INF210: Modelling of Computing
Oblig 2

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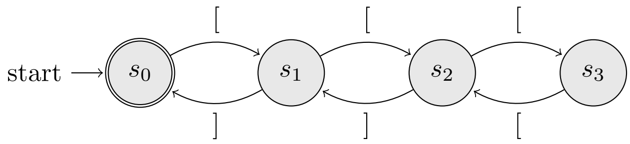
4) This exercise is about the language of balanced parentheses

$D = \{\mathbf{u} \in \Sigma^* \mid$ all prefixes contain at least as many ['s as]'s, and the total number of ['s equal the number of]'s$\}$

a) List all words in D of length 6.
There are 5: [[[]]], [][[]], [[]][], [[][]] and [][][].

b) Show that the language of all words of D with length at most n is regular for any fixed n.
All finite languages are regular since they can be constructed by a finite number of concatenations and unions of singleton languages. This language is finite and therefore regular.

c) Is the Dyck language of depth at most 3 regular? (Depth is the maximal number of nested parentheses)
This restricted language is recognized by the following finite automaton:



Since it is recognized by a finite automaton, it is regular.

d) Is the Dyck language of depth at most n for any fixed n regular?
We can construct a finite automaton like the one above for any fixed n with $n + 1$ states. By the same argument as before, this language is also regular.

e) Is the Dyck language regular?
No. The language consisting of simply nested parentheses $\{[^n]^n \mid n \in \mathbb{N}\}$ is contained in the Dyck language. This is isomorphic to $\{a^n b^n \mid n \in \mathbb{N}\}$ which is known to be non-regular so the Dyck language is also non-regular.

f) An inductive definition of the Dyck language is given by:

$$\lambda \in D'$$

$$\mathbf{u}, \mathbf{v} \in D' \implies \mathbf{uv} \in D'$$

$$\mathbf{w} \in D' \implies [\mathbf{w}] \in D'$$

Show that these two definitions are equal.

We will show the equality in two steps. First $D' \subseteq D$ by induction on the length of words k. As a base step for $k = 0$, $\lambda \in D$ and also in D' by definition. We form the induction hypothesis $|\mathbf{w}| \leq k, \mathbf{w} \in D' \implies \mathbf{w} \in D$. Now we assume it holds for $k-1$ and show it also holds for k.
Consider $\mathbf{w} \in D'$ of length k. Since it is in D', either $\mathbf{w} = [\mathbf{w}']$ or $\mathbf{w} = \mathbf{uv}$ for some $\mathbf{u}, \mathbf{v}, \mathbf{w}' \in D'$. In the first case $|\mathbf{w}'| \leq k$, so $\mathbf{w}'$ is in D by the induction hypothesis and so $\mathbf{w}$ is as well. In the second case $|\mathbf{u}|, |\mathbf{v}| < k$ (since we have used the rule to construct $\mathbf{w}$ from smaller parts) and hence in D by the induction hypothesis. Since both are in D, they have an equal number of opening and closing parentheses, and so does $\mathbf{w}$ so it is in D.

Secondly we show $D \subseteq D'$. First note that we can determine if a word is in D by maintaining a counter as we read from left to right. Start the counter at 0, increase by one for each [and decrease by one for each]. If this counter always remains non-negative and is 0 at the end of the word, then the word is in the Dyck language.
First we look at a word $\mathbf{w}$ in D such that the counter always monotonically increases, then monotonically decreases to 0. (This corresponds to simply nested parentheses.) Looking at the sequence of counter values we can reconstruct the word by concatenating a [whenever it increases, and a] whenever it decreases. For a sequence that increases n times and then decreases n times, this construction is like taking $[\mathbf{w}]$ n times and so the word is in D' as well.

Now we let the counter decrease to 0. If it ever reaches 0 after i steps, then we know $\mathbf{u} = w_0...w_i \in D$ and by the preceding argument also in D'. Since the whole word is in D it must again decrease to 0 and so $\mathbf{u} = w_{i+1}...w_n$ is also in D' and by construction $\mathbf{w} = \mathbf{uv}$ is in D' as well.

This argument is not entirely complete since we could have words like [[][]] in which the counter is neither going straight up and down, nor reaching 0 in the middle. However we note that [][] is covered by the second possibility and we can therefore surround it by parentheses and still have a word in D'.

