Flocking in Fixed-Wing Agents
SLP Report

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Abstract

The report investigates the problem of achieving coordinated flocking behavior in swarm agents while accommodating realistic constraints, particularly in adverse conditions such as bad weather or hostile environments. While models for general flocking behavior have been extensively studied, they often rely on assumptions that are ill-suited for real-world applications, leading to limitations.

The primary objective of this report is to establish the mathematical principles of flocking algorithms and adapt them to specific application constraints by drawing upon existing research. It delves into key concepts of flocking, including Olfati Saber flocking and the integration of barrier functions. These concepts serve as building blocks for developing a model that attains coordinated flocking behavior while adhering to critical constraints, including fixed-wing dynamics, collision avoidance, actuator saturation, and limited communication range.

The report presents a few potential approaches to address these challenges, supported by simulation results. These solutions seek to offer insight into the adaptation of control laws to meet such constraints effectively.

1 Introduction

In this report, I detail my learnings of existing algorithms for flocking and collision avoidance in multi-agent systems, or "swarms" of agents.

Much like a flock of birds, swarm agents consist of individual entities, such as drones or robots, that exhibit coordinated behavior. This coordination often mirrors avian dynamics and is achieved through algorithms and rules governing how agents interact to attain collective objectives efficiently.

A fundamental technical challenge in this domain is collision avoidance. Birds can navigate crowded skies without colliding due to their innate awareness. In swarm robotics, collision avoidance requires the development of intricate algorithms that enable agents to detect and evade obstacles, whether they are stationary barriers or other agents.

Implementing flocking behavior in real-world systems presents practical challenges. Actuator saturation limits an agent's ability to precisely adjust its position and to react quickly enough to avoid disaster. Scalability is essential for accommodating more agents without compromising performance, and striking the right balance between intelligence and simplicity is crucial. Additionally, agents often have limited sensing capabilities, which can pose obstacles to timely and accurate information about the environment, impacting both collision avoidance and flocking coordination. These practical concerns are central to the successful application of flocking behavior in real-world scenarios.

2 Background

By definition[CR19], there are three kinds of MAS: centralised, decentralised and distributed. Practically, the distributed system is most favourable as long as the agents can be made simple enough. The algorithms considered implicitly consider the system to be distributed. We will see this further ahead.

2.1 Olfati-Saber Flocking

In this paper, the author presents a theoretical framework for design and analysis of distributed flocking algorithms. **Note:** The contents of this section are from [Olf06].

Reynold's rules of flocking consist of three heuristics:

- Flock Centering: attempt to stay close to nearby flockmates
- Collision Avoidance: avoid collisions with nearby flockmates
- Velocity Matching: attempt to match velocity with nearby flockmates

For the purpose of flocking, we will work with graphs that depict spatial information. A graph G is a pair (V, E) that consists of vertices $V = \{1, 2..., n\}$ and edges $E \subseteq \{(i, j) : i, j \in V, j \neq i\}$. The adjacency matrix, denoted as A, is a square matrix representing the connections or interactions between individual agents in the swarm. It is often used to describe the topology or network structure of the swarm. Each entry A_{ij} of the adjacency matrix is defined as follows:

$$A_{ij} = \begin{cases} 1 & \text{if } (i,j) \in E \\ 0 & \text{otherwise} \end{cases}$$

Adjacency matrices can be used as weighted graphs by replacing the 0,1 elements with varying weights. For an undirected graph G, the adjacency matrix is symmetric i.e, $A^T = A$. Using the adjacency matrix, we can define the in-degree, out-degree and neighbour set of the i^{th} agent. The row sum $(\sum_{j=1}^{N} a_{ij})$ indicates the out-degree of the i^{th} agent, the column sum $(\sum_{j=1}^{N} a_{ji})$ indicates the in-degree, and the neighbour set is defined as follows:

$$N_i = \{j \in V : a_{ij} \neq 0\} = \{j \in V : (i,j) \in E\} \tag{1}$$

The Laplacian matrix, denoted as L, is derived from the adjacency matrix and is a key component in analyzing the collective behavior of a swarm. It is defined as:

$$L = D - A$$

where D is the degree matrix, a diagonal matrix where D_{ii} represents the degree of agent i, i.e., the number of connections or interactions it has. The Laplacian matrix L captures the relationship between an agent and its neighbors, and it plays a crucial role in characterizing the dynamics of the swarm.

Let $q_i, p_i \in \mathcal{R}^m$ denote the position and velocity of the i^{th} agent. Thus the dynamics of the system are given as

$$\begin{cases} \dot{q_i} = p_i \\ \dot{p_i} = u_i \end{cases}$$

where u_i is the control input, or the acceleration applied to the agent.

Now we bake the distributed nature of the network into the equations using the neighbour set N_i. We define an interaction range r between two agents.

$$N_i = \{j \in V : ||q_j - q_i|| < r\} \tag{2}$$

where $||\cdot||$ is the Euclidean norm in R^m. Thus we are able to define our adjacency matrix in a similar manner.

$$A_{ij} = \begin{cases} 1, & \text{if } ||q_j - q_i|| < r \\ 0, & \text{otherwise} \end{cases} \tag{3}$$

The author goes on to define an "α-lattice" as a structure wherein all neighbouring agents are at a distance d from each other. The scale and ratio of the graph are defined as d and $\kappa = r/d$.

$$||q_j - q_i|| = d \ \forall \ j \in N_i(q) \tag{4}$$

As a measurement of the deviation of a given flock structure from an alpha lattice, we define a deviation energy

$$E_\delta(q) = \frac{1}{|E|+1} \sum_{i=1}^{N} \sum_{j \in N_i} \psi(||q_j - q_i|| - d) \tag{5}$$

For a quasi-α-lattice, with $|||q_j - q_i|| - d| < \delta$, we get that the deviation energy is bounded

$$E_\delta(q) \leq \frac{|E(q)|}{|E(q)|+1}\delta^2 \leq \delta^2 = \epsilon^2 d^2, \quad \epsilon << 1 \quad (6)$$

Thus, we see that the alpha lattices are low-energy configurations of the N agents. Thus, we look for a smooth collective potential of the form

$$V(q) = \frac{1}{2}\sum_i \sum_{j \neq i} \psi_\alpha(||q_j - q_i||_\sigma) \quad (7)$$

where $\psi_\alpha(z)$ is a smooth pairwise attractive/repulsive potential.

