_{n-3k-3} ways.



Identity 6. $\ell_n = f_n + 2\sum_{i=1}^{n-2} \ell'_{n-i-1} f_{i-1} = f_n + 2\sum_{i=1}^{n-2}\sum_{j=0}^{n-2-i} \ell_j f_{i-1}$ for $n \geq 3$.

Proof. Condition on the location of the first tromino (notice that this tromino must have one of two different orientations). This tromino occurs at column i for some $1 \leq i \leq n-2$. There are f_{i-1} way to tile the board from columns 1 to $i-1$ and ℓ'_{n-i-1} ways to tile the board from columns $i+2$ to n. Finally, if there is no tromino in the tiling, we have f_n to tile the board.



Note: Identity 5 can also be reinterpreted by considering the form

$$\sum_{i=1}^{n-2}\sum_{j=3}^{n-i+1} f_{i-1}\ell_{n-i-j+1} = \sum_{i=1}^{n-2}\sum_{j=0}^{n-2-i} \ell_j f_{i-1} = \frac{\ell_n - f_n}{2}.$$

We now define the notion of breakability. A tiling is *breakable* at column i, for $1 \le i \le n-1$, if it can be decomposed into two disjoint tilings, one covering columns 1 through i and the other covering columns $i+1$ through n.

Identity 7. $\ell_{n+m} = \ell_m\ell_n + \ell_{m-1}\ell_{n-1} + 2\ell'_{m-1}\ell'_{n-1} + 2\ell_{m-1}\ell'_{n-1} + 2\ell_{n-1}\ell'_{m-1}$ for $m, n \ge 1$.

Proof. Consider the breakability of a 2 x $(m+n)$ board at column m.

I) The tiling is breakable at column m, giving rise to $\ell_m\ell_n$ ways to tile the board.

II) The tiling is not breakable at column m.

(II.1) Two horizontal dominoes are covering columns m and $m+1$, giving rise to $\ell_{m-1}\ell_{n-1}$ ways of tiling the board.

(II.2) One horizontal domino is covering columns m and $m+1$ $(2\ell'_{m-1}\ell'_{n-1})$.

(II.3) A tromino covering column m and $m+1$ $(2\ell_{m-1}\ell'_{n-1} + 2\ell_{n-1}\ell'_{m-1}$ ways for four different orientations).

Note: If $n = m$, we have $\ell_{2n} = \ell_n^2 + \ell_{n-1}^2 + 2\ell'^2_{n-1} + 4\ell_{n-1}\ell'_{n-1}$.

For convenience, we define a *prime tiling* as an $(m$ x $n)$ tiling which is not breakable at any of its columns. There are two orientations of a 2 x n prime tiling for $n \ge 3$. For $n \in \{1, 2\}$, a prime 2 x n tiling is composed of either a single vertical domino or two horizontal dominoes.

Identity 2 (reinterpretation) $\ell_n = \ell_{n-1} + \ell_{n-2} + 2\sum_{i=0}^{n-3}\ell_i$ for $n \ge 3$.

Proof II. Condition on the first column i, for $1 \le i \le n-1$, at which the tiling is breakable. If it not breakable at any of its columns, we will say that $i = n$. The board is composed of two parts: 1) a prime 2 x i tiling followed by an arbitrary 2 x $(n-i)$ tiling. Therefore $\ell_n = \sum_{i=1}^{n} e_i\ell_{n-i}$, where e_i represents the number of 2 x i prime tilings. The identity follows immediately.

This reinterpretation naturally gives rise to a certain generalization. If we let $T_{m,n}$ denote the number of ways to tile an m x n board with some set of polyominoes, we get $T_{m,n} = \sum_{i=1}^{n} t_i T_{m,n-i}$, where t_i denotes the number of prime m x i tilings. Notice that we have $f_n = T_{1,n} = \sum_{i=1}^{n} t_i T_{1,n-i} = T_{1,n-1} + T_{1,n-2} = f_{n-1} + f_{n-2}$.