5) In this exercise we are asked to show that two PDA models are equivalent by showing each model *emulates* the other. $M = (\Sigma, \Gamma, Q, q_0, \Delta, F)$ is as defined in the textbook and $M' = (\Sigma', \Gamma', !, Q', q_0' \Delta')$ is a PDA with empty-stack acceptance and the designated start-of-stack symbol !.

First we construct an M' emulating M. Since the PDAs should recognize the same language $\Sigma' = \Sigma$. We want to store the execution of M on the stack of M' so $\Gamma' = \{s\gamma t \mid s, t \in Q, \gamma \in \Gamma\} \cup \{!\}$. Our construction will have two states $Q' = \{q_0', q_1'\}$ where q_0' is the starting state and q_1' will serve as a faux final state. We define Δ in three steps:

1. First the pushing rules $(q_0', a, s\alpha r; t\beta r, q_0')$ for each $(s, a, \alpha; \beta, t) \in \Delta, a \in \Sigma \cup \{\lambda\}$ and $r \in Q$. These rules emulate all the rules in M that push something on the stack and store the current state on stack. r serves as a fallback state. Note that α and β may be the empty word.

2. Second the popping rules $(q_0', a, s\gamma t; \lambda, q_0')$ for $(s, a, \gamma; \lambda, t) \in \Delta$. These rules emulate popping the symbol γ from the stack and moving to a new state. Because of the fallback states added by the pushing rules and the non-determinism of our model we avoid getting stuck.

3. Finally we add the rules $(q_0', a, s\gamma f; \lambda, q_1')$ for each $f \in F$ to move into our "accepting" state and the single rule $(q_1', \lambda, !; \lambda, q_1)$ to pop the bottom-of-stack symbol. Since execution is non-

deterministic we will move to this "accepting" state with only ! on the stack when M would move to a final state with an empty stack. Then we pop the bottom-of-stack marker and accept.

Finally we construct an M emulating a given M'. From lectures (CFL.pdf slide 13) we know that such an M' can be reduced to a single state, so we may assume without loss of generality that M' is such a reduced PDA. Rules in Δ' are given as triples omitting the state.

Again we want to decide the same languages so $\Sigma = \Sigma'$. This time we need not store any extra information in the stack so $\Gamma = \Gamma'$ as well. Instead we need extra states to emulate pushing words, so $Q = \Gamma' \cup \{q_0\} \cup \{f\}$ where q_0 is the starting state and f is a final state. To avoid confusion q_γ denotes the state associated with the stack symbol γ.

To emulate a rule $(l, \gamma'; \boldsymbol{\omega}) \in \Delta'$ with $\boldsymbol{\omega} = \omega_1 ... \omega_n$ we add the rule $(q_0, l, \gamma'; \lambda, q_{\omega_1})$ to reach a "pushing state" and then $(q_{\omega_i}, \lambda, \lambda; \omega_{i+1}, q_{\omega_{i+1}})$ for each symbol ω_i. Finally the rule $(q_{\omega_n}, \lambda, \lambda; \lambda, q_0)$ moves us back to where we started. This set of rules emulates pushing a word by moving through a series of "pushing" states corresponding to the symbols of the word and pushing them one by one.
If $\boldsymbol{\omega} = \lambda$ we have $(q_0, l, \gamma'; \lambda, q_0)$ instead.

To emulate accepting with an empty stack we add the rules $(q_0, l, \lambda; \lambda, f)$ for each $(l, !; \lambda) \in \Delta'$. This moves us to a final state if M' would pop the bottom-of-stack symbol. We also add $(q_0, l, \gamma; \lambda, f)$ for each $(l, \gamma; \lambda)$ in Δ' in case M' is not well behaved and replaces !. Since execution is non-deterministic and the only place to go from f is to pop symbols and stay in f we will end up in f with an empty stack only if M' would have an empty stack.

Textbook 3.3

Exercise 3) and 4) use the procedure described by the text book in section 3.3 to mark pairs of states that cannot be collapsed, and then collapsing the remaining pairs.

3) with one step labelled "a" from s4 to s2

Step one marks $\{s_0, s_2\}, \{s_0, s_4\}, \{s_1, s_2\}, \{s_1, s_4\}, \{s_2, s_3\}, \{s_3, s_4\}$.
Step two marks $\{s_0, s_1\}$ and $\{s_0, s_3\}$. The final two pairs are not marked in step three and we collapse $\{s_1, s_3\}$ and $\{s_2, s_4\}$ resulting in the minimal automata in figure 1.

