A Concrete Introduction to Number Theory and Algebra–群、子群、循环群

Libin Wang

Shool of Computer Science, South China Normal University

October 7, 2023

Table of contents

Motivation.

Motivation.

抽象：发现已知的世界中事物的共性与区别，忽略掉某些细节，得到一种更通用、更宽泛、可以描述更广阔世界的框架或语言，从而探求未知世界的知识。

Motivation.

Motivation.

抽象：发现已知的世界中事物的共性与区别，忽略掉某些细节，得到一种更通用、更宽泛、可以描述更广阔世界的框架或语言，从而探求未知世界的知识。

Question.

更给出（或找出）更适合你自己的“抽象”的定义。

Motivation.

Motivation.

Last chapter, we extremely rely on *Cancellation Law*.
If $gcd(c, m) = 1$ and $ac \equiv bc \pmod{m}$, then

$$a \equiv b \pmod{m}.$$

Motivation.

Motivation.

Last chapter, we extremely rely on *Cancellation Law*.
If $gcd(c, m) = 1$ and $ac \equiv bc \pmod{m}$, then

$$a \equiv b \pmod{m}.$$

Question.

Actually, what is cancellation?

Motivation.

Ideas.

From $ac \equiv bc \pmod{m}$ to $a \equiv b \pmod{m}$, seemingly, we need division, actually we need multiplication:

$$acc^{-1} \equiv bcc^{-1} \pmod{m}.$$

Hence by $cc^{-1} \equiv 1 \pmod{m}$, we have

$$a \equiv b \pmod{m}.$$

Why c^{-1} exists? Because of $gcd(c, m) = 1$!

Motivation.

Additional conditions.

Need more conditions?

Motivation.

Additional conditions.

Need more conditions?

Yes, we need *association*:

$$(ac)c^{-1} \equiv a(cc^{-1}) \pmod{m},$$

and *closure*, which means we only consider numbers from 1 to $m-1$.

Motivation.

Additional conditions.

Need more conditions?

Yes, we need *association*:

$$(ac)c^{-1} \equiv a(cc^{-1}) \pmod{m},$$

and *closure*, which means we only consider numbers from 1 to $m-1$.

Question.

什么数学操作不满足结合律?

Motivation.

Wrap up.

Wrap these up. For a set $\mathbb{G}$ and an operator $\cdot$ on the elements, we need:

- *Closure*: $\forall a, b \in \mathbb{G}, \quad a \cdot b \in \mathbb{G}$.
- *Association*: $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.
- An element "1" called *identity*, s.t. $1 \cdot a = a \cdot 1 = a$.
- $\forall a \in G$, there exists $a^{-1} \in G$, such that $a \cdot a^{-1} = 1 = a^{-1} a$, called *inverse*.

Group（群）.

Definition.

Definition(Group). A group is a set $\mathbb{G}$ and an operator $\cdot$ on the elements, satisfies the following axioms:

- *Closure(封闭性)*: $\forall a, b \in \mathbb{G}, \quad a \cdot b \in \mathbb{G}$.
- *Association(结合律)*: $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.
- There is an element "$e \in \mathbb{G}$" called *identity(单位元)*, s.t. $e \cdot a = a \cdot e = a$.
- $\forall a \in G$, there exists $a^{-1} \in G$ called *inverse(逆元)*, such that $a \cdot a^{-1} = e = a^{-1} \cdot a$.

Examples of groups.

Groups.

- $(\mathbb{Z}, +)$ is a group, while $(\mathbb{Z}, \times)$ is not a group.
- $(\mathbb{Q}, +)$ and $(\mathbb{R}, +)$ are groups.
- $(\mathbb{Q}^*, \times)$ and $(\mathbb{R}^*, \times)$ are groups.

Examples of groups.

Groups.

- $(\mathbb{Z}, +)$ is a group, while $(\mathbb{Z}, \times)$ is not a group.
- $(\mathbb{Q}, +)$ and $(\mathbb{R}, +)$ are groups.
- $(\mathbb{Q}^*, \times)$ and $(\mathbb{R}^*, \times)$ are groups.

Check.

Please check and know why.

Examples of some important groups.

