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Nonlinear Dynamical Systems: ACMS 403630 — 01
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Homework 1

Question 1

Question: Consider a differential equation $\dot{x} = f(x) = ax - x^3$ for some $a \in \mathbb{R}$. Determine fixed points and their stability for cases where a is positive, negative, and zero.

Answer: In order to determine the fixed points, we must set $f(\bar{x}) = 0$ and solve for $\bar{x}$. This yields:

$$f(\bar{x}) = 0 = a\bar{x} - \bar{x}^3 = x(a - \bar{x}^2)$$

$$\bar{x} = 0, \pm\sqrt{a}$$

Thus, our fixed points are going to be located at $\bar{x} = 0$ and $\pm\sqrt{a}$. In order to determine the stability at each of these points for the three cases of a, we will graph them using $a = 4, a = 0$ and $a = -4$ as representative cases.

Our first case $a = 4$, represents the case where $a > 0$. Seen in the graphs below, this scenario yields stable fixed points at $\pm\sqrt{(a)}$ and an unstable node at $a = 0$. For $a = 0$, we have one stable node at the origin. Finally, for the $a < 0$ case, we use $a = -4$ as a proxy. This graph reveals a similar one to the $a = 0$ case in that there is a stable node at the origin.
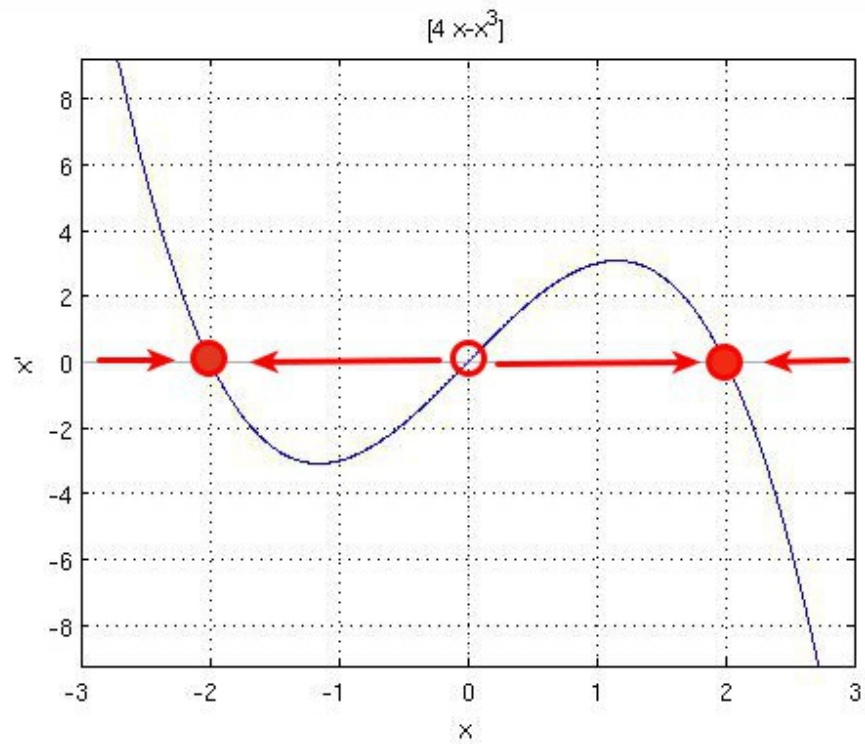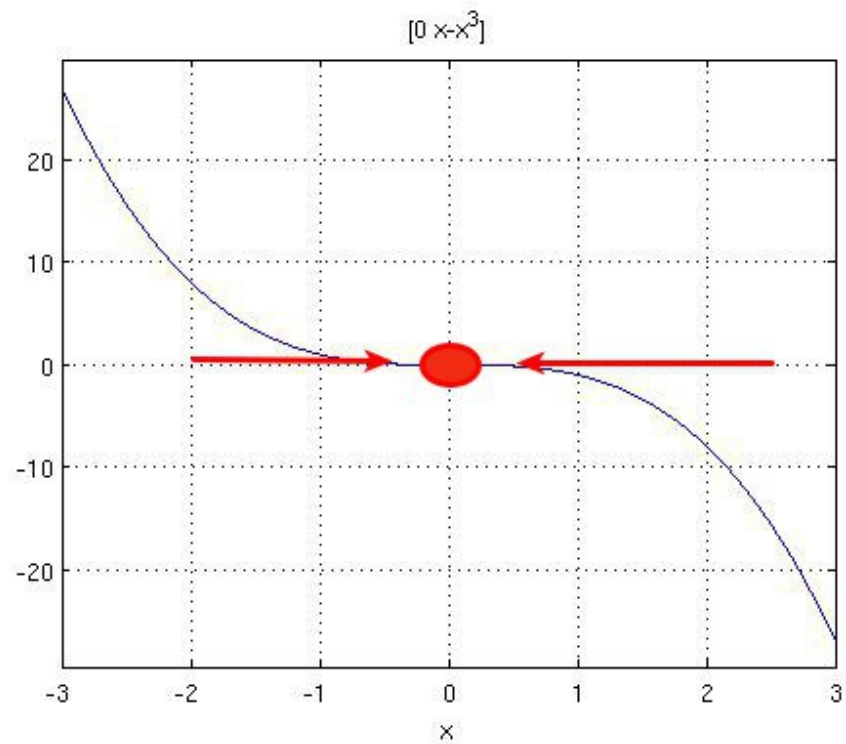
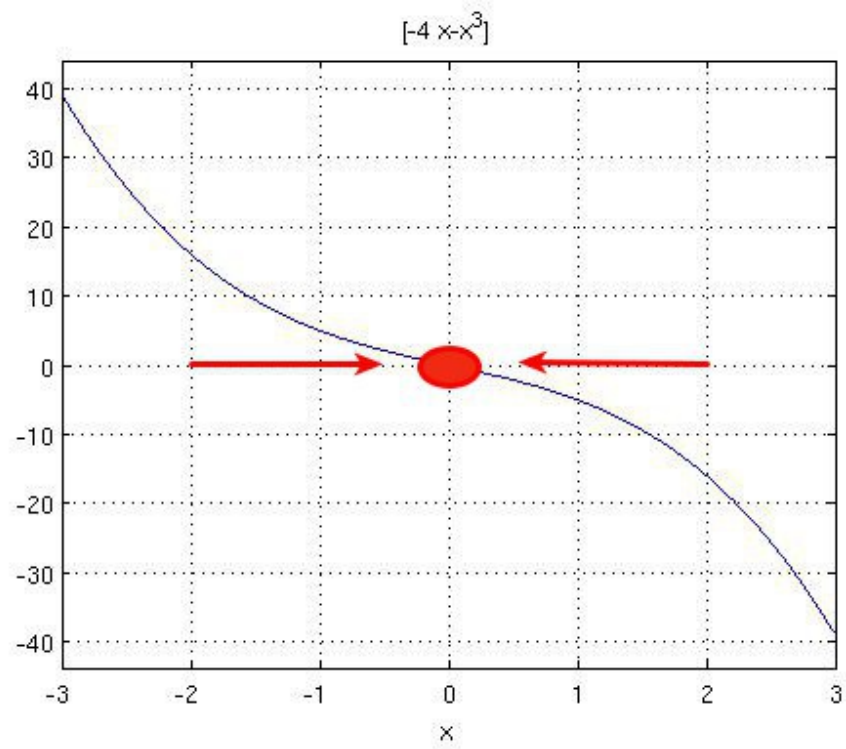Figure 1: $a > 0$ represented by $a = 4$