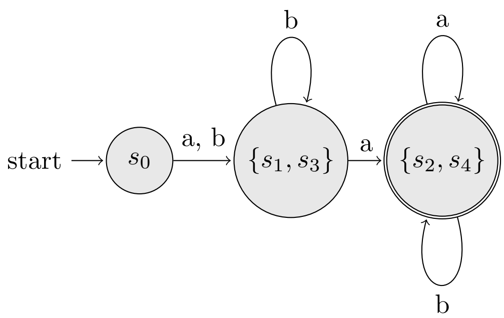


Figure 1: Minimal automata with "a"-step from s_4 to s_2

3) as given

This time all pairs are marked and no states can be collapsed.

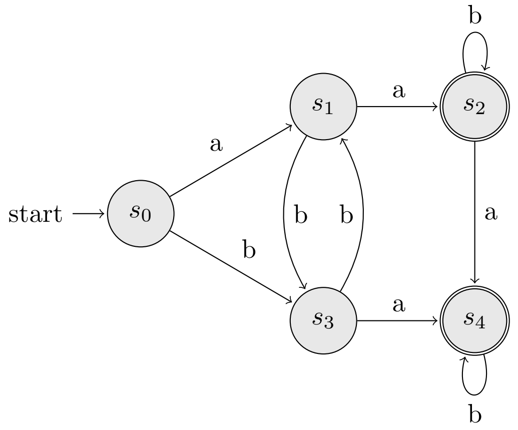


Figure 2: Minimal automata as given

4)

Step one marks $\{s_0, s_1\}, \{s_0, s_2\}, \{s_1, s_3\}, \{s_2, s_3\}$.
Step two marks $\{s_1, s_3\}$ and $\{s_0, s_3\}$, so no pair of states can be collapsed. The FSM is minimal as given.

6) Find minimal automaton for $a(b \mid c)^*bb^*$

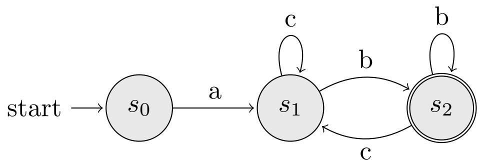


Figure 3: Minimal automaton for $a(b \mid c)^*bb^*$

To show this FSM is minimal we perform the same procedure as in 3) and 4).
Step 1 marks $\{s_0, s_2\}$ and $\{s_1, s_2\}$, and $\{s_0, s_1\}$ cannot be collapsed since $\Upsilon(s_1, a)$ is not defined.

Textbook 3.4

6) Determine whether the language $\{\mathbf{ww} \mid \mathbf{w} \in \Sigma^*, |\Sigma| = 2\}$ is regular.

Assume regular. Then by pumping lemma there exists an $n > 0$ such that for any word $\mathbf{xyz}$ in the language of length n or greater, $\mathbf{xy}^k\mathbf{z}$ is also in the language for all k.
Let $\mathbf{w} = 0^n1^n$ (with $\Sigma = \{0, 1\}$, but isomorphic to any other such word over two letters). Now $\mathbf{ww}$ is certainly long enough for the pumping lemma and in the language by construction. Now let $\mathbf{ww} = \mathbf{xyz}$ in order to apply the pumping lemma.
The pumping must occur within the first n letters, so $\mathbf{xy}$ is a subword of $\mathbf{w}$ and $\mathbf{ww} = \mathbf{xyvw}$ for some $\mathbf{v}$. Then $\mathbf{xy}^0\mathbf{v}$ is either $a^{n-|\mathbf{y}|}b^n, a^nb^{n-|\mathbf{y}|}$ or a^jb^k for some $j, k \neq n$. Either way $\mathbf{xy}^0\mathbf{v} \neq \mathbf{w}$ and $\mathbf{xy^0vw}$ is not in the language and the pumping lemma does not hold.
We conclude that the language is not regular.

7) Determine whether the language $\{a^{2n} \mid n \geq 1\}$ is regular.

We note that the language is described by the regular expression $(aa)^+$ and conclude it is regular.