$\mathbb{Z}_n$

Let n be an integer, $\mathbb{Z}_n = \{0, 1, 2, \cdots, n-1\}$ forms a group under the operation of addition. However, $(\mathbb{Z}_n, \times)$ is not a group.

Examples of some important groups.

$\mathbb{Z}_n$

Let n be an integer, $\mathbb{Z}_n = \{0, 1, 2, \cdots, n-1\}$ forms a group under the operation of addition. However, $(\mathbb{Z}_n, \times)$ is not a group.

$\mathbb{Z}_p^*$

Let p be a prime number, $\mathbb{Z}_p^* = \{1, 2, \cdots, p-1\}$ forms a group under the operation of multiplication. (Recall Fermat's Little Theorem.)

Examples of some important groups.

$\mathbb{Z}_n$

Let n be an integer, $\mathbb{Z}_n = \{0, 1, 2, \cdots, n-1\}$ forms a group under the operation of addition. However, $(\mathbb{Z}_n, \times)$ is not a group.

$\mathbb{Z}_p^*$

Let p be a prime number, $\mathbb{Z}_p^* = \{1, 2, \cdots, p-1\}$ forms a group under the operation of multiplication. (Recall Fermat's Little Theorem.)

$\mathbb{Z}_n^*$

Let n be an integer, $\mathbb{Z}_n^* = \{a \in [1..n-1] \text{ and } gcd(a, n) = 1\}$ forms a group under the operation of multiplication. (Recall Euler's Theorem.)

Basic Properties of Groups.

Proposition

Proposition 1. The identity element in a group $\mathbb{G}$ is unique; that is, there exists only one element $e \in \mathbb{G}$ s.t. $eg = ge = g$ for all $g \in \mathbb{G}$.

Basic Properties of Groups.

Proposition

Proposition 1. The identity element in a group $\mathbb{G}$ is unique; that is, there exists only one element $e \in \mathbb{G}$ s.t. $eg = ge = g$ for all $g \in \mathbb{G}$.

Proof.

Suppose $\exists e, e' \in \mathbb{G}$ are identities. Then:

- $ee' = e'$
- $ee' = e$

Combining these two equations, we have $e = ee' = e'$. $\square$

Proposition 2.

Proposition

Proposition 2. If $\forall g \in \mathbb{G}$, *then the inverse of* g, g^{-1}, *is unique.*

Proof.

如果 g^{-1} 和 g' 都是 g 的逆元，则有

$$g' = g'e = g'(gg^{-1}) = (g'g)g^{-1} = eg^{-1} = g^{-1}.$$

$\square$

Proposition 3.

Proposition

Proposition 3. Let $\mathbb{G}$ *be a group. If* $a, b \in \mathbb{G}$, *then* $(ab)^{-1} = b^{-1}a^{-1}$.

Proof.

By construction.

- $ab(b^{-1}a^{-1}) = e$.
- $(b^{-1}a^{-1})ab = e$.

Combining these two equations, we know, the inverse of (ab) is $b^{-1}a^{-1}$. $\square$

Proposition 4.

Proposition

Proposition 4. Let $\mathbb{G}$ *be a group,* $\forall g \in \mathbb{G}$, $(g^{-1})^{-1} = g$.

Proposition 4.

Proposition

Proposition 4. Let $\mathbb{G}$ be a group, $\forall g \in \mathbb{G}$, $(g^{-1})^{-1} = g$.

Proof.

By definition, $gg^{-1} = e$ and $g^{-1}(g^{-1})^{-1} = e$. Hence:

$$(g^{-1})^{-1} = (gg^{-1})(g^{-1})^{-1} = g(g^{-1}(g^{-1})^{-1}) = ge = g.$$

□

Proposition 4.

Proposition

Proposition 4. Let $\mathbb{G}$ be a group, $\forall g \in \mathbb{G}$, $(g^{-1})^{-1} = g$.

Proof.

By definition, $gg^{-1} = e$ and $g^{-1}(g^{-1})^{-1} = e$. Hence:

$$(g^{-1})^{-1} = (gg^{-1})(g^{-1})^{-1} = g(g^{-1}(g^{-1})^{-1}) = ge = g.$$

□

另一种思路.

