CHAPTER 5

# Stability in the sense of Lyapunov

Stability is one of the most important properties characterizing a system's qualitative behavior. There are a number of stability concepts used in the study of dynamical systems. But perhaps the most important stability concept is that of *stability in the sense of Lyapunov* or simply *Lyapunov stability*. This concept is not a property of a system, it is a property of the system's equilibrium point which says that we can make the future states of system remain arbitrarily close to the equilibrium by simply taking the initial condition close enough. In this regard, it is ultimately a statement about the continuity of the flows in the neighborhood of the equilibrium point.

From an application's standpoint, this notion of remaining "close" to an equilibrium lies at the heart of the *regulation* problem. A control system is designed to regulate its response in the neighborhood of an operating point. In other words, whether or not that operating point can be maintained under small input disturbances is ultimately related to Lyapunov stability in that the system cannot diverge from that neighborhood for sufficiently small disturbances. In this regard, Lyapunov stability provides a necessary condition that must be satisfied for one to be able to solve the regulation problem.

By itself, however, Lyapunov stability does not exactly capture what many systems engineers are interested in. From a more pragmatic standpoint, one wants to be able to say that if the disturbance is $\epsilon$-large, then the output will be no larger than $\delta$-large. This $\epsilon$-$\delta$ relationship is built into Lyapunov stability, but the explicit dependence of system outputs and inputs are not addressed in the Lyapunov stability concept, for the concept is defined for unforced systems of the form $\dot{x}(t) = f(t, x(t))$. Nonetheless, when one develops useful stability concepts for input/output systems there is usually a close relationship between these two concepts; so much so that one might say Lyapunov stability is the root concept needed to adequately explain all other concepts.

This chapter focuses on elementary Lyapunov stability theory for nonlinear dynamical systems. We open by defining local Lyapunov stability for time-invariant systems of the form $\dot{x}(t) = f(x(t))$ where we show that the existence of a *Lyapunov function* or what we sometimes called a *Lyapunov stability certificate* is sufficient for the stability of the system's equilibrium point. We establish a number of well known results for the Lyapunov stability of time-invariant systems; including Chetaev's instability theorem, theorems regarding Global Lyapunov stability and LaSalle's invariance principle. We take a close look at Lyapunov stability for LTI systems and discuss how to relate chapter 4's linearization theorem to Lyapunov stability through Lyapunov's indirect method. An interesting aspect of Lyapunov theory for LTI systems is that the existence of a Lyapunov function is both necessary and sufficient for stability. This also holds for certain nonlinear systems and we establish two such converse theorems for these systems. We cover an important extension of Lyapunov stability to time-varying systems where we introduce some important notions of "uniform" stability that will be needed when we begin studying nonlinear control. Finally, we review a computational approach to certifying system stability that uses convex programming algorithms to find Lyapunov stability certificates.

## 1. Lyapunov Stability

This section defines the notion of Lyapunov stability for time-invariant systems. Consider a time-invariant dynamical system

$$\dot{x}(t) = f(x(t))$$

where $f : D \to \mathbb{R}^n$ is locally Lipschitz on a domain $D \subset \mathbb{R}^n$. We say that $x^* \in D$ is an *equilibrium point* of the system if $0 = f(x^*)$. We assume without loss of generality that $x^* = 0$. For if $x^* \neq 0$, we can introduce a change of variables $y = x - x^*$ such that the differential equation for $y$ is

$$\dot{y} = \dot{x} - \dot{x}^* = f(y + x^*) = g(y)$$

The ODE for the transformed state, $\dot{y} = g(y)$, is clearly equivalent to the original system and it has an equilibrium at $y^* = 0$.

We say that the equilibrium, $x^* = 0$ is *stable* in the sense of Lyapunov if for all $\epsilon > 0$ there exists $\delta > 0$ such that $|x(0)| < \delta$ implies $|x(t)| < \epsilon$ for all $t \geq 0$. We say the equilibrium is *unstable* if it is NOT stable. The equilibrium point is *asymptotically stable* if it is stable and if $x(t) \to 0$ as $t \to \infty$.

As an example consider the nonlinear pendulum described by the state equation

$$m\ell\ddot{\theta} = -mg\sin\theta - k\ell\dot{\theta}$$

where $m$ is the mass of the pendulum bob, $\ell$ is the pendulum's length, $g$ is gravitational acceleration, and $k$ is a damping constant related to the dissipation of the pendulum's energy. The function $\theta : \mathbb{R} \to \mathbb{R}$ represents the time evolution of the pendulum bob's angle with respect to the vertical. This is a second order differential equation, but when we defined the notion of Lyapunov stability it was with respect to a first order differential equation. So we will need to transform our second order representation into a first order representation.

This is done by introducing a state variable for each time derivative of the pendulum bob's angle. In particular, note that if the $n$th order differential equation may be written as

$$\frac{d^n y}{dt^n} = h\left(y, \frac{dy}{dt}, \frac{d^2 y}{dt^2}, \cdots, \frac{d^{n-1} y}{dt^{n-1}}\right)$$

then we can transform it into a "set" of $n$ first order differential equations through the variable assignments

$$x_1 = y, \quad x_2 = \frac{dy}{dt}, \quad \ldots, \quad x_n = \frac{d^{n-1} y}{dt^{n-1}}$$

to obtain the following set of first order differential equations

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 \\
&\vdots \\
\dot{x}_n &= h(x_1, x_2, \ldots, x_n)
\end{aligned}$$

which we rewrite as the vector system

$$\dot{x}(t) = f(x(t))$$

with vector field $f : \mathbb{R}^n \to \mathbb{R}^n$.

If we do this for the pendulum system, we obtain the following set of ordinary differential equations

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\frac{g}{\ell}\sin x_1 - \frac{k}{m}x_2
\end{aligned}$$

The equilibria are obtained by setting the right hand side of the above ODE to zero to obtain a system of algebraic equations whose roots are the equilibria for the system. Doing this for the first ODE yields,

$$\dot{x}_1 = 0 = x_2$$

which implies that $x_2^* = 0$. If we insert this into the second equation and set $\dot{x}_2 = 0$, we obtain

$$0 = \dot{x}_2 = -\frac{g}{\ell}\sin x_1$$

whose roots are $x_1^* = n\pi$ for $n = 0, \pm 1, \pm 2, \cdots$. Physically speaking these two equilibrium correspond to the two potential "balancing" solutions for the pendulum; one occurring when the pendulum is hanging down and the other when it is balanced upright. Mathematically, however, we notice that there are an infinite number of equilibrium if we view this system as evolving over Euclidean 2-space. However, since we know that the angle actually wraps around itself, we can see that the system state is actually evolving on a manifold which is formed by the cylinder shown in Fig. 1. The right pane in the figure shows the orbits of the pendulum on this cylinder, whereas the left pane shows the orbits after that pendulum has been mapped onto $\mathbb{R}^2$.

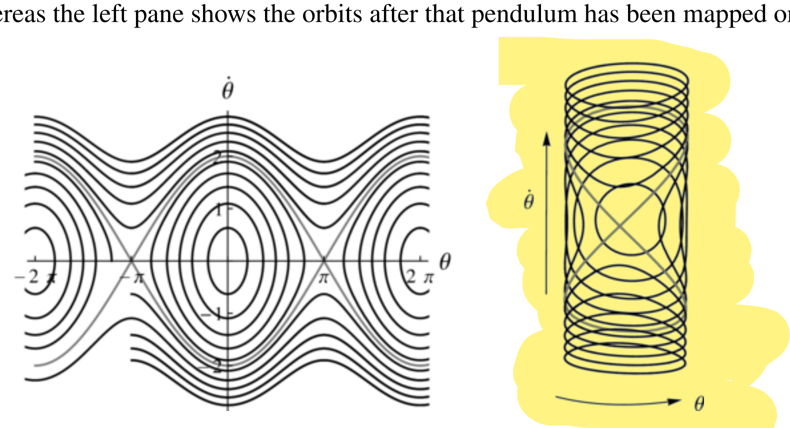


FIGURE 1. Phase Space of Pendulum in $\mathbb{R}^2$ and on Cylindrical Manifold

A natural way to investigate whether an equilibrium is stable is to see whether it dissipates "energy". Since our pendulum is a mechanical system, we can use the standard notions of potential and kinetic energy. If the total energy in the system is "decreasing" over time, then we may suspect that the equilibrium is stable. Let us first consider the case when the damping coefficient is zero (i.e. $k = 0$). The total energy is the sum of the pendulum's potential and kinetic energies and can be written as

$$\begin{aligned} E(x) &= \int_0^x \frac{g}{\ell} \sin y dy + \frac{1}{2} x_2^2 \\ &= \frac{g}{\ell}(1 - \cos(x_1)) + \frac{1}{2} x_2^2 \end{aligned}$$

The first term in the first equation is the potential energy and the second term is the kinetic energy for the pendulum bob. To see what happens to the energy over time we compute the *directional derivative* of $E$ as the system state evolves,

$$\begin{aligned} \frac{dE(x(t))}{dt} &= \dot{E} = \frac{\partial E}{\partial x_1}\dot{x}_1 + \frac{\partial E}{\partial x_2}\dot{x}_2 \\ &= \frac{g}{\ell}\sin x_1 \frac{dx_1}{dt} + x_2 \frac{dx_2}{dt} \\ &= \frac{g}{\ell} x_2 \sin x_1 + x_2 \left( -\frac{g}{\ell}\sin x_1 \right) \\ &= 0 \end{aligned}$$

In other words, when $k = 0$ (no damping) the time rate of change in the system's total energy is zero.

Now let us consider the case with positive damping $k > 0$. Repeating the computation for $\dot{E}$ now yields,

$$\dot{E} = -\frac{k}{m} x_2^2 \leq 0$$

This is always negative when $x_2 \neq 0$, which implies that that the energy is decreasing. Since it is decreasing and we known $E(t) \geq 0$ we can conclude that it must converge to a constant $E^*$ level. This $E^*$ is then the minimum energy of the physical system's "stable" configuration.

The notion of Lyapunov stability may be seen as a generalization of this "energy" view of the system. What we need is an "energy" function such that when it is decreasing we can formally prove the "stability" of the equilibrium in the formal sense defined above. The following "direct" theorem of Lyapunov provides the result we are looking for.

THEOREM 49. **(Lyapunov Direct)** *Let* $0$ *be an equilibrium point for* $\dot{x} = f(x)$ *where* $f : D \to \mathbb{R}^n$ *is locally Lipschitz on domain* $D \subset \mathbb{R}^n$*. Assume there exists a continuously differentiable function* $V : D \to \mathbb{R}$ *such that*

- $V(0) = 0$ *and* $V(x) > 0$ *for all* $x \in D$ *not equal to zero.*
- $\dot{V} = \frac{dV(x(t))}{dt} = \frac{\partial V}{\partial x} = [D_f V](x) \leq 0$ *for all* $x \in D$*.*

*Then* $x = 0$ *is stable in the sense of Lyapunov.*

Advised to go through the proofs: going through proofs builds intuition for further proofs



**Proof:** For any $\epsilon > 0$ consider the open ball

$$B_r = \{x \in \mathbb{R}^n \,:\, |x| < r\} \subset D$$

whose radius $r \in (0, \epsilon]$. Let $\alpha = \min_{|x|=r} V(x)$ be the minimum value that the function $V$ attains on the boundary of $B_r$. Because the boundary is a closed and bounded set and because $V$ is continuous, we know it must attain its minimum on the boundary.

Now consider a subset of $B_r$

$$\Omega_\beta = \{x \in B_r \,:\, V(x) \leq \beta\}$$

where $\beta \in (0, \alpha)$. The set $\Omega_\beta$, therefore, is a set contained in an $\epsilon$-neighborhood of the equilibrium $0$ where the function $V$ takes values less than the minimum, $\alpha$, attained on the boundary of $B_r$.

So let $x : \mathbb{R} \to \mathbb{R}^n$ be any solution to $\dot{x} = f(x(t))$ where $x(0) \in \Omega_\beta$. By assumption,

$$\dot{V}(x) = \frac{\partial V}{\partial x} f(x) \leq 0$$

Since $\dot{V}$ is not positive, we know that $V(x(t))$ is a monotone decreasing function of time and we can therefore conclude that

$$V(x(t)) \leq V(x(0)) \leq \beta$$

for all $t \geq 0$.

Since $V$ is continuous and $V(0) = 0$, there must exist a $\delta > 0$ such that $|x| \leq \delta$ implies $V(x) < \beta$ so there exists another open ball

$$B_\delta = \{x \in \mathbb{R}^n \,:\, |x| < \delta\}$$

such that $B_\delta \subset \Omega_\beta \subset B_r$. This means that if $x(0) \in B_\delta$, then $x(0) \in \Omega_\beta$ which implies $x(t) \in \Omega_\beta$ for all $t \geq 0$. Since $\Omega_\beta$ is a subset of $B_r$ which is an open ball about the origin of radius less than $\epsilon$, we can conclude $|x(t)| < \epsilon$ for all $t \geq 0$. This is precisely the definition of Lyapunov stability and so the proof is complete. $\diamondsuit$

The following theorem shows that by slightly strengthening the condition on $\dot{V}$, one can establish that the equilibrium is asymptotically stable.

THEOREM 50. **(Asymptotic Stability)** *Under the hypotheses of theorem 49, if $\dot{V}(x) < 0$ for all $x \in D - \{0\}$, then the equilibrium is asymptotically stable.*

**Proof:** Since $V(x(t))$ is a monotone decreasing function of time and bounded below, we know there exists a real $c \geq 0$ such that $V(x(t)) \to c$ as $t \to \infty$. Let us assume that $c$ is strictly greater than zero. By the continuity of $V$ there exists $d > 0$ such that the open ball $B_d = \{x : |x| < d\}$ such that

$$B_d \subset \Omega_c \equiv \{x \,:\, V(x) \leq c\}$$

By our assumption, however, $x(t)$ cannot enter the ball $B_d$ since $V(x(t))$ is always greater than $c$. So let

$$-\gamma = \max_{x \in \Omega_c - B_d} \dot{V}(x) < 0$$

This implies that

$$V(x(t)) = V(x(0)) - \int_0^t \dot{V}(x(\tau))d\tau \leq V(x(0)) - \gamma t$$

This last equation, however, means that $V$ must eventually become negative for large enough $t$ which cannot happen since $V(x(t)) > 0$. So $c$ cannot be positive and we have to conclude that $c = 0$. So the state asymptotically converges to zero and we can conclude that the equilibrium at zero is asymptotically stable. $\diamondsuit$

A $C^1$ function $V : D \rightarrow \mathbb{R}$ that satisfies the conditions in theorems 49 or 50 is called a *Lyapunov funtion*. These theorems, therefore, state that the "existence" of a Lyapunov function "certifies" that the equilibrium is stable or asymptotically stable. One may also refer to these functions, $V$, as Lyapunov stability *certificates*.

We say that $V : D \rightarrow \mathbb{R}^n$ is *positive definite* (PD) if $V(0) = 0$ and $V(x) > 0$ for $x \in D - \{0\}$. The function is *positive semi-definite* (PSD) if $V(x) \geq 0$ for all $x \in D - \{0\}$. The function is *negative definite* if $-V$ is positive definite.

The conditions in theorems 49 and 50 can also be stated in terms of these function characterizations defined above. So we say the equilibrium at 0 is stable if there exists a $C^1$ *positive definite* function, $V$, whose directional derivative is *negative semi-definite*. The equilibrium at 0 is asymptotically stable if there exists a $C^1$ *positive definite* function, $V$ whose directional derivative is *negative definite*.

It can, in general, by very difficult to find a Lyapunov function for a given system. Often such functions are determined by using a *candidate Lyapunov function* that is known to be a Lyapunov function for a system closely related to the one under study. One may then parameterize that functional form and try to "search" for the parameters that render the candidate function an actual Lyapunov function for the system.

**Example:** As an example, let us consider the differential equation

$$\dot{x}(t) = -g(x(t))$$

where $g : \mathbb{R} \rightarrow \mathbb{R}$ is locally Lipschitz on $(-a, a)$ with $g(0) = 0$. We also assume that $g$ is "odd" in the sense that its graph lies in the first and third quadrants, so that $xg(x) > 0$ for all $x \neq 0$. Clearly, $x = 0$ is an equilibrium point for this system.

For such systems, let us consider a function, $V : \mathbb{R} \rightarrow \mathbb{R}$ that takes values

$$V(x) = \int_0^x g(y)dy \tag{73}$$

Note that $V \in C^1$ and $V(0) = 0$. Also observe that if $x > 0$, then $V(x) > 0$ since $g(y) > 0$ for any $y > 0$. If $x < 0$, then we observe

$$\int_0^{-|x|} g(y)dy = -\int_{-|x|}^0 g(y)dy > 0$$

since $g(y) < 0$ for $y < 0$. This means, therefore, that $V(x) > 0$ for $x \neq 0$ and so $V$ is a positive definite function.