10) Determine whether the language $\{\mathbf{ww}^R \mid \mathbf{w} \in \{a,b\}^*, |\mathbf{w}| \leq 3\}$ is regular.

The language is finite, and therefore regular. It is described by the (exhaustive) regular expression $aaaaaa \mid bbbbbb \mid abaaba \mid baaaab \mid aabbaa \mid abbbba \mid aaaa \mid bbbb \mid abba \mid baab \mid aa \mid bb \mid \lambda$.

Textbook 3.5

1) Prove there is an algorithm to decide if a regular language $L = \Sigma^*$

We construct a 3-step algorithm based on the observation that $\Sigma^* \setminus \Sigma^* = \emptyset$

1. Construct a finite automaton M_L accepting L
2. Swap final and non-final states of M_L to obtain an automaton for the complement of $L = \Sigma^* \setminus L$.
3. For each final state in $M_{\Sigma^* \setminus L}$, check if reachable from initial state. If yes, then $L \neq \Sigma^*$. If no final states are reachable then $L = \Sigma^*$.

This algorithm will terminate since M_L is finite and hence $M_{\Sigma^* \setminus L}$ is finite. It produces the correct answer because $\Sigma^* \setminus L = \emptyset$ if and only if $L = \Sigma^*$

6) Prove there is an algorithm for determining if there is a word in a regular language that begins with a given letter.

Assume we have a finite automaton M_L accepting L. Then for each state q_i reachable from q_0 by the given letter and each final state f_i: if f_i is reachable from q_i, return "yes". Then if we exhaust the search, return "no".
This algorithm will terminate since M_L is final. It gives the correct answer because any accepting path for a word starting with the given letter must start with that letter and end in a final state.

7) Prove there is an algorithm to determine if a regular language contains a word of even length.

We construct a 2-step algorithm based on the observation that any even number is a multiple of 2, and the assumption that we can find a path between two nodes in a graph. We also assume a finite automaton M_L accepting L:

1. construct a graph G such that $V(G) = Q$ and $E(G) = \{(q_i, q_j) \mid \text{there is a path of length 2 from} q_i \text{to} q_j \text{in} M_L\}$
2. for each final state f_i, check if it can be reached from q_0 in G.

This algorithm gives the right answer because a final state can be reached from the start state in G iff a word of even length is accepted by M_L. To determine whether there is a path from q_0 to f_i we can use a simple breadth-first search, requiring at most $|V| + |E|$ steps. Doing this naively for each $f_i \in F$ (where F is the set of final states in M_L) results in at most $|F|(|V| + |E|)$ steps.

Textbook 3.6

18) Construct a pushdown automaton accepting $L = \{\mathbf{w}c\mathbf{w}^r \mid \mathbf{w} \in \{a,b\}^*\}$.

Use PDA model with empty stack-acceptance and stack alphabet $\{\alpha, \beta\}$.

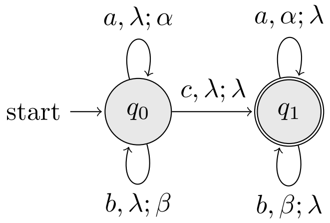


Figure 4: pushdown automaton accepting $L = \{\mathbf{w}c\mathbf{w}^r \mid \mathbf{w} \in \{a, b\}^*\}$

Show that the class of languages accepted by PDA's is closed under union, concatenation and and the Kleene star.

We will show this by construction of new PDA's. Assume the PDA $M_L = \langle \Sigma, \Gamma, Q, q_0, \Delta, F\rangle$ accepts the language L and $M_{L'} = \langle \Sigma', \Gamma', Q', q_0', \Delta', F'\rangle$ accepts L'.

First we construct a PDA which accepts $L \cup L'$. $M_{L\cup L'} = \langle \Sigma \cup \Sigma', \Gamma \cup \Gamma', Q \uplus Q' \cup \{q_0''\}, q_0'', \Delta'', F \cup F' \cup \{q_0''\}$ if q_0 or q_0' is final$\rangle$ where $\Delta'' = \Delta \cup \Delta' \cup \{(q_0'', a, \alpha; \beta, q_i) \mid (q_0, a, \alpha; \beta, q_i) \in Q \vee (q_0', a, \alpha; \beta, q_i) \in Q'\}$. That is a PDA whose alphabet and stack alphabet are simply the union of the original PDAs'. In the set of states take a disjoint union and add a new initial state q_0''. This is also in F'' if either original initial state was final. Finally we add rules to go from the new initial state to all states reachable from original initial states. This is straight-forwardly analogous to the construction of $M_{L\cup L'}$ for finite automata.
This new PDA accepts all the words in L because paths through M_L are also in $M_{L\cup L'}$. It accepts the words in L' by the same argument. It does not accept any other words because any path from q_0'' to a final state is also a path from either q_0 or q_0' to a final state, with the first swapped for one of the new added steps and hence must be in L or L'.