该命题要证明的是 g^{-1} 的逆元是 g。根据 g^{-1} 的定义，“交换” g 与 g^{-1} 的位置，即得!

Proposition 5.

Proposition

Proposition 5. Let $\mathbb{G}$ *be a group, for any two elements* $a, b \in \mathbb{G}$*. Then the equation* $ax = b$ *and* $xa = b$ *have unique solutions in* $\mathbb{G}$*.*

Proposition 5.

Proposition

Proposition 5. Let $\mathbb{G}$ be a group, for any two elements $a, b \in \mathbb{G}$. Then the equation $ax = b$ and $xa = b$ have unique solutions in $\mathbb{G}$.

Proof.

- Existence. Such an x exists.
- Uniqueness. Suppose that x_1 and x_2 are both solutions......

Left as an exercise. $\square$

Proposition 6.

Proposition

Cancellation Law. Let $\mathbb{G}$ be a group, and $a, b, c \in \mathbb{G}$. Then $ba = ca$ implies $b = c$ and $ab = ac$ implies $b = c$.

Proposition 6.

Proposition

Cancellation Law. Let $\mathbb{G}$ be a group, and $a, b, c \in \mathbb{G}$. Then $ba = ca$ implies $b = c$ and $ab = ac$ implies $b = c$.

Proof.

Left as an exercise. $\square$

Proposition 6.

Proposition

Cancellation Law. Let $\mathbb{G}$ *be a group, and* $a, b, c \in \mathbb{G}$*. Then* $ba = ca$ *implies* $b = c$ *and* $ab = ac$ *implies* $b = c$*.*

Proof.

Left as an exercise. □

A little thought.

Where does "*Cancellation Law*" come from?

思考.

置换与消去律

在费尔马小定理和欧拉定理的证明中，依赖消去律可得：
对任意素数 p 和与 p 互素的正整数 a，
$\mathbb{Z}_p^* = a\mathbb{Z}_p^* = \{ai : \forall i \in \mathbb{Z}_p^*\}$;
对任意合数 n 和与 n 互素的正整数 a，$a\mathbb{Z}_n^* = \mathbb{Z}_n^*$。
请问，对任意的群 $\mathbb{G}$ 和群元 a，是否有
$\mathbb{G} = a\mathbb{G} = \{ag : \forall g \in \mathbb{G}\}$? 为什么?

Notations.

Let $\mathbb{G}$ be a group, and $g \in \mathbb{G}$. For $n \in \mathbb{N}$.

Notations.

- $g^0 = e$
- $g^n = \underbrace{g \cdot g \cdots g}_{\text{n times}}$
- $g^{-n} = \underbrace{g^{-1} \cdot g^{-1} \cdots g^{-1}}_{\text{n times}}$

Order.

The order of a finite group is the number of elements that it contains. If $\mathbb{G}$ is a group containing n elements, we write $|\mathbb{G}| = n$.

Defintions

Definition

(Subgroup.)（子群）

Let $\mathbb{G}$ be a group and $\mathbb{H}$ a subset of $\mathbb{G}$. If $\mathbb{H}$ is a group under group operation in $\mathbb{G}$, then $\mathbb{H}$ is said to be a subgroup of $\mathbb{G}$, denoted by $\mathbb{H} \leq \mathbb{G}$.

Examples of Subgroup.

Examples of Subgroup.

- For any group $\mathbb{G}$, there is a trivial subgroup $\{e\}$.
- The additive groups: $\mathbb{Z} \leq \mathbb{Q} \leq \mathbb{R} \leq \mathbb{C}$.
- $\forall n \in \mathbb{Z}$, $n\mathbb{Z} = \{kn | k \in \mathbb{Z}\}$ is a subgroup of $\mathbb{Z}$.

Examples of Subgroup.

Subgroup of $\mathbb{Z}_p^*$.

Let p be a prime, for all $i \in \mathbb{Z}_p^*$, compute $i^2 \bmod p$, form a set $\mathbb{S} = \{i^2 \bmod p, \forall i \in \mathbb{Z}_p^*\}$. Check that $\mathbb{S}$ is a group under the operation of multiplication, namely $\mathbb{S}$ is a subgroup of $\mathbb{Z}_p^*$. What is the order of $\mathbb{S}$?