2.1.1 Smooth Collective Potentials

Here, we devise the form of the potential $V(q)$. To construct a smooth potential, the author defines a nonnegative map called a σ-norm.

The σ-norm is defined as

$$||z||_\sigma = \frac{1}{\epsilon}(\sqrt{1+\epsilon||z||^2} - 1) \quad (8)$$

with a parameter $\epsilon > 0$ and a gradient $\sigma_\epsilon(z) = \nabla||z||_\sigma$ given by

$$\sigma_\epsilon(z) = \frac{z}{\sqrt{1+\epsilon||z||^2}} = \frac{z}{1+\epsilon||z||_\sigma} \quad (9)$$

The author defines a bump function $\rho_h(z)$ which is a smooth non-increasing function from $[1, 0]$, in the domain $[0, 1]$. $\rho_h(z)$ is defined as follows:

$$\rho_h(z) = \begin{cases} 1, & z \in [0, h) \\ \frac{1}{2}[1 + cos(\pi\frac{z-h}{1-h})], & z \in [h, 1] \\ 0, & \text{otherwise} \end{cases} \quad (10)$$

where $h \in (0, 1)$. This bump function is used to define the adjacency matrix so that the multi-agent system is of a distributed nature.

$$a_{ij}(q) = \rho_h(||q_j - q_i||_\sigma / r_\alpha) \in [0, 1], j \neq i \quad (11)$$

This weighted adjacency matrix allows us to define a potential such that the attractive/repulsive force depends on the distance between the two agents.

Using the above, the author defines

$$\phi_\alpha(z) = \rho_h(z/r_\alpha)\phi(z - d_\alpha) \quad (12)$$

$$\phi(z) = \frac{1}{2}[(a+b)\sigma_1(z+c) + (a-b)] \quad (13)$$

where $r_\alpha = ||r||_\sigma, d_\alpha = ||d||_\sigma$. We get the pairwise attractive/repulsive potential between agents to be of the from

$$\psi_\alpha(z) = \int_{d_\alpha}^{z} \phi_\alpha(s) ds \quad (14)$$

Here, the plot for $\psi_\alpha(z)$ has been taken from the reference.

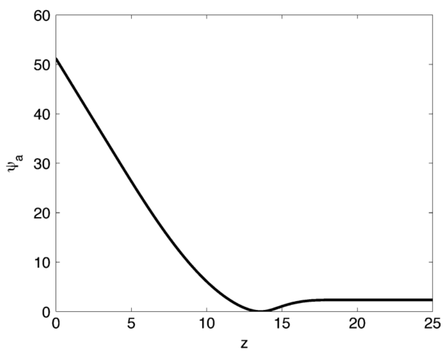


Fig. 3. Smooth pairwise potential $\psi_\alpha(z)$ with a finite cut-off.

2.1.2 Flocking Algorithms

In general, recollecting Reynold's laws of flocking, to each agent, a control input is applied that consists of three terms

$$u_i = {f_i}^g + {f_i}^d + {f_i}^\gamma \quad (15)$$

The first term ${f_i}^g = -\nabla_{q_i} V(q)$ corresponds to a gradient-based term, which represents the flock centering and collision avoidance properties, the second term corresponds ${f_i}^d$ to a velocity matching term, and the final term ${f_i}^\gamma$ corresponds to a navigational control input.

One can think of this system as that of a damped spring in an external force field. The gradient term corresponds to the force due to displacement, the velocity matching term corresponds to the damping in the spring, and the navigational term acts to displace the equilibrium position.

The presence of the γ force is similar to the flock experiencing a force towards a "leader agent", because of which the paper deals with the navigational term in the context of γ-agents.

Let us now explicitly calculate the form of ${f_i}^g = -\nabla_{q_i} V(q)$. We have

$$V(q) = \frac{1}{2}\sum_{i=1}^{N}\sum_{j\neq i}\psi_\alpha(||q_j - q_i||_\sigma) \tag{16}$$

$$V(q) = \sum_{i=1}^{N}\sum_{j>i}\psi_\alpha(||q_j - q_i||_\sigma) \tag{17}$$

$$\frac{\partial V}{\partial q_i} = \sum_{j\neq i}\frac{\partial \psi_\alpha(||q_j - q_i||_\sigma)}{\partial q_i} \tag{18}$$

Let us substitute $||q_j - q_i||_\sigma$ by z, and $a_{ij}(q)$ by $a_{ij}(z)$.

$$\frac{\partial V}{\partial q_i} = \sum_{j\neq i}\frac{\partial}{\partial q_i}\int_{d_\alpha}^{z}\rho_h(\frac{s}{r_\alpha})\phi(s - d_\alpha)ds \tag{19}$$

$$\frac{\partial V}{\partial q_i} = \sum_{j\neq i}\frac{\partial}{\partial q_i}\int_{d_\alpha}^{z}a_{ij}(s)\phi(s - d_\alpha)ds \tag{20}$$

$$\frac{\partial V}{\partial q_i} = \sum_{j\neq i}\frac{\partial}{\partial ||q_j - q_i||_\sigma}\int_{d_\alpha}^{z}a_{ij}(s)\phi(s - d_\alpha)ds \cdot \frac{\partial ||q_j - q_i||_\sigma}{\partial q_i} \tag{21}$$

$$\frac{\partial V}{\partial q_i} = \sum_{j\neq i}\frac{\partial}{\partial s}\int_{d_\alpha}^{z}a_{ij}(s)\phi(s - d_\alpha)ds \cdot \frac{\partial ||q_j - q_i||_\sigma}{\partial q_i} \tag{22}$$

$$\frac{\partial ||q_j - q_i||_\sigma}{\partial q_i} = -\nabla ||q_j - q_i||_\sigma = \sigma_\epsilon(q_j - q_i) \tag{23}$$

$$\frac{\partial V}{\partial q_i} = \sum_{j\neq i}a_{ij}(z)\phi(z - d_\alpha)\cdot\sigma_\epsilon(q_j - q_i) \tag{24}$$

$$\frac{\partial V}{\partial q_i} = \sum_{j\neq i}\phi_\alpha(||q_j - q_i||_\sigma)\cdot\sigma_\epsilon(q_j - q_i) \tag{25}$$

