A Concrete Introduction to Number Theory and Algebra–群、子群、循环群

Libin Wang

Shool of Computer Science, South China Normal University

October 21, 2020

Table of contents

Motivation.

Motivation.

Last chapter, we extremely rely on *Cancellation Law*.
If $gcd(c, m) = 1$ and $ac \equiv bc \pmod{m}$, then

$$a \equiv b \pmod{m}.$$

Motivation.

Motivation.

Last chapter, we extremely rely on *Cancellation Law*.
If $gcd(c, m) = 1$ and $ac \equiv bc \pmod{m}$, then

$$a \equiv b \pmod{m}.$$

Question.

Actually, what is cancellation?

Motivation.

Ideas.

From $ac \equiv bc \pmod{m}$ to $a \equiv b \pmod{m}$, seemingly, we need division, actually we need multiplication:

$$acc^{-1} \equiv bcc^{-1} \pmod{m}.$$

Hence by $cc^{-1} \equiv 1 \pmod{m}$, we have

$$a \equiv b \pmod{m}.$$

Why c^{-1} exists? Because of $gcd(c, m) = 1$!

Motivation.

Additional conditions.

Need more conditions?

Motivation.

Additional conditions.

Need more conditions?

Yes, we need *association*:

$$(ac)c^{-1} \equiv a(cc^{-1}) \pmod{m},$$

and *closure*, which means we only consider numbers from 1 to $m-1$.

Motivation.

Wrap up.

Wrap these up. For a set $\mathbb{G}$ and an operator $\cdot$ on the elements, we need:

- *Closure*: $\forall a, b \in \mathbb{G}, \quad a \cdot b \in \mathbb{G}$.
- *Association*: $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.
- An element "1" called *identity*, s.t. $1 \cdot a = a \cdot 1 = a$.
- $\forall a \in G$, there exists $a^{-1} \in G$, such that $a \cdot a^{-1} = 1 = a^{-1}a$, called *inverse*.

Group（群）.

Definition.

Definition(Group). A group is a set $\mathbb{G}$ and an operator $\cdot$ on the elements, satisfies the following axioms:

- *Closure(封闭性)*: $\forall a, b \in \mathbb{G}, \quad a \cdot b \in \mathbb{G}$.
- *Association(结合律)*: $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.
- There is an element "$e \in \mathbb{G}$" called *identity(单位元)*, s.t. $e \cdot a = a \cdot e = a$.
- $\forall a \in G$, there exists $a^{-1} \in G$ called *inverse(逆元)*, such that $a \cdot a^{-1} = e = a^{-1} \cdot a$.

Examples of groups.

Groups.

- $(\mathbb{Z}, +)$ is a group, while $(\mathbb{Z}, \times)$ is not a group.
- $(\mathbb{Q}, \times)$ and $(\mathbb{R}, \times)$ are groups.

Examples of groups.

Groups.

- $(\mathbb{Z}, +)$ is a group, while $(\mathbb{Z}, \times)$ is not a group.
- $(\mathbb{Q}, \times)$ and $(\mathbb{R}, \times)$ are groups.

Check.

Please check and know why.

Examples of some important groups.

$\mathbb{Z}_n$

Let n be an integer, $\mathbb{Z}_n = \{0, 1, 2, \cdots, n-1\}$ forms a group under the operation of addition. However, $(\mathbb{Z}_n, \times)$ is not a group.

Examples of some important groups.

$\mathbb{Z}_n$

Let n be an integer, $\mathbb{Z}_n = \{0, 1, 2, \cdots, n-1\}$ forms a group under the operation of addition. However, $(\mathbb{Z}_n, \times)$ is not a group.

$\mathbb{Z}_p^*$

Let p be a prime number, $\mathbb{Z}_p^* = \{1, 2, \cdots, p-1\}$ forms a group under the operation of multiplication. (Recall Fermat's Little Theorem.)

Examples of some important groups.