Properties of Subgroup.

Exercise.

Write a program to play with $\mathbb{Z}_n^*$.

- Given an integer n, construct the multiplicative group $\mathbb{Z}_n^*$;
- Find a subgroup of the group $\mathbb{Z}_n^*$;
- Find a relation between the size of subgroup and the size of $\mathbb{Z}_n^*$.

Properties of subgroup.

Proposition

(Subgroup.) A nonempty subset $\mathbb{H}$ *of a group* $\mathbb{G}$ *is a subgroup of* $\mathbb{G}$ *if and only if* $\mathbb{H} \neq \emptyset$, *and* $ab^{-1} \in \mathbb{H}$ *for all* $a, b \in \mathbb{H}$.

Proof.

Two directions. The $\rightarrow$ part is easy. For $\leftarrow$ part, you need to check that $\mathbb{H}$ satisfies all the axioms of a group. $\square$

Cyclic Groups（循环群）.

Example of cyclic group.

Consider the following computation: Choose a number g from Z_p^* randomly, p is a prime, and compute:

$$\mathbb{S} = \{g, g^2, g^3, \cdots, g^i, \cdots\}$$

Cyclic Groups（循环群）.

Example of cyclic group.

Consider the following computation: Choose a number g from Z_p^* randomly, p is a prime, and compute:

$$\mathbb{S} = \{g, g^2, g^3, \cdots, g^i, \cdots\}$$

Questions:

- May $\mathbb{S}$ be finite?
- May $\mathbb{S}$ be a group? Why or why not?
- May $\mathbb{S}$ equals Z_p^*?

Cyclic Groups.

Example of cyclic group.

For example: For $p = 11$, choose $g = 4$, and compute:

$$\mathbb{S} = \{4, 4^2, 4^3, \cdots, g^i, \cdots\}$$

Cyclic Groups.

Example of cyclic group.

For example: For $p = 11$, choose $g = 4$, and compute:

$$\mathbb{S} = \{4, 4^2, 4^3, \cdots, g^i, \cdots\}$$

We will have:

$$\mathbb{S} = \{4, 5, 9, 3, 1\}$$

Certainy, it is finite and it is a group.

Cyclic Groups.

Example of cyclic group.

For example: For $p = 11$, choose $g = 4$, and compute:

$$\mathbb{S} = \{4, 4^2, 4^3, \cdots, g^i, \cdots\}$$

We will have:

$$\mathbb{S} = \{4, 5, 9, 3, 1\}$$

Certainy, it is finite and it is a group.Questions:

- What will we get if $g = 2$?
- What will we get if $g = 3$?

Cyclic Groups.

Theorem

Let $\mathbb{G}$ be a group and g be any element in $\mathbb{G}$. Then the set

$$\langle g \rangle = \{g^k : k \in \mathbb{Z}\}$$

is a subgroup of $\mathbb{G}$. We call $\langle g \rangle$ the cyclic group generated by g, and g is a generator of the group.

Proof.

Check the axioms. $\square$

Examples of Cyclic groups.

Some cyclic groups.

- $(\mathbb{Z}, +)$ is a cyclic group, while 1 is the generator.
- $(\mathbb{Z}_n, +)$ is a cyclic group, while 1 is the generator.
- $\langle i \rangle$ is a cyclic group, while i is the generator.

Examples of Cyclic groups.

Some cyclic groups.

- $(\mathbb{Z}, +)$ is a cyclic group, while 1 is the generator.
- $(\mathbb{Z}_n, +)$ is a cyclic group, while 1 is the generator.
- $\langle i \rangle$ is a cyclic group, while i is the generator.

Check.

Please check and know why.

Examples of Cyclic groups.

Some cyclic groups.

- Z_p^* is a cyclic group, while p is a prime.
- $\mathbb{Z}_n^*$ is *NOT* a cyclic group, while n is composite.

Examples of Cyclic groups.

Some cyclic groups.

- Z_p^* is a cyclic group, while p is a prime.
- $\mathbb{Z}_n^*$ is *NOT* a cyclic group, while n is composite.