Figure 2: $a = 0$

Figure 3: $a = 0$

Question 2

Question: Find fixed points and sketch nullclines, vector field and plausible phase portrait of the system:

$$\dot{x} = x(x - y) = f(x, y)$$
$$\dot{y} = y(2x - y) = g(x, y)$$

Answer: In order to find the fixed points, we need to find $(\bar{x}, \bar{y})$ such that $f(\bar{x}, \bar{y}) = 0$ and $g(\bar{x}, \bar{y}) = 0$. Thus, the only fixed point occurs at $(0, 0)$.

The Jacobian matrix at a general point in this system is given as:

$$J = \begin{Bmatrix} 2x - y & -x \\ 2y & -2y + 2x \end{Bmatrix}$$

Evaluating the Jacobian at the fixed point $(0, 0)$ yields:

$$J = \begin{Bmatrix} 0 & 0 \\ 0 & 0 \end{Bmatrix}, \qquad \Lambda^2 = 0, \qquad \Lambda = 0, \qquad det = 0, \qquad trace = 0$$

This analysis tells us that linearization of our system predicts a center at $(0, 0)$ and this is our fixed point, as can be seen by the intersection of the nullclines in the graphs below.
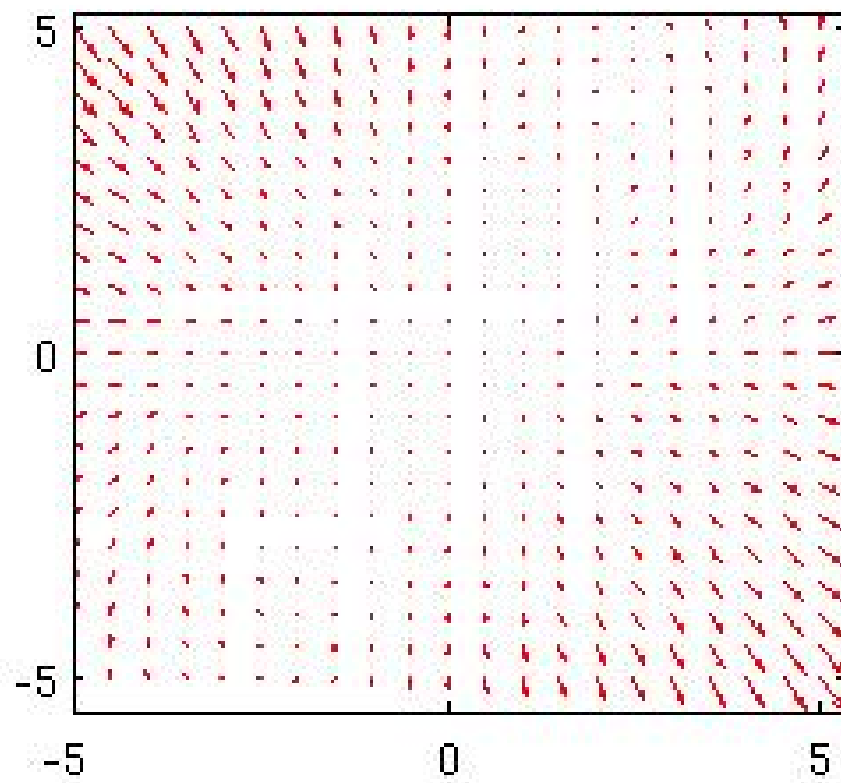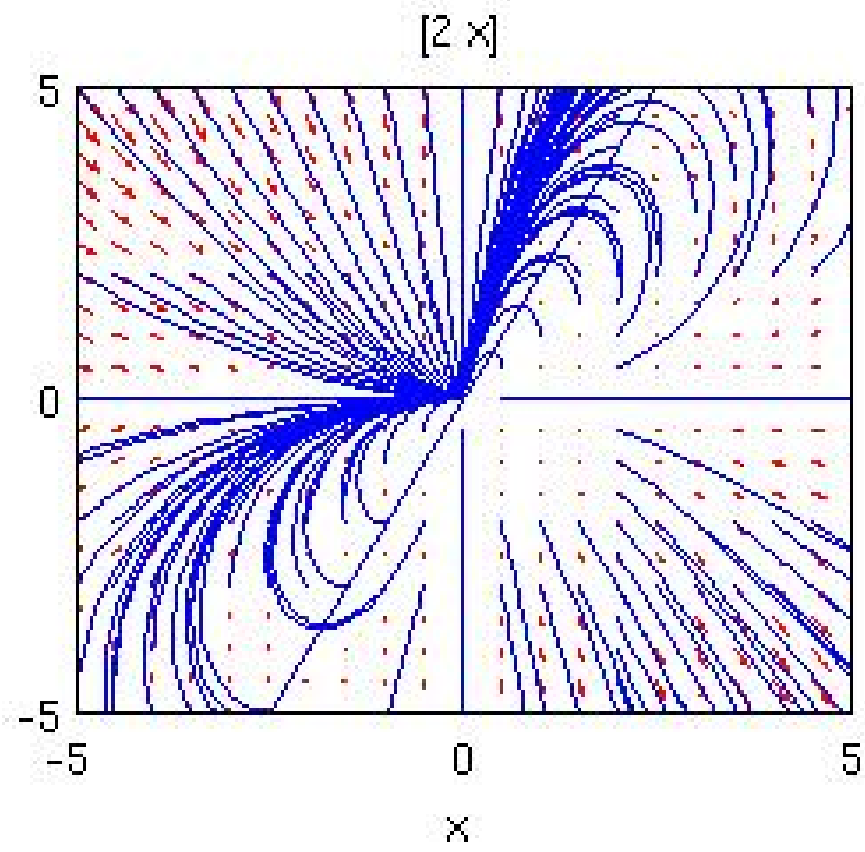
Figure 4: Vector Field and Nullclines