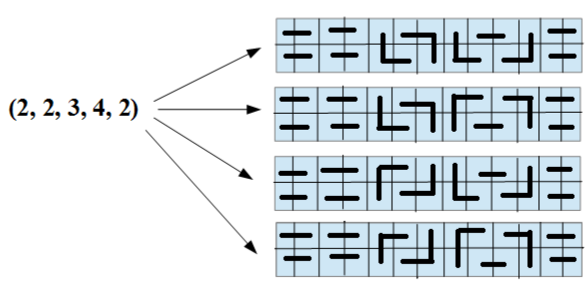
For the purpose of identity 8, we define the following:

Definition 3. For $n \geq 2$, let $l_{n,k}$ denote the number of ways to tile a (2 x n) board with only trominoes and horizontal dominoes such that the tiling is breakable at k of the columns $1, 2, .., n-1$.

***identity 8.1.** $l_{n,k} = \begin{cases} \sum_{p=0}^{k} \binom{k+1}{p} \binom{n-2k-3}{k-p} 2^{k-p+1} \text{ if } n \neq 2(k+1) \\ \sum_{p=0}^{k} \binom{k+1}{p} \binom{n-2k-3}{k-p} 2^{k-p+1} + 1 \text{ if } n = 2(k+1) \end{cases}$

for $0 \leq k \leq \lfloor \frac{n}{2} \rfloor - 1$

Proof. Since the tiling is be breakable at k columns, it must be composed of $k+1$ prime tilings. Now consider the equation (*) $n = x_1 + x_2 + .. + x_{k+1}$ where $x_i \geq 2$ are integers. An ordered tuple $(x_1, .., x_{k+1})$ which satisfies (*) and contains exactly p 2's corresponds to 2^{k-p+1} valid tilings. Conversely, if we represent a valid (2 x n) tiling as an ordered tuple of integers which correspond to the lengths of its prime tilings, we get a solution to (*) which satisfies the required constraints. Hence we have a correspondence and it suffices to count the number of valid ordered tuples. If $n = 2(k+1)$, we have the "good" tuple $(x_1, .., x_{k+1}) = (2, 2, 2, ..., 2)$. Otherwise, assume $p \leq k$. We have $n - 2p = x'_1 + ... + x'_{k+1-p}$ for $x'_i \geq 3$ i.e. $n - 2p - 3(k+1-p) = n - 3k + p - 3 = x''_1 + x''_2 + ..x''_{k+1-p}$ where $x''_i \geq 0$. Now, by standard methods (ex. stars and bars), we get that the number of solutions is $\sum_{p=0}^{k} \binom{k+1}{p} \binom{n-2k-3}{k-p}$. Notice that we do not need to care about lower bound issues for p. If $n - 2p - 3(k+1-p) = n - 3k + p - 3 \leq -1$, we have $n - 2k - 3 \leq k - p - 1$, so the binomial coefficient is 0. The identity follows from the correspondence established.

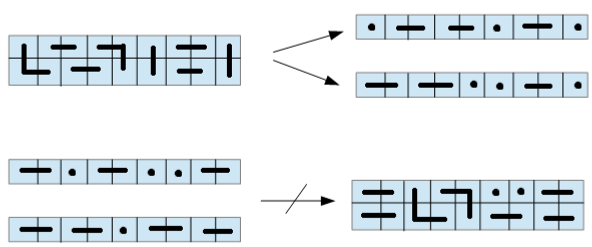


***Identity 8.2.** $\ell_n = \sum_{i=0}^{n-2} \sum_{j=0}^{\lfloor \frac{n-i-2}{2} \rfloor} \binom{j+i+1}{i} l_{n-i,j} + 1$ for $n \geq 2$.

Proof. Condition on the number of vertical dominoes in the tiling. Suppose that there are i such dominoes, $0 \le i \le n$. Then $\ell_n = \sum_{i=0}^{n} S_i$, where S_i is the number of tilings with exactly i vertical dominoes. $S_n = 1$ and $S_{n-1} = 0$, so assume $0 \le i \le n-2$ and therefore $2 \le n-i \le n$. To calculate S_i, first count the number of 2 x $(n-i)$ tilings with no vertical dominoes. Such a tiling is breakable at j columns for some $0 \le j \le \lfloor \frac{n-i}{2} \rfloor - 1$. The i vertical dominoes can thus be inserted into $j+2$ points — simply decompose the tiling into $j+1$ pieces. This can be done in $\binom{j+1+i}{i}$ ways. Hence $S_i = \sum_{j=0}^{\lfloor \frac{n-i-2}{2} \rfloor} \binom{j+i+1}{i} l_{n-i,j}$.