To determine if $V$ is a Lyapunov function one needs to compute its directional derivative. This derivative is

$$\dot{V} = \frac{\partial V}{\partial x}\dot{x} = \frac{\partial V}{\partial x}(-g(x)) = -g^2(x)$$

which is clearly negative definite. So from theorem 50 we can conclude that the equilibrium (origin) is asymptotically stable.

So let us return to our earlier pendulum system with state equations

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\left(\frac{g}{\ell}\right)\sin x_1 \end{aligned}$$

We cannot directly use the $V$ from equation 73 as a Lyapunov function for the pendulum since the pendulum system has two states and our candidate function only has one state. However, we can use equation (73) for both states as shown below

$$\begin{aligned} V(x) &= \int_0^{x_1}\left(\frac{g}{\ell}\right)\sin y dy + \int_0^{x_2} y dy \\ &= \left(\frac{g}{\ell}\right)(1-\cos(x_1)) + \frac{1}{2}x_2^2 \end{aligned} \tag{74}$$

Note that this is precisely the "energy" function we examined initially. $V$ is clearly positive definite that we can establish by noting $V(0) = 0$ and that

$$0 \leq \frac{1}{2}x_2^2 < V(x)$$

for all $x_1 \neq n\pi$ and all $x_2$. This clearly means that for $D = (-2\pi, 2\pi) - \{0\}$ that $V(x) > 0$ (i.e. is positive definite).

We now check to see if $\dot{V}$ is negative definite. Computing its partial derivative yields

$$\begin{aligned} \frac{\partial V}{\partial x}f(x) &= \left(\frac{g}{\ell}\right)\dot{x}_1\sin x_1 + x_2\dot{x}_2 \\ &= \left(\frac{g}{\ell}\right)x_2\sin x_1 - \left(\frac{g}{\ell}\right)x_2\sin x_1 = 0 \end{aligned}$$

This shows that $\dot{V} = 0$ for all $x$ and so we can only conclude that $\dot{V}$ is negative semi-definite. From theorem 49 we can then conclude that the equilibrium at $(0,0)$ is *stable* in the sense of Lyapunov.

We cannot, however, conclude that the equilibrium is asymptotically stable. To force the equilibrium to be asymptotically zero, we need to add damping by letting $k > 0$. In this case, the equations of motion for the pendulum become,

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\left(\frac{g}{\ell}\right)\sin x_1 - \left(\frac{k}{m}\right)x_2 \end{aligned}$$

If we now compute the directional derivative of $V$ for the damped system we obtain

$$\begin{aligned}\dot{V}(x) &= \frac{g}{\ell}\dot{x}_1 \sin x_1 + x_2 \dot{x}_2 \\ &= \frac{g}{\ell} x_2 \sin x_1 - \frac{g}{\ell} x_2 \sin x_1 - \frac{k}{m} x_2^2 \\ &= -\frac{k}{m} x_2^2\end{aligned}$$

Note that $\dot{V} \leq 0$ and so is still only negative semi-definite since $\dot{V} = 0$ for any state of the form $x = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}$. So again all we can conclude is that the equilibrium is Lyapunov stable.

We know, however, that this system's equilibrium at $(0,0)$ is actually asymptotically stable. But to prove that we need to take the original Lyapunov function and parameterize it in a way that gives us more freedom. One way of doing this is to make the second term in equation (74) a quadratic function of both states, not just $x_2$. The resulting candidate Lyapunov function is now

(75) $$V(x) = \frac{g}{\ell}(1 - \cos(x_1)) + \frac{1}{2} x^T \mathbf{P} x$$

where $\mathbf{P} = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}$ is a symmetric and *positive definite* matrix. We require $\mathbf{P}$ to be positive definite since this will ensure that $V$ also remains positive definite. The matrix $\mathbf{P}$ is positive definite if and only if

(76) $$p_{11} > 0, \quad p_{22} > 0, \quad p_{11}p_{22} - p_{12}^2 > 0$$

So let us recompute the directional derivative

$$\begin{aligned}\dot{V} &= \left[ p_{11}x_1 + p_{12}x_2 + \frac{g}{\ell}\sin x_1 \right] x_2 + (p_{12}x_1 + p_{22}x_2)\left[ -\frac{g}{\ell}\sin x_1 - \frac{k}{m}x_2 \right] \\ &= \frac{g}{\ell}(1 - p_{22})x_2 \sin x_1 - \frac{g}{\ell} p_{12} x_1 \sin x_1 \\ &\quad + \left[ p_{11} - p_{12}\frac{k}{m} \right] x_1 x_2 + \left[ p_{12} - p_{22}\frac{k}{m} \right] x_2^2\end{aligned}$$

We are free to select the $p_{ij}$ terms to be anything that satisfy equation (76). In particular, we choose them to try and cancel out the indefinite cross terms. Selecting $p_{22} = 1$ cancels the term $x_2 \sin x_1$. Selecting $p_{11} = \frac{k}{m} p_{12}$ cancels the cross term $x_1 x_2$. With these selections, we see that

$$\dot{V} = -\frac{g}{\ell} p_{12} x_1 \sin x_1 + \left[ p_{12} - p_{22}\frac{k}{m} \right] x_2^2$$

If we let $p_{12} > 0$, then the first term $-\frac{g}{\ell} p_{12} x_1 \sin x_1$ is negative about the origin. If we let $p_{22} = 1$ and $p_{12} < \frac{k}{m}$, then the second term $\left[ p_{12} - p_{22}\frac{k}{m} \right] x_2^2$ is negative. So combining these considerations we select a $\mathbf{P}$ matrix in which

$$0 < p_{12} < \frac{k}{m}, \quad p_{22} = 1, \quad p_{11} = \frac{k}{m} p_{12} < \frac{k^2}{m^2}$$

which ensures $V$ is positive definite and $\dot{V}$ is negative definite about the origin. We can then use theorem 50 to conclude that the origin is asymptotically stable.

This example illustrates an important limitation of Lyapunov's stability theorem. It is only a *sufficient* condition for stability. This means our inability to find a Lyapunov function does not imply the equilibrium is unstable. The last example suggests that the appropriate use of Lyapunov theory often involves proposing a parameterized family of positive definite functions and then searching for the parameters in that family that verify $\dot{V}$ is negative definite. In particular, we sometimes say that this is a search for a function that *certifies* the asymptotic stability of the equilibrium and so these $V$ functions are sometimes called Lyapunov stability *certificates*.

The *quadratic* form we used to augment our earlier $V$ often appears as a Lyapunov function for linear dynamical systems. In light of the invariant manifold theorem, therefore, ones first attempt to find a stability certificate starts with a $V$ of the form,

$$V(x) = x^T \mathbf{P} x \tag{77}$$

where $\mathbf{P} = \mathbf{P}^T > 0$. If we were to do this for our pendulum example, however, the directional derivative would become

$$\begin{aligned} \dot{V} &= -p_{22}\frac{g}{\ell}x_2 \sin x_1 - \frac{g}{\ell}p_{12}x_1 \sin x_1 \\ &\quad + \left[p_{11} - p_{12}\frac{k}{m}\right] x_1 x_2 + \left[p_{12} - p_{22}\frac{k}{m}\right] x_2^2 \end{aligned}$$

Note that if we try to remove the first cross term $x_2 \sin x_1$ by setting $p_{22} = 0$, then the resulting $\mathbf{P}$ would not be positive definite and so we cannot directly use the quadratic form.

However, one may also notice that

$$x_2 \sin x_1 \approx x_2(x_1 + o(|x_1|^2))$$

If we then insert this into our directional derivative we obtain

$$\begin{aligned} \dot{V} &= \left(p_{11} - p_{12}\frac{k}{m} - p_{22}\frac{g}{\ell}\right) x_1 x_2 \\ &\quad -p_{22}\frac{g}{\ell}x_2 o(|x_1|^2) - \frac{g}{\ell}p_{12}x_1 \sin x_1 + \left[p_{12} - p_{22}\frac{k}{m}\right] x_2^2 \end{aligned}$$

So let us select $p_{11}$, $p_{12}$, and $p_{22}$ so that

$$\begin{aligned} p_{11} - p_{22}\frac{k}{m} - p_{22}\frac{g}{\ell} &= 0 \\ p_{12} &> 0 \\ p_{12} - p_{22}\frac{k}{m} &< 0 \end{aligned}$$

which would allow us to infer that

$$\dot{V} = -W(x_1, x_2) + \text{error term}$$

in which $W$ is a positive definite function. The first term in the preceding equation is therefore negative definite and due to continuity of $V$ we know the error term will be bounded if we can stay within a sufficiently small neighborhood such that $-W < |\text{error term}|$. If we can establish this for some neighborhood about the origin, then $\dot{V}$ is dominated by the $-W$ term and we can conclude that the origin is *locally asymptotically stable*.

The preceding discussion showed that what one can conclude about the equilibrium's stability is highly dependent on the type of Lyapunov function we choose. The use of a quadratic candidate Lyapunov function only allowed us to find a certificate that ensured the local asymptotic stability of the equilibrium. The use of the candidate Lyapunov function in equation (75) however, was much stronger in that it allowed us to certify the *global* asymptotic stability of the origin.

## 2. Chetaev's Instability Theorem

Note that theorems 49 and 50 are only *sufficient* conditions for the equilibrium to be stable. Our inability to find a Lyapunov function, $V$, does not certify the instability of the equilibrium. To certify an unstable equilibrium, the following theorem can be used.

THEOREM 51. **(Chetaev Instability Theorem)** *Let $x = 0$ be an equilibrium for $\dot{x} = f(x(t))$ where $f : D \to \mathbb{R}^n$ is locally Lipschitz on domain $D \subset \mathbb{R}^n$. Let $V : D \to R$ by a $C^1$ function such that $V(0) = 0$ and $V(x_0) > 0$ for some $x_0$ with arbitrarily small $|x_0|$. If there exists $\delta > 0$ such that $\dot{V}(x) > 0$ for all $x$ in*

$$U = \{x : |x| \le \delta, \quad V(x) > 0\}$$

*then $x = 0$ is unstable.*

**Proof:** Suppose there exists a $\delta$ such that $\dot{V}(x) > 0$ for all $x \in U$. Let $x_0$ be a point in the interior of $U$ so that $V(x_0) > 0$. Now let $x : \mathbb{R} \to \mathbb{R}^n$ denote the trajectory generated by the system with $x_0$ as its initial point. Since $\dot{V}(x) > 0$ for all $x \in U$, and since $U$ is compact there is a minimum rate of growth

$$\gamma = \min\{\dot{V}(x) : x \in U \text{ and } V(x) \ge V(x_0)\}$$

?

and we can see that

$$V(x(t)) = V(x_0) + \int_0^t \dot{V}(x(s))ds \ge V(x_0) + \int_0^t \gamma ds = V(x_0) + \gamma t$$

there exists a time $T > 0$ when $x(T)$ is on the boundary, $\partial U$, of $U$. There are two parts to that boundary

$$\begin{aligned} \partial_B U &= \{x : |x| = \delta, \quad V(x) < 0\} \\ \partial_A U &= \{x : |x| < \delta, \quad V(x) = 0\} \end{aligned}$$

Note that $x(T)$ cannot cannot be in $\partial_A U$, for $V = 0$ on the boundary and $V(x) > 0$ for $x \in U$. Since $\dot{V}(x) > 0$ for all $x \in U$, it is impossible for $V(x(t))$ to decrease to zero. This means that $x(T)$ must be in $\partial_B U$. So for the given $\delta$, we see that if $x_0$ is arbitrarily close to 0, then the state trajectory leaves $U$ and hence leaves a $\delta$-neighborhood of the origin. This establishes the instability of the equilibrium. $\diamondsuit$

Fig. 2 shows an example of a Chetaev function of the form $V(x) = \frac{1}{2}(x_1^2 - x_2^2)$. The ball $N_\delta$ is shown in the Fig. 2. For this Chetaev function, the set $U$ in which $V(x) > 0$ and $\dot{V}(x) > 0$ is shown by the shaded region. This shaded region consists of two boundaries, $\partial_A$ and $\partial_B$. The theorem's proof argues that any trajectory starting within $U$ must leave through $\partial_B$ and since the origin is contained in $U$ we can readily see that no matter how close we start to $x_0$, the state trajectory will leave a $\delta$ neighborhood of the origin.

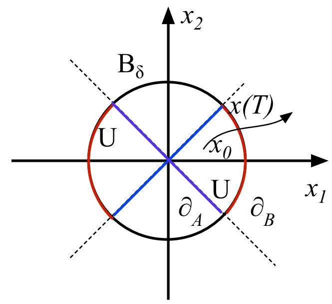


FIGURE 2. Chetaev Function - trajectory starting at $x_0$ in $U$ exits at time $T$ at boundary $\partial_B U$.

As an example, consider the system

$$\begin{aligned} \dot{x}_1 &= x_1 + g_1(x) \\ \dot{x}_2 &= -x_2 + g_2(x) \end{aligned}$$

where $g_1 : \mathbb{R}^2 \to \mathbb{R}$ and $g_2 : \mathbb{R}^2 \to \mathbb{R}$ are locally Lipschitz such that there exists a constant $k > 0$ where

$$|g_1(x)| \leq k|x|^2, \quad |g_2(x)| \leq k|x|^2$$

We consider a candidate Chetaev function of the form

$$V(x) = \frac{1}{2}(x_1^2 - x_2^2)$$

Choose a point along the $x_2 = 0$ since $V > 0$ at all such points. Now look at $\dot{V}$ and note that

$$\dot{V}(x) = x_1^2 + x_2^2 + x_1 g_1(x) - x_2 g_2(x)$$

with the last term satisfying

$$|x_1 g_1(x) - x_2 g_2(x)| \leq |x_1||g_1(x)| + |x_2||g_2(x)| \leq 2k|x|^3$$

We can therefore see that

$$\dot{V} \geq |x|^2 - 2k|x|^3 = |x|^2(1 - 2k|x|)$$

If we choose $\epsilon < \frac{1}{2k}$, then $\dot{V} > 0$ and so $V$ is a Chetaev function and the origin is unstable.

As another example, let us consider the system

$$\begin{aligned} \dot{x}_1 &= f_1(x) = -x_1^3 + x_2 \\ \dot{x}_2 &= f_2(x) = x_1^6 - x_2^3 \end{aligned}$$

This system has two equilibria at $(0,0)$ and $(1,1)$. Consider the set

$$\Omega = \{0 \leq x_1 \leq 1\} \cap \{x_2 \geq x_1^3\} \cap \{x_2 \leq x_1^2\} \tag{78}$$

which is shown below in Fig. 3.

From Fig. 3, we see that the set $\Omega$ in equation (78) has two boundaries. On the boundary marked "A", one knows that $x_2 = x_1^2$, $f_2(x) = 0$, and $f_1(x) > 0$. The resulting vector field directions shown along boundary $A$ point into the set $\Omega$. On the boundary marked "B", one knows that $x_2 = x_1^3$, $f_2(x) > 0$ and $f_1(x) = 0$. This vector field direction is also shown in the figure and they also point in $\Omega$. One can therefore see that for any state $x(0) \in \Omega$, the resulting trajectory cannot leave through either boundary "A" or "B". So if we select a candidate Chetaev function $V(x) = (x_1^2 - x_2)(x_2 - x_1^3)$, then $V(x) \geq 0$ on $\Omega$ and $\dot{V}(x) \geq 0$ on $\Omega$. We can therefore conclude that $V$ is a Chetaev function and so the origin cannot be stable.

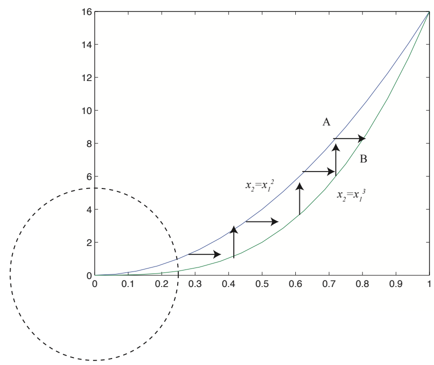


FIGURE 3. Example of Chetaev Function

## 3. Global Lyapunov Stability

The preceding section introduced two theorems (49 and 50) that provided sufficient characterizations of Lyapunov stability certificates. As stated, however, the Lyapunov stability concept is primarily a "local" concept in that it asserts the stability in a sufficiently small neighborhood of the equilibrium (origin). In many applications we would like some assurance that the asymptotic stability property applies *globally*. In other words, for any initial $x(0) \in \mathbb{R}^n$ one can guarantee that the resulting state trajectories asymptotically converge to the origin. Note that even if $\dot{V} < 0$ for all $x \in \mathbb{R}^n$, it may not be possible to deduce global asymptotic stability. The following example provides an example of this problem.