$\mathbb{Z}_n$

Let n be an integer, $\mathbb{Z}_n = \{0, 1, 2, \cdots, n-1\}$ forms a group under the operation of addition. However, $(\mathbb{Z}_n, \times)$ is not a group.

$\mathbb{Z}_p^*$

Let p be a prime number, $\mathbb{Z}_p^* = \{1, 2, \cdots, p-1\}$ forms a group under the operation of multiplication. (Recall Fermat's Little Theorem.)

$\mathbb{Z}_n^*$

Let n be an integer, $\mathbb{Z}_n^* = \{a \in [1..n-1] \text{ and } gcd(a, n) = 1\}$ forms a group under the operation of multiplication. (Recall Euler's Theorem.)

Basic Properties of Groups.

Proposition

Proposition 1. The identity element in a group $\mathbb{G}$ is unique; that is, there exists only one element $e \in \mathbb{G}$ s.t. $eg = ge = g$ for all $g \in \mathbb{G}$.

Basic Properties of Groups.

Proposition

Proposition 1. The identity element in a group $\mathbb{G}$ is unique; that is, there exists only one element $e \in \mathbb{G}$ s.t. $eg = ge = g$ for all $g \in \mathbb{G}$.

Proof.

Suppose $\exists e, e' \in \mathbb{G}$ are identities. Then:

- $ee' = e'$
- $ee' = e$

Combining these two equations, we have $e = ee' = e'$. □

Proposition 2-3.

Proposition

Proposition 2. If $\forall g \in \mathbb{G}$, then the inverse of g, g^{-1}, is unique.

Proposition 2-3.

Proposition

Proposition 2. If $\forall g \in \mathbb{G}$, then the inverse of g, g^{-1}, is unique.

Proposition

Proposition 3. Let $\mathbb{G}$ be a group. If $a, b \in \mathbb{G}$, then $(ab)^{-1} = b^{-1}a^{-1}$.

Proof.

By construction.

- $ab(b^{-1}a^{-1}) = e$.
- $(b^{-1}a^{-1})ab = e$.

Combining these two equations, we know, the inverse of (ab) is $b^{-1}a^{-1}$. □

Proposition 4.

Proposition

Proposition 4. Let $\mathbb{G}$ be a group, $\forall g \in \mathbb{G}$, $(g^{-1})^{-1} = g$.

Proposition 4.

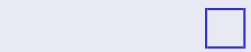
Proposition

Proposition 4. Let $\mathbb{G}$ be a group, $\forall g \in \mathbb{G}$, $(g^{-1})^{-1} = g$.

Proof.

By definition, $gg^{-1} = e$ and $g^{-1}(g^{-1})^{-1} = e$. Hence:

$$(g^{-1})^{-1} = (gg^{-1})(g^{-1})^{-1} = g(g^{-1}(g^{-1})^{-1}) = ge = g.$$



Proposition 5.

Proposition

Proposition 5. Let $\mathbb{G}$ be a group, for any two elements $a, b \in \mathbb{G}$. Then the equation $ax = b$ and $xa = b$ have unique solutions in $\mathbb{G}$.

Proposition 5.

Proposition

Proposition 5. Let $\mathbb{G}$ be a group, for any two elements $a, b \in \mathbb{G}$. Then the equation $ax = b$ and $xa = b$ have unique solutions in $\mathbb{G}$.

Proof.

- Existence. Such an x exists.
- Uniqueness. Suppose that x_1 and x_2 are both solutions.....

Left as an exercise. □

Proposition 6.

Proposition

Cancellation Law. Let $\mathbb{G}$ be a group, and $a, b, c \in \mathbb{G}$. Then $ba = ca$ implies $b = c$ and $ab = ac$ implies $b = c$.

Proposition 6.

Proposition

Cancellation Law. Let $\mathbb{G}$ be a group, and $a, b, c \in \mathbb{G}$. Then $ba = ca$ implies $b = c$ and $ab = ac$ implies $b = c$.