***Identity 9.** $\ell_n = f_n^2 - 2 \sum_{i=0}^{n-2} \ell_i f_{n-2-i}^2$ for $n \ge 2$.

For this identity, decompose the (2 x n) board into two (1 x n) boards by cutting it horizontally in the middle. Any (2 x n) tiling of the board gives rise to two (1 x n) boards tiled with squares and dominoes. However, not all pairs of (1 x n) boards tiled with squares and dominoes give rise to a valid (2 x n) board. Call such (1 x n) board pairs *incompatible*.



Unless there are two adjacent squares which are above (or below) a domino after combining the two (1 x n) tilings, we get a valid, unique (2 x n) tiling. Otherwise, only one of these squares can be part of a tromino, i.e. the resulting (2 x n) board is not valid.

Proof. By conditioning on the location of the two squares which make a pair of (1 x n) boards incompatible (their first occurrence in the tiling), we get that there are exactly $2 \sum_{i=0}^{n-2} \ell_i f_{n-2-i}^2$ incompatible pairs. It follows immediately that $\ell_n = f_n^2 - 2 \sum_{i=0}^{n-2} \ell_i f_{n-2-i}^2$.

Identity 10.

$\ell_{n+1} + 2\ell_{n-1} + 6\sum_{i=0}^{n-2} \ell_i 4^{n-i-2} + 3\sum_{i=1}^{n-1} \ell_{i-1} 4^{n-i-1} + 4\sum_{i=2}^{n-1} \ell'_{i-1} 4^{n-i-1} = 4^n$, for $n \geq 3$.

This identity is similar to the identity $f_n + f_{n-1} + \sum_{i=0}^{n-2} f_i 2^{n-i-2} = 2^n$, for $n \geq 0$ (see [1]). To prove identity 10, we convert (2 x n) tilings into length $n-1$ quaternary strings (base 4) as follows. The kth integer in the string will equal:

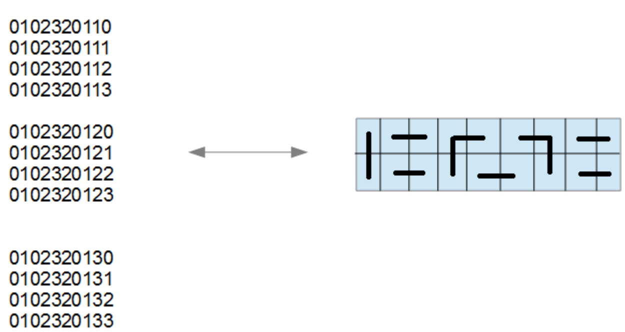
a) 0 if the corresponding tiling is breakable at the kth column.

b) 1 if in the corresponding tiling there are two horizontal dominoes crossing the kth column.

c) 2 if in the corresponding tiling a figure is crossing only the top half of the kth column.

d) 3 if in the corresponding tiling a figure is crossing only the bottom half of the kth column.

Every tiling maps to a unique quaternary string. Call a string of length k *valid* if it can be mapped to a (2 x k+1) tiling (notice that a valid string always maps to a unique tiling). If a string is not valid, extract from it its initial valid segment of maximum length (possibly $\emptyset$), which we call the *tail*. By taking its tail, any quaternary string of length n can be mapped to a (2 x i) tiling, for some $0 \leq i \leq n+1$.



Finally, notice that the set of all quaternary string of length n can be partitioned into disjoint subsets, each of which can be identified with a unique tiling (possibly $\emptyset$). We now proceed to the proof of identity 10.

Proof. There are ℓ_{n+1} valid strings. Now compute the number of invalid strings. Suppose the tail is of length i and ends with a '1'. Clearly, there are ℓ_{i-1} such tails. The tail must be followed by either a '1' a '2' or a '3'. This gives a total of $3\ell_{i-1}4^{n-i-1}$ strings. Since i ranges from 1 to $n-1$, we get $3\sum_{i=1}^{n-1}\ell_{i-1}4^{n-i-1}$ strings of length n for which the tail ends in a '1'. Now compute the number of strings for which the tail of length i ends in a '0'. For $0 \leq i \leq n-2$, the tail must be followed by one of the following: '22', '33', '23', '32', '21', or '31' ($i = 0$ corresponds to strings having tails of length 0). If $i = n-1$, the tail is followed by '2' or '3'. Hence we get a total of $2\ell_{n-1} + 6\sum_{i=0}^{n-2}\ell_i 4^{n-i-2}$. Similarly, compute the number of strings for which the tail of length i ends in a '2' (by symmetry equal to the number of strings with tails ending in a '3'). Such a tail is followed by a '1' or a '2'. We get a total of $2\sum_{i=2}^{n-1}\ell'_{i-1}4^{n-i-1}$.