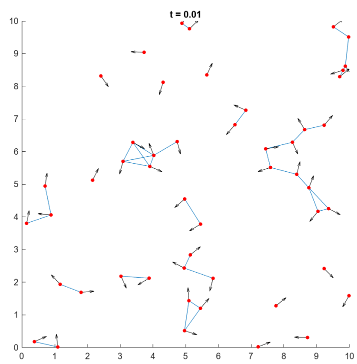
$$\nabla V(q) = \sum_{i}\sum_{j\in N_i}\phi_\alpha(||q_j - q_i||_\sigma)\cdot\mathbf{n_{ij}} \tag{26}$$

where $\mathbf{n_{ij}}$ is $\sigma_\epsilon(q_j - q_i)$. Also, note that are able to change the domain of the second summation from $j \neq i$ to $j \in N_i$ as the $a_{ij}(q)$ term is zero for all pairs of agents that are not within a distance r_α of each other.

The consensus term, in order to generate a force which will cause the agents to align directions, is generally constructed as

$${f_i}^d = \sum_{j\in N_i}a_{ij}(q)(p_j - p_i) \tag{27}$$

In Fig. 1, the boids are simulated using the control law produced as a result of equations (16) through (27), $u_i = {f_i}^g + {f_i}^d$. We see that the flock undergoes fragmentation, where the flock breaks up into smaller flocks in the absence of a "central" navigation force given by ${f_i}^\gamma$.



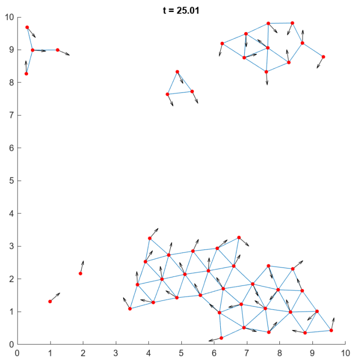


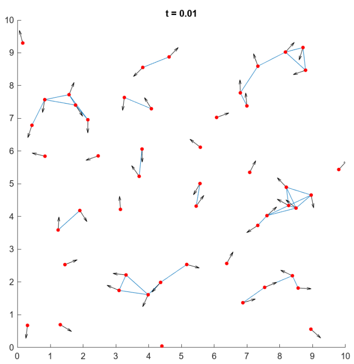
Fig. 1: Flocking without a Central Force

Similarly, the navigation term can be assumed to come from a γ-agent which may be static or dynamic, and acts as a rendezvous point for the system given the linear form

$$f_i{}^{\gamma} = -c_1(q_1 - q_r) - c_2(p_i - p_r), \quad c_1, c_2 > 0 \quad (28)$$

where q_r, p_r are the position and velocity of the γ-agent respectively.

In Fig. 2, the boids are simulated using the above control law, along with the navigational term which always acts towards the center of the region, namely the point $(5, 5)$. This gives the flock the tendency to stay together. This can be generalised to set a navigational control input towards a moving point, as noted in the section on Flocking Algorithms.



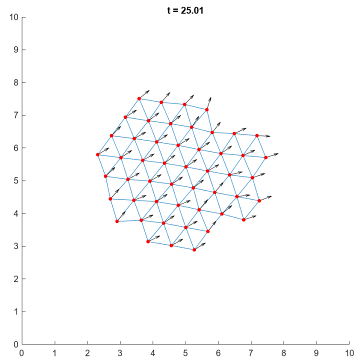


Fig. 2: Flocking with a Central Force

We note that in the absence of this term, the flock does not converge to an α-lattice despite the α-lattice being the lowest energy configuration for the flock due to its distributed nature. Once an agent A is no longer within the sensing region of any other agents, there is no reason for agent A to act in a way to minimise the deviation energy E_δ.

The simulations are documented here.

2.2 Barrier Functions

The contents of this section are from [SPE18] and [Squ+19]. These papers comprise of a special treatment of barrier functions, namely ZCBFs. A more general description of barrier functions can be found here[Ame+17].

Barrier functions are often useful when discussing the safety of a given system. In this case, a safe system would mean having no collisions. In these papers, the authors find a way to minimally alter a nominal control input $\hat{u}_i$ to ensure that the system remains safe.

Note: [Squ+19] apparently implements this idea of barrier function based collision avoidance for agents in a 3D space. However, they assume that small pitch angles of the aircraft allow us to consider translation in the z-direction independently from translation along the x and y directions. This is not the most practical consideration, and thus I will only be detailing the results in 2D.

We start off assuming the agent's states to be of the form $x_i = [p_{i,x}, p_{i,y}, \theta_i]^T$ where $p_{i,x}, p_{i,y}$ represent the position of the agent and θ_i represents the heading of the agent. The system's state is simply a collation of all the agents' states. Thus, the system's state is given by $x = [{x_a}^T, {x_b}^T]^T$. For generality, we also assume that the dynamics can be represented by a control affine system i.e, dynamics which are linear in their response to the control input.

$$\dot{x} = f(x) + g(x)u \tag{29}$$

Note that here, the control inputs are specified by velocities, $u = [v_x, v_y, \omega]$, and not accelerations. Thus, we require $u(t)$ to be a Lipschitz continuous controller.