Check.

Please check and know why.

Primitive Root（原根）

Definition

Let a and n be relatively prime integers with $n > 0$. The order of a modulo n is the smallest exponent $e \geq 1$ such that $a^e \equiv 1 \pmod{n}$. If the order of a modulo n equals to the largest possible order modulo n, then a is called a primitive root modulo n.

Primitive Root

Example

From last example, we know the order of 4 *modulo* 11 *is* 5, *and the order of* 2 *and* 3 *modulo n is* 10. *Since the largest possible order modulo* 11 *is* 10, *thus* 2 *and* 3 *are two primitive roots modulo* 11. *Using language of group, we may say that* $\mathbb{Z}_{11}$ *is a cyclic group generated by* 2 *or* 3, *and* 2 *and* 3 *are generators of* $\mathbb{Z}_{11}$.

Properties of Cyclic Groups.

Theorem

Let $\mathbb{G} = \langle g \rangle$ be a cyclic group of order n. Then $g^k = e$ if and only if n divides k.

Properties of Cyclic Groups.

Theorem

Let $\mathbb{G} = \langle g \rangle$ be a cyclic group of order n. Then $g^k = e$ if and only if n divides k.

Proof.

Note that, n is the least positive number s.t. $g^n = e$.

Properties of Cyclic Groups.

Theorem

Let $\mathbb{G} = \langle g \rangle$ be a cyclic group of order n. Then $g^k = e$ if and only if n divides k.

Proof.

Note that, n is the least positive number s.t. $g^n = e$.

1. The $\leftarrow$ part is trivial, since $g^k = g^{ns} = e$.

Properties of Cyclic Groups.

Theorem

Let $\mathbb{G} = \langle g \rangle$ be a cyclic group of order n. Then $g^k = e$ if and only if n divides k.

Proof.

Note that, n is the least positive number s.t. $g^n = e$.
1. The $\leftarrow$ part is trivial, since $g^k = g^{ns} = e$.
2. The $\rightarrow$ part. Suppose $g^k = e$. By division algorithm, $k = nq + r$, where $0 \leq r < n$. Hence,

$$e = g^k = g^{nq+r} = g^{nq} g^r = g^r.$$

Thus, $r = 0$. $\square$

Properties of Cyclic Groups.

Theorem

Let $\mathbb{G} = \langle g \rangle$ be a cyclic group of order n. If $h = g^k$ then the order of h is n/d, where $d = gcd(k, n)$.

Properties of Cyclic Groups.

Theorem

Let $\mathbb{G} = \langle g \rangle$ be a cyclic group of order n. If $h = g^k$ then the order of h is n/d, where $d = gcd(k, n)$.

Proof.

Let m be the least positive number s.t. $h^m = g^{km} = e$.

Properties of Cyclic Groups.

Theorem

Let $\mathbb{G} = \langle g \rangle$ be a cyclic group of order n. If $h = g^k$ then the order of h is n/d, where $d = gcd(k, n)$.

Proof.

Let m be the least positive number s.t. $h^m = g^{km} = e$.

1. Then $n \mid km$, equivalently, $(n/d) \mid (k/d)m$.

Properties of Cyclic Groups.

Theorem

Let $\mathbb{G} = \langle g \rangle$ be a cyclic group of order n. If $h = g^k$ then the order of h is n/d, where $d = gcd(k, n)$.

Proof.

Let m be the least positive number s.t. $h^m = g^{km} = e$.
1. Then $n \mid km$, equivalently, $(n/d) \mid (k/d)m$.
2. Since $d = gcd(k, n)$, n/d and k/d are relatively prime. Thus, $(n/d) \mid (k/d)m$ implies $(n/d) \mid m$. The smallest such m is n/d. $\square$

Properties of Cyclic Groups.

通过生成元找生成元

已知 2 是群 $\mathbb{Z}_{11}^*$ 的生成元，群 $\mathbb{Z}_{11}^*$ 的阶是 10，$2^3 = 8 \in \mathbb{Z}_{11}^*$，且 $gcd(3, 10) = 1$，所以 8 的阶是 10，即 8 也是一个生成元。5 不是生成元，因为 $5 = 2^4 \bmod 11$，$gcd(4, 10) = 2$。请读者自行验证以上结论。以上命题告诉我们，在知道某个元是生成元时，如何找到另一个生成元。

Properties of Cyclic Groups.