Figure 5: Phase Portrait

Question 3

Question: Show that the following system is reversible:

$$\dot{x} = y(1 - x^2) = f(x, y)$$
$$\dot{y} = 1 - y^2 \quad = g(x, y)$$

Answer: We can show that this system is reversible by showing that $f(x, -y) = -f(x, y)$ and $g(x, -y) = g(x, y)$, as seen below.

$$f(x, -y) = -y(1 - x^2) = -f(x, y)$$
$$g(x, -y) = 1 - (-y)^2 = 1 - y^2 = g(x, y)$$

Question 4

Question: Is the origin a nonlinear center of the system?

$$\dot{x} = -y - x^2 = f(x, y)$$
$$\dot{y} = x \quad\quad = g(x, y)$$

Answer: Maybe, the origin might be a nonlinear center to this system. Reversibility would show that this is a nonlinear center, but this system does not seem to be reversible in the y direction (despite the answer in the back of the book). The phase portrait in the graph below confirm the suspicion that this is not a nonlinear center, as there is no robust center at $(0, 0)$

The linearization process consists in verifying that $(0, 0)$ is a fixed point and then evaluating the Jacobian of this system at the $(0, 0)$ and calculating the resulting eigenvalues.

The Jacobian matrix at a general point in this system is given as:

$$J = \begin{Bmatrix} -2x & -1 \\ 1 & 0 \end{Bmatrix}$$

Evaluating the Jacobian at the fixed point $(0, 0)$ yields:

$$J = \begin{Bmatrix} 0 & -1 \\ 1 & 0 \end{Bmatrix}, \quad \Lambda^2 = -1, \quad \Lambda = \pm i, \quad det = 1, \quad trace = 0$$

Since the $det > 0$ and the $trace = 0$, by linearization the fixed point $(0, 0)$ would be classified as a center. In order to verify this, we overlay the phase portrait on the vector field. With initial conditions starting at $(.1, .1)$ the system returns a closed orbit around the fixed point, which is expected behavior for a fixed point. This closed orbit verifies that the fixed point at $(0, 0)$ is indeed a center.

This system can explicitly be shown to have a *nonlinear center* by showing that this system is reversible under the transformations $x \mapsto -x$ and $y \mapsto -y$.

$$f(-x, y) = -y - (-x)^2 = -y - (x)^2 = f(x, y)$$
$$g(-x, y) = (-x) = -g(x, y)$$

And (the back of the book agrees that this system is reversible. I cannot figure out how $f(x, -y) = -f(x, y)$, however.)

$$f(x, -y) = -(-y) - x^2 =? -f(x, y) = -(-y - x^2) = y + x^2$$
$$g(x, -y) = x = g(x, y)$$

If this system is indeed reversible, then we can use Thm 6.61 (Strogatz) to prove that this is a robust *nonlinear center*.