To construct a PDA which accepts $L \cdot L'$ we "append" $M_{L'}$ to M_L. Let

$$M_{L\cdot L'} = \langle \Sigma \cup \Sigma', (\Gamma \uplus \Gamma') \cup \{!\}, Q \uplus Q', \{q_o\}, \Delta'', F'\rangle$$

where $\Delta'' = \Delta \cup \Delta' \cup \{(q_i, \lambda, !; \lambda, q_0') \mid q_i \in F\}$ and ! is a new symbol to signify bottom of the stack. We start with this ! on the stack.
This new machine is like executing M_L and the, if we reach a final state with ! on the stack, popping ! and executing M_L' on the remainder of the input. It accepts $L \cdot L'$

Finally, the Kleene star of a language L^* is accepted by "looping" M_L. To do this we add a new initial and final state q_0', a bottom of stack symbol !, and an "empty" rule $(q_0', \lambda, \lambda; !, q_0)$, allowing our new machine to accept the empty word (zero loops) and pushing our bottom stack symbol before starting M_L. Then we add rules $\{(q_i, \lambda, !; \lambda, q_0') \mid q_i \in F\}$ to go from any final state to the start state. These rules pop ! allowing for acceptance in q_0' so we can remove them from F leaving q_0' as the only accepting state.

Textbook 4.1

14) Construct a grammar for the language $(ab)^* \mid (ac)^*$

$$\begin{array}{lcl} S & \rightarrow & abA \mid acB \\ A & \rightarrow & abA \mid \lambda \\ B & \rightarrow & acB \mid \lambda \end{array}$$

21) Construct a grammar for the language $(a|b)^*(aa|bb)(a|b)^*$

$$\begin{array}{lcl} S & \rightarrow & aA \mid bA \mid B \\ A & \rightarrow & aA \mid bA \mid B \\ B & \rightarrow & aaC \mid bbC \\ C & \rightarrow & aC \mid bC \mid \lambda \end{array}$$

24) Find an automaton which accepts the language generated by $S \rightarrow aB$, $A \rightarrow aB$, $B \rightarrow bA \mid b$

We construct an automaton with states $N \cup \{f\}$ and initial state S following the procedure:

- $A \rightarrow aB$ becomes the rule (A, a, B)
- $A \rightarrow a$ becomes the rule (A, a, f)
- $A \rightarrow \lambda$ makes A final

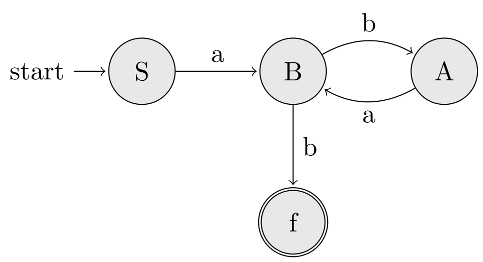


Figure 5: The finite automaton for **24)**

36) Construct a grammar that generates the language accepted by a given finite automaton.

We apply the procedure from the previous question in reverse.

- the rule (Q, a, Q') becomes $Q \rightarrow aQ$'
- if $Q' \in F$, add $Q \rightarrow a$ to our grammar rules
- if $Q \in F$, add $Q \rightarrow \lambda$ to our rules

Also note that the state S_4 is a dead end, and will not contribute to the accepted language. The resulting grammar has $N = \{S_0, S_1, S_2, S_3\}$, $\Sigma = \{a, b\}$ and production rules:

$S_0 \rightarrow aS_1 \mid bS_2$

$S_1 \rightarrow bS_1 \mid aS_3 \mid a$

$S_2 \rightarrow aS_1 \mid bS_3 \mid b$

$S_3 \rightarrow \lambda$

Textbook 4.2

4) and **5)** concern the grammar G with $N = \{A, B, S\}$, $\Sigma = a, b$, R :

$$\begin{array}{lcl} S & \rightarrow & ABABABA \\ A & \rightarrow & Aa \mid \lambda \\ B & \rightarrow & b \end{array}$$