Proof.

Left as an exercise. □

Proposition 6.

Proposition

Cancellation Law. Let $\mathbb{G}$ be a group, and $a, b, c \in \mathbb{G}$. Then $ba = ca$ implies $b = c$ and $ab = ac$ implies $b = c$.

Proof.

Left as an exercise. □

A little thought.

Where does "*Cancellation Law*" come from?

思考.

置换与消去律

在费尔马小定理和欧拉定理的证明中，依赖消去律可得：
对任意素数 p 和与 p 互素的正整数 a，
$\mathbb{Z}_p^* = a\mathbb{Z}_p^* = \{ai : \forall i \in \mathbb{Z}_p^*\}$;
对任意合数 n 和与 n 互素的正整数 a，$a\mathbb{Z}_n^* = \mathbb{Z}_n^*$。
请问，对任意的群 $\mathbb{G}$ 和群元 a，是否有
$\mathbb{G} = a\mathbb{G} = \{ag : \forall g \in \mathbb{G}\}$? 为什么?

Notations.

Let $\mathbb{G}$ be a group, and $g \in \mathbb{G}$. For $n \in \mathbb{N}$.

Notations.

- $g^0 = e$
- $g^n = \underbrace{g \cdot g \cdots g}_{\text{n times}}$
- $g^{-n} = \underbrace{g^{-1} \cdot g^{-1} \cdots g^{-1}}_{\text{n times}}$

Order.

The order of a finite group is the number of elements that it contains. If $\mathbb{G}$ is a group containing n elements, we write $|\mathbb{G}| = n$.

Defintions

Definition

(Subgroup.)（子群）
Let $\mathbb{G}$ be a group and $\mathbb{H}$ a subset of $\mathbb{G}$. If $\mathbb{H}$ is a group under group operation in $\mathbb{G}$, then $\mathbb{H}$ is said to be a subgroup of $\mathbb{G}$, denoted by $\mathbb{H} \leq \mathbb{G}$.

Examples of Subgroup.

Examples of Subgroup.

- For any group $\mathbb{G}$, there is a trivial subgroup $\{e\}$.
- The additive groups: $\mathbb{Z} \leq \mathbb{Q} \leq \mathbb{R} \leq \mathbb{C}$.
- $\forall n \in \mathbb{Z}$, $n\mathbb{Z} = \{kn | k \in \mathbb{Z}\}$ is a subgroup of $\mathbb{Z}$.

Examples of Subgroup.

Subgroup of $\mathbb{Z}_p^*$.

Let p be a prime, for all $i \in \mathbb{Z}_p^*$, compute $i^2 \bmod p$, form a set $\mathbb{S} = \{i^2 \bmod p, \forall i \in \mathbb{Z}_p^*\}$. Check that $\mathbb{S}$ is a group under the operation of multiplication, namely $\mathbb{S}$ is a subgroup of $\mathbb{Z}_p^*$. What is the order of $\mathbb{S}$?

Properties of Subgroup.

Exercise.

Write a program to play with $\mathbb{Z}_n^*$.

- Given an integer n, construct the multiplicative group $\mathbb{Z}_n^*$;
- Find a subgroup of the group $\mathbb{Z}_n^*$;
- Find a relation between the size of subgroup and the size of $\mathbb{Z}_n^*$.

Properties of subgroup.

Proposition

(Subgroup.) A nonempty subset $\mathbb{H}$ *of a group* $\mathbb{G}$ *is a subgroup of* $\mathbb{G}$ *if and only if* $\mathbb{H} \neq \emptyset$*, and* $ab^{-1} \in \mathbb{H}$ *for all* $a, b \in \mathbb{H}$*.*

Proof.

Two directions. The $\rightarrow$ part is easy. For $\leftarrow$ part, you need to check that $\mathbb{H}$ satisfies all the axioms of a group. $\square$

Cyclic Groups（循环群）.