Consider the dynamical system

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2) = -\frac{6x_1}{(1+x_1^2)^2} + 2x_2 \\ \dot{x}_2 &= f_2(x_1, x_2) = -2\frac{x_1 + x_2}{(1+x_1^2)^2} \end{aligned}$$

We use the following positive definite function

$$V(x) = \frac{x_1^2}{1+x_1^2} + x_2^2$$

as a candidate Lyapunov function. We then certify that this is indeed a Lyapunov function for the system by computing the directional derivative

$$\begin{aligned} \dot{V}(x) &= \frac{2x_1(1+x_1^2) - 2x_1^3}{(1+x_1^2)^2}\dot{x}_1 + 2x_2\dot{x}_2 \\ &= -\frac{12x_1^2}{(1+x_1^2)^4} - \frac{4x_2^2}{(1+x_1^2)^2} \end{aligned}$$

This is negative definite for all $x \in \mathbb{R}^2$ and so we can conclude that the origin is asymptotically stable.

Even though, $\dot{V}$, is negative definite for all $\mathbb{R}^2$, however, we cannot conclude that all trajectories will asymptotically go to the origin. To be asymptotically stable requires 1) that the equilibrium is stable and 2) that

$x(t) \to 0$ as $t \to \infty$. Recall, however, that the property of Lyapunov stability requires that for any $\epsilon$, there exists a $\delta$ such that $|x(0)| < \delta$ implies $|x(t)| < \epsilon$ for all $\epsilon$. The problem we face with this characterization is that $\delta$ may be function of the $\epsilon$ we choose. In particular, as $\epsilon$ gets smaller and smaller, the capture neighborhood size $\delta(\epsilon)$ may go to a constant value , $\overline{\delta}$. For the neighborhood $N_{\overline{\delta}}(0)$, Lyapunov stability is guaranteed, but it may not necessarily be guaranteed for $x(0)$ outside of $N_{\overline{\delta}}(0)$. That set $N_{\overline{\delta}}(0)$ represents an inner approximation to equilibrium's *region-of-attraction* (RoA), namely all initial states from which Lyapunov stability is assured.

The origin of the above system is asymptotically stable, but its region of attraction is not all of $\mathbb{R}^2$. An inner approximation for the RoA can be determined by an examination of $\dot{V}$. In particular, consider a hyperbola,

$$x_2 = \frac{2}{x_1 - \sqrt{2}}$$

The tangents to the hyperbola are given by the equation

$$\frac{dx_2}{dx_1} = \frac{-1}{\frac{1}{2}x_1^2 - 2\sqrt{2}x_1 + 1}$$

For $x_1 > \sqrt{2}$, we see that

$$2x_1^2 + 2\sqrt{2}x_1 + 1 > \frac{1}{2}x_1^2 - \sqrt{2}x_1 + 1$$

so that it is apparent that the slopes of the vector field and the tangent to the hyperbola satisfy

$$\left.\frac{f_2}{f_1}\right|_{\text{hyperbola}} > \text{slope of hyperbola's tangents}$$

The hyperbola and its slopes are sketched in Fig. 4 in the phase space of the system. The lines of constant $V$ are also shows in this figure. Finally the ratio of the vector fields, $f_2/f_1$, are also shown. We can therefore see that $f_2/f_1$ is always a vector pointing away from the hyperbola. This means, therefore, that any state starting above the hyperbola will be directed away from the hyperbola and not into it. In other words, any initial state starting above the hyperbola cannot asymptotically converge to the origin and so we cannot assert that the origin is asymptotically stable, even though $\dot{V} < 0$ is for all of $\mathbb{R}^2$. From the Figure we can see that the region above the hyperbola is a divergent region for which $x_1$ asymptotically goes to infinity. The boundary of the divergent region provides an outer approximation to the RoA. We can also use Fig. 4 to form an inner approximation of the region of attraction for which all trajectories are guaranteed to asymptotically approach the origin.

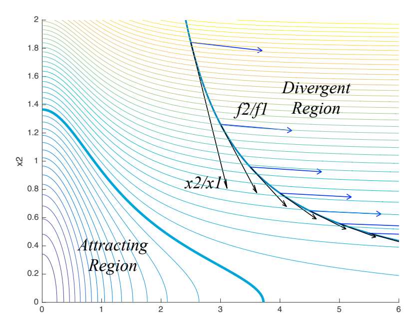


FIGURE 4. Example showing finite region of attraction

The preceding example showed that even if $\dot{V}(x) < 0$ for all $x \neq 0$, that we cannot guarantee the region of attraction is the entire region. The following theorem, provides a sufficient condition for the equilibrium to be globally asymptotically stable. The main condition is that the Lyapunov function $V$ is *radially unbounded*. In particular, we say $V : \mathbb{R}^n \to \mathbb{R}$ is radially unbounded if $V(x) \to \infty$ as $|x| \to \infty$.

THEOREM 52. **(Barbashin-Krasovskii Theorem)** *Let $x = 0$ be an equilibrium point for $\dot{x} = f(x)$ where $f$ is locally Lispchitz. Let $V : \mathbb{R}^n \to \mathbb{R}$ be a radially unbounded $C^1$ function such that $V(0) = 0$, $V(x) > 0$ for all $x \neq 0$ and $\dot{V}(x) < 0$ for all $x \neq 0$. Then the equilibrium point $x = 0$ is globally asymptotically stable.*

**Proof:** Consider any $x \in \mathbb{R}^n$ and let $c = V(x)$. Since $V$ is radially unbounded we know that for any $c > 0$ there exists $\epsilon > 0$ such that $V(x) > c$ whenever $|x| > \epsilon$. So the set $\Omega_c = \{x : V(x) \leq c\}$ must be a subset of the closed ball $B_\epsilon = \{x : |x| \leq \epsilon\}$ and it follows that $V(x(t)) \leq V(x_0) = c$ for all $t \geq 0$. This implies that $x(t) \in N_\epsilon(0)$ for all $t \geq 0$. Since $c$ can be made arbitrarily large, this implies that $x(0)$ can be taken arbitrarily large and still lie within a bounded $\Omega_c$. $\diamondsuit$

As an example of the application of the Barbashin-Krasovskii theorem, let us consider the second order system,

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -h(x_1) - x_2
\end{aligned}$$

where $a > 0$, $h : \mathbb{R} \to \mathbb{R}$ is locally Lispchitz, $h(0) = 0$, and $yh(y) > 0$ for all $y \neq 0$. Let us consider a candidate Lyapunov function of the form

$$V(x) = \frac{1}{2} x^T \begin{bmatrix} k & k \\ k & 1 \end{bmatrix} x + \int_0^{x_1} h(y)dy$$

where $0 < k < 1$. This is positive definite for all $x \in \mathbb{R}^2$ and is radially unbounded. The derivative is

$$\dot{V}(x) = -(1-k)x_2^2 - kx_1 h(x_1)$$

which is negative definite for all $x \in \mathbb{R}^2$ since $0 < k < 1$. The origin therefore is globally asymptotically stable.

## 4. Invariance Principle

Theorem 50 only established the equilibrium's asymptotic stability if $\dot{V}$ was negative definite. Finding such a Lyapunov function may actually be very difficult in many applications and yet this may not necessarily mean that the system is not asymptotically stable. The *invariance principle* provides a useful tool for determining whether an equilibrium is asymptotically stable when all we can guarantee is that $\dot{V}$ is negative semi-definite. The proof of this principle rests on some useful concepts in topological dynamical systems theory, specifically the concept of a *positive limit set*.

Let $\phi$ be a dynamical system (i.e. the flow) over domain $D \subset \mathbb{R}^n$. Recall that an orbit of $\phi$ with respect to state $x \in \mathbb{R}^n$ is the set of points

$$\gamma(x) = \{y \in \mathbb{R}^n : y = \phi(t, x), \ t \in \mathbb{R}^n\}$$

A point $p$ is said to be an $\omega$ or *positive limit point* of $x \in \mathbb{R}^n$ if there exists a sequence of times $\{t_n\}$ such that $\lim_{t_n \to \infty} \phi(t, x) = p$. This point is said to be an $\alpha$ or *negative limit point* of $x$ if $\lim_{t_n \to -\infty} \phi(t, x) = p$. The set of all $\omega$ limit points of $x$ is denoted as $\omega(x)$ and the union of all such limit points for all $x \in D$ is called the *system's* positive limit set $\Omega = \bigcup_{x \in D} \omega(x)$.

Consider a dynamical system $\phi$ on phase space $\mathbb{R}^n$. An set $M \subset \mathbb{R}^n$ is said to be *invariant* if $x \in M$ implies $\phi(t, x) \in M$ fo all $t \geq 0$. We say that $M$ is *attracting* if for all $\epsilon > 0$ and any $x \in D$ there exists $T > 0$ such that $\inf_{y \in M} |\phi(t, x) - y| < \epsilon$ for all $t > T$. With these definitions we can now state and prove the following theorem.

THEOREM 53. **(Positive Limit Set Theorem)** *Consider the dynamical system $\dot{x}(t) = f(x(t))$ where $f : D \to \mathbb{R}^n$ is locally Lipschitz on a compact set $D \subset \mathbb{R}^n$. Assume that all $x : \mathbb{R} \to \mathbb{R}^n$ satisfying $\dot{x} = f(x)$ belong to $D$ for all $t \geq 0$. Then the system's $\omega$-limit set, $\Omega$, is non-empty, compact, invariant and attracting.*

**Proof:** For any $x_0 \in D$, let $x(t; x_0)$ denote the orbit generated by $x_0$. Since $x(t; x_0)$ is within a compact set, $D$, for all $t \geq 0$, it has by the Weierstrass theorem a limit point which serves as the positive limit point and so the limit set $\Omega$ is non-empty.

So since $x(t; x_0)$ lies in a compact set of $\mathbb{R}^n$, there exists an $M > 0$ such that $|x(t; x_0)| \leq M$ for all $t \geq 0$. This bound therefore is uniform and so if we consider any convergent sequence $\{x(t_i)\}_i$ that converges to $y \in \Omega$, then the limit point of this sequence is also bounded by $M$ and so $\Omega$ is also bounded.

We now show that $\Omega$ is closed. Let $\{y_i\}$ be a sequence of points in $\Omega$ such that $y_i \to y$. We need to prove that $y$ is also an $\omega$-limit point. Since $y_i \in \Omega$, there is a sequence $\{t_{i_j}\}$ such that $x(t_{i_j}) \to y_i$ as $t_{i_j} \to \infty$. So we now construct another sequence of time $\{\tau_i\}$ such that

$$\begin{array}{rcl} \tau_2 > t_{2_2} & \text{where} & |x(\tau_2) - y_2| < 1/2 \\ \tau_3 > t_{3_3} & \text{where} & |x(\tau_3) - y_3| < 1/3 \\ \vdots & & \vdots \\ \tau_i > t_{i_i} & \text{whre} & |x(\tau_i) - y_i| < 1/i \\ \vdots & & \vdots \end{array}$$

Clearly $\tau_i \to \infty$ as $i \to \infty$. So for any $\epsilon > 0$ there exists $N_1, N_2 > 0$ such that

$$\begin{aligned} |x(\tau_i) - y_i| &< \frac{\epsilon}{2} \quad \text{for all} \quad i > N_1 \\ |y_i - y| &< \frac{\epsilon}{2} \quad \text{for all} \quad i > N_2 \end{aligned}$$

and so

$$|x(\tau_i) - y| \leq |x(\tau_i) - y_i| + |y_i - y| \leq \epsilon$$

for all $i > \max(N_1, N_2)$. This implies that $x(\tau_i) \to y$ as $\tau_i \to \infty$ and so $y$ is a positive limit point and lies in $\Omega$. This implies $\Omega$ is closed and since we already know it is bounded this implies that it is compact by the Heine-Borel theorem.

We now show that $\Omega$ is invariant with respect to $f$. Let $y \in \Omega$ and let $x(t; y)$ denote the orbit starting from $y$. Since $y \in \Omega$, there is a state trajectory that we denote as $\overline{x}(t, x_0)$, an $x_0 \in D$, and a sequence $\{t_i\}$ such that $\overline{x}(t_i; x_0) \to y$ a $t_i \to \infty$. So consider

$$\overline{x}(t + t_i, x_0) = \overline{x}(t, \overline{x}(t_i, x_0))$$

Because the trajectory is continuous with respect to $t$,

$$\begin{aligned} \lim_{i\to\infty} \overline{x}(t + t_i; x_0) &= \lim_{i\to\infty} \overline{x}(t, \overline{x}(t_i, x_0)) \\ &= \overline{x}(t, \lim_{i\to\infty} \overline{x}(t_i; x_0)) \\ &= \overline{x}(t, y) \end{aligned}$$

which shows that $\overline{x}(t; y)$ is in $\Omega$ for all $t > 0$ and so $\Omega$ is positively invariant with respect to $f$.

Finally we show that $\Omega$ is attracting. Suppose that this is not the case, then there exists $\epsilon > 0$ and a sequence $\{t_i\}$ with $t_i \to \infty$ such that

$$\inf_{y\in\Omega} |x(t_i) - y| > \epsilon$$

We know that $\{x(t_i)\}$ is compact (bounded), so it contains a convergent subsequence $x(t_{i_j}) \to x^*$ as $t_{i_j} \to \infty$. This would imply that $x^* \in \Omega$ which contradicts the assumption that $x(t_i)$ stays an $\epsilon$ distance away from $\Omega$ and so $\Omega$ must be attracting. $\diamondsuit$

The preceding theorem 53 establishes the existence of a steady-state qualitative behavior (as represented by the $\omega$-limit set) whenever we know the state trajectories are confined to a compact set. This greatly limits what the trajectories can do and we can use this restriction to establish asymptotic stability even when $\dot{V}(x)$ is only negative semi-definite. This result is known as the *Invariance Principle*.

THEOREM 54. **(Invariance Principle)** *Consider the system $\dot{x} = f(x)$ where $f : D \to \mathbb{R}^n$ is locally Lipschitz on $D \subset \mathbb{R}^n$. Let $K \subset D$ be a compact invariant set with respect to $f$. Let $V : D \to \mathbb{R}$ be a $C^1$ function such that $\dot{V}(x) \leq 0$ on $K$. Let $M$ be the largest invariant set in $E = \{x \in K \,:\, \dot{V}(x) = 0\}$. Then $M$ is attracting for all trajectories starting in $K$.*

**Proof:** Since $\dot{V} \leq 0$ and $V(x(t))$ is bounded, we know that $V(x(t)) \to a \geq 0$. For any $p$ in the system's positive limit set $\Omega$, there exists $\{t_n\}$ such that $x(t_n) \to p$ as $t_n \to \infty$. This implies that

$$a = \lim_{n\to\infty} V(x(t_n)) = V(\lim_{n\to\infty} x(t_n))$$

We can therefore conclude that $V(x) = a$ for all $x \in \Omega$. From the earlier limit-set theorem (53), we know that $\Omega$ is attracting and invariant. Since $V(x) = a$ on $\Omega$ this implies that $\dot{V}(x) = 0$ on $\Omega$ so we can conclude that $\Omega \subset M$ (since it is invariant) and $\Omega \subset E$ (since $\dot{V} = 0$). So we can conclude that

$$\Omega \subset M \subset E \subset K$$

and since $\Omega$ is attracting the set $M$ must also be attracting. $\diamondsuit$

Note that $V$ need not be positive definite since the trajectories are confined to a compact set. We also note that $\dot{V}$ does not need to be negative definite to establish that trajectories asymptotically approach the largest invariant set $M$ in $E$. So theorem 54 greatly relaxes the earlier conditions in Lyapunov's direct theorem 49. All we require for this relaxation on $V$ is that we know the trajectory is confined to a compact set. If this is the case, then the following theorem can be used to establish the asymptotic stability of the equilibrium without requiring $\dot{V}$ be negative definite.

THEOREM 55. **(Asymptotic Stability - invariance theorem)** *Let $x = 0$ be an equilibrium point for $\dot{x} = f(x)$ where $f : D \to \mathbb{R}^n$ is locally Lipschitz on $D \subset \mathbb{R}^n$. Let $V : D \to \mathbb{R}$ be a $C^1$ positive definite function on $D$ containing $x = 0$ such that $\dot{V}(x) \leq 0$ on $D$. If the origin, $\{0\}$, is the largest invariant set in the set $\{x \in D : \dot{V}(x) = 0\}$, then the origin is asymptotically stable.*

**Proof:** This is a direct consequence of the invariance principle in theorem 54. This theorem's hypothesis states that the origin is the largest invariant set in the set where $\dot{V}(x) = 0$ and so the origin is attracting all trajectories in $D$. $\diamondsuit$

Consider the system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -g(x_1) - h(x_2) \end{aligned}$$

where $g$ and $h$ are locally Lipschitz such that $g(0) = 0$, $h(0) = 0$, $yg(y) > 0$ and $yh(y) > 0$. Let us first determine the equilibria that satisfy the following algebraic equations

$$\begin{aligned} 0 &= x_2 \\ 0 &= -g(x_1) - h(x_2) \end{aligned}$$

which can be readily seen to have a single solution for $(x_1^*, x_2^*) = (0, 0)$. So the origin is our equilibrium point.