Corollary

Let $\mathbb{G} = \langle g \rangle$ be a cyclic group of order n then there are exactly $\phi(n)$ generators in $\mathbb{G}$.

Properties of Cyclic Groups.

Corollary

Let $\mathbb{G} = \langle g \rangle$ be a cyclic group of order n then there are exactly $\phi(n)$ generators in $\mathbb{G}$.

Proof.

There are n elements in $\mathbb{G}$ with the form g^i, for all $i \in \mathbb{Z}_n$. For arbitrary g^i, its order is n/d, where $d = gcd(i, n)$, then g^i is a generator when $d = 1$ which means i is relatively prime to n. There are $\phi(n)$ elements in $\mathbb{Z}_n$ are relatively prime to n, therefore there are $\phi(n)$ generators in $\mathbb{G}$. $\square$

Properties of Cyclic Groups.

Corollary

Let $\mathbb{G} = \langle g \rangle$ be a cyclic group of order p, where p is a prime, then all elements in $\mathbb{G}$ except e are generators.

Proof.

Trivially from Corollary 14. $\square$

Primitive Root Theorem

Theorem

(Primitive Root Theorem.) Every prime p has a primitive root modulo p, and there are exactly $\phi(p-1)$ primitive roots modulo p.

General Primitive Root Theorem

Theorem

If $n \in \mathbb{Z}$ are 2, 4, p^e and $2p^e$, for all primes $p > 2$ and all possitive integers e, then $\mathbb{Z}_n^*$ is cyclic.

Coset（陪集）

Definition of Coset.

Let $\mathbb{G}$ be a group and $\mathbb{H}$ a subgroup of $\mathbb{G}$. Define a left coset of $\mathbb{H}$ with representative $g \in \mathbb{G}$ to be the set

$$g\mathbb{H} = \{gh : h \in \mathbb{H}\}.$$

Right coset can be defined similarly by

$$\mathbb{H}g = \{hg : h \in \mathbb{H}\}.$$

Coset

Examples of Coset.

Recall our previous proof of Fermat's Little Theorem, we randomly choose a number $a \in \mathbb{Z}_p^*$, and prove

$$a\mathbb{Z}_p^* = \mathbb{Z}_p^*$$

It is similar in Eurler's Theorem. $\forall a \in \mathbb{Z}_n^*$,

$$a\mathbb{Z}_n^* = \mathbb{Z}_n^*$$

Coset

Examples of Coset.

Let $p = 11$, let $g = 4$, then $\mathbb{H} = \{g^i : i \in \mathbb{Z}\}$ is a subgroup of a $\mathbb{Z}_p^*$. Actually, $\mathbb{H} = \{1, 3, 4, 5, 9\}$. Compute:

- $\forall a \in \mathbb{H}$, what is $a\mathbb{H}$?
- $\forall a \notin \mathbb{H}$ and $a \in \mathbb{Z}_p^*$, what is $a\mathbb{H}$?

Properties of Coset.

The number of the elements in a coset.

Let $\mathbb{G}$ be a group and $\mathbb{H}$ a subgroup of $\mathbb{G}$. $\forall g \in \mathbb{G}$, the number of elements in $\mathbb{H}$ is the same as the number of elements in $g\mathbb{H}$.

Proof.

Define a map $\psi : \mathbb{H} \rightarrow g\mathbb{H}$ by $\psi(h) = gh$. Show the map is one-to-one and onto. (Please recall what we have done in the proof of Fermat's Little theorem.) □

Properties of Coset.

Identical or isolation.

Let $\mathbb{G}$ be a group and $\mathbb{H}$ a subgroup of $\mathbb{G}$. $\forall g_1, g_2 \in \mathbb{G}$, then $g_1\mathbb{H} = g_2\mathbb{H}$ or $g_1\mathbb{H} \cap g_2\mathbb{H} = \emptyset$.

Properties of Coset.