Example of cyclic group.

Consider the following computation: Choose a number g from Z_p^* randomly, p is a prime, and compute:

$$\mathbb{S} = \{g, g^2, g^3, \cdots, g^j, \cdots\}$$

Cyclic Groups（循环群）.

Example of cyclic group.

Consider the following computation: Choose a number g from Z_p^* randomly, p is a prime, and compute:

$$\mathbb{S} = \{g, g^2, g^3, \cdots, g^j, \cdots\}$$

Questions:

- May $\mathbb{S}$ be finite?
- May $\mathbb{S}$ be a group? Why or why not?
- May $\mathbb{S}$ equals Z_p^*?

Cyclic Groups.

Example of cyclic group.

For example: For $p = 11$, choose $g = 4$, and compute:

$$\mathbb{S} = \{4, 4^2, 4^3, \cdots, g^j, \cdots\}$$

Cyclic Groups.

Example of cyclic group.

For example: For $p = 11$, choose $g = 4$, and compute:

$$\mathbb{S} = \{4, 4^2, 4^3, \cdots, g^j, \cdots\}$$

We will have:

$$\mathbb{S} = \{4, 5, 9, 3, 1\}$$

Certainy, it is finite and it is a group.

Cyclic Groups.

Example of cyclic group.

For example: For $p = 11$, choose $g = 4$, and compute:

$$\mathbb{S} = \{4, 4^2, 4^3, \cdots, g^j, \cdots\}$$

We will have:

$$\mathbb{S} = \{4, 5, 9, 3, 1\}$$

Certainy, it is finite and it is a group.Questions:

- What will we get if $g = 2$?
- What will we get if $g = 3$?

Cyclic Groups.

Theorem

Let $\mathbb{G}$ be a group and g be any element in $\mathbb{G}$. Then the set

$$\langle g \rangle = \{g^k : k \in \mathbb{Z}\}$$

is a subgroup of $\mathbb{G}$. We call $\langle g \rangle$ the cyclic group generated by g, and g is a generator of the group.

Proof.

Check the axioms. □

Primitive Root（原根）

Definition

Let a and n be relatively prime integers with $n > 0$. The order of a modulo n is the smallest exponent $e \geq 1$ such that $a^e \equiv 1 \pmod{n}$. If the order of a modulo n equals to the largest possible order modulo n, then a is called a primitive root modulo n.

Primitive Root

Example

From last example, we know the order of 4 *modulo* 11 *is* 5*, and the order of* 2 *and* 3 *modulo n is* 10*. Since the largest possible order modulo* 11 *is* 10*, thus* 2 *and* 3 *are two primitive roots modulo* 11*. Using language of group, we may say that* $\mathbb{Z}_{11}$ *is a cyclic group generated by* 2 *or* 3*, and* 2 *and* 3 *are generators of* $\mathbb{Z}_{11}$*.*

Properties of Cyclic Groups.

Theorem

Let $\mathbb{G} = \langle g \rangle$ be a cyclic group of order n. Then $g^k = e$ if and only if n divides k.

Properties of Cyclic Groups.

Theorem

Let $\mathbb{G} = \langle g \rangle$ be a cyclic group of order n. Then $g^k = e$ if and only if n divides k.

Proof.

Note that, n is the least positive number s.t. $g^n = e$.

Properties of Cyclic Groups.

Theorem

Let $\mathbb{G} = \langle g \rangle$ be a cyclic group of order n. Then $g^k = e$ if and only if n divides k.

Proof.

Note that, n is the least positive number s.t. $g^n = e$.

1. The $\leftarrow$ part is trivial, since $g^k = g^{ns} = e$.

Properties of Cyclic Groups.

Theorem

Let $\mathbb{G} = \langle g \rangle$ be a cyclic group of order n. Then $g^k = e$ if and only if n divides k.

Proof.