The safety of a given state of the system is evaluated using an evaluation function h. We define the safe set C_h to be the superlevel set of h.

$$\begin{aligned} C_h =& \{x \in \mathcal{R}^n : h(z) \geq 0\} \quad (30)\\ \partial C_h =& \{x \in \mathcal{R}^n : h(x) = 0\} \quad (31)\\ {C_h}^C =& \{x \in \mathcal{R}^n : h(x) < 0\} \quad (32) \end{aligned}$$

The significance of a barrier function is that it can be used to show that the safe set is forward invariant, meaning that any state that starts in the safe set remains in the safe set for all future time.

The authors define a continuously differentiable function h to be a zeroing control barrier function (ZCBF) on a set D, such that $C \subseteq D \subseteq \mathcal{R}^n$ where C is the set defined in (29), if there exists an extended class κ function α[1] such that

$$\sup_{u \in U} L_f h(x) + L_g h(x) u + \alpha(h(x)) \geq 0 \tag{33}$$

The admissible control space is defined as

$$\begin{aligned} K_h = \{u \in U :\ & L_f h(x) + L_g h(x) u \\ & + \alpha(h(x)) \geq 0\} \end{aligned} \tag{34}$$

This paper[Ame+17] gives us a very significant result for ZCBFs:

Theorem 1: Given a set $C_h \subseteq \mathcal{R}^n$ for a continuously differentiable function h, if h is a ZCBF on D, then any Lipschitz continuous controller $u : D \rightarrow U$ such that $u(x) \in K_h(x)$ will render the set C forward invariant.

Essentially, this theorem states that if an initial state is safe, then there will always exist a control input for which the system will continue to remain safe. Of course, this is a theoretical result and has practical complications, such as actuator saturation, which would be represented by an upper bound on the slope of $u(t)$.

The authors now proceed to define a candidate ZCBF

$$h(x(t), \rho, \gamma) = \inf_{\tau \in [0, \infty)} \rho(\hat{x}(t+\tau)) \tag{35}$$

$$\hat{x}(t+\tau) = x(t) + \int_0^\tau \dot{\hat{x}}(t+\eta) d\eta \tag{36}$$

$$\begin{aligned} \dot{\hat{x}}(t+\tau) = & f(\hat{x}(t+\tau)) + \\ & g(\hat{x}(t+\tau))\gamma(\hat{x}(t+\tau)) \end{aligned} \tag{37}$$

Here, ρ is a distance metric given by $\rho(x(t)) = ||x_a - x_b||^2 - {D_s}^2$, where D_s is a minimum threshold distance. γ is a control law that all the agents are assumed to follow in order to calculate the safety of the set.

$h(x)$ evaluates the safety of the state by measuring the closest any two agents will come in all future time assuming they all follow a given control law γ for all future time.

They then prove that this $h(x)$ can be used as a ZCBF for a system if the following holds

- h has a unique x minimiser for all $x \in D$
- ρ is continuously differentiable
- γ is such that $f(x) + g(x)\gamma(x)$ is continuously differentiable

The case when h has multiple x minimisers has to be dealt with separately, for which a reference is provided[SPE18].

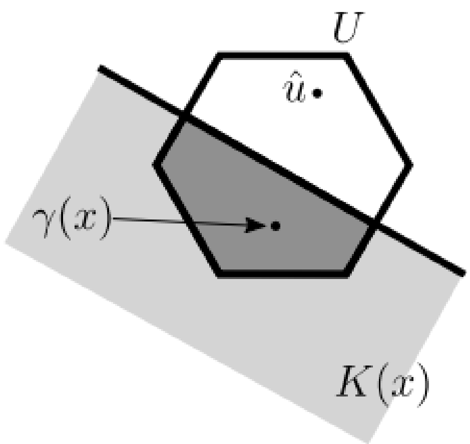
Now that we have a ZCBF, recalling Theorem 1, we can find the control input u which will keep the state safe in the future. This is done by

[1] A function α is an extended κ class function if α is such that $\alpha : (-c, d) \rightarrow (-c, \infty)$ is a strictly increasing function with $\alpha(0) = 0$

considering the set of all control inputs U, and its intersection with the set of all admissible control inputs K_h. The minimal deviation from the nominal control law $\hat{u}$ is ensured as the control input given to the system is calculated as the solution to the following QP (Quadratic Program)

$$\begin{aligned} u^* = \min_{u \in \mathcal{R}^m} \frac{1}{2}||u - \hat{u}||^2, \\ u \in U, u \in K_h \end{aligned} \tag{38}$$

This has been depicted below for clarity.



h defined in (35) can be a barrier function because γ or a small offset from γ lies in $U \cap K_h \; \forall \; x \in D$

This method provides a nice framework to calculate control inputs for fixed-wing aircrafts using the idea of safe sets, however, this method does not consider bounds on $\frac{du}{dt}$, as mentioned previously, the agents are considered to be kinematic agents, and the safe set is limited by the γ chosen for evaluation.

For example, assume our state is $x(0) = [-\infty, 0, 0, \infty, 0, \pi]^T$. Thus, we have two agents at $x = -\infty$, $x = \infty$ that are flying at each in a head-on fashion. If we choose $\gamma = [1, 0, -1, 0]^T$ (units/s), then the given $h(x) = -{D_s}^2$. We see that even though the agents have infinite time to react and avoid a collision, the given $h(x)$ deems the state to be unsafe. This problem is dealt with in this paper[Squ+22], where they produce an algorithm which iteratively expands the safe set to include states like the one mentioned in the example above.