Now consider the candidate Lyapunov function,

$$V(x) = \int_0^{x_1} g(y)dy + \frac{1}{2}x_2^2$$

We know that $V$ is positive definite and its directional derivative is

$$\begin{aligned} \dot{V}(x) &= g(x_1)x_2 + x_2(-g(x_1) - h(x_2)) \\ &= -x_2 h(x_2) \leq 0 \end{aligned}$$

So $\dot{V} \leq 0$ and is only negative semi-definite. The set $E$ is

$$E = \{x\,;\, \dot{V}(x) = 0\} = \{x : x_2 = 0\}$$

So let $x(t;x_0)$ be any trajectory starting in $E$. This implies that $\dot{x}_1(0) = x_2(0) = 0$ which means $x_1(t)$ is constant for all time. But if $x_1(t) \neq 0$ then

$$\dot{x}_2(t) = -g(x_1(t)) - h(x_2(t)) \neq 0$$

which would force $x(t)$ to leave the set $E$ unless $x_1(t)$ were also zero. So we can conclude that the origin is the largest invariant set in $E$ and from theorem 55 we can conclude that the origin is asymptotically stable. Note that the set attracted to $E$ is the entire state space, so we can actually conclude global asymptotic stability.

As another example, consider the Lyapunov equation

$$\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} = -\mathbf{C}^T\mathbf{C}$$

associated with the linear dynamical system

$$\begin{aligned} \dot{x} &= \mathbf{A}x(t) \\ y &= \mathbf{C}x(t) \end{aligned}$$

Let $V(x) = x^T\mathbf{P}x$ where $\mathbf{P}$ is positive definite and symmetric. Note that $\dot{V}(x) = -x^T\mathbf{C^T}\mathbf{C}x \leq 0$ This means that $\dot{V}$ is only negative semi-definite unless $C^TC$ has full rank. We will use the invariance principle to determine under what conditions the origin of this linear system is asymptotically stable.

The set where $\dot{V} = 0$ is the set where $\mathbf{C}x = 0$. So our set $E$ is

$$E = \{x \,:\, \mathbf{C}x = 0\}$$

This means that $E$ is the null space of $\mathbf{C}$. If the null space is trivial, then clearly $E$ is just the origin, which would mean that the origin is asymptotically stable.

In most cases, however, we would not expect $\ker(\mathbf{C})$ to be trivial. So in general, one could not conclude the origin is asymptotically stable unless we were to introduce some other restrictions. In particular, let us assume that the pair $(\mathbf{A},\mathbf{C})$ is observable. The state trajectory would then be

$$\mathbf{C}x(t) = \mathbf{C}e^{\mathbf{A}t}x_0$$

and this is identically zero (i.e.remains in $E$) if $(\mathbf{A},\mathbf{C})$ is observable and $x_0 = 0$. So the only trajectory that can remain in $E$ for all time is the one that starts at the origin. We can therefore conclude under the observability condition that the largest invariant set in $E$ and hence the origin is asymptotically stable.

Our last example illustrates what happens when the origin is not the largest invariant set in $E$. We consider a simple example drawn from model reference adaptive control (MRAC). Let the plant be

$$\dot{y}_p = a_p y_p + u$$

where $u$ is the control and $a_p$ is an unknown constant. We take the standard control suggested by the MIT rule

$$\begin{aligned}\dot{\theta} &= -\gamma y_p^2 \\ u &= \theta y_p\end{aligned}$$

In this case, we see $\theta$ as a control gain that is adaptively adjusted by the size of the plant state. If we let $x_1 = y_p$ and $x_2 = \theta$, then the associated state space equations for this nonlinear system become

$$\begin{aligned}\dot{x}_1 &= -(x_2 - a_p)x_1 \\ \dot{x}_2 &= -\gamma x_1^2\end{aligned}$$

We now consider a candidate Lyapunov function of the form

$$V(x) = \frac{1}{2}x_1 2 + \frac{1}{2\gamma}(x_2 - b_p)^2$$

where we let $b_p$ be large enough so that $b_p > a_p$. This means we know $a_p$ has a known upper bound, even though we may not know its exact value. Now let us compute $\dot{V}$ to obtain

$$\begin{aligned}\dot{V}(x) &= x_1\dot{x}_1 + \frac{1}{\gamma}(x_2 - b_p)\dot{x}_1 \\ &= -x_1^2(x_2 - a_p) + x_1^2(x_2 - b_p) \\ &= -x_1^2(b_p - a_p)\end{aligned}$$

This is less than zero whenever $x_1 \neq 0$ and so the set $E$ where $\dot{V}(x) = 0$ is

$$E = \{x \in \mathbb{R}^2 \,:\, x_1 = 0\}$$

Clearly $\dot{V}$ is only negative semidefinite. The origin is not the largest invariant set in $E$. It is trivial to see that $E$ is itself an invariant set, which means that all we can conclude is that the trajectories are convergent to a gain $\theta^*$. That gain may not be the best gain for this system and it may drift under noisy input data. In particular, we saw how such a drift in the MRAC example from chapter 1 could give rise to a " bursting" phenomenon. This is "drift" was an important issue in the early use of MRAC schemes which ultimately was addressed by requiring that a "persistently exciting" input to the adaptive control algorithm that would perturb the system out of $E$.

## 5. Linear Time-Invariant Systems

Lyapunov's direct method only provides a sufficient condition for stability unless the system is linear. For LTI systems, one can show that the existence of a Lyapunov function is both necessary and sufficient for asymptotic stability.

Let us use theorem 50 to study the stability of the LTI system's equilibrium point. We consider a candidate Lyapunov function of the form,

$$V(x) = x^T\mathbf{P}x$$

where $\mathbf{P}$ is a real symmetric positive definite matrix. The directional derivative of $V$ now becomes

$$\begin{aligned}\dot{V} &= \dot{x}^T\mathbf{P}x + x^T\mathbf{P}\dot{x} \\ &= x^T(\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A})x\end{aligned}$$

Let $\mathbf{Q}$ by a symmetric positive definite matrix such that $\mathbf{P}$ satisfies

$$\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} + \mathbf{Q} = 0$$

Then for this *Lyapunov equation* one can easily conclude that

$$\dot{V} = x^T(\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A})x = -x^T\mathbf{Q}x < 0$$

for $x \neq 0$. From theorem 50 we can immediately conclude that the origin is asymptotically stable. This result is summarized in the following theorem.

THEOREM 56. **(LTI Lyapunov Equation)** *If there exist symmetric positive definite matrices $\mathbf{P}$ and $\mathbf{Q}$ such that*

$$\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} + \mathbf{Q} = 0$$

*then the origin of the LTI system $\dot{x} = \mathbf{A}x$ is asymptotically stable.*

Using theorem 56 to certify that an LTI system's equilibrium is asymptotically stable is not very practical since we already have easier tests for asymptotic stability. In particular, we already know that the state trajectory for an LTI system with initial state $x_0$ can always be expressed as

$$x(t) = \sum_{i=1}^{\sigma}\sum_{k=1}^{n_i}\mathbf{A}_{ik}t^k e^{\lambda_i t}x_0$$

where $\mathbf{A}_{ik}$ is a suitable residue matrix, $\sigma$ represents the number of distinct eigenvalues of $\mathbf{A}$ and $n_i$ is the multiplicity of the $i$th distinct eigenvalue. A quick inspection of the terms in the sum reveals that these terms asymptotically go to zero if and only if the real parts of the eigenvalues are all negative. This test for asymptotic stability can be summarized in the following theorem

THEOREM 57. **(Eigenvalue Test for LTI Stability)** *The origin of the LTI system $\dot{x}(t) = \mathbf{A}x(t)$ is asymptotically stable if and only if all of the eigenvalues of $\mathbf{A}$ have negative real parts.*

Note that theorem 57 is a necessary and sufficient condition for asymptotic stability whereas our Lyapunov based result in theorem 56 is only a sufficient condition. This suggests that the existence of a Lyapunov function may also be necessary and sufficient; a suggestion that is formalized and proven in the following theorem

THEOREM 58. *Given matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$ the following statements are equivalent*

*(1) All eigenvalues of $\mathbf{A}$ have negative real parts.*
*(2) There exists a positive definite matrix $\mathbf{Q}$ such that the Lyapunov equation has a unique solution*
*(3) For every positive definite matrix $\mathbf{Q}$, the Lyapunov equation has a unique solution*

**Proof:** That (3) implies (2) is obvious. That (2) implies (1) follows from Lyapunov direct method in theorem 50. So all we need to prove is that (1) implies (3). So suppose all eigenvalues of $A$ have negative real parts, then we can show that the Lyapunov equation has a unique symmetric positive definite solution of the form

$$\mathbf{P} = \int_0^\infty e^{\mathbf{A}^T t}\mathbf{Q}e^{\mathbf{A}t}dt$$

We can verify this as follows. Note that

$$\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} = \int_0^\infty \left(\mathbf{A}^T e^{\mathbf{A}^T t}\mathbf{Q}e^{\mathbf{A}t} + e^{\mathbf{A}^T t}\mathbf{Q}e^{\mathbf{A}t}\mathbf{A}\right) dt$$

But

$$\frac{d}{dt}\left(e^{\mathbf{A}^T t}\mathbf{Q}e^{\mathbf{A}t}\right) = \mathbf{A}^T e^{\mathbf{A}^T t}\mathbf{Q}e^{\mathbf{A}t} + e^{\mathbf{A}^T t}\mathbf{Q}e^{\mathbf{A}t}\mathbf{A}$$

so we can conclude that

$$\begin{aligned}\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} &= \int_0^\infty \frac{d}{dt}\left(e^{\mathbf{A}^T t}\mathbf{Q}e^{\mathbf{A}t}\right) = \left[e^{\mathbf{A}^T t}\mathbf{Q}e^{\mathbf{A}t}\right]_0^\infty \\ &= \lim_{t\to\infty}\left(e^{\mathbf{A}^T t}\mathbf{Q}e^{\mathbf{A}t}\right) - e^{\mathbf{A}^T 0}\mathbf{Q}e^{\mathbf{A}0}\end{aligned}$$

Because we know that $x(t) \to 0$ as $t \to \infty$, we can conclude that the first term above is zero. The second term is clearly $\mathbf{Q}$. So for this proposed $\mathbf{P}$, we can conclude that $\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} + \mathbf{Q} = 0$.

This particular $\mathbf{P}$ is symmetric since

$$\mathbf{P}^T = \int_0^\infty \left(e^{\mathbf{A}^T t}\mathbf{Q}e^{\mathbf{A}t}\right)^T dt = \int_0^\infty \left(e^{\mathbf{A}t}\right)^T \mathbf{Q}^T \left(e^{\mathbf{A}^T t}\right)^T = \int_0^\infty e^{\mathbf{A}^T t}\mathbf{Q}e^{\mathbf{A}t}dt = \mathbf{P}$$

This $\mathbf{P}$ is also positive definite since

$$x^T\mathbf{P}x = \int_0^\infty x^T e^{\mathbf{A}^T t}\mathbf{Q}e^{\mathbf{A}t}x dt = \int_0^\infty z^T(t)\mathbf{Q}z(t)dt$$

where $z(t) = e^{\mathbf{A}t}x$. Since $\mathbf{Q}$ is positive definite we see that $z^T(t)\mathbf{Q}z(t) > 0$ for all $z(t) \neq 0$. The only $x$ for which $z(t)$ can be zero is $x = 0$ since $e^{\mathbf{A}t}$ is nonsingular. Therefore $\mathbf{P}$ is positive definite.

Finally, we see that $\mathbf{P}$ is unique. Let us assume there is another matrix $\overline{\mathbf{P}}$ such that

$$\mathbf{A}^T\overline{\mathbf{P}} + \overline{\mathbf{P}}\mathbf{A} = -\mathbf{Q}$$

If we subtract this from the original Lyapunov equation $\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} = -\mathbf{Q}$, we obtain

$$\mathbf{A}^T(\mathbf{P} - \overline{\mathbf{P}}) + (\mathbf{P} - \overline{\mathbf{P}})\mathbf{A} = 0$$

Multiplying both sides by $e^{\mathbf{A}^T t}$ and $e^{\mathbf{A}t}$ yields,

$$e^{\mathbf{A}^T t}\mathbf{A}^T(\mathbf{P} - \overline{\mathbf{P}})e^{\mathbf{A}t} + e^{\mathbf{A}^T t}(\mathbf{P} - \overline{\mathbf{P}})\mathbf{A}e^{\mathbf{A}t} = 0$$

for all $t \geq 0$. On the other hand,

$$\frac{d}{dt}\left(e^{\mathbf{A}^T t}(\mathbf{P} - \overline{\mathbf{P}})e^{\mathbf{A}t}\right) = e^{\mathbf{A}^T t}\mathbf{A}^T(\mathbf{P} - \overline{\mathbf{P}})e^{\mathbf{A}t} + e^{\mathbf{A}^T t}(\mathbf{P} - \overline{\mathbf{P}})\mathbf{A}e^{\mathbf{A}t} = 0$$

which implies $e^{\mathbf{A}^T t}(\mathbf{P}-\overline{\mathbf{P}})e^{\mathbf{A}t}$ is constant for all time. Because of the stability conditions we know that $e^{\mathbf{A}t} \to 0$ as $t \to \infty$ and so this function must be zero for all time. Since $e^{\mathbf{A}t}$ is nonsingular this implies $\mathbf{P}-\overline{\mathbf{P}} = 0$ and so the solution is unique. $\diamondsuit$

What we have just shown is that a solution to the Lyapunov equation is necessary and sufficient for the eigenvalue condition ensuring $\mathbf{A}$ is a Hurwitz matrix. Since we already know that $\mathbf{A}$ being Hurwitz is necessary and sufficient for the asymptotic stability of the equilibrium, we have just established that the Lyapunov equation's solution is also necessary and sufficient for asymptotic stability. This final result is summarized in the following theorem.

THEOREM 59. *The origin of system $\dot{x} = \mathbf{A}x$ is asymptotically stable if and only if for any positive definite symmetric matrix $\mathbf{Q}$ there exists a symmetric positive definite matrix $\mathbf{P}$ that satisfies the Lyapunov equation $\mathbf{A}^T\mathbf{P}+\mathbf{P}\mathbf{A}+\mathbf{Q}=0$.*

For linear systems, one can actually say something stronger about the rate at which the state asymptotically approaches the origin. We can use the Lyapunov function to characterize the rate of convergence. In particular, for a given system $\dot{x} = f(x)$, with an equilibrium at the origin, one says the origin is *exponentially stable* if there exists a real constant $\gamma$ and $K$ such that $|x(t)| \leq Ke^{-\gamma t}$ for all $t \geq 0$. The actual values of $\gamma$ and $K$ can be obtained using Lyapunov based methods. Not all nonlinear systems are exponentially stable, but we will show below that if any LTI system is asymptotically stable it must also be exponentially stable.

