SEIR model with age-classes

March 19, 2020

We want to take a SEIR model with 3 classes of infection and modify it to account for different population ages. In the following, unless specified otherwise, bold quantities are n-dimensional vectors, n being the number of classes of age. *E.g.* $\mathbf{S}$ is composed by elements S_i, with $i \in \{1, \ldots, n\}$. To maintain lower indeces as vectorial indeces, I also defined $\mathbf{I}^{(k)}$, with $k \in \{1, 2, 3\}$ the three infection classes. Hence, *e.g.* $I_1^{(2)}$ is the number of individuals of classe of age 1 in the second infection class (severe infections). The first assumption that could simplify the problem by a lot is that individuals who progress to further infection stages (as well as individuals recovering and dying) do not change class of age. With respect to the original model with 3 classes of age, we will have terms that represent transmission between age classes. The equation for $\dot{\mathbf{S}}$ is then

$$\dot{\mathbf{S}} = -\sum_{k=1}^{3} \mathbf{B}^{(k)} \mathbf{I}^{(k)} \ , \tag{1}$$

where $\mathbf{B}^{(k)}$ are the 3 matrices of transmission from infected to susceptible individuals, with elements $\beta_{ij}^{(k)}$ indicating the probability of an individual of age j in the severity class k to infect a susceptible individual of age i. At this point we can of course define

$$\mathbf{B}^{(k)} = \beta^{(k)} \mathbf{C} \ ,$$

with $\mathbf{C}$ the contact matrix between different classes of age with elements c_{ji}, and $\beta^{(k)}$ the probability of any individual in the severity class k to infect a susceptible individual. Notice that this is a specific case in which the infection rate is not dependent on age; we will continue with this assumption. *E.g.*, for an individual of class of age 1 in a model with only two classes of age, this would look as

$$\dot{S}_1 = -c_{11}\beta^{(1)} I_1^{(1)} - c_{12}\beta^{(1)} I_2^{(1)} - c_{11}\beta^{(2)} I_1^{(2)} - c_{12}\beta^{(2)} I_2^{(2)} - c_{11}\beta^{(3)} I_1^{(3)} - c_{12}\beta^{(3)} I_2^{(3)} \ .$$

With the same logic, let's define E:

$$\dot{\mathbf{E}} = -\mathbf{a}\mathbf{E} + \sum_{k=1}^{3} \beta^{(k)} \mathbf{C} \mathbf{I}^{(k)} \ , \tag{2}$$

where $\mathbf{a}$ is the diagonal matrix with the rates of progression per class of age, with diagonal elements a_i. For the same example showed before, an element of class of age 1 would look as

$$\dot{E}_1 = -E_1 a_1 + c_{11}\beta^{(1)} I_1^{(1)} + c_{12}\beta^{(1)} I_2^{(1)} + c_{11}\beta^{(2)} I_1^{(2)} + c_{12}\beta^{(2)} I_2^{(2)} + c_{11}\beta^{(3)} I_1^{(3)} + c_{12}\beta^{(3)} I_2^{(3)} \ .$$

The three infectious classes look, you guessed it, as

$$\dot{\mathbf{I}}^{(1)} = \mathbf{a}\mathbf{E} - (\mathbf{\Gamma}^{(1)} + \mathbf{p}^{(1)}) \mathbf{I}^{(1)} \ , \tag{3}$$

$$\dot{\mathbf{I}}^{(2)} = \mathbf{p}^{(1)}\mathbf{I}^{(1)} - (\mathbf{\Gamma}^{(2)} + \mathbf{p}^{(2)})\mathbf{I}^{(2)} \ , \tag{4}$$

$$\dot{\mathbf{I}}^{(3)} = \mathbf{p}^{(2)}\mathbf{I}^{(2)} - (\mathbf{\Gamma}^{(3)} + \mathbf{M})\mathbf{I}^{(3)} \ , \tag{5}$$

with $\mathbf{\Gamma}^{(k)}$ the diagonal matrices (diagonal elements $\gamma_i^{(k)}$, $i \in \{1, \dots n\}$) of rates at which infected individuals of infection class k recover, and $\mathbf{p}^{(k)}$ are the diagonal matrices of the rates at which infected individuals in class k progress to class $k+1$. $\mathbf{M}$, is the matrix of mortality rates of individuals in the most severe stage of infection, with elements μ_i. *E.g.*, an element of $\dot{\mathbf{I}}^{(1)}$ would look as

$$\dot{I}_1^{(1)} = E_1 a_1 - (\gamma_1^{(1)} + p_1) I_1^{(1)} \ .$$

Finally, let's take a look at $\mathbf{R}$ and $\mathbf{M}$:

$$\dot{\mathbf{R}} = \sum_{k=1}^{3} \mathbf{\Gamma}^{(k)}\mathbf{I}^{(k)} \ , \tag{6}$$

$$\dot{\mathbf{D}} = \mathbf{M}\mathbf{I}^{(3)} \ , \tag{7}$$

with respective example elements

$$R_1 = \gamma_1^{(1)} I_1^{(1)} + \gamma_1^{(2)} I_1^{(2)} + \gamma_1^{(3)} I_1^{(3)} \ ,$$

$$D_1 = \mu_1 I_1^{(3)} \ .$$

The only equations that couple among classes of age are those for $\mathbf{S}$ and $\mathbf{E}$, while the others behave as n uncoupled equations. Computational problems can be addressed by reducing n, of course, or by assuming many elements of the cross-age matrices $\mathbf{B}^{(k)}$ to be null, in order to reduce coupling. I'm not sure about the feasability of the whole model though, hence any idea of how to approximate the model would be welcome.