2.3 Fixed-Wing Dynamics

The Olfati-Saber flocking law generates forces that do not account for the dynamics of fixed-wings. To model fixed-wing agents, the velocity of the agent must lie between v_{min} and v_{max}, the linear acceleration, in the direction parallel to the heading, is capped by a maximum value of a_{max}, and the turning rate ω is limited by $\frac{gtan(\phi_{max})}{v}$ where ϕ, v specifies the roll angle of the agent and the velocity of the agent respectively. Thus, we need to resolve the forces in the frame of the agent in the parallel and perpendicular directions and saturate them in a suitable manner. Let us first consider the transform from cartesian axes to the agent's frame.

$$\begin{bmatrix} v_x \\ v_y \end{bmatrix} = \begin{bmatrix} v\cos(\theta) \\ v\sin(\theta) \end{bmatrix} \tag{39}$$

$$\begin{bmatrix} \dot{v}_x \\ \dot{v}_y \end{bmatrix} = \begin{bmatrix} a\cos(\theta) - v\sin(\theta)\dot{\theta} \\ a\sin(\theta) + v\cos(\theta)\dot{\theta} \end{bmatrix} \tag{40}$$

$$\begin{bmatrix} \dot{v}_x \\ \dot{v}_y \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -v\sin(\theta) \\ \sin(\theta) & v\cos(\theta) \end{bmatrix} \begin{bmatrix} a \\ \omega \end{bmatrix} \tag{41}$$

$$\begin{bmatrix} a \\ \omega \end{bmatrix} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\frac{\sin(\theta)}{v} & \frac{\cos(\theta)}{v} \end{bmatrix} \begin{bmatrix} \dot{v}_x \\ \dot{v}_y \end{bmatrix} \tag{42}$$

$$\begin{bmatrix} a \\ \omega \end{bmatrix} = \mathcal{R} \cdot u \tag{43}$$

The ϕ required to achieve the particular ω is uniquely specified as we have a bijective transformation between ω, ϕ given by $\omega = \frac{gtan(\phi)}{v}$. The constraints of using fixed-wings are now clear: for a given linear velocity, there will always be a minimum, finite turning radius, specified by $r = \frac{v^2}{gtan(\phi_{max})}$.

The dynamics must also account for delays in control inputs i.e, the agent cannot switch from

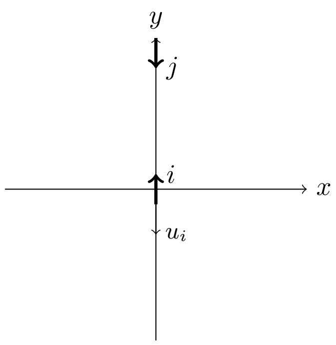
$-\phi_{max}$ to ϕ_{max} within one time step, thus we assume for simplicity that the acceleration and roll follow first order dynamics

$$\dot{a} = \frac{a_c - a}{\tau_a} \tag{44}$$

$$\dot{\phi} = \frac{\phi_c - \phi}{\tau_\phi} \tag{45}$$

3 Fixed-Wing Flocking

In Vignesh's BTP, he attempted to use the Olfati-Saber flocking law to model the flocking of 2D fixed-wing agents. Initially, using just the regular flocking law, he observed collisions in around 40% of all simulations. Making a simple observation, we can deduce very simply why this might be the case. Consider the case of a head-on collision:



The given u_i in this case of a head-on collision is directly opposite to the heading of the agent. This control input is such that $u_x = 0$ for $\theta = \pi/2$. Thus the transform to $[a, \omega]^T = [u_i, 0]^T$. We see that the turning rate induced in the agent is 0, which cannot prevent collisions in the case of fixed-wing agents. Instead, we want a higher turning rate for a greater force in the backward direction. As a solution to such a high collision rate, he proposed two modifications to the control law:

- **Collision Avoidance:** For collision avoidance we define a sector behind every agent. If the commanded acceleration from saber flocking is in this arc and has magnitude greater than max deceleration available, then max rate turns are commanded
- **Velocity Control:** To ensure that velocity does not exceed limits, we use proportional feedback $a_c = K_{P_{min}}(v_{min} - v_i)$ when velocity limits are exceeded

Using the above, the collision rate drops to 0.4%, which is a very significant improvement in performance, however, 0.4% is still a very prevalent failure rate for practical implementation, so I propose some solutions to the problems I was able to identify.

The first modification proposed, which implements maximum turning in the agent if the force on the agent is greater than a certain value and within a certain region behind the agent is only implemented in the control allocation scheme. The allocator is a module local to every agent which generates the required control input, (a, ω) in this case, so as to generate the force specified by the control (flocking) law on the agent.

Hence, a suitable modification would be, in contrast to the if/else condition in the control law allocation, to add to the flocking law a force which always acts perpendicularly to the heading of the agent, which is a strictly decreasing function dependent on the cosine of the angle the relative velocity of agent j with respect to agent i makes with the heading of agent i. An extension of this term would be to scale this force with the magnitude of the relative velocity vector. This could be done by applying two different forces along and perpendicular to the heading instead of a $p_j - p_i$ term, both of which scale as $||p_j - p_i||$.

$$a_{ij,||} = a_{ij}(q)(p_j - p_i) \cdot \hat{\theta}_i \tag{46}$$

$$a_{ij,\perp} = \omega_{ij} = a_{ij}(q)\frac{\alpha}{||p_i||} tanh(\beta sin(\frac{\eta_{ij}}{2})) \tag{47}$$

where η is the angle between the relative velocity vector $(p_j - p_i)$ and the heading $\frac{p_i}{||p_i||}$.

Note: This is not a final flocking law. This is merely an initial guess which incorporates the principles mentioned in the above paragraph to avoid fixed-wing collisions. I have not yet done any mathematical analysis on this proposed law (at the time of submission of the report).

Due to the modifications not being reflected in the final flocking law, the proof for stability of the flock was not straightforward in Vignesh's work. My idea is to incorporate this force directly into the flocking law, so that it falls under the purview of mathematical analysis. Stability in the context of flocks here, means that the agents will tend towards the α-lattice structure. It does not provide guarantees on having zero collisions. However, using a sufficiently large d (α-lattice parameter) should allow us to achieve zero collisions for a given v_{min}, v_{max}.