Note that, n is the least positive number s.t. $g^n = e$.

1. The $\leftarrow$ part is trivial, since $g^k = g^{ns} = e$.
2. The $\rightarrow$ part. Suppose $g^k = e$. By division algorithm, $k = nq + r$, where $0 \leq r < n$. Hence,

$$e = g^k = g^{nq+r} = g^{nq} g^r = g^r.$$

Thus, $r = 0$. □

Properties of Cyclic Groups.

Theorem

Let $\mathbb{G} = \langle g \rangle$ be a cyclic group of order n. If $h = g^k$ then the order of h is n/d, where $d = gcd(k, n)$.

Properties of Cyclic Groups.

Theorem

Let $\mathbb{G} = \langle g \rangle$ be a cyclic group of order n. If $h = g^k$ then the order of h is n/d, where $d = gcd(k, n)$.

Proof.

Let m be the least positive number s.t. $h^m = g^{km} = e$.

Properties of Cyclic Groups.

Theorem

Let $\mathbb{G} = \langle g \rangle$ be a cyclic group of order n. If $h = g^k$ then the order of h is n/d, where $d = gcd(k, n)$.

Proof.

Let m be the least positive number s.t. $h^m = g^{km} = e$.
1. Then $n \mid km$, equivalently, $(n/d) \mid (k/d)m$.

Properties of Cyclic Groups.

Theorem

Let $\mathbb{G} = \langle g \rangle$ be a cyclic group of order n. If $h = g^k$ then the order of h is n/d, where $d = gcd(k, n)$.

Proof.

Let m be the least positive number s.t. $h^m = g^{km} = e$.

1. Then $n \mid km$, equivalently, $(n/d) \mid (k/d)m$.
2. Since $d = gcd(k, n)$, n/d and k/d are relatively prime. Thus, $(n/d) \mid (k/d)m$ implies $(n/d) \mid m$. The smallest such m is n/d. □

Properties of Cyclic Groups.

通过生成元找生成元

已知 2 是群 $\mathbb{Z}_{11}^*$ 的生成元，群 $\mathbb{Z}_{11}^*$ 的阶是 10，$2^3 = 8 \in \mathbb{Z}_{11}^*$，且 $gcd(3, 10) = 1$，所以 8 的阶是 10，即 8 也是一个生成元。5 不是生成元，因为 $5 = 2^4 \bmod 11$，$gcd(4, 10) = 2$。请读者自行验证以上结论。以上命题告诉我们，在知道某个元是生成元时，如何找到另一个生成元。

Properties of Cyclic Groups.

Corollary

Let $\mathbb{G} = \langle g \rangle$ be a cyclic group of order n then there are exactly $\phi(n)$ generators in $\mathbb{G}$.

Properties of Cyclic Groups.

Corollary

Let $\mathbb{G} = \langle g \rangle$ be a cyclic group of order n then there are exactly $\phi(n)$ generators in $\mathbb{G}$.

Proof.

There are n elements in $\mathbb{G}$ with the form g^i, for all $i \in \mathbb{Z}_n$. For arbitrary g^i, its order is n/d, where $d = gcd(i, n)$, then g^i is a generator when $d = 1$ which means i is relatively prime to n. There are $\phi(n)$ elements in $\mathbb{Z}_n$ are relatively prime to n, therefore there are $\phi(n)$ generators in $\mathbb{G}$. □

Properties of Cyclic Groups.

Corollary

Let $\mathbb{G} = \langle g \rangle$ be a cyclic group of order p, where p is a prime, then all elements in $\mathbb{G}$ except e are generators.

Proof.

Trivially from Corollary 14. □

Primitive Root Theorem

Theorem

(Primitive Root Theorem.) Every prime p has a primitive root modulo p, and there are exactly $\phi(p-1)$ primitive roots modulo p.