4) convert the grammar to Chomsky Normal Form (CNF).
We proceed in 3 steps:

1. Remove $A \to Aa$ by constructing the new non-terminal N_a and rules $N_a \to a,\ A \to AN_a$
2. Eliminate rules with more than two non-terminals on the right. We replace $S \to ABABABA$ with

$$\begin{array}{ll} S & \to AS_1 \\ S_1 & \to BS_2 \\ S_2 & \to AS_3 \\ S_3 & \to BS_4 \\ S_4 & \to AS_5 \\ S_5 & \to BA \end{array}$$

3. Get rid of the null production $A \to \lambda$. We also have to add rules $B \to X$ for any existing rule $B \to AX$ or $B \to XA$

The resulting grammar $G' = \langle \Sigma, N \cup \{N_a, S_1, ..., S_5\}, S, R' \rangle$ with $R' =$

$$\begin{array}{ll} A & \to AN_a \\ N_a & \to a \\ B & \to b \\ S & \to AS_1 \\ S_1 & \to BS_2 \\ S_2 & \to AS_3 \\ S_3 & \to BS_4 \\ S_4 & \to AS_5 \\ S_5 & \to BA \\ A & \to N_a \\ S & \to S_1 \\ S_2 & \to S_3 \\ S_4 & \to S_5 \\ S_5 & \to B \end{array}$$

5) convert the grammar to Greibach normal form
Starting with the grammar G' already in CNF we proceed in two steps:

1. Remove the left recursive rule $A \to AN_a$ by replacing it with $A \to A',\ A' \to aA'$
2. Get rid of non-terminals on the left of productions by replacing them with the terminals they generate.

We obtain a new grammar with $N'' = \{N \cup \{A', S_1...S_6\}\}$ and $R'' =$

$$\begin{array}{ll} S & \to aA'S_1 \mid bS_2 \\ S_1 & \to bS_2 \\ S_2 & \to aA'S_3 \mid bS_4 \\ S_3 & \to bS_4 \mid bS_6 \\ S_4 & \to aA'S_5 \\ S_5 & \to b \mid bS_6 \\ S_6 & \to a \mid aA' \\ A & \to a \mid aA' \mid \lambda \end{array}$$

8) Convert the following grammar to CNF:

$$\begin{array}{lcl} S & \rightarrow & AbaB \\ A & \rightarrow & bAa \mid \lambda \\ B & \rightarrow & AAb \mid aabA \end{array}$$

Following the same procedure as exercise **4)** we obtain:

$$\begin{array}{ll} N_a & \rightarrow a \\ N_b & \rightarrow b \\ S & \rightarrow AS_1 \mid S_1 \\ S_1 & \rightarrow N_bS_2 \\ S_2 & \rightarrow N_aB \\ A & \rightarrow N_bA_1 \\ A_1 & \rightarrow AN_a \mid N_a \\ B & \rightarrow B_1 \mid AB_1 \mid N_aB_2 \\ B_1 & \rightarrow N_b \\ B_2 & \rightarrow N_aN_b \end{array}$$

9) convert the preceding grammar to GNF This time there are no left-recursive rules to eliminate, so the only necessary step is replacing leading non-terminals with the terminals they generate. We note $A \rightarrow^* bA_1,\ A_1 \rightarrow^* bA_1a|a,\ B_1 \rightarrow^* bA_1N_b,\ B_2 \rightarrow^* ab,\ N_a \rightarrow^* a,\ N_b \rightarrow^* b$ and construct the production rules:

$$\begin{array}{ll} N_a & \rightarrow a \\ N_b & \rightarrow b \\ S & \rightarrow bA_1B_1 \mid b\ S_2 \\ S_1 & \rightarrow bS_2 \\ S_2 & \rightarrow aB \\ A & \rightarrow bA_1 \\ A_1 & \rightarrow BA_1N_a \mid a \\ B & \rightarrow bA_1B_1 \mid aB_2 \mid bA_1N_b \mid b \\ B_1 & \rightarrow bA_1N_b \mid b \\ B_2 & \rightarrow aN_b \end{array}$$