For an LTI system, $\dot{x} = \mathbf{A}x$, we know that if this system's origin is asymptotically stable there exists symmetric positive definite matrices $\mathbf{P}$ and $\mathbf{Q}$ that satisfy the Lyapunov equation $\mathbf{A}^T\mathbf{P}+\mathbf{P}\mathbf{A}+\mathbf{Q}=0$. So the directional derivative of $V$ may be bounded as

$$\dot{V} = x^T(\mathbf{A}^T\mathbf{P}+\mathbf{P}\mathbf{A})x \leq -x^T\mathbf{Q}x$$

We also know that

$$\underline{\lambda}(\mathbf{P})|x|^2 \leq x^T\mathbf{P}x \leq \overline{\lambda}(\mathbf{P})|x|^2$$

where $\underline{\lambda}(\mathbf{P})$ and $\overline{\lambda}(\mathbf{P})$ are the minimum and maximum eigenvalues of $\mathbf{P}$, respectively. So one can rearrange the above inequality to show that

$$|x(t)|^2 \leq \frac{x^T\mathbf{P}x}{\underline{\lambda}(\mathbf{P})} = \frac{V(x(t))}{\underline{\lambda}(\mathbf{P})} \leq \frac{V(0)}{\underline{\lambda}(\mathbf{P})}$$

So we know that $x(t)$ is uniformly bounded (which implies stability in the sense of Lyapunov). In a similar way we can bound $|x(t)|^2$ from below using the fact that $V(x) = x^T\mathbf{P}x \leq \overline{\lambda}(\mathbf{P})|x|^2$ to conclude that

$$|x|^2 \geq \frac{V(x(t))}{\overline{\lambda}(\mathbf{P})}$$

Now going back to the $\dot{V}$ equation and using the lower bound on $|x|^2$ to obtain

$$\dot{V} = -x^T\mathbf{Q}x \leq -\underline{\lambda}(\mathbf{Q})|x|^2 \leq -\frac{\underline{\lambda}(\mathbf{Q})}{\overline{\lambda}(\mathbf{P})}V(x(t))$$

This is a linear differential inequality constraint and so we can use the comparison principle to conclude that

$$V(t) \leq e^{-\frac{\underline{\lambda}(\mathbf{Q})}{\overline{\lambda}(\mathbf{P})}} V(0)$$

Since

$$\underline{\lambda}(\mathbf{P})|x(t)|^2 \leq V(t) \leq \overline{\lambda}(\mathbf{P})|x(t)|^2$$

we can conclude

$$|x(t)| \leq \frac{\overline{\lambda}(\mathbf{P})}{\underline{\lambda}(\mathbf{P})}|x(0)|^2 e^{-\frac{\underline{\lambda}(\mathbf{Q})}{\overline{\lambda}(\mathbf{P})}t}$$

which completes the proof and shows that an asymptotically stable linear system is also exponentially stable.

So we have two necessary and sufficient conditions for the asymptotic stability an LTI system's equilibrium. Which one do we use? Clearly if all we want is a yes/no declaration then the computation of the eigenvalue condition is easiest. Lyapunov methods, however, can be useful because they not only tell us whether the system is exponentially stable, the can also tell us how close the system might be to instability. In other words, we can use them to evaluate how robust the stability property might be.

As an example, let us consider the perturbed LTI system

$$\dot{x}(t) = (\mathbf{A} + \mathbf{\Delta})x(t)$$

where we know that the origin of $\dot{x} = \mathbf{A}x$ is asymptotically stable and $\mathbf{\Delta}$ is some real perturbation matrix. Since we know the unperturbed system is asymptotically stable, we also know there exist symmetric positive definite matrices $\mathbf{P}$ and $\mathbf{Q}$ that satisfy the Lyapunov equation $\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} + \mathbf{Q} = 0$. So let us compute $\dot{V}$ for the perturbed system to obtain

$$\dot{V} = x^T(\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} + \mathbf{\Delta}^T\mathbf{P} + \mathbf{P}\mathbf{\Delta})x \leq x^T(-\mathbf{Q} + \mathbf{\Delta}^T\mathbf{P} + \mathbf{P}\mathbf{\Delta})x$$

This will be negative definite provided

$$x^T(\mathbf{\Delta}^T\mathbf{P} + \mathbf{P}\mathbf{\Delta})x < x^T\mathbf{Q}x$$

for all $x$. Note that since $\mathbf{P}$ is a real symmetric matrix then

$$\underline{\lambda}(\mathbf{P})|x|^2 \leq x^T\mathbf{P}x \leq \overline{\lambda}(\mathbf{P})|x|^2$$

where $\underline{\lambda}(\mathbf{P})$ is the minimum eigenvalue of $\mathbf{P}$ and $\overline{\lambda}(\mathbf{P})$ is the largest eigenvalue of $\mathbf{P}$. One can then see that the above requirement for robust stability will hold if

$$2\overline{\lambda}(\mathbf{P})\overline{\lambda}(\mathbf{\Delta}) < \underline{\lambda}(\mathbf{Q})$$

which can be restated as a condition on the eigenvalues of the perturbation matrix

$$\overline{\lambda}(\mathbf{\Delta}) \leq \frac{\underline{\lambda}(\mathbf{Q})}{2\overline{\lambda}(\mathbf{P})}$$

This last inequality provides an approximation of how "large" the perturbation can be before there is a chance of the system losing asymptotic stability. This bound can therefore be seen as a *robust stability* condition for it shows how *robust* the stability property will be to perturbations of the system matrix $A$.

## 6. Lyapunov's Indirect Method

We now turn to investigate how the Lyapunov stability of a nonlinear system's equilibrium point is related to the stability of its linearization. This theorem asserts that if the nonlinear system's equilibrium is hyperbolic we can use the stability of the linearization's equilibrium to infer the stability of the nonlinear system's equilibrium. This theorem is sometimes known as Lyapunov's indirect method.

THEOREM 60. **(Lyapunov's Indirect Method)** *Let $\dot{x} = \mathbf{A}x$ be the linearization of nonlinear system $\dot{x} = f(x)$ about the nonlinear system's equilibrium point. Let $\{\lambda_i\}_{i=1}^n$ denote the eigenvalues of matrix $\mathbf{A}$. If $\mathrm{Re}(\lambda_i) < 0$ for all $i$ then the nonlinear system's equilibrium is asymptotically stable. If there exists $i$ such that $\mathrm{Re}(\lambda_i) > 0$, then the origin is unstable.*

**Proof:** Consider a candidate Lyapunov function of the form $V(x) = x^T\mathbf{P}x$ where $\mathbf{P} = \mathbf{P}^T > 0$. Under the first condition when $\mathbf{A}$ is Hurwitz, we can take $\mathbf{P}$ to satisfy the Lyapunov equation $\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} + \mathbf{Q} = 0$ for some $\mathbf{Q} = \mathbf{Q}^T > 0$. Take the directional derivative of $V$ with respect to the nonlinear system's vector field, $f$,

$$\begin{aligned}
\dot{V} &= x^T\mathbf{P}f(x) + f^T(x)\mathbf{P}x \\
&= x^T\mathbf{P}(\mathbf{A}x + g(x)) + [x^T\mathbf{A}^T + g^T(x)]\mathbf{P}x \\
&= x^T(\mathbf{P}\mathbf{A} + \mathbf{A}^T\mathbf{P})x + 2x^T\mathbf{P}g(x) \\
&= -x^T\mathbf{Q}x + 2x^T\mathbf{P}g(x)
\end{aligned}$$

So the first term above is negative definite. The second term is indefinite. We know, however, that $\frac{|g(x)|}{|x|} \to 0$ as $|x| \to 0$ so for all $\gamma > 0$ there exists $r > 0$ such that

$$|g(x)| < \gamma|x|$$

when $|x| < r$. This means that

$$\begin{aligned}
\dot{V} &< -x^T\mathbf{Q}x + 2\gamma\|\mathbf{P}\||x|^2 \\
&< -(\underline{\lambda}(\mathbf{Q}) - 2\gamma\|\mathbf{P}\|)|x|^2
\end{aligned}$$

where $\underline{\lambda}(\mathbf{Q})$ is the minimum eigenvalue of $\mathbf{Q}$. So if we choose $\gamma < \frac{\lambda(\mathbf{Q})}{2\|\mathbf{P}\|}$ then we can guarantee $\dot{V} < 0$ for $|x| < r$ which establishes the asymptotic stability of the origin when $\mathbf{A}$ is Hurwitz.

We now turn to the instability part of the theorem where $\mathbf{A}$ has at least one eigenvalue with a positive real part and no center eigenvalues. Split $\mathbf{A}$ into its stable and unstable parts and expose that split using a similarity transformation

$$\mathbf{T}\mathbf{A}\mathbf{T}^{-1} = \begin{bmatrix} -\mathbf{A}_1 & 0 \\ 0 & \mathbf{A}_2 \end{bmatrix}$$

where both $\mathbf{A}_1$ and $\mathbf{A}_2$ have eigenvalues with negative real parts (i.e. are Hurwitz). So the system's equations become

$$\begin{aligned}\dot{z}_1 &= -\mathbf{A}_1 z_1 + g_1(z)\\ \dot{z}_2 &= \mathbf{A}_2 z_2 + g_2(z)\end{aligned}$$

Clearly from the first part of the theorem we know $z_2 \to 0$. To see what happens to $z_1$ let $\mathbf{Q}_1$ and $\mathbf{Q}_2$ be two symmetric positive definite matrices and consider the Lyapunov equations

$$\mathbf{P}_i\mathbf{A}_i + \mathbf{A}_i^T\mathbf{P}_i = -\mathbf{Q}_i$$

for $i = 1, 2$. Symmetric positive definite matrices $\mathbf{P}_1$ and $\mathbf{P}_2$ can be found to satisfy these equations since we know $\mathbf{A}_1$ and $A_2$ are Hurwitz. We now consider the function of the form,

$$V(z) = z_1^T\mathbf{P}_1 z_1 - z_2^T\mathbf{P}_2 z_2 = z^T \begin{bmatrix} \mathbf{P}_1 & 0 \\ 0 & -\mathbf{P}_2 \end{bmatrix} z$$

which can be shown to be a Chetaev function (finish this). So we can use Chetaev's theorem 51 to conclude the origin is unstable. $\diamondsuit$

We summarize the preceding theorem's findings below

- If the equilibrium of the linearization of $\dot{x} = f(x)$ is asymptotically stable, then the the original nonlinear system is also locally asymptotically stable.
- If linearization has any eigenvalue that has a positive real part, then we can conclude the origin of the nonlinear system is unstable.
- If the linearization of the nonlinear system has a system matrix whose eigenvalues have nonpositive real parts and there exists at least one eigenvalue with a zero real part, then nothing can be concluded about the asymptotic stability of the equilibrium.

There is therefore a huge hole in our linearization's ability to deduce the stability properties of the nonlinear system's equilibrium. This occurs when the linearization has a center eigensubspace. Nonetheless, Lyapunov's indirect method still provides an extremely powerful tool in verifying the stability/instability of certain nonlinear systems. We now show some examples.

Consider the dynamical system

$$\ddot{\theta} + \dot{\theta} + \frac{g}{\ell}\sin\theta = 0$$

which is a simple pendulum with damping. The associated state space realization takes the form

$$\begin{aligned}\dot{x}_1(t) &= x_2(t)\\ \dot{x}_2(t) &= -\frac{g}{\ell}\sin x_1(t) - x_2(t)\end{aligned}$$

This model has countably many equilibria given by $(n\pi, 0)$ for $n = 0, \pm 1, \pm 2, \ldots$. We examine the stability of the physical equilibrium at $(0, 0)$ and $(\pi, 0)$ using the indirect method in theorem 60. This requires that we first compute the Jacobian of $f$

$$\left[\frac{\partial f}{\partial x}\right] = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{\ell}\cos x_1 & -1 \end{bmatrix}$$

At the two equilibria, $(0,0)$ and $(\pi, 0)$ we end up with the matrices

$$\begin{aligned} \mathbf{A}_1 &= \left[\frac{\partial f}{\partial x}\right]_{x=(0,0)} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{\ell} & -1 \end{bmatrix} \\ \mathbf{A}_2 &= \left[\frac{\partial f}{\partial x}\right]_{x=(\pi,0)} = \begin{bmatrix} 0 & 1 \\ \frac{g}{\ell} & -1 \end{bmatrix} \end{aligned}$$

The characteristic polynomial of $\mathbf{A}_1$ is $s(s+1) + \frac{g}{\ell} = 0$ which has roots,

$$\lambda_{1,2} = -\frac{1}{2} \pm \sqrt{1 - 4\frac{g}{\ell}}$$

both of which have negative real parts for all $g/\ell > 0$. So we know by theorem 60 that the equilibrium at $(0,0)$ is asymptotically stable. On the other hand the characteristic polynomial for $\mathbf{A}_2$ has roots

$$\lambda_{1,2} = -\frac{1}{2} \pm \sqrt{1 + 4\frac{g}{\ell}}$$

One root has a positive real part and the other has a negative real part. There are no center eigenvalues so by theorem 60 the equilibrium at $(\pi, 0)$ is unstable.

Let us consider another example using the van der pol oscillator.

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= -\mu(1 - x_1^2)x_2 - x_1 \end{aligned}$$

where $\mu \in \mathbb{R}$. This system has a unique equilibrium at $(0,0)$. The Jacobian at $(0,0)$ is

$$\mathbf{A} = \left[\frac{\partial f}{\partial x}\right]_{x=(0,0)} = \begin{bmatrix} 0 & 1 \\ -1 & -\mu \end{bmatrix}$$

it follows that if $\mu < 0$ then the zero solution $(0,0)$ is unstable and if $\mu > 0$ the zero solution is locally exponentially stable.

Let us now consider the following system

$$\begin{aligned} \dot{x}_1 &= a x_1^3 + x_1^2 x_2 \\ \dot{x}_2 &= -x_2 + x_2^2 + x_1 x_1 - x_1^3 \end{aligned}$$

where $a > 0$ is a parameter. This system has an equilibrium at $(0,0)$ and we'd like to investigate the asymptotic stability of this equilibrium. The linearized system matrix

$$\mathbf{A} = \left[\frac{\partial f}{\partial x}\right]_{x=0} = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$$

This $\mathbf{A}$ matrix has a zero eigenvalue and a negative eigenvalue, so we cannot use theorem 60 to say anything about the stability of its equilibrium point. The zero or center eigenvalue has associated with it a center manifold. Recall from chapter 4 that if we can rewrite the system in the following form $\dot{x} = f(x)$ as

$$\begin{aligned} \dot{z} &= \mathbf{A}z + f(z, y) \\ \dot{y} &= \mathbf{B}y + g(z, y) \end{aligned} \tag{79}$$

where $\mathbf{A}$ is a $c \times c$ matrix that has eigenvalues with zero real parts and $\mathbf{B}$ is an $s \times s$ matrix whose eigenvalues have negative real parts where $f(0,0) = 0$, $f'(0,0) = 0$, $g(0,0) = 0$ and $g'(0,0) = 0$, we define a *center manifold* for the equilibrium at $(0,0)$ as

$$W^c_{\mathrm{loc}}(0) = \{(z,y) \in \mathbb{R}^c \times \mathbb{R}^s \,:\, z = h(y),\ |y| < \delta,\ h(0) = 0, h'(0) = 0\}$$

where $h : \mathbb{R}^c \to \mathbb{R}^s$ is a $C^2$ function for $\delta$ sufficiently small. The existence of the function $h$ which generates the graph $(y, h(y))$ for $W^c_{\mathrm{loc}}$ was established in chapter 4 theorem 46. For sufficiently small neighborhood of the origin, we showed in chapter 4 theorem 47 that the dynamics of the system restricted to the center manifold for $u$ sufficient small are given by the $c$-dimensional vector field

$$\dot{u} = \mathbf{A}u + f(u, h(u)) \tag{80}$$

where $u \in \mathbb{R}^c$ and $h : \mathbb{R}^c \to \mathbb{R}^s$ characterizes the graph of the center manifold. We can then use these results to establish the following theorem which essentially states that the for this case, the stability of the equilibrium is completely determined by the center manifold's dynamics.

Theorem 47 asserted that the stability of the reduced system (80) zero solution would determine in the zero solution of the full system (79) was also stable. That notion of stability is equivalent to our notion of asymptotic stability. This is important for it means that one only needs to study the stability of the reduced order system in equation (80). Since the dimension of the center manifold is much lower than that of the original system, the use of the center manifold theorem represents a significant way of reducing the complexity of studying the stability of nonlinear systems.

## 7. Lyapunov Stability for Time-Varying Systems

We now extend our study of Lyapunov stability to time-varying systems of the form

$$\dot{x}(t) = f(t, x(t)), \quad x(t_0) = x_0$$

We say that the origin is an equilibrium point if $f(t, 0) = 0$ for all $t \geq t_0$. This equilibrium is said to be *stable* in the sense of Lyapunov if for all $\epsilon > 0$ and for all $t_0$ there exists $\delta(\epsilon, t_0) > 0$ such that

$$|x(t_0)| \leq \delta \text{ implies } |x(t)| < \epsilon \text{ for all time } t \geq t_0$$

We say the equilibrium is *asymptotically stable* if there exists $c(t_0) > 0$ such that $|x(t)| \to 0$ as $t \to \infty$ for all $|x(t_0)| < c$.

It is important to note that the radius of the initial neighborhood $N_\delta(0)$ is a function of $\epsilon$ and more importantly a function of the initial time $t_0$. What this means it that it is quite possible that as $t_0$ changes then the radius of our starting neighborhood varies as well. What we are concerned about is that as $t_0$ approaches some limit, the resulting sequence of $\delta$'s approaches zero. This means that while the system is Lyapunov stable for all finite time, $t_0$, it may asymptotically tend to become less stable as $t_0$ increases. The following example illustrates this point more clearly.