Secondly, the Monte Carlo simulations done did not consider any constraints on initial conditions. One could thus fathom applying the idea of safe sets to make sure the starting conditions of the simulation are necessarily not such that two agents will inevitably collide.

If we can find a suitable barrier function h for the above, then applying theorem 1, and utilising the control model from (38), we can prove the stability of the system, which in the context of barrier functions implies zero collisions, while minimally altering the nominal control applied to the agent. The nominal control here is the control input applied for the specified force from the modified flocking law.

4 Stability Analysis

Given the flocking laws from section 2.1, we have

$$\dot{p} = -\nabla V(q) - \hat{L}p \tag{48}$$

where $\hat{L}$ is the laplacan matrix such that $\hat{L}_{ij}(q) = a_{ij}(q)$.

Thus defining a hamiltonian for our system given as:

$$H(q,p) = V(q) + \frac{1}{2}\sum_{i=1}^{N} ||p_i||^2 \tag{49}$$

$$\dot{H} = \nabla_q V \dot{q} + p^T \dot{p} \tag{50}$$

Noting $\dot{q} = p$ and since V is a pairwise potential, we have $V_{ij} = V_{ji}$

$$\dot{H} = \nabla V p - p^T(\nabla V + \hat{L}p) \tag{51}$$

$$\dot{H} = -p^T \hat{L} p = -\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(q)||p_j - p_i||^2 \tag{52}$$

Since $\hat{L}$ is semi-positive definite, $\dot{H} \leq 0$. Recall that a negative $\dot{H}$ only represents that the structure formed by the agents tends towards an α-lattice. All points on the line $p_i = p_j \forall i, j$ are equilibria for H. All such points are therefore (qualitatively) stable. The consensus term in $\dot{p} = -\hat{L}p$ has the effect of driving the velocities of all nearby agents to some common $\bar{p}$.

To show stability however, this frame of reference is rather inconvenient, as the "stable" manifold is a 1D line. The dynamics in this frame of reference could be analysed separately.

Performing the coordinate transform to the CM frame,

$$v_i \leftarrow p_i - \bar{p}, \ \bar{p} = \frac{1}{n}\sum_{i=1}^{n} p_i \tag{53}$$

allows us to apply LaSalle's invariance principle [Olf06], which requires $\dot{\mathbf{x}} = f(\mathbf{x})$ such that $f(\mathbf{0}) = \mathbf{0}$. This is true in the CM frame, which proves asymptotic stability of the origin i.e, all the agents will move at some common velocity $\bar{v}$. Doing this, we get

$$\dot{H} = -v^T \hat{L} v = -\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} ||v_j - v_i||^2 \tag{54}$$

$$\dot{H} = -\frac{1}{2}\sum_{i=1}^{n}\sum_{ij1}^{n} ||p_j - p_i||^2 \tag{55}$$

Adding the turning term, as previously mentioned, I present the following argument:

Since we have the explicit form of H in the CM frame, we attempt to add a new term to $\dot{p}$ such that $\dot{H} \leq 0$ still holds. Let us call this term $\mathbf{x}$ for now.

$$\dot{H} = -v^T \hat{L} v + v^T \cdot \mathbf{x} \tag{56}$$

Therefore, for $\dot{H}$ to remain negative,

$$v^T \cdot \mathbf{x} \leq 0 \tag{57}$$

$$\sum_{i=1}^{n} p_i \mathbf{x}_i - \bar{p} \sum_{i=1}^{n} \mathbf{x}_i \leq 0 \tag{58}$$

From this equation, we can attempt to formulate an $\mathbf{x}$ such that the agents try to form an alpha lattice.

5 A New Consensus Term

Here, the approach lies in "baking" the fixed-wing model into the control law. Instead of adding a new term to Olfati's control law, let's try replacing terms in the flocking law. Before we go on, let us first motivate this choice of modification.

Having looked at simulation results, we know for a fact that simply applying Olfati's flocking law to fixed-wings is a flawed approach. Qualitatively, as previously mentioned, this is due to the fact that the fixed-wing model in constrained in its motion along the perpendicular direction. There are two flaws in this approach.

Firstly, suppose two agents are about to make a head-on collision. For a fixed-wing, to avoid a collision would mean to turn as much as possible. However, Olfati's flocking would only cause the agents to decelerate, as would be appropriate for boids, but not for fixed-wings. This is depicted below, where u_{ij} represents the force exerted by agent j on agent i.

$$u_{12} \dashleftarrow\!\!\longrightarrow \quad \longleftarrow\!\!\dashrightarrow u_{21}$$

Secondly, the maximum turning rate of a fixed-wing is dependent on its velocity, i.e, it is not a constant upper bound, while the linear acceleration or retardation is bounded by some constant. Due to this, the olfati force $[a_x, a_y]^T$ is capped separately along and perpendicular to the velocity of the agent. Since the flocking law doesn't account for this behaviour that is particular to the fixed wing, it may not be able to cause the agent to turn enough in time to avoid a collision.

Our purpose here is to find the form of the a flocking law pertaining specifically to a fixed-wing model such that the two problems mentioned above are dealt with i.e, a flocking law that induces a large amount of turning for the case of a head-on collision, and bounds the turning of the agent appropriately.

Since we know that a_i, ω_i must depend entirely on the agents neighbouring agent i, they must contain terms of the form

$$a_i = \hat{\theta}_i (\sum_{j=1}^{n} a_{ij}(q)\alpha_{ij}) \tag{59}$$

$$\omega_i = \hat{\theta}_{i\perp} (\sum_{j=1}^{n} a_{ij}(q)\beta_{ij}) \tag{60}$$

We shall determine α_{ij}, β_{ij} based on heuristics, and analyse the stability of the system.