Tiling with Horizontal Dominoes and Squares

Call a board *simple* if it is tiled with only horizontal dominoes (1 x 2) and monominoes/squares (1 x 1).

Definition 4. Let $w_{k,n}$ denote the number of (k x n) simple boards.

Definition 5. Let $p_{k,i}$ denote the number of prime (k x i) tilings associated with a (k x n) simple board, where $1 \leq i \leq n$.

Identity 11. $w_{k,n} = f_n^k$.

Identity 12. $f_n^2 = f_{n-1}^2 + 3f_{n-2}^2 + 2\sum_{i=0}^{n-3} f_i^2$, for $n \geq 3$.

Proof. It is not hard to verify that $p_{2,i} = 2$ if $i \geq 3$, $p_{2,2} = 3$ and $p_{2,1} = 1$. From the generalization which followed from identity 2, we have $w_{2,n} = \sum_{i=1}^{n} p_{2,i} \cdot w_{2,n-i}$. Together with identity 11, this gives the required result.

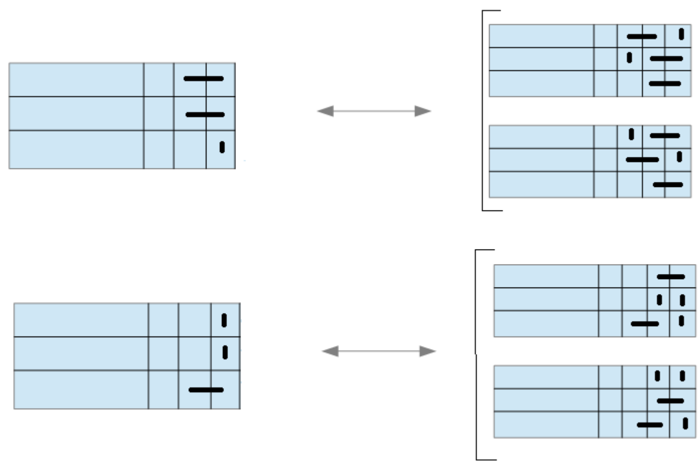
Identity 13. $p_{3,i} = 2p_{3,i-1} + p_{3,i-2}$, for $i \geq 5$.

Proof. Assume $i \geq 5$. Let A, B, and C denote the set of [3 x i] prime tilings, [3 x $(i-1)$] prime tilings, and [3 x $(i-2)$] prime tilings, respectively. Map every element of B to two elements in A as follows:

1) If the last column of the tiling in B contains two squares, extend one of

them to form a domino. Either one of the squares can be extended, corresponding to two distinct elements in A.

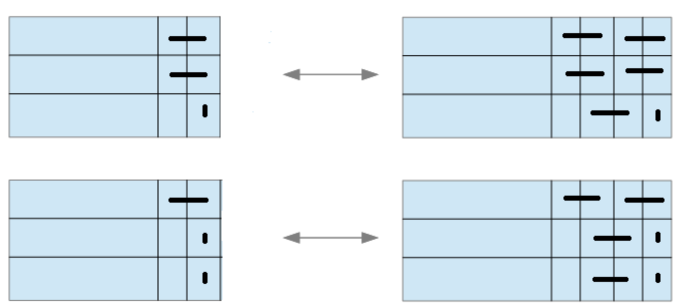
2) If the last column of the tiling in B contains one square, extend it. Move one of the dominoes in that column one unit forward, leaving a square behind (see diagram below).



Map every element in C to one element in A as follows:

1) If the last column in the tiling in C has two squares, extend both of them to form dominoes.

2) If the last column has one square, extend it to form a domino. Insert a pair of dominoes at the end (see below).



Notice that every tiling in A has been mapped to, and the inverse is unique. Indeed, number the columns from 1 to i. If column i of the tiling in A contains one square, look at the number of dominoes crossing the vertical line left of column $i-2$. If column i contains two squares, look at the number of dominoes crossing the vertical line left of column $i-1$. The conclusion follows.