Consider the system

$$\dot{x} = (6t \sin t - 2t)x$$

The solution to this may be obtained by a separation of variables

$$\frac{dx}{x} = (6t \sin t - 2t)dt$$

that can be integrated from $t_0$ to $t$ to obtain

$$\begin{aligned} x(t) &= x(t_0) \exp\left( \int_{t_0}^{t} (6s \sin s - 2s) ds \right) \\ &= x(t_0) \exp\left( 6 \sin t - 6t \cos t - t^2 - 6 \sin t_0 + 6t_0 \cos t_0 + t_0^2 \right) \end{aligned}$$

It is obvious that for fixed $t_0$ that the quadratic term $-t^2$ will eventually dominate so we can see there is a constant $c(t_0)$ that is a function of the initial time where

$$|x(t)| < c(t_0)|x(t_0)|, \quad \text{for all } t \geq t_0$$

So let us consider any $\epsilon > 0$ and let us select $\delta = \frac{\epsilon}{c(t_0)}$. From the above equation we can therefore conclude that the origin is stable in the sense of Lyapunov.

The problem we see, however, is that $\delta$ is a function of the initial time $t_0$. In particular, let us consider a sequence $\{t_{0n}\}_{n=0}^{\infty}$ of initial times where $t_{0n} = 2n\pi$ for $n = 0, 1, 2, \ldots, \infty$. Let us evaluate $x(t)$ exactly $\pi$ seconds after $t_{0n}$. This shows us that

$$x(t_{0n} + \pi) = x(t_0) \exp((4n + 2)(6 - \pi)\pi)$$

for $x(t_0) \neq 0$, we see that

$$\tfrac{x(t_{0n}+\pi)}{x(t_{0n})} \to \infty \text{ as } n \to \infty$$

In other words, as one goes further out in time, the system tends to become "less" stable.

As another example let us consider the system

$$\dot{x} = -\frac{x}{1+t}$$

In this case we again separate variables and integrate to obtain

$$x(t) = x(t_0) \frac{1 + t_0}{1 + t}$$

Note that the origin is stable in the sense of Lyapunov. It is clear that for any $t_0$ that $|x(t)| \leq |x(t_0)|$ for all $t \geq t_0$. So for any $\epsilon > 0$ we can choose $\delta$ that is independent of $t_0$. Note however, that the origin is also asymptotically stable. In other words, for any $\epsilon > 0$ we can find $T > 0$ such that $|x(t)| < \epsilon$ for all $t \geq t_0 + T$. In particular, given $\epsilon$ we see

$$|x(t)| < |x(t_0)| \frac{1+t_0}{1+t_0+T} < \epsilon$$

which we can rearrange to see that

$$t_0 \left( \frac{|x(t_0)|}{\epsilon} - 1 \right) - 1 < T$$

The left hand side of the above inequality becomes a lower bound on $T$. This bound is a function of $\epsilon$ and the initial time $t_0$. In particular as $t_0 \to \infty$ we see that $T$ goes to infinity also. In other words, the system's rate of approach to the origin becomes arbitrarily slow for later initial times.

The concerns illustrated in the above examples motivate the following modifications to the Lyapunov stability concept for time-varying systems. In particular, we say the equilibrium point at the origin is *uniformly stable* if for all $\epsilon > 0$ there exists $\delta > 0$ that is *independent* of $t_0$ such that

$$|x(t_0)| \leq \delta \text{ implies } |x(t)| < \epsilon \text{ for all } t \geq t_0$$

We say the equilibrium is *uniformly asymptotically stable* or UAS if it is uniformly stable (US) and there exists a $\delta$ that is independent of $t_0$ such that for all $\epsilon > 0$, there exists $T > 0$ that is also independent of $t_0$ where $|x(t)| \leq \epsilon$ for all $t \geq t_0 + T(\epsilon)$ and for all $|x(t_0)| < \delta$. Finally one says the equilibrium is *globally uniformly asymptotically stable* if it is 1) uniformly stable, 2) $\delta$ can be chosen so that $\delta(\epsilon) \to \infty$ as $\epsilon \to \infty$, and there exists constants $T$ and $c$ (independent of $t_0$) such that for any choice of $\epsilon$ we know $|x(t)| < \epsilon$ for $t \geq t_0 + T$ when $|x(t_0)| < c$.

It will be convenient to introduce the notion of a *comparison function* to state and prove theorems similar to the direct theorem (49) for time-invariant systems. In particular,

- A continuous function $\alpha : [0, a) \to [0, \infty)$ belongs to $\mathcal{K}$ if it is strictly increasing and if $\alpha(0) = 0$.
- A function $\alpha : [0, a) \to [0, \infty)$ belongs to class $\mathcal{K}_\infty$ if it belongs to class $\mathcal{K}$ and it is radially unbounded (i.e. $\alpha(r) \to \infty$ as $r \to \infty$)
- A continuous function $\beta : [0, a) \times [0, \infty) \to [0, \infty)$ is class $\mathcal{KL}$ if
  - for all fixed $s$, then $\beta(r, s)$ is class $\mathcal{K}$ with respect to $r$.
  - for all fixed $r$, $\beta(r, s)$ is decreasing with respect to $s$ such that $\beta(r, s) \to 0$ as $s \to \infty$.

We refer to class $\mathcal{K}$ and $\mathcal{KL}$ functions as *comparison functions*. Class $\mathcal{K}$ functions generalize our usual notion of a positive definite function. We use class $\mathcal{K}$ functions to bound the behavior of $V$ as it varies with respect to the initial time $t_0$. Class $\mathcal{KL}$ functions are used to upper bound the behavior of a state trajectory that asymptotically approaches the origin. This comparison function is used to bound trajectories with different initial times in a uniform manner.

Because of their usefulness, it will be useful to summarize some important properties of comparison functions in the following theorem.

THEOREM 61. **(Comparison Function Properties)**

*(1) If $\alpha \in \mathcal{K}$ over $[0, a)$ then $\alpha^{-1} \in \mathcal{K}$ over $[0, \alpha(a))$.*
*(2) If $\alpha \in \mathcal{K}_\infty$, then $\alpha^{-1} \in \mathcal{K}_\infty$.*
*(3) If $\alpha_1, \alpha_2 \in \mathcal{K}$, then $\alpha_1 \circ \alpha_2 \in \mathcal{K}$.*
*(4) If $\alpha_1, \alpha_2 \in \mathcal{K}$ and $\beta \in \mathcal{KL}$ then $\alpha_1(\beta(\alpha_2(r), s)) \in \mathcal{KL}$.*

**Proof:** Part (1): Since $\alpha$ is continuous and strictly increasing on $[0, a)$, for each $y \in [0, \alpha(a))$ there is a unique $x$ such that $\alpha(x) = y$. Define $x = \alpha^{-1}(y)$. By the continuity of $\alpha$ we know that for all $\epsilon > 0$ there exists $\delta$ such that $|\alpha(x_1) - \alpha(x_2)| \leq \epsilon$ for all $|x_1 - x_2| \leq \delta$. Simply reverse the roles of $\epsilon$ and $\delta$ to show that $\alpha^{-1}$ is continuous. To establish the increasing nature of $\alpha^{-1}$, consider $x_1 = \alpha^{-1}(y_1)$ and $x_2 = \alpha^{-1}(y_2)$ where $y_1 < y_2$. Applying $\alpha$ to both sides yields $\alpha(x_1) = y_1$ and $\alpha(x_2) = y_2$. Since $\alpha$ is class $\mathcal{K}$ (increasing) and since $y_1 < y_2$, we know $x_1$ must also be less than $x_2$, thereby proving that $\alpha^{-1}$ is increasing. Finally proving $0 = \alpha^{-1}(0)$ is established by applying $\alpha$ to both sides.

Part(3): Let $\alpha(r) = \alpha_1(\alpha_2(r))$, then $\alpha(0) = \alpha_1(\alpha_2(0)) = 0$ and $\alpha(r)$ is clearly non-negative. Moreover, if we let $r_2 > r_1$, this implies $\alpha_2(r_2) > \alpha_2(r_1)$ which means $\alpha_1(\alpha_2(r_2)) > \alpha_1(\alpha_2(r_1))$, which implies $\alpha$ is increasing. Finally the composition of any two continuous functions is again continuous.

Part (4): For each fixed $s$, $\beta(\alpha_2(r), s)$ is a class $\mathcal{K}$ function of $r$. Therefore $\alpha_1(\beta(\alpha_2(r), s))$ is also a clas $\mathcal{K}$ function of $r$. For each fixed $r$, $\beta(\alpha_2(r), s)$ is decreasing as $s$ increases. Therefore $\alpha_1(\beta(\alpha_2(r), s))$ decreases as $s$ increases. Moreover we know that $\beta(\alpha_2(r), s) \to 0$ as $s \to \infty$ for fixed $r$. So $\alpha_1(\beta(\alpha_2(r), s)) \to 0$ as $s \to \infty$, thereby showing that $\alpha_1(\beta(\alpha_2(r), s))$ is class $\mathcal{KL}$. $\diamondsuit$

Class $\mathcal{K}$ and $\mathcal{KL}$ functions can be used to restate the definitions that were given earlier for uniform stability and uniform asymptotic stability. In particular, these restatements are

- The equilibrium is *uniformly stable* if and only if there exists a class $\mathcal{K}$ function $\alpha$ and a positive constant $c$ that is independent of $t_0$ such that $|x(t)| \leq \alpha(|x(t_0)|)$ for all $t \geq t_0$ and all $|x(t_0)| < c$.
- The equilibrium is *uniformly asymptotically stable* if and only if there exists a class $\mathcal{KL}$ function $\beta$ and a positive constant $\delta$ independent of $t_0$ such that $|x(t)| \leq \beta(|x(t_0)|, t - t_0)$ for all $t \geq t_0$ and all $|x(t_0)| < \delta$.

In many cases we can tighten our notion of asymptotic stability to *exponential stability*. In particular, we say that the equilibrium point $x = 0$ of $\dot{x}(t) = f(t, x(t))$ is *exponentially stable* if there exists $k > 0$ and $\gamma > 0$ such that $|x(t)| \leq k|x(t_0)|e^{-\gamma(t-t_0)}$. In this case we see that the class $\mathcal{KL}$ function given in our restatement of UAS becomes $\beta(r, s) = kre^{-\gamma s}$.

We now use the preceding notions to extend Lyapunov's direct method to time-varying systems. In particular, we will make use of the following version of the comparison principle.

THEOREM 62. **(Class $\mathcal{K}$ version of Comparison Principle)** *Let $\dot{y} = -\alpha(y)$ where $t \in \mathbb{R}$ and $\alpha : \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz class $\mathcal{K}$ function on $[0, a)$. For all $0 \leq y_0 < a$, this equation has a unique solution* *satisfying the initial value problem* $\dot{y} = -\alpha(y)$ *with* $y(0) = y_0$ *that we denote as* $y(t) = \phi(y, t_0)$ *where* $\phi$ *is class* $\mathcal{KL}$*.*

**Proof:** Since $\alpha$ is locally Lipschitz, the equation has a solution for every initial state $y_0 \geq 0$. Since $\dot{y} < 0$ whenever $y(t) > 0$, the solution has the property that $y(t) \leq y_0$ for all $t \geq t_0$. This solution, therefore, is bounded and can be extended for all $t \geq t_0$.

By integration we have

$$-\int_{y_0}^{y} \frac{dx}{\alpha(x)} = \int_{t_0}^{t} ds$$

For notational convenience we let

$$\eta(y) = -\int_{b}^{y} \frac{dx}{\alpha(x)}$$

for any $b < a$. This function is a strictly decreasing differentiable function on $(0, a)$. Since it is strictly decreasing its inverse $\eta^{-1}$ exists. Moreover, we can see that $\lim_{y\to 0} \eta(y) = \infty$. This limit follows from two facts. First the solution of the differential equation $y(t) \to 0$ as $t \to \infty$ (since $\dot{y} < 0$ and $y > 0$). Second, the limit at 0 can only be reached asymptotically (i.e. as $t \to \infty$) rather than at a finite time. This follows form the uniqueness of solutions.

For any $y_0 > 0$, the solution $y(t)$ satisfies

$$\eta(y(t)) - \eta(y_0) = t - t_0$$

Therefore

$$y(t) = \eta^{-1}(\eta(y_0) + t - t_0)$$

So we can define the function

$$\sigma(r, s) = \begin{cases} \eta^{-1}(\eta(r) + s) & r > 0 \\ 0 & r = 0 \end{cases}$$

Clearly $y(t) = \sigma(t, t_0)$ for all $t \geq t_0$ and $y_0 \geq 0$. This function is continuous since $\eta$ and $\eta^{-1}$ are continuous. It is strictly increasing in $r$ for fixed $s$ because

$$\frac{\partial \sigma}{\partial r} = \frac{\alpha(\sigma(r, s))}{\alpha(r)} > 0$$

and strictly decreasing in $s$ for fixed $r$ because

$$\frac{\partial \sigma}{\partial s} = -\alpha(\sigma(r, s)) < 0$$

Finally we can show $\sigma(r, s) \to 0$ as $s \to \infty$, thereby establishing that $\sigma$ is class $\mathcal{KL}$. $\diamondsuit$

The characterization of UAS may be used to prove the following Lyapunov theorem for time-varying systems.

THEOREM 63. **(Lyapunov Stability for Time-Varying System)** *Let $x = 0$ be an equilibrium point for $\dot{x} = f(t, x)$ and let $V : [0, \infty \times D \to \mathbb{R}$ be a $C^1$ function over $D \subset \mathbb{R}^n$ such that*

$$\underline{\alpha}(|x|) \leq V(t, x) \leq \overline{\alpha}(|x|)$$

*If*

$$\dot{V} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) \leq -\alpha(x)$$

*for all $t \geq 0$ for all $x \in D$ where $\underline{\alpha}$, $\overline{\alpha}$, and $\alpha$ are class $\mathcal{K}$ functions on $D$, then $x = 0$ is uniformly asymptotically stable.*

**Proof:** Consider any $\epsilon > 0$ and define

$$\delta = \overline{\alpha}^{-1}(\underline{\alpha}(\epsilon))$$

Then using the bound on $V$ and $\dot{V}$ we can deduce that $|x(t_0)| < \delta$ implies

$$\begin{aligned} \underline{\alpha}(|x(t)|) &\leq V(x(t)) \leq V(x(t_0)) \\ &\leq \overline{\alpha}(|x(t_0)|) \leq \underline{\alpha}(|x(t_0)|) \leq \overline{\alpha}(\delta) = \underline{\alpha}(\epsilon) \end{aligned}$$

So we can conclude that $|x(t)| \leq \epsilon$ for all $t \geq t_0$ which implies the Lyapunov stability of the equilibrium point.

Now let $V(t) = V(x(t))$ where $x$ is a trajectory satisfying the system's differential equation. Let

$$\theta(r) = \alpha(\overline{\alpha}^{-1}(r))$$

and observe that $\dot{V} \leq -\alpha(|x|)$ implies that

$$\dot{V} \leq -\alpha(\overline{\alpha}(V(t)) = -\theta(V(t))$$

One can easily deduce from the properties of class $\mathcal{K}$ functions that $\theta$ is also class $\mathcal{K}$. One can also show that $\theta$ has to be locally Lipschitz. There exists a unique solution $y(t)$ to the differential equation $\dot{y} = -\theta(y)$ such that $y(t_0) = V(t_0)$ and $y(t) = \phi(V(t_0), t - t_0)$ for some $\mathcal{KL}$ function $\phi$. By the comparison lemma we therefore know that

$$V(t) \leq \phi(V(t_0), t - t_0)$$

which implies that

$$|x(t)| \leq \alpha^{-1}(\phi(\overline{\alpha}(|x(t_0)|), t - t_0))$$

Since the right hand side is a class $\mathcal{KL}$ function, we know $|x(t)| \to 0$ as $t \to \infty$, which is sufficient to assure that the equilibrium is UAS. $\diamondsuit$

We now introduce a lemma relating the comparison functions to the usual positive definite functions used in developing Lyapunov based methods.