Supposing a hamiltonian defined as

$$H = \frac{1}{2} \sum_{i=1}^{n} ||\hat{\theta}_i - \bar{\theta}||^2 \tag{61}$$

$$\dot{H} = \sum_{i=1}^{n} (\hat{\theta}_i - \bar{\theta}) \cdot (\omega_i - \bar{\omega}) \tag{62}$$

$$\dot{H} = -\frac{1}{n} \sum_{i,j} \hat{\theta}_i \cdot \omega_j \tag{63}$$

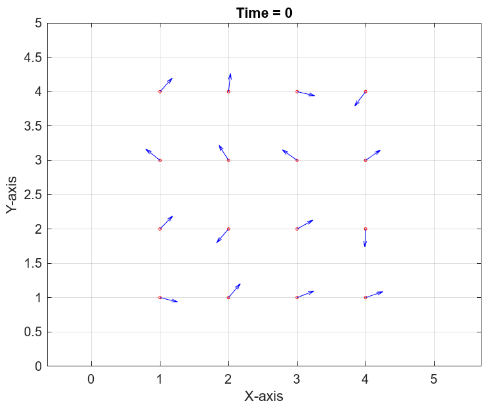
$$\dot{H} = -\frac{1}{n} \sum_{i,j} \hat{\theta}_i \cdot \hat{\theta}_{j\perp} \cdot \kappa_j$$

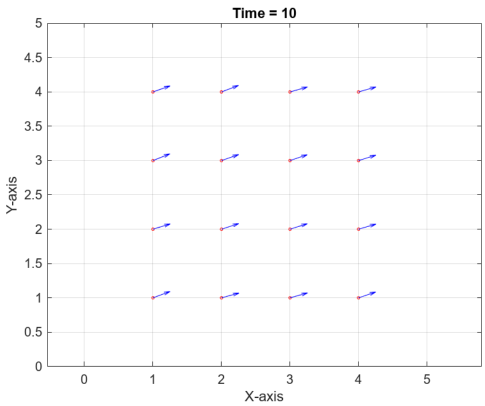
$$\kappa_j = -\frac{A}{||p_j||} \cdot tanh(B \sum_k a_{jk}(q) sin(\frac{\eta_{jk}}{2})) \tag{64}$$

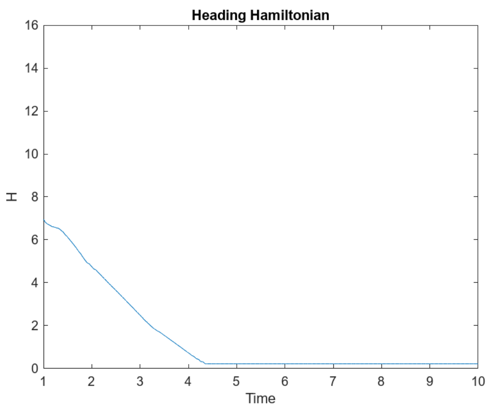
where A, B are constants and η_{jk} is the angle between the relative velocity vector $p_k - p_j$ and agent j's heading $\hat{\theta}_j$. Refer to the appendix to understand the intention behind this heuristic. For the given κ_j, $\dot{H} \leq 0$ is a trivial proof for only two agents interacting.

$$\dot{H} = -\frac{1}{2} (\hat{\theta}_1 \cdot \hat{\theta}_{2\perp} \kappa_2 + \hat{\theta}_2 \cdot \hat{\theta}_{1\perp} \kappa_1)$$

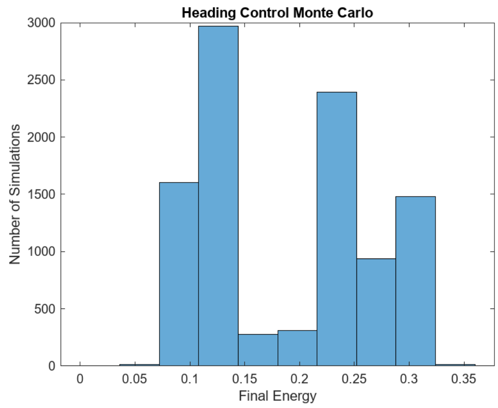
Notice that $\hat{\theta}_{\perp} = \hat{k} \text{ x } \hat{\theta}$, and $\frac{\kappa_1}{|\kappa_1|} = -\frac{\kappa_2}{|\kappa_2|}$ i.e, their signs are reversed such that both terms are positive. The proof for n agents is slightly more involved i.e, I have not been able to do it yet. Here are the simluation results for 4 agents, such that all agents can detect all other agents equally $(a_{ij}(q) = 1 \; \forall \; i, j)$, and since the agents are not moving, we set $\frac{A}{||p_i||} = 1 \; \forall \; i$.





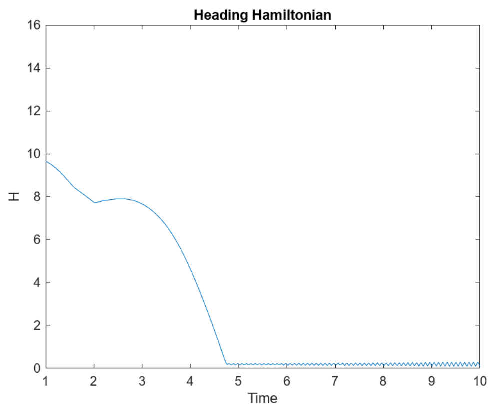


A monte carlo of 10000 simulations was performed resulting in 100% convergence!