Identical or isolation.

Let $\mathbb{G}$ be a group and $\mathbb{H}$ a subgroup of $\mathbb{G}$. $\forall g_1, g_2 \in \mathbb{G}$, then $g_1\mathbb{H} = g_2\mathbb{H}$ or $g_1\mathbb{H} \cap g_2\mathbb{H} = \emptyset$.

Proof.

Suppose $\exists h_1, h_2 \in \mathbb{H}$ s.t. $g_1 h_1 = g_2 h_2$, we prove the $g_1\mathbb{H} \subseteq g_2\mathbb{H}$. Similarly, $g_2\mathbb{H} \subseteq g_1\mathbb{H}$. Then $g_1\mathbb{H} = g_2\mathbb{H}$. Note that:

$$\forall g_1 h \in g_1\mathbb{H}, g_1 h = g_1(h_1 h_1^{-1})h = g_2(h_2 h_1^{-1} h) \in g_2\mathbb{H}$$

□

Properties of Coset.

Partitioning of group $\mathbb{G}$.

Let $\mathbb{G}$ be a group and $\mathbb{H}$ a subgroup of $\mathbb{G}$. Then the left cosets of $\mathbb{H}$ in $\mathbb{G}$ partition $\mathbb{G}$.

Properties of Coset.

Partitioning of group $\mathbb{G}$.

Let $\mathbb{G}$ be a group and $\mathbb{H}$ a subgroup of $\mathbb{G}$. Then the left cosets of $\mathbb{H}$ in $\mathbb{G}$ partition $\mathbb{G}$.

Proof.

Nothing! Convince yourself that the cosets $g\mathbb{H}$ cover $\mathbb{G}$, and then recall the last proposition. Why cover? Note that $e \in \mathbb{H}$! $\square$

Lagrange's Theorem

Notation.

Let $\mathbb{G}$ be a finite group and $\mathbb{H}$ a subgroup of $\mathbb{G}$. Define the index of $\mathbb{H}$ in $\mathbb{G}$ to be the number of left cosets of $\mathbb{H}$ in $\mathbb{G}$. We denote the index by $[\mathbb{G} : \mathbb{H}]$.

Lagrange's Theorem.

Let $\mathbb{G}$ be a finite group and $\mathbb{H}$ a subgroup of $\mathbb{G}$. Then $|\mathbb{G}|/|\mathbb{H}| = [\mathbb{G} : \mathbb{H}]$ is the number of distinct left cosets of $\mathbb{H}$ in $\mathbb{G}$.

Proof.

The group $\mathbb{G}$ is partitioned in $[\mathbb{G} : \mathbb{H}]$ distinct left cosets. Each left coset has $|\mathbb{H}|$ elements; therefore, $|\mathbb{G}| = [\mathbb{G} : \mathbb{H}]|\mathbb{H}|$ $\square$

Corollaries from Lagrange's Theorem

Corollary

Suppose that $\mathbb{G}$ is a finite group and $g \in \mathbb{G}$. Then the order of g must divide $|\mathbb{G}|$.

Corollary

Let $\mathbb{G}$ be a group and $|\mathbb{G}| = p$ where p is a prime. Then $\mathbb{G}$ is cyclic and any $g \in \mathbb{G}$ such that $g \neq e$ is a generator.

Corollary

Let $\mathbb{H}$ and $\mathbb{K}$ be subgroups of a finite group $\mathbb{G}$ such that $\mathbb{K} \subset \mathbb{H} \subset \mathbb{G}$. Then

$$[\mathbb{G} : \mathbb{K}] = [\mathbb{G} : \mathbb{H}][\mathbb{H} : \mathbb{K}]$$

Corollaries from Lagrange's Theorem

Corollary

Fermat's Little Theorem.

$$a^{p-1} \equiv 1 \pmod{p}.$$

Corollary

Euler's Theorem.

$$a^{\phi(n)} \equiv 1 \pmod{n}.$$

Abstract Fermat's Little Theorem.

Theorem

(Abstract Fermat's Little Theorem.) Let $\mathbb{G}$ be a finite group with order n. Then for any $a \in \mathbb{G}$, $a^n = e$.