SOME PROPERTIES AND APPLICATION OF CUTSET OF A GRAPH

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Abstract. Graph theory is one of the most important and basic topics of discrete mathematics in Mathematics. In all sectors of science graph theory has a great impact. The common use of graphs occurs in Computer Science, Physic, Biology, Finance and Chemistry except within Mathematics itself. Our main objective is to represent the cut-set, another type of subgraph of a connected graph. If deleting a certain number of edges from a graph makes it disconnected, then those set of deleted edges are called the cutset of the graph. Properties of cut-sets and its application will be discussed. When examining the characteristics of communication and transportation networks, cut-sets are crucial, for instance if we want to know if there are any network weak points that may be strengthened.

1 Introduction

One approach used in the study of connected graphs is to ask 'how connected is a connected graph? In this paper we shall study the cutset that is a type of subgraph of a connected graph G whose removal from G disconnects G [1]. Cutsets are of great importance in studying properties of communication and transportation networks. When examining the characteristics of communication and transportation networks, cutsets are crucial.

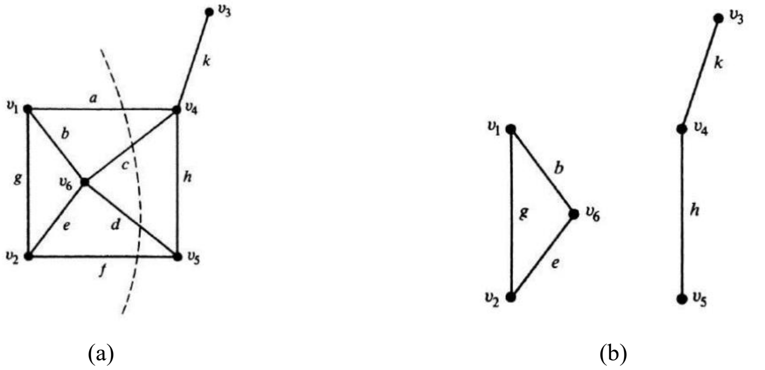

Fig. 1. Removal of a cut-set $\{a, c, d, f\}$ from a graph "cuts" it into two.

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Consider the six vertices in Fig.1 (a), for instance, as six cities connected by telephone lines. If there are any areas where the network is vulnerable and needs to be strengthened with more phone lines, we want to know about them. The cutset of the graph with the fewest edges is the most vulnerable when we examine all of the cutsets. In Fig. 4-1(a), the removal of just one edge can completely cut off the city represented by vertex v_3 from the rest of the network. In this chapter we shall study the *cutset***.** Properties of cutsets and their applications will be covered.

A traffic control problem at an intersection can be efficiently modeled as a graph. Cut-set of a graph can be used to study the most efficient route or the traffic control system to direct the traffic flow to its maximum capacity using the minimum number of edges.[2]

As an example of cutset application that will be discussed here is when we are given n stations that are to be connected by means of e lines (telephone lines, bridges, railroads, tunnels, or highways) where $e \geq n - 1$. What is the best way of connecting? By "best" we mean that the network should be as invulnerable to destruction of individual stations and individual lines as possible. In other words, construct a graph with *n* vertices and *e* edges that has the maximum possible edge connectivity and vertex connectivity.

2 Method and Terminology

The method used is literature study with the initial step being to study properties of cutsets and then its applications in traffic control problem.

In this section, some definitions related to the graph theory have been discussed which are important for representing our main objective in the later sections.

A graph G consists of a non-empty finite set V(G) of elements called vertices, and a finite family E(G) of unordered pairs of (not necessarily distinct) elements of V(G) called edges. A subgraph of a graph G is a graph, each of whose vertices belongs to V(G) and each of whose edges belongs to E{G). A walk consists of sequence of edges. A walk in which no vertex appears more than once is called a path.[1] A graph is connected if there is a path between each pair of vertices. and disconnected otherwise. Any disconnected graph G can be expressed as the union of connected graphs, each of which is a component of G. Given a graph G, a walk in G is a finite sequence of edges of the form also denoted by in which any two consecutive edges are adjacent or identical. Such a walk determines a sequence of vertices. We call the initial vertex and the final vertex of the walk, and speak of a walk from to. A disconnecting set in a connected graph G is a set of edges whose removal disconnects G. We further define a cutset to be a disconnecting set, no proper subset of which is a disconnecting set. It is possible for a walk to begin and end at the same vertex. Such a walk is called a closed walk. A closed walk in which no vertex (except the initial and the final vertex) appears more than once is called a circuit or cycle. A tree is a connected graph without any circuits. A tree T is said to be a spanning tree of a connected graph G if T is a subgraph of G and T contains all vertices of G.

3 Results and Discussion

In this section, it is explained the results of research and at the same time is given the comprehensive discussion. The discussion can be made in several sub-sections.

3.1 . Some Properties of Cutset

In Figure 1, the set of edges {*a*, *c*, *d*, *f*} is a cutset. There are many other cutsets, such as {*a*, *b*, *g*} , {*a*, *b*, *e*, *f*} , and {*d*, *h*, *f*}. Edge {*k*} alone is also a cutset. The set of edges {*a*, *c*, *h*, *d*} ,on the other hand, is not a cutset, because one of its proper subsets, {*a*, *c*, *h*}, is a cutset.