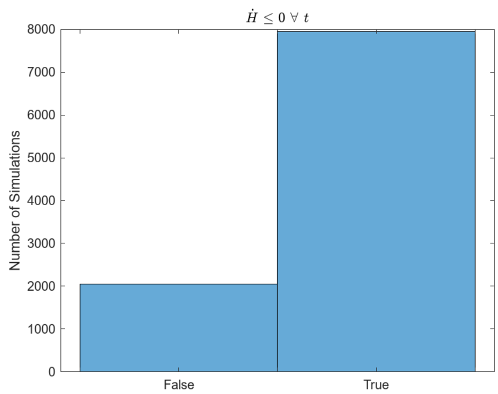


We see that the maximum value the Hamiltonian obtains is less than 0.36! It works!

However, we do see the occasional edge case where $\dot{H} > 0$, as shown below.



We note that for most initial conditions, $\dot{H} \leq 0 \; \forall \; t$. In the same monte carlo, we observe that this is true for approimately 80% of simulations.



Motivated by this, we might later aim to find some constraints on the initial conditions that will ensure the above condition.

Let's try to apply this to the old framework (where $v_i = p_i - \bar{p}$)!

$$H(x,v) = V(x) + \frac{1}{2}\sum_{i=1}^{n} ||v_i||^2$$

$$\dot{H} = \dot{V} + v^T(-\nabla V + \dot{v})$$

We can break $\dot{v}$ into accelerations along and perpendicular to the heading.

$$\dot{v} = a + \omega \tag{65}$$

where a is along $\hat{\theta}$, and ω is along $\hat{\theta}_\perp$.

$$\dot{H} = \sum_{i=1}^{n}(p_i - \bar{p}) \cdot ((a_i - \bar{a}) + (\omega_i - \bar{\omega}))$$

A similar reduction takes place as in (62), (63).

$$\dot{H} = (\sum_i p_i \cdot a_i - \frac{1}{n}\sum_{i,j} p_i \cdot a_j) - \frac{1}{n}\sum_{i,j} p_i \cdot \omega_j \tag{66}$$

However, $-\frac{1}{n}\sum_{i,j} p_i \cdot \omega_j$ is merely the same as (63) but scaled by positive weights $||p_i||$. Thus the same analysis as that of (61) applies here. We are then left with the problem of proving the stability of the first term on the RHS of (66) i.e, we must show that the speeds of the agents also converge.

Similar to the heuristic for ω, we would like to formulate a in such a way that allows us to a) account for actuator saturation and b) maintain a minimum and maximum speed for the agent. For now, I ignore (b).

$$a_i = Ctanh(\sum_j a_{ij}(q)(p_j - p_i) \cdot \hat{\theta}_i)\ \hat{\theta}_i \tag{67}$$

Once again, we see that the proof for $\dot{H} \leq 0$ is trivial for only 2 agents.

$$\dot{H} = \frac{1}{2}(p_1 \cdot (a_1 - a_2) + p_2 \cdot (a_2 - a_1))$$

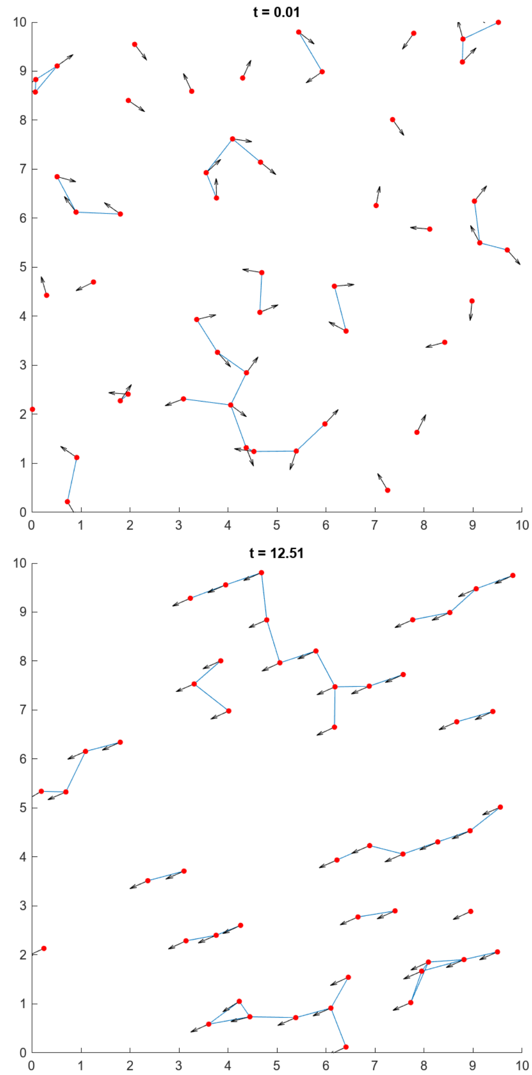
$$\dot{H} = \frac{1}{2}(a_1 \cdot (p_1 - p_2) + a_2 \cdot (p_2 - p_1))$$

Let us denote $(p_j - p_i) \cdot \hat{\theta}_i$ by ζ_{ij}, and thus $(p_i - p_j) \cdot \hat{\theta}_i$ by $-\zeta_{ij}$.

$$\dot{H} = -\frac{C}{2}(\zeta_{12}tanh(a_{12}(q)\zeta_{12}) + \zeta_{21}tanh(a_{21}(q)\zeta_{21})$$

Since $tanh(x)$ has the same sign as x, and since $a_{ij}(q)$ is non-negative, we deduce that both terms inside the brackets are non-negative.

Applying these laws with some very rough handtuning, we obtain the following



6 Further Scope

Extending fixed-wing flocking to 3D in a generalised sense would not be so straightforward, as the dynamics of fixed-wings are more involved when one does not treat the z-component of the agent independently. However, the greater number of dimensions give us more degrees of freedom to avoid collisions.

Heading towards practical applications, problems regarding limited sensing and communication would have to be addressed. Consider the case where for a given v_{min}, v_{max} we find a threshold distance d for the fixed-wing agents. This d may not be suitable for usecases such as that of limited sensing. Hence, a barrier function based approach might be more successful there. However, this is still an interesting avenue of research.

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