INF210: Modelling of Computing

Oblig 2

Åsmund Aqissiaq Arild Kløvstad

November 7, 2020

main.pdf 5.7

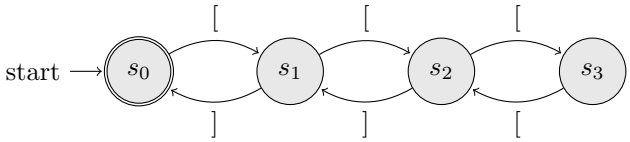
4) This exercise is about the language of balanced parentheses

$D = \{\mathbf{u} \in \Sigma^* \mid$ all prefixes contain at least as many ['s as]'s, and the total number of ['s equal the number of]'s$\}$

a) List all words in D of length 6.
There are 5: [[]], []][], [[]][], [[][]] and [][][].

b) Show that the language of all words of D with length at most n is regular for any fixed n.
All finite languages are regular. This language is finite and therefore regular.

c) Is the Dyck language of depth at most 3 regular? (Depth is the maximal number of nested parentheses)
This restricted language is recognized by the following finite automaton:



Since it is recognized by a finite automaton, it is regular.

d) Is the Dyck language of depth at most n for any fixed n regular?
We can construct a finite automaton like the one above for any fixed n with $n+1$ states. By the same argument as before, this language is also regular.

e) Is the Dyck language regular?
No. The language consisting of simply nested parentheses $\{[^n]^n \mid n \in \mathbb{N}\}$ is contained in the Dyck language. This is isomorphic to $\{a^n b^n \mid n \in \mathbb{N}\}$ which is known to be non-regular so the Dyck language is also non-regular.

f) An inductive definition of the Dyck language is given by:

$$\lambda \in D'$$

$$\mathbf{u}, \mathbf{v} \in D' \implies \mathbf{uv} \in D'$$

$$\mathbf{w} \in D' \implies [\mathbf{w}] \in D'$$

Show that these two definitions are equal.

We will show the equality in two steps. First $D' \subseteq D$ by induction on the length of words k. As a base step for $k = 0$, $\lambda \in D$ and also in D' by definition. We form the induction hypothesis $|\mathbf{w}| \leq k, \mathbf{w} \in D' \implies \mathbf{w} \in D$. Now we assume it holds for $k-1$ and show it also holds for k. Consider $\mathbf{w} \in D'$ of length k. Since it is in D', either $\mathbf{w} = [\mathbf{w}']$ or $\mathbf{w} = \mathbf{uv}$ for some $\mathbf{u}, \mathbf{v}, \mathbf{w}' \in D'$. In the first case $|\mathbf{w}'| \leq k$, so $\mathbf{w}'$ is in D by the induction hypothesis and so $\mathbf{w}$ is as well. In the second case $|\mathbf{u}|, |\mathbf{v}| < k$ (since we have used the rule to construct $\mathbf{w}$ from smaller parts) and hence in D by the induction hypothesis. Since both are in D, they have an equal number of opening and closing parentheses, and so does $\mathbf{w}$ so it is in D.

Secondly we show $D \subseteq D'$.

Textbook 3.3

Exercise 3) and 4) use the procedure described by the text book in section 3.3 to mark pairs of states that cannot be collapsed, and then collapsing the remaining pairs.

3) with one step labelled "a" from s4 to s2

Step one marks $\{s_0, s_2\}, \{s_0, s_4\}, \{s_1, s_2\}, \{s_1, s_4\}, \{s_2, s_3\}, \{s_3, s_4\}$.
Step two marks $\{s_0, s_1\}$ and $\{s_0, s_3\}$. The final two pairs are not marked in step three and we collapse $\{s_1, s_3\}$ and $\{s_2, s_4\}$ resulting in the minimal automata in figure 1.

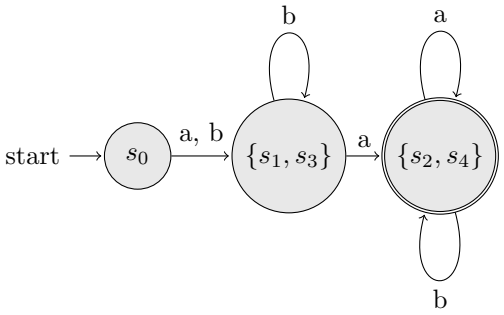


Figure 1: Minimal automata with "a"-step from s_4 to s_2

3) as given

Same as above, but this time step two also marks $\{s_2, s_4\}$ and only $\{s_1, s_3\}$ is collapsed resulting in figure 2