The following are some properties [3] about a cutset .

Theorem 1. Every edge of a tree is a cut-set.

Proof : Since removal of any edge from a tree breaks the tree into two parts, so every edge of a tree is a cut-set.

Theorem 2. Every cutset in a connected graph *G* must contain at least one branch of every spanning tree of *G*.

Proof : Let *T* is a spanning tree of *G* and *S* is arbitrary subset in *G* that is not to have any edge in common with *T*. Then removal *S* from *G* would not disconnect the graph. So, *S* is not a cutset in *G*.

Theorem 3. Every circuit has an even number of edges in common with any cutset.

Proof : Consider a cut-set *S* in graph *G* (Figure 2). Let the removal of *S* partition the vertices of *G* into two (mutually exclusive or disjoint) subsets V_1 and V_2. Consider a circuit Γ in *G*. If all the vertices in Γ are entirely within vertex set V_1 (or V_2), the number of edges common to *S* and Γ is zero; that is, $N(S \cap \Gamma) = 0$, an even number. If, on the other hand, some vertices in Γ are in V_1 and some in V_2, we traverse back and forth between the sets *V*1 and *V*2 as we traverse the circuit (see Figure 2).

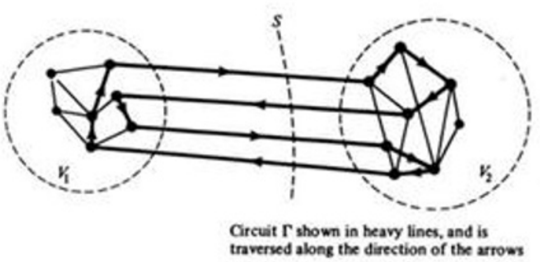

Fig.2. Circuit and a cutset in *G*.

3.2 . Application of cutset on communication and transportation networks

Each cutset of a connected graph G consists of a certain number of edges. The number of edges in the smallest cutset (i.e., cutset with fewest number of edges) is defined as the **edge connectivity** of G. Equivalently, the edge connectivity of a connected graph can be defined as the minimum number of edges whose removal (i.e., deletion) reduces the rank of the graph by one. The edge connectivity of a tree, for instance, is one. The edge connectivities of the graphs in Figure 1(a), 3, and 4 are one, two, and three, respectively.

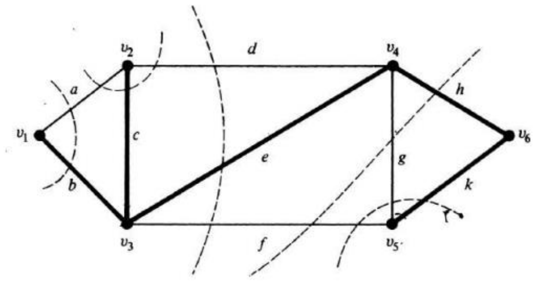

Fig.3. Graph with edge connectivity two and vertex connectivity two.

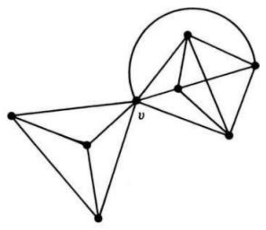

Fig.4. Graph with edge connectivity three and vertex connectivity one.

On examining the graph in Figure 4, we find that although removal of no single edge (or even a pair of edges) disconnects the graph, the removal of the single vertex v does. Therefore, we define another analogous term called vertex connectivity. The vertex connectivity (or simply connectivity) of a connected graph G is defined as the minimum number of vertices whose removal from G leaves the remaining graph disconnected. Again, the vertex connectivity of a tree is one. The vertex connectivities of the graphs in Figure 1(a), 3, and 4 are one, two, and one, respectively. Note that from the way we have defined it vertex connectivity is meaningful only for graphs that have three or more vertices and are not complete.

For example, the graph in Figure 4 has n = 8, e = 16, and has vertex connectivity of one and edge connectivity of three. Another graph with the same number of vertices and edges (8 and 16, respectively) can be drawn as shown in Figure 5. It can easily be seen that the edge connectivity as well as the vertex connectivity of this graph is four. Consequently, even after any three stations are bombed, or any three lines destroyed, the remaining stations can still continue to "communicate" with each other. Thus the network of Figure 5 is better connected than that of Fig. 4 (although both consist of the same number of lines, that are 16).

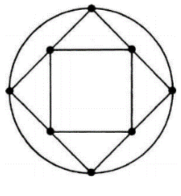

Fig.5. Graph with edge connectivity and vertex connectivity four.

4 Conclusion

In this paper we have proved some properties of a cutset in a graph and also show its application on communication or transportation networks. The properties are that every edge of a tree is a cut-set, every cutset in a connected graph *G* must contain at least one branch of every spanning tree of *G*, and that every circuit has an even number of edges in common with any cutset. Whereas with the application of cutset we can see which graph is better among graphs with the same number of nodes and lines.

References

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