From identity 13, we can write $f_n^3 = \sum_{i=1}^{n} p_{3,i} \cdot f_{n-i}^3$, which gives

$f_5^3 = f_4^3 + 7f_3^3 + 12f_2^3 + 30f_1^3 + 72f_0^3$

$f_6^3 = f_5^3 + 7f_4^3 + 12f_3^3 + 30f_2^3 + 72f_1^3 + 174f_0^3$

$f_7^3 = f_6^3 + 7f_5^3 + 12f_4^3 + 30f_3^3 + 72f_2^3 + 174f_1^3 + 420f_0^3$

$f_8^3 = f_7^3 + 7f_6^3 + 12f_5^3 + 30f_4^3 + 72f_3^3 + 174f_2^3 + 420f_1^3 + 1014f_0^3$

$f_9^3 = f_8^3 + 7f_7^3 + 12f_6^3 + 30f_5^3 + 72f_4^3 + 174f_3^3 + 420f_2^3 + 1014f_1^3 + 2448f_0^3$

. . .

3 Exercises and Problems

Identity 14. $p_{k,i} = \sum_{x_{i-1}\geq 1} \sum_{x_{i-2}\geq 1} \cdots \sum_{x_2\geq 1} \sum_{x_1\geq 1} \binom{k}{x_1}\binom{k-x_1}{x_2}\binom{k-x_2}{x_3}\cdots\binom{k-x_{i-2}}{x_{i-1}}$.

Identity 15. $\sum_{i=1}^{n} (-1)^i \ell_i = (-1)^n(\ell_{n-1} + \ell'_{n-2}) + \sum_{i=0}^{n-3} (-1)^{i+1}\ell_i$, for $n \geq 3$.

Identity 16. For $k \geq 2$, $f_k f_{k+1} = \sum_{i=0}^{k} \ell_i + 2\sum_{j=2}^{k}\sum_{i=0}^{j-2} \ell_i f_{j-2-i}^2$.

Consider the sequence defined by $a_0 = a_1 = a_2 = 1$ and
$a_n = a_{n-1} + 2\sum_{i=0}^{\lfloor\frac{n-3}{2}\rfloor} a_{n-3-2i}$, for $n \geq 3$. Prove the following identities:

Identity 17. $\ell_n = a_n + \sum_{i=0}^{n-2} a_i X_i$, for $n \geq 4$, where $X_i = \begin{cases} \ell_{n-i-2} + 2\sum_{j=2}^{\lfloor\frac{n-i}{2}\rfloor} \ell_{n-i-2j} \text{ if } i \in [0, n-4] \\ \ell_{n-i-2} \text{ if } i \in \{n-2, n-3\} \end{cases}$.

Identity 18.
$a_{n+1} + 2a_{n-1} + 8\sum_{i=2}^{n-2}\sum_{j=1}^{\lfloor\frac{i}{2}\rfloor} a_{i-2j}\cdot 3^{n-i-2} + 4\sum_{i=0}^{n-2} a_i \cdot 3^{n-i-2} + 4\sum_{j=1}^{\lfloor\frac{n}{2}\rfloor} a_{n-2j} = 3^n$,
for $n \geq 4$.

Consider the sequence $a_0 = 1$, $a_1 = k$, $a_n = \begin{cases} pa_{n-1} + a_{n-2} \text{ if } n \text{ is even} \\ ka_{n-1} + a_{n-2} \text{ if } n \text{ is odd} \end{cases}$, for $n \geq 2$ and positive integers p and k. Prove the following:

Identity 19. $a_n a_{n+1} = k \sum a_i^2 + p \sum a_i^2$, for $n \geq 0$,

where the first sum is taken over all even $i \leq n$, and the second sum over all odd $i \leq n$.

Definition. For $n \geq 0$, let L_n denote the number of ways to tile a circular n-board with dominoes and L-trominoes.

We let $L_0 = 2$, for convenience, so that $\{L_i\}_{i \geq 0} : 2, 1, 5, 10, 25, 51, 116, ...$

Identity 20. $L_n = L_{n-1} + L_{n-2} + 2 \sum_{i=0}^{n-3} L_i + 4\delta_n$, for $n \geq 3$, where we have

$$\delta_n = \begin{cases} 1 \text{ if } n \text{ is even} \\ 0 \text{ if } n \text{ is odd} \end{cases} .$$