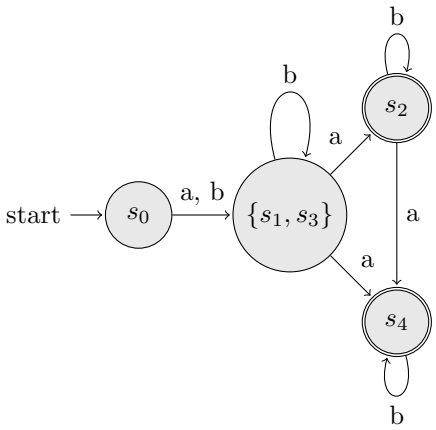


Figure 2: Minimal automata as given

4)

Step one marks $\{s_0, s_1\}, \{s_0, s_2\}, \{s_1, s_3\}, \{s_2, s_3\}$.
Step two marks $\{s_1, s_3\}$ and $\{s_0, s_3\}$ is collapsed.

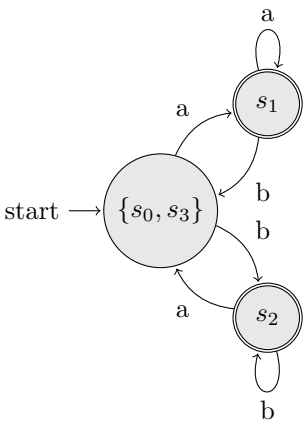


Figure 3: Minimal automata 4)

6) Find minimal automaton for $a(b \mid c)^*bb^*$

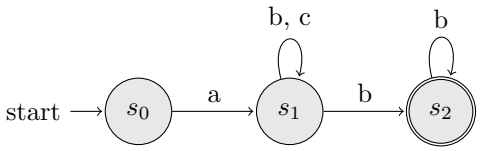


Figure 4: Minimal automaton for $a(b \mid c)^*bb^*$

Textbook 3.4

6) Determine whether the language $\{\mathbf{ww} \mid \mathbf{w} \in \Sigma^*, |\Sigma| = 2\}$ is regular.

Assume regular. Then by pumping lemma there exists an $n > 0$ such that for any word $\mathbf{xyz}$ in the language of length n or greater, $\mathbf{xy}^k\mathbf{z}$ is also in the language for all k.
We consider $\mathbf{w} \in \Sigma^*$ such that $|\mathbf{w}| = n$ and say $\mathbf{w} = \mathbf{uv}$ with distinct $\mathbf{u}, \mathbf{v}$ without loss of generality. Then $\mathbf{ww}$ is large enough for the pumping lemma and in the language by construction. Now let $\mathbf{ww} = \mathbf{xyz}$ in order to apply the pumping lemma. There are two ways this could split $\mathbf{ww}$: either $\mathbf{x} = \mathbf{w} = \mathbf{yz}$ or $\mathbf{y} = \mathbf{vu}$, splitting $\mathbf{w}$.
In the first case $\mathbf{y}^k\mathbf{z} \neq \mathbf{w}$ for $k \neq 1$ and the pumping lemma does not hold. (More precisely they are not *necessarily* equal for an arbitrary $\mathbf{w}$).
In the second case $\mathbf{xy}^k\mathbf{z} = \mathbf{uvu}^k\mathbf{v}$. For $k = 0$ we have $\mathbf{uv}$, which is certainly not in the language (assuming distinct $\mathbf{u}, \mathbf{v}$) and so the pumping lemma does not hold.
We conclude that the language is not regular.

7) Determine whether the language $\{a^{2n} \mid n \geq 1\}$ is regular.

We note that the language is described by the regular expression $(aa)^+$ and conclude it is regular.

10) Determine whether the language $\{\mathbf{ww}^R \mid \mathbf{w} \in \{a, b\}^*, |\mathbf{w}| \leq 3\}$ is regular.

The language is finite, and therefore regular. It is described by the (exhaustive) regular expression $aaaaaa \mid bbbbbb \mid abaaba \mid baaaab \mid aabbaa \mid aaaa \mid bbbb \mid abba \mid baab \mid aa \mid bb$.

Textbook 3.5

1) Prove there is an algorithm to decide if a regular language $L = \Sigma^*$

We construct a 3-step algorithm based on the observation that $\Sigma^* \setminus \Sigma^* = \emptyset$

1. Construct a finite automaton M_L accepting L
2. Swap final and non-final states of M_L to obtain an automaton for the complement of $L = \Sigma^* \setminus L$.
3. For each final state in $M_{\Sigma^* \setminus L}$, check if reachable from initial state. If yes, then $L \neq \Sigma^*$. If no final states are reachable then $L = \Sigma^*$.