General Primitive Root Theorem

Theorem

If $n \in \mathbb{Z}$ are 2, 4, p^e and $2p^e$, for all primes $p > 2$ and all possitive integers e, then $\mathbb{Z}_n^*$ is cyclic.

Coset（陪集）

Definition of Coset.

Let $\mathbb{G}$ be a group and $\mathbb{H}$ a subgroup of $\mathbb{G}$. Define a left coset of $\mathbb{H}$ with representative $g \in \mathbb{G}$ to be the set

$$g\mathbb{H} = \{gh : h \in \mathbb{H}\}.$$

Right coset can be defined similarly by

$$\mathbb{H}g = \{hg : h \in \mathbb{H}\}.$$

Coset

Examples of Coset.

Recall our previous proof of Fermat's Little Theorem, we randomly choose a number $a \in \mathbb{Z}_p^*$, and prove

$$a\mathbb{Z}_p^* = \mathbb{Z}_p^*$$

It is similar in Eurler's Theorem. $\forall a \in \mathbb{Z}_n^*$,

$$a\mathbb{Z}_n^* = \mathbb{Z}_n^*$$

Coset

Examples of Coset.

Let $p = 11$, let $g = 4$, then $\mathbb{H} = \{g^i : i \in \mathbb{Z}\}$ is a subgroup of a $\mathbb{Z}_p^*$. Actually, $\mathbb{H} = \{1, 3, 4, 5, 9\}$. Compute:

- $\forall a \in \mathbb{H}$, what is $a\mathbb{H}$?
- $\forall a \notin \mathbb{H}$ and $a \in \mathbb{Z}_p^*$, what is $a\mathbb{H}$?

Properties of Coset.

The number of the elements in a coset.

Let $\mathbb{G}$ be a group and $\mathbb{H}$ a subgroup of $\mathbb{G}$. $\forall g \in \mathbb{G}$, the number of elements in $\mathbb{H}$ is the same as the number of elements in $g\mathbb{H}$.

Proof.

Define a map $\psi : \mathbb{H} \to g\mathbb{H}$ by $\psi(h) = gh$. Show the map is one-to-one and onto.(Please reall what we have done in the proof of Fermat's Little theorem.) □

Properties of Coset.

Identical or isolation.

Let $\mathbb{G}$ be a group and $\mathbb{H}$ a subgroup of $\mathbb{G}$. $\forall g_1, g_2 \in \mathbb{G}$, then $g_1\mathbb{H} = g_2\mathbb{H}$ or $g_1\mathbb{H} \cap g_2\mathbb{H} = \emptyset$.

Properties of Coset.

Identical or isolation.

Let $\mathbb{G}$ be a group and $\mathbb{H}$ a subgroup of $\mathbb{G}$. $\forall g_1, g_2 \in \mathbb{G}$, then $g_1\mathbb{H} = g_2\mathbb{H}$ or $g_1\mathbb{H} \cap g_2\mathbb{H} = \emptyset$.

Proof.

Suppose $\exists h_1, h_2 \in \mathbb{H}$ s.t. $g_1 h_1 = g_2 h_2$, we prove the $g_1\mathbb{H} \subseteq g_2\mathbb{H}$. Similarly, $g_2\mathbb{H} \subseteq g_1\mathbb{H}$. Then $g_1\mathbb{H} = g_2\mathbb{H}$. Note that:

$$\forall g_1 h \in g_1\mathbb{H}, g_1 h = g_1(h_1 h_1^{-1})h = g_2(h_2 h_1^{-1} h) \in g_2\mathbb{H}$$

Properties of Coset.

Partitioning of group $\mathbb{G}$.

Let $\mathbb{G}$ be a group and $\mathbb{H}$ a subgroup of $\mathbb{G}$. Then the left cosets of $\mathbb{H}$ in $\mathbb{G}$ partition $\mathbb{G}$.

Properties of Coset.

Partitioning of group $\mathbb{G}$.