THEOREM 64. **(Comparison and Positive Definite Functions)** *Let $V(\cdot) : D \rightarrow \mathbb{R}$ be a continuous positive definite function defined on a domain $D \subset \mathbb{R}^n$ that includes the origin. Let $N_r(0) \subset D$ be an open neighborhood of the origin for some $r > 0$. Then there exist class $\mathcal{K}$ functions $\underline{\alpha}$ and $\overline{\alpha}$ defined on $[0, r]$ such that*

$$\underline{\alpha}(\|x\|) \leq V(x) \leq \overline{\alpha}(\|x\|)$$

*for all $x \in B_r$. Moreover if $D = \mathbb{R}^n$ and $V(x)$ is radially unbounded, then $\underline{\alpha}$ and $\overline{\alpha}$ can be chosen to be class $\mathcal{K}_\infty$ with the preceding inequality still holding for all $x \in \mathbb{R}^n$.*

**Proof:** Define the function $\psi : \mathbb{R} \rightarrow \mathbb{R}$ as

$$\psi(s) = \inf_{s \leq |x| \leq r} V(x), \quad \text{for } 0 \leq s \leq r$$

The function $\psi(\cdot)$ is continuous, positive definite and increasing and $V(x) \geq \psi(|x|)$ for $0 \leq |x| \leq r$. This function, however, is not strictly increasing. So let $\underline{\alpha}(s)$ be a class $\mathcal{K}$ function such that $\underline{\alpha}(s) \leq k\psi(s)$ with $0 < k < 1$. Then $V(x) \geq \psi(|x|) \geq \underline{\alpha}(|s|)$ for $|x| \leq r$. A similar argument can be used to obtained a class $\mathcal{K}$ lower bounding $V$. In this case, our comparison function is chosen to be

$$\phi(s) = \sup_{|x| \leq s} V(x), \quad \text{for } 0 \leq s \leq r$$

and we select $\overline{\alpha} \in \mathcal{K}$ to satsify $\overline{\alpha}(s) \geq k\phi(s)$ with $k > 1$. $\diamondsuit$

Theorem 63 characterizes the Lyapunov stability of a time-varying system using comparison functions. With the preceding theorem 64, we can now readily show that

THEOREM 65. **(Lyapunov Direct Method for Time-Varying System)** *Let $x = 0$ be an equilibrium for $\dot{x}(t) = f(t, x)$, let $V : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$ be a continuously differentiable function, and let $\underline{W} : \mathbb{R}^n \rightarrow \mathbb{R}$, $\overline{W} : \mathbb{R}^n \rightarrow \mathbb{R}$, and $W : \mathbb{R}^n \rightarrow \mathbb{R}$ be continuous positive definite functions such that*

$$\underline{W}(x) \leq V(t, x) \leq \overline{W}(x)$$
$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) \leq -W(x)$$

*for all $t \geq 0$ and $x \in D$, then the origin is uniformly asymptotically stable.*

If the comparison functions $\underline{\alpha}$, $\overline{\alpha}$, and $\alpha$ can be bounded by $k|x|^c$ for some constants $k$ and $c$, then it is possible to sharpen the prior results to deduce *uniform exponential stability*. Let's assume there exist positive constants, $c$, $k_1$, $k_2$, and $k_3$ such that

$$\begin{aligned} \underline{\alpha}(\|x\|) &\geq k_1\|x\|^c \\ \overline{\alpha}(\|x\|) &\leq k_2\|x\|^c \\ \alpha(\|x\|) &\geq k_3\|x\|^c \end{aligned}$$

Then we can see that the first part of our theorem requires

$$k_1\|x\|^c \leq V(t, x) \leq k_2\|x\|^c$$

The bound on the directional derivative therefore becomes

$$\begin{aligned}\dot{V}(t,x) &\leq -\alpha(\|x\|) \leq -k_3\|x\|^c \\ &\leq -\frac{k_3}{k_2}V(t,x)\end{aligned}$$

By the comparison lemma we see that

$$V(t,x(t)) \leq V(t_0,x(t_0))e^{-\frac{k_3}{k_2}(t-t_0)}$$

which implies that

$$\|x(t)\| \leq \left(\frac{V(t,x)}{k_1}\right)^{1/c} \leq \left(\frac{k_2}{k_1}\right)^{1/c}\|x(t_0)\|e^{-\frac{k_3}{k_2}(t-t_0)}$$

which implies that the equilibrium at 0 is uniformly exponentially stable.

## 8. Converse Theorems

Most of the preceding results have only been sufficient conditions for Lyapunov stability. In the study of linear systems, however, we were able to show that the existence of a Lyapunov function was also necessary for stability. The question is whether these type of *converse theorems* exist for other nonlinear systems. This section briefly summarizes some of the more important results regarding these converse theorems. The first theorem stated and proven below establishes a converse theorem for exponentially stable systems.

THEOREM 66. **(Converse Theorem - exponential stability:)** *Let $x = 0$ be an equilibrium point for $\dot{x}(t) = f(t,x)$ where $f$ is a $C^1$ function whose Jacobian, $\left[\frac{\partial f}{\partial x}\right]$ is bounded on $D$. Let $k$ and $\gamma$ be positive constants such that*

$$\|x(t)\| \leq k\|x(t_0)\|e^{-\gamma(t-t_0)}$$

*for all $t \geq t_0$. Then there is a function $V : [0,\infty) \times D \rightarrow \Re$ such that*

$$\begin{aligned}c_1\|x\|^2 \leq V(t,x) \leq c_2\|x\|^2 & \\ \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}f(t,x) &\leq -c_3\|x\|^2 \\ \left\|\frac{\partial V}{\partial x}\right\| \leq c_4\|x\| &\end{aligned}$$

*for some positive constants $c_1$, $c_2$, $c_3$, and $c_4$.*

**Proof:** So let us consider the function

$$V(t,x) = \int_t^{t+T} \phi^T(\tau;t,x)\phi(\tau;t,x)d\tau$$

Since the solutions are exponentially decaying we can see that

$$\begin{aligned} V(t,x) &= \int_t^{t+T} \|\phi(\tau;t,x)\|^2 d\tau \\ &\leq \int_t^{t+T} k^2 e^{-\lambda(t-\tau)} d\tau \|x\|^2 \\ &= \frac{k^2}{2\lambda}(1-e^{-2\lambda T})\|x\|^2 \end{aligned}$$

Since the Jacobian is bounded there exists $L > 0$ such that

$$\left\| \frac{\partial f}{\partial x}(t,x) \right\| \leq L$$

This means that $\|f(t,x)\| \leq L\|x\|$ which suggests

$$\|\phi(\tau;t,x)\|^2 \geq \|x\|^2 e^{-2L(\tau-t)}$$

Inserting this into our earlier expression for $V(t,x)$ yields,

$$\begin{aligned} V(t,x) &\geq \int_t^{t+T} e^{-2L(\tau-t)} d\tau \|x\|^2 \\ &= \frac{1}{2L}(1-e^{-2LT})\|x\|^2 \end{aligned}$$

So we can conclude that

$$c_1\|x\|^2 \leq V(t,x) \leq c_2\|x\|^2$$

with

$$\begin{aligned} c_1 &= \frac{1-e^{-2LT}}{2L} \\ c_2 &= \frac{k^2(1-e^{-2LT})}{2\lambda} \end{aligned}$$

Now let

$$\begin{aligned} \phi_t(\tau;\tau,x) &= \frac{\partial}{\partial t}\phi(\tau;t,x) \\ \phi_x(\tau;\tau,x) &= \frac{\partial}{\partial x}\phi(\tau;t,x) \end{aligned}$$

Note that

$$\phi_t(\tau;t,x) + \phi_x(\tau;t,x)f(t,x) = 0$$

So the directional derivative of $V$ is

$$\begin{aligned}\dot{V} &= \phi^T(t+T;t,x)\phi(t+T;t,x) - \phi^T(t;t,x)\phi(t;t,x) \\ &\quad + \int_t^{t+T} 2\phi^T(\tau;t,x)\phi_t(\tau;t,x)d\tau \\ &\quad + \int_t^{t+T} 2\phi^T(\tau;t,x)\phi_x(\tau;t,x)d\tau \\ &= \phi^T(t+T;t,x)\phi(t+T;t,x) - \|x\|^2 \\ &\leq -(1-k^2e^{2\lambda T})\|x\|^2 = -c_3\|x\|^2\end{aligned}$$

which holds if we choose $T = \ln \frac{2k^2}{2\lambda}$.

The last part bounding the matrix norm of $\left[\frac{\partial V}{\partial x}\right]$ is obtained by direct computation and then using the uniform bound $\left\|\frac{\partial f}{\partial x}\right\| \leq L$. Taken altogether, these results establish that $V$ has the stated properties of a Lyapunov function. $\diamondsuit$

We can actually prove a much stronger converse theorem which establishes that the existence of a Lyapunov function is always necessary for the trivial solution $x(t) = 0$ to be uniformly asymptotically stable. The proof for this theorem is much more involved and requires a technical result known as Massera's lemma. For completeness, we simply state the converse theorem for UAS systems. The prove is not given as it is not essential for the remaining material.

THEOREM 67. **(UAS - Converse Theorem)** *Let $x(t) = 0$ be the equilibrium point of $\dot{x} = f(t,x)$ where $f$ is continuously differentiable on $D = \{x \in \mathbb{R}^n : \|x\| < r\}$ and the Jacobian matrix is bounded uniformly in $t$. Let $\beta$ be a class $\mathcal{KL}$ function and let $r_0$ be a constant such that $\beta(r_0, 0) < r$. Assume that the system trajectory satisfies,*

$$\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0)$$

*for all $x(t_0) \in D$ and all $t \geq t_0 > 0$. Then there is a continuously differentiable function $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ and class $\mathcal{K}$ functions $\underline{\alpha}$, $\overline{\alpha}$, $\alpha$, and $\omega$ such that*

$$\begin{aligned}\underline{\alpha}(\|x\|) \leq V(t,x) &\leq \overline{\alpha}(\|x\|) \\ \dot{V} &\leq -\alpha(\|x\|) \\ \left\|\frac{\partial V}{\partial x}\right\| &\leq \omega(\|x\|)\end{aligned}$$

## 9. Computational Methods for Stability Certificates

Lyapunov's direct method certifies the stability of a system's equilibrium by checking if a function, $V : \mathbb{R}^n \to \mathbb{R}$, is positive definite $V(x) > 0$ with a directional derivative that is negative semidefinite (i.e. $-\dot{V} \geq 0$). Lyapunov's result provides no guidance in how to "search" for such $V$. One method for finding candidate Lyapunov functions is to start with a function that is already known to be a stability certificate for a closely related system and then introduce a parameterization of that function. The problem is then reduced to searching for those parameters that satisfy the two conditions. That search is usually conducted as part of an optimization problem, that seeks to minimize some measure of those parameter's strength over a feasible set for which those parameters ensure $V > 0$ and $-\dot{V} \geq 0$. This means, therefore, that one can transform the problem of certifying system stability to a computational problem. This section examines this computational approach to finding Lyapunov stability certificates.

One of the main stumbling blocks we face in developing such a computational approach is that the problem of deciding whether a multi-variate function $V$ is positive semidefinite is undecidable. If we restrict our attention to $V$ that are polynomial, then that search becomes NP-hard. So at the outset, our problem of simply certifying whether a candidate Lyapunov function is indeed a stability certificate is computationally intractable. We can get around this issue by relaxing the Lyapunov conditions to a criterion that is sufficient for positivity and yet is computationally easy to verify. The particular relaxation we consider searches for certificates, $V$, that are *sum-of-square* or SOS polynomials.

Let $\mathbb{R}[x]$ denote the set of all polynomials in the indeterminate variables $x = \{x_1, \ldots, x_n\}$ with real valued coefficients. If a polynomial $V \in \mathbb{R}[x]$ is positive semidefinite, then an obvious necessary condition is that its degree is even. A simple sufficient condition for $V$ to be positive semidefinite, therefore, is the existence of a SOS decomposition of the form,

$$V(x) = \sum_i v_i^2(x) \tag{81}$$

where $v_i \in \mathbb{R}[x]$ for all $i = 1, 2, \ldots, m$. If we can find an SOS decomposition then one can conclude that $V$ is positive semidefinite. The obvious questions are 1) how conservative is this SOS decomposition and 2) how easy is it to find such decomposition? The first question is known as Hilbert's 17th problem for which a good review of existing results will be found in [**Rez00**]. In particular, it can be shown that the SOS and nonnegative polynomials are equivalent for polynomials of one variable, quadratic polynomials, and quartic polynomials in two variables [**Par03**].

To answer the second question regarding finding SOS decompositions, let us consider a polynomial $V \in \mathbb{R}[x]$ of degree $2d$ and let us assume it can be written as a quadratic form in all monomials of degree less than equal $d$ given by the different products of the $x$ variables. In particular, this means we can write

$$V(x) = v^T \mathbf{Q} v, \quad v^T = \left[1, x_1, x_2, \ldots, x_n, x_1x_2, \ldots, x_n^d\right] \tag{82}$$

where $\mathbf{Q}$ is a constant matrix. The length of the monomial vector, $v$, is $\begin{pmatrix} n+d \\ d \end{pmatrix}$. If the matrix $\mathbf{Q}$ is positive semidefinite, then $V(x)$ has an SOS decomposition and so is nonnegative. Note that the matrix $\mathbf{Q}$ is not unique and so $\mathbf{Q}$ may be PSD for some representations and not for others. By expanding out the right hand side of equation (82) and matching coefficients of $x$, one can readily show that the set of matrices that satisfy equation (82) will form an affine variety of a linear subspace (in the space of symmetric matrices). If the intersection of this affine subspace with the positive semidefinite matrix cone is nonempty, then the function $V$ is guaranteed to be SOS and so is also nonnegative.

As an example, consider a function $V$ of the form

$$V(x,y) = 2x^4 + 2x^3y - x^2y^2 + 5y^4 \tag{83}$$

If we take $v^T = \left[x^2, y^2, xy\right]$, then $V$ may be written as a quadratic form,

$$\begin{aligned} V(x,y) &= \begin{bmatrix} x^2 \\ y^2 \\ xy \end{bmatrix}^T \begin{bmatrix} q_{11} & q_{12} & q_{13} \\ q_{12} & q_{22} & q_{23} \\ q_{13} & q_{23} & q_{33} \end{bmatrix} \begin{bmatrix} x^2 \\ y^2 \\ xy \end{bmatrix} \\ &= q_{11}x^4 + q_{22}y^4 + (q_{33} + 2q_{12})x^2y^2 + 2q_{13}x^3y + 2q_{23}xy^3 \end{aligned}$$

If we then equate coefficients, we obtain the following system of linear equations

$$\begin{bmatrix} 2 \\ 5 \\ -1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} q_{11} \\ q_{22} \\ q_{33} \\ q_{12} \\ q_{13} \\ q_{23} \end{bmatrix}$$

The set of all solutions to this system of linear inequalities can be readily shown to be

$$\begin{bmatrix} q_{11} & q_{22} & q_{33} & q_{12} & q_{13} & q_{23} \end{bmatrix} = \begin{bmatrix} 2 & 5 & -1-2\lambda & \lambda & 1 & 0 \end{bmatrix}$$

where $\lambda \in \mathbb{R}$ is any real value and so our expression for $V$ take the form,

$$V(x,y) = v^T \begin{bmatrix} 2 & \lambda & 1 \\ \lambda & 5 & 0 \\ 1 & 0 & -1-2\lambda \end{bmatrix} v = v^T \mathbf{Q}(\lambda) v = v^T(\mathbf{Q}_0 + \lambda \mathbf{Q}_1)v \tag{84}$$

where $\mathbf{Q}_0 = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 5 & 0 \\ 1 & 0 & -1 \end{bmatrix}$ and $\mathbf{Q}_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix}$. To see if $V$ has an SOS decomposition, we need to find $\lambda$ such that $\mathbf{Q}(\lambda) = \mathbf{Q}_0 + \lambda\mathbf{Q}_1$ is a positive semidefinite matrix. Note that this takes the form of a nonstrict linear matrix inequality or LMI.

The "standard form" for a "strict" linear matrix inequality (LMI) is an affine matrix-valued function of the form,

$$\mathbf{Q}(\lambda) = \mathbf{Q}_0 + \sum_{i=1}^{m} \lambda_i \mathbf{Q}_i > 0$$

where $\lambda \in \mathbb{R}^m$ are decision variables and $\mathbf{Q}_i = \mathbf{Q}_i^T \in \mathbb{R}^{n\times n}$ are symmetric matrices for $i = 1, 2, \ldots, m$. The LMI feasibility problem is given symmetric matrices, $\{\mathbf{Q}_i\}_{i=1}^m$, determine where there exists a vector $\lambda \in \mathbb{R}^m$ such that the LMI $\mathbf{Q}(\lambda) > 0$. We've stated the strict version of this problem. The nonstrict version requires us to verify that $\mathbf{Q}(\lambda) \geq 0$ which is actually the form of the problem we gave in our example.

The LMI feasibility problem is one of those matrix problems which are computationally tractable. This problem is efficiently solved using "interior-point" techniques that revolutionized the solution of linear programs back in the mid 1980's [**ARVK89**]. The development of interior-point solvers for strict LMI problems appeared in the early 1990's [**GNLC94**]. These solvers are recursive algorithms with polynomial time-complexity. Surprisingly, the number of recursions is relatively constant with respect to the number of problem decision variables, which makes these methods extremely efficient. Algorithms that solve the nonstrict LMI problems are sometimes called semidefinite programs [**VB96**]. Freely available SDP solvers such as SDPT3 began to appear around 2000 [**TTT99**].