This algorithm will terminate since M_L is finite and hence $M_{\Sigma^* \setminus L}$ is finite. It produces the correct answer because $\Sigma^* \setminus L = \emptyset$ if and only if $L = \Sigma^*$

6) Prove there is an algorithm for determining if there is a word in a regular language that begins with a given letter.

Assume we have a finite automaton M_L accepting L. Then for each state q_i reachable from q_0 by the given letter and each final state f_i: if f_i is reachable from q_i, return "yes". Then if we exhaust the search, return "no".
This algorithm will terminate since M_L is final. It gives the correct answer because any accepting path for a word starting with the given letter must start with that letter and end in a final state.

7) Prove there is an algorithm to determine if a regular language contains a word of even length.

We construct a 2-step algorithm based on the observation that any even number is a multiple of 2, and the assumption that we can find a path between two nodes in a graph. We also assume a finite automaton M_L accepting L:

1. construct a graph G such that $V(G) = Q$ and $E(G) = \{(q_i, q_j) \mid \text{there is a path of length 2 from} q_i \text{to} q_j \text{in} M_L\}$

2. for each final state f_i, look for a path from q_0 to f_i. If such a path exists return "yes", otherwise return "no".

This algorithm gives the right answer because if a path exists between q_i and q_j in G, then it also exists in M_L and a path of length n exists in G if and only if a path of length $2n$ exists in M_L

Textbook 3.6

18) Construct a pushdown automaton accepting $L = \{\mathbf{w}c\mathbf{w}^r \mid \mathbf{w} \in \{a, b\}^*\}$.

Use PDA model with empty stack-acceptance and stack alphabet $\{\alpha, \beta\}$.

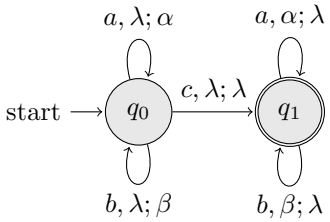


Figure 5: pushdown automaton accepting $L = \{\mathbf{w}c\mathbf{w}^r \mid \mathbf{w} \in \{a, b\}^*\}$

union, concat and kleene star go here

Textbook 4.1

14) Construct a grammar for the language $(ab)^* \mid (ac)^*$

$$\begin{array}{lcl} S & \to & abA \mid acB \\ A & \to & abA \mid \lambda \\ B & \to & acB \mid \lambda \end{array}$$

21) Construct a grammar for the language $(a|b)^*(aa|bb)(a|b)^*$

$$\begin{array}{lcl} S & \to & aA \mid bA \mid B \\ A & \to & aA \mid bA \mid B \\ B & \to & aaC \mid bbC \\ C & \to & aC \mid bC \mid \lambda \end{array}$$

24) Find an automaton which accepts the language generated by $S \to aB$, $A \to aB$, $B \to bA \mid b$

We construct an automaton with states $N \cup \{f\}$ and initial state S following the procedure:

- $A \to aB$ becomes the rule (A, a, B)
- $A \to a$ becomes the rule (A, a, f)
- $A \to \lambda$ makes A final

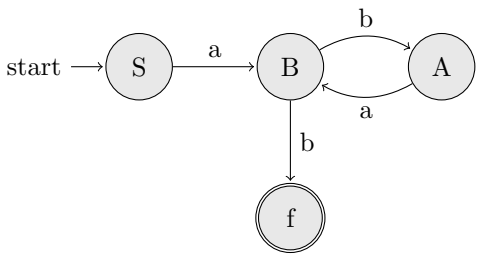


Figure 6: The finite automaton for **24)**

36) Construct a grammar that generates the language accepted by a given finite automaton.

We apply the procedure from the previous question in reverse.

- the rule (Q, a, Q') becomes $Q \to aQ'$
- if $Q' \in F$, add $Q \to a$ to our grammar rules
- if $Q \in F$, add $Q \to \lambda$ to our rules

Also note that the state S_4 is a dead end, and will not contribute to the accepted language. The resulting grammar has $N = \{S_0, S_1, S_2, S_3\}$, $\Sigma = \{a, b\}$ and production rules:

$S_0 \to aS_1 \mid bS_2$
$S_1 \to bS_1 \mid aS_3 \mid a$
$S_2 \to aS_1 \mid bS_3 \mid b$
$S_3 \to \lambda$