Let $\mathbb{G}$ be a group and $\mathbb{H}$ a subgroup of $\mathbb{G}$. Then the left cosets of $\mathbb{H}$ in $\mathbb{G}$ partition $\mathbb{G}$.

Proof.

Nothing! Convince yourself that the cosets $g\mathbb{H}$ cover $\mathbb{G}$, and then recall the last proposition. Why cover? Note that $e \in \mathbb{H}$! □

Lagrange's Theorem

Notation.

Let $\mathbb{G}$ be a group and $\mathbb{H}$ a subgroup of $\mathbb{G}$. Define the index of $\mathbb{H}$ in $\mathbb{G}$ to be the number of left cosets of $\mathbb{H}$ in $\mathbb{G}$. We denote the index by $[\mathbb{G} : \mathbb{H}]$.

Lagrange's Theorem.

Let $\mathbb{G}$ be a group and $\mathbb{H}$ a subgroup of $\mathbb{G}$. Then $|\mathbb{G}|/|\mathbb{H}| = [\mathbb{G} : \mathbb{H}]$ is the number of distinct left cosets of $\mathbb{H}$ in $\mathbb{G}$.

Proof.

The group $\mathbb{G}$ is partitioned in $[\mathbb{G} : \mathbb{H}]$ distinct left cosets. Each left coset has $|\mathbb{H}|$ elements; therefore, $|\mathbb{G}| = [\mathbb{G} : \mathbb{H}]|\mathbb{H}|$ □

Corollaries from Lagrange's Theorem

Corollary

Suppose that $\mathbb{G}$ is a finite group and $g \in \mathbb{G}$. Then the order of g must divide $|\mathbb{G}|$.

Corollary

Let $\mathbb{G}$ be a group and $|\mathbb{G}| = p$ where p is a prime. Then $\mathbb{G}$ is cyclic and any $g \in \mathbb{G}$ such that $g \neq e$ is a generator.

Corollary

Let $\mathbb{H}$ and $\mathbb{K}$ be subgroups of a finite group $\mathbb{G}$ such that $\mathbb{K} \subset \mathbb{H} \subset \mathbb{G}$. Then

$$[\mathbb{G} : \mathbb{K}] = [\mathbb{G} : \mathbb{H}][\mathbb{H} : \mathbb{K}]$$

Corollaries from Lagrange's Theorem

Corollary

Fermat's Little Theorem.

$$a^{p-1} \equiv 1 \pmod{p}.$$

Corollary

Euler's Theorem.

$$a^{\phi(n)} \equiv 1 \pmod{n}.$$

Abstract Fermat's Little Theorem.

Theorem

(Abstract Fermat's Little Theorem.) Let $\mathbb{G}$ be a finite group with order n. Then for any $a \in \mathbb{G}$, $a^n = e$.

Exercises.

(Exercise 0.)

1. $\forall a, b \in \mathbb{G}$, prove that $ab^n a^{-1} = (aba^{-1})^n$ for $n \in \mathbb{Z}$.
2. $\forall a, b \in \mathbb{G}$, $(ab)^2 = a^2 b^2$, prove that the group $\mathbb{G}$ is abelian.
3. Prove that the inverse of $g_0 g_1 \cdots g_n = g_n^{-1} g_{n-1}^{-1} \cdots g_0^{-1}$.

Exercises.

(Exercise 1.)

1. Prove the Fermat's Little Theorem using Group Theory.
2. Prove the Euler's Theorem using Group Theory.

Exercises.

(Exercise 2.)

Suppose that q is a prime and $p = 2 * q + 1$ is also a prime. Let $g = h^2$ is not equal to 1, where h is a random number choosen from $\mathbb{Z}_p$. Certainly, $\langle g \rangle$ is a cyclic group.

(a) Write a python(or Sage) program to generate the cyclic group $\langle g \rangle$.

(b) What is the order of $\langle g \rangle$, and why?

(c) How many generators are there in the group $\langle g \rangle$? Why?