One of the main issues in using such SDP solvers is that their user interfaces are not in a form that is easy to use directly. This has led to the development of a number of toolkits that essentially translate LMI expressions that are in the form of matrix inequalities, into the standard form that the solvers then work with. One of the first widely used toolkits that was developed specifically for SOS programming was SOSTOOLS [**PPP02**]. The interface for SOSTOOLS can be somewhat clumsy to work with and so a more recent interface toolkit known as YALMIP [**Lof04**] has been gaining widespread acceptance across the community. The examples that we show below use YALMIP as the interface to the SDP solver.

We will now use YALMIP to see if the polynomial in equation (83) has an SOS decomposition. Recall that this involves finding a real $\lambda$ such that $\mathbf{Q}(\lambda)$ in equation (84) is positive semidefinite. We start by declaring the state variables and forming the polynomial we want to check,

```
x = sdpvar(1,1); y = sdpvar(1,1);
V = (2*x^4)+(2*x^3*y)-(x^2*y^2)+(5*y^4);
```

We then form the vector of monomials, $v$, in equation (82) and then construct the quadratic form, $v^T\mathbf{Q}v$. The command `monolist` constructs a list of all monomials with degree less than 2. For this problem that means

$$v^T = \begin{bmatrix} 1 & x & y & x^2 & xy & y^2 \end{bmatrix} \tag{85}$$

You could have also specified a specific list of monomials.

```
v         = monolist([x y],degree(V)/2);
Q        = sdpvar(length(v));
V_sos = v'*Q*v;
```

We then form the set of SOS constraints that are passed on to the solver. These constraints require $\mathbf{Q}$ to be PSD and the coefficients of the SOS polynomial to match the coefficients of the specified $V$. Once this is done we can call the SOS solver that computes the SOS decomposition of $V$ (if it exists). Since we did not formally declare any SOS-type constraint, we use the solver `optimize` which returns the desired answer (if it exists) in the matrix $\mathbf{Q}$.

```
F = [coefficients(V-V_sos,[x y])==0, Q>=0];
```

```
sol=optimize(F);
if sol.problem==0
value(Q);
end
```

The diagnostics from `optimize` are contained in the structure `sol` and if member `sol.problem` is zero, then the SDP solver was able to find a positive semidefinite matrix $\mathbf{Q}$ that satisfied the problem's constraints. For this particular example that matrix is

$$\mathbf{Q}=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 & -1.4476 \\ 0 & 0 & 0 & 1 & 1.8952 & 0 \\ 0 & 0 & 0 & -1.4476 & 0 & 5 \end{bmatrix}$$

which is defined with respect to the monomial ordering in equation (85).

The original ordering we used in defining our problem in equation (84) had a monomial ordering of $v^T = \left[x^2, y^2, xy\right]$. If we extract out these rows and columns of the $\mathbf{Q}$ computed using YALMIP, then we obtain

$$\mathbf{Q}(\lambda)=\begin{bmatrix} 2 & -1.4476 & 1 \\ -1.4476 & 5 & 0 \\ 1 & 0 & 1.8952 \end{bmatrix} = \begin{bmatrix} 2 & \lambda & 0 \\ \lambda & 5 & 0 \\ 1 & 0 & -1-2\lambda \end{bmatrix}$$

YALMIP asserted that for $\lambda=-1.4476$, this $\mathbf{Q}(\lambda)$ is positive semidefinite. This observation is verified by computing the eigenvalues of $\mathbf{Q}(-1.4476)$ to find they ( $0.9633$, $3.2922$, and $5.6398$) are all nonnegative. The SOS decomposition can be obtained by taking the square root of $\mathbf{L}^T\mathbf{L}=\mathbf{Q}$, to obtain

$$\begin{aligned}
\mathbf{L}=\begin{bmatrix} 1.2927 & -0.4202 & 0.3903 \\ -0.4202 & 2.1957 & 0.0467 \\ 0.3903 & 0.0467 & 1.3193 \end{bmatrix} \Rightarrow V(x) &= \sum_{i=1}^{3} v_i(x) \\
&= (1.2927x^2-0.4202y^2-0.3903xy)^2 \\
&\quad +(-0.4202x^2+2.1957y^2+0.0467xy)^2 \\
&\quad +(0.3903x^2+0.0467y^2+1.3193xy)^2 \\
&= 2x^4+2x^3y-x^2y^2+5y^4
\end{aligned}$$

which verifies that $V$ is SOS.

The preceding discussion steps through with YALMIP what we did in forming the LMI $\mathbf{Q}(\lambda)=\mathbf{Q}_0+\lambda\mathbf{Q}_1$. YALMIP also provides a more direct way of doing this through the command `sos` that streamlines the task of forming an SOS constraint and then using the command `solvesos` to compute the decomposition and actually find the $\mathbf{Q}$ matrices. Alternatively, one could use the command `sosd` to just return the SOS decomposition.

```
 x = sdpvar(1,1); y = sdpvar(1,1);
V = (2*x^4)+(2*x^3*y)-(x^2*y^2)+(5*y^4);
F = sos(V);
[sol,u,Q,res] = solvesos(F);
if sol.problem==0
    sdisplay(u{1})
    value(Q{1})
    v = sosd(F);
    sdisplay(v)
end;
```

This returns a slightly different decomposition than we obtained doing the long way, but it still forms an SOS decomposition for $V$, merely emphasizing the fact that these decompositions are not unique.

With the preceding introduction to using YALMIP in finding SOS decompositions, we now proceed to show how to use it in finding Lyapunov functions for nonlinear dynamical systems. We'll show three examples. The first one looks at finding the SOS decomposition for a Lyapunov function of an LTI system. This is done, because we can readily check the answer by solving the system's Lyapunov equation. The second example looks at a more complex nonlinear system example whose right hand side is only rational (not polynomial). The last example looks at a somewhat larger polynomial system that was presented in [**PP02**].

**Example 1:** Consider the linear dynamical system

$$\begin{aligned}\dot{x}_1 &= -ax_1 + x_2 \\ \dot{x}_2 &= x_1 - x_2\end{aligned}$$

We can check the stability of this system by looking at the eigenvalues of $A = \begin{bmatrix} -a & 1 \\ 1 & -1 \end{bmatrix}$. These eigenvalues will all have nonpositive real parts for $a \geq 1$. At $a = 1$, it has a zero eigenvalue and for $a < 1$, the system is unstable. We will use this to check the calculation made by YALMIP. As before, our YALMIP script starts by cleaning up the workspace and declaring the state variables,

```
clear all;
yalmip('clear');
sdpvar x1 x2;
x = [x1 ; x2];
```

We then declare the vector field with $a = 2$,

```
a = 2;
f = [  -a*x1+x2; x1-x2];
```

We form the first SOS constraint that requires $V \geq 0$. This constraint uses `sos` to form the SOS constraint

```
P = sdpvar(length(x));
V = x'*P*x;
F = [P>=0]+[sos(V)];
```

We then form the second SOS constraint that requires $-\dot{V} > 0$. This constraint uses `jacobian` to symbolically compute the Jacobian of $V$. Note that the actual constraint we are checking to be SOS is $-\dot{V} - \epsilon(x_1^2 + x_2^2) \geq 0$. The second part of this inequality forces $-\dot{V}$ to be strictly positive definite since the SDP solver only works with nonstrict inequality constraints.

```
negVdot = -jacobian(V,x)*f;
eps = 0.1;
F = F + [sos(negVdot-eps*(x'*eye(2,2)*x))];
```

We then use `solvesos` to compute the SOS decomposition of these constraints. The function returns the solution status `sol`, a vector `u` of the monomials, and the symmetric matrix $\mathbf{Q}$ associated with those monomials. The returned vector `u` and `Q` are data structures that contain two members; one for the first sos constraint on $V$ and another for the second SOS constraint on $\dot{V}$. We're interested in the first one. In particular if `sol.problem` equals 0, then SDPT3 found a feasible solution and we can then display it

```
[sol,u,Q] = solvesos(F);
if sol.problem == 0
    disp('Constraints are SOS');
    sdisplay(u{1}'*Q{1}*u{1})
else
    disp('Constraints FAILED');
end
```

The Lyapunov function returned from this has the form

$$
\begin{aligned}
V(x_1, x_2) &= 5.2779x_1^2 + 6.7222x_2^2 + 2.8886x_1x_2 \\
&= \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \begin{bmatrix} 5.2779 & 1.443 \\ 1.443 & 6.722 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\
&= x^T \mathbf{P} x
\end{aligned}
$$

We can readily check to see that $\mathbf{P}$ is indeed positive definite and symmetric with real eigenvalues $4.3852$ and $7.6148$. We can also verify that it satisfies the Lyapunov equation

$$\begin{aligned} \mathbf{A}^T\mathbf{P} + \mathbf{PA} &= \begin{bmatrix} -2 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 5.2779 & 1.443 \\ 1.443 & 6.722 \end{bmatrix} + \begin{bmatrix} 5.2779 & 1.443 \\ 1.443 & 6.722 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 1 & -1 \end{bmatrix} \\ &= \begin{bmatrix} -24 & 7.6671 \\ 7.6671 & -7.6671 \end{bmatrix} \end{aligned}$$

which has eigenvalues $-27.0352$ and $-4.6320$ and so is negative definite as expected.

To double check our answer, let us see what happens if we let $a = 1$, so that the system has a zero eigenvalue. In this case, running the same script yields `problem.sol=1`, which implies that SDPT3 could not find an SOS decomposition. In particular, we fail to find a positive definite $V$ for this problem. If we relaxed the requirement for $-\dot{V} > 0$ and simply required it to be $\dot{V} \geq 0$, we would be able to get a solution. We can relax this restriction by simply changing the SOS constraint on $\dot{V}$ to

```
F =  F + [sos(negVdot)];
```

With this change, SDPT3 does find a solution, but the resulting $V(x)$ matrix is now

$$V(x) = 5.7114x_1^2 + 5.7114x_2^2 + 1.6364x_1x_2 = x^T \begin{bmatrix} 5.7114 & 1.6364 \\ 1.6364 & 5.7114 \end{bmatrix} x$$

which is positive definite. But now when we look at the Lyapunov equation we see that,

$$\begin{aligned} \mathbf{A}^T\mathbf{P} + \mathbf{PA} &= \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 5.1174 & 0.8182 \\ 0.8182 & 5.7114 \end{bmatrix} + \begin{bmatrix} 5.1174 & 0.8182 \\ 0.8182 & 5.7114 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \\ &= \begin{bmatrix} -9.7864 & 9.7864 \\ 9.7864 & -9.7864 \end{bmatrix} \end{aligned}$$

which has eigenvalues $-19.5729$ and $0$. So this did not yield an asymptotically stable system, as expected. The reason why our SOS decomposition failed was because we were forcing $-\dot{V}$ to be positive definite, not just positive semidefinite.

**Example 2:** Let us now look at some examples with nonlinear dynamics. In particular we consider the system

$$\begin{aligned} \dot{x}_1 &= -x_1^3 - x_1x_3^2 \\ \dot{x}_2 &= -x_2 - x_1^2x_2 \\ \dot{x}_3 &= -x_3 - \frac{3x_3}{x_3^2+1} + 3x_1^2x_3 \end{aligned}$$

which has an equilibrium at the origin. The issue we face with this system is that the right hand side of the ODE now consists of *rational functions*, not just polynomials. Since our use of SOS decompositions is restricted to polynomials, this means we cannot directly use it to find a Lyapunov equation for this system.

In order to use the SOS decomposition to find a Lyapunov function, we need to transform this into a realization that has polynomial right hand side. This is easily done by introduce a time scaling of the form $dt = d\tau(x_3^2+1)$. Since the factor $x_3^2+1$ is always positive, if the system is stable with respect to time scale $\tau$, it should also be stable with respect to $t$. This change of time scale transforms the original ODEs into a polynomial set of the form

$$
\begin{aligned}
\dot{x}_1 &= (-x_1^3 - x_1 x_3^2)(x_3^2+1) \\
\dot{x}_2 &= (-x_2 - x_1^2)(x_3^2+1) \\
\dot{x}_3 &= (-x_3 + 3x_1^2 x_3)(x_3^2+1) - 3x_3
\end{aligned}
$$

which is polynomial and so we can apply the SOS approach to searching for a Lyapunov function. The YALMIP script used for this system is similar to what we used for the earlier LTI system example. The candidate Lyapunov function we use is a quadratic function of $x$.

```
  %clean up the workspace
clear all;
yalmip('clear');

%declare state variable
sdpvar x1 x2 x3;
x = [x1;x2;x3];

%declare vector field
f = [(-x1^3-x1*x3^2)*(x3^2+1);
     (-x2-x1^2)*(x3^2+1);
     (-x3+3*x1^2*x3)*(x3^2+1)-3*x3];

%declare V>0 constraint
%z = [x1;x2;x3];
P = sdpvar(length(x));
V = x'*P*x;
eps = 0.001;
F = [P>=0]+[sos(V-eps*(x'*eye(3,3)*x))];

%declare -Vdot>=0 constraint
negVdot = -jacobian(V,x)*f;
F = F + [sos(negVdot)];

%compute sos decomposition
[sol,u,Q] = solvesos(F);
if sol.problem==0;
    disp('Constraints are SOS');
    sdisplay(u{1}'*Q{1}*u{1});
```

```
else
    disp('Constraints FAILED');
end;
```

In this example, we required $V > 0$ and $-\dot{V} \leq 0$. The script returns with the following Lyapunov function

$$V(x) \quad = \quad 16.1618x_1^2 + 10.2512x_2^2 + 3.8279x_3^2$$

which is clearly positive definite.

**Example 3:** The last example from [PP02] considers a 6 state polynomial system of the form,

$$\begin{aligned}
\dot{x}_1 &= -x_1^3 + 4x_2^3 - 6x_3x_4 \\
\dot{x}_2 &= -x_1 - x_2 + x_5^3 \\
\dot{x}_3 &= x_1x_4 - x_3 + x_4x_6 \\
\dot{x}_4 &= x_1x_3 + x_3x_6 - x_4^3 \\
\dot{x}_5 &= -2x_2^3 - x_5 + x_6 \\
\dot{x}_6 &= -3x_3x - 4 - x_5^3 - x_6
\end{aligned}$$

Let us start by looking for a Lyapunov function of the form $V(x) = x^T\mathbf{P}x$. In this case, the YALMIP script will not be able to find a positive definite solution. So we change the nature of the candidate Lyapunov function to have the form,

$$V(x) = z^T\mathbf{P}z$$

where

$$z^T = \left[\begin{array}{cccccc} x_1 & x_3 & x_4 & x_6 & x_2^2 & x_5^2 \end{array}\right]$$

Note that we've added fourth order terms to some of the components. In this case, YALMIP returns with the following Lyapunov function

$$V(x) = +19.2951x_1^2 + 42.5312x_3^2 + 51.2484x_4^2 + 32.1599x_6^2 + \left(32.2599x_2^4 + 15.6299x_5^4 + 21.7675x_2^2x_5^2\right)$$

which is positive definite.

## 10. Concluding Remarks

This chapter gave a whirlwind introduction to Lyapunov stability concepts. It essentially covers the main material on Lyapunov stability in chapter 3 of [Kha96]. We started by defining the notion for time-invariant systems and then explored the relationship of Lyapunov stability for a time-invariant nonlinear system and its linearization. We then examined how these concepts can be extended to time-varying nonlinear systems. A novel feature of this chapter is its discussion of SOS programming as a tool for computing Lyapunov functions for systems.

The main finding was that for cases when the system's linearization system matrix was Hurwitz, then the nonlinear system's origin was asymptotically stable. We saw that if the linearized system matrix had at least one eigenvalue with a positive real part, then we can immediately infer the instability of the origin. The only "interesting" case occurs when the linearized system matrix has eigenvalues that have real parts that are less than or *equal* to zero. These zero-eigenvalues are associated with a center manifold for the nonlinear system and using the Center Manifold theorems from chapter 4, we can use the behavior of the system restricted to that manifold to infer the stability of the origin. This last point is useful, for it means that to evaluate stability, we only need to examine the behavior of the system of the center manifold and since that manifold has a much smaller dimension than the full state space, this provides a useful reduction method that is often used in the study of the stability of nonlinear dynamical systems.

The notion of Lyapunov stability represents a central organizing concept in nonlinear control theory. The next chapter investigates how we extend these stability concepts to input-output systems.

There are a number of fine textbooks that focus primarily on Lyapunov theory for nonlinear systems. This chapter only identifies the most relevant parts of those earlier textbooks for our purposes. For those interested in following up on many aspects of Lyapunov stability (especially with regard to time-varying systems and the converse theorems) it is suggested you consult [**HB67, Kha96, Vid02, HC08**].