However, I am still not convinced that this system is reversible in the y direction. I cannot figure it out explicitly and the phase portrait below also fails to show $(0,0)$ as a stable center.
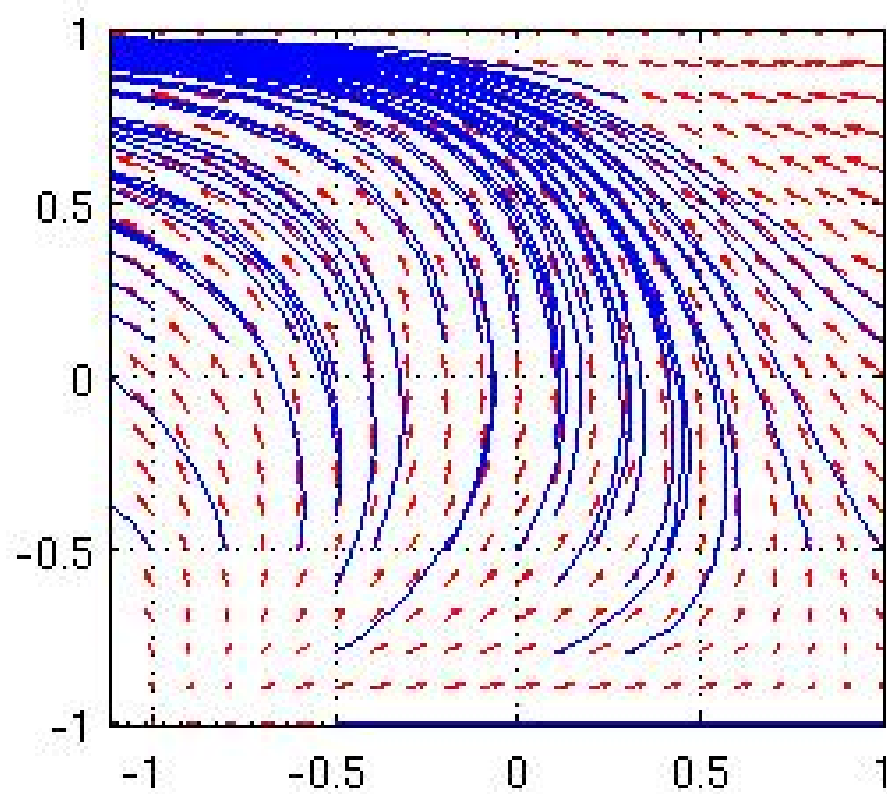
Figure 6: Phase Portrait

Question 5

Question: Study the Lotka-Voltera predator-prey model given by the system of equations:

$$\dot{R} = aR - bRF$$
$$\dot{F} = -cF + dRF$$

a) Show that the model can be recast in a non-dimensional form:

$$\dot{x} = x(1 - y)$$
$$\dot{y} = \mu y(x - 1)$$

b) Plot the vector field of the system.

c) Discuss the biological relevance of solutions of the system.

Answer:

a) In order to show that this model can be recast in a non-dimensional form, we factor both of the equations to get an intuition as to what our substitution will be:

$$\dot{R} = aR - bRF \quad = aR(1 - \frac{b}{a}F)$$
$$\dot{F} = -cF + dRF = cF(\frac{d}{c}R - 1)$$

Intuition tells us that our substitution will be $x = \frac{d}{c}R$ and $y = \frac{b}{a}F$. We then find a $\tau(t)$, where $\tau = t/T$ and T is our characteristic time scale to be found. To find T, we use the chain rule in reverse and then solve $\frac{dt}{d\tau}$ for both $\dot{x}$ and *doty*.

The chain rule yields:

$$\dot{x} = \frac{dx}{d\tau} = \frac{dx}{dR}\frac{dR}{dt}\frac{dt}{d\tau} = x(1 - y)$$

$$\dot{y} = \frac{dy}{d\tau} = \frac{dy}{dF}\frac{dF}{dt}\frac{dt}{d\tau} = \mu y(x - 1)$$

We are able to substitute for everything in these equations except for $\frac{dt}{d\tau}$, using the fact that:

$$\frac{dx}{dR} = \frac{d}{c}$$

$$\frac{dy}{dF} = \frac{b}{a}$$

10

Substitution yields:

$$\dot{x} = \frac{d}{c}Ra(1-y)\frac{dt}{d\tau} = x(1-y)$$

From this, we can see that $\frac{dt}{d\tau} = \frac{1}{a}$. We solve this by separation of variables and find that $\tau = at$. Substituting $\frac{dt}{d\tau} = \frac{1}{a}$ back into $\dot{y}$, we will see that $\mu = \frac{c}{a}$.

$$\dot{y} = \frac{b}{a}cF(x-1)\frac{dt}{d\tau} = \mu y(x-1) \qquad\qquad = \frac{c}{a}y(x-1)$$

Ultimately, with these substitutions, we arrive at the desired result:

$$\dot{x} = x(1-y)$$
$$\dot{y} = \mu y(x-1)$$

b) A vector field can be seen in the graph below. Note: this vector field used $\mu = \frac{c}{a} = \frac{.3}{.7}$.

c) The vector field below shows the interesting result that the species represented by $\dot{x}$, which is the rabbit species in this model, almost always gets near extinction only to recover. We can also see that the fixed point at $(1,1)$ yields a linear center which is predicted by eigenvalues of $\lambda = \pm\mu i$. This center would appear to be nonlinear, as well, because of the vector field, although this cannot be shown by reversibility because this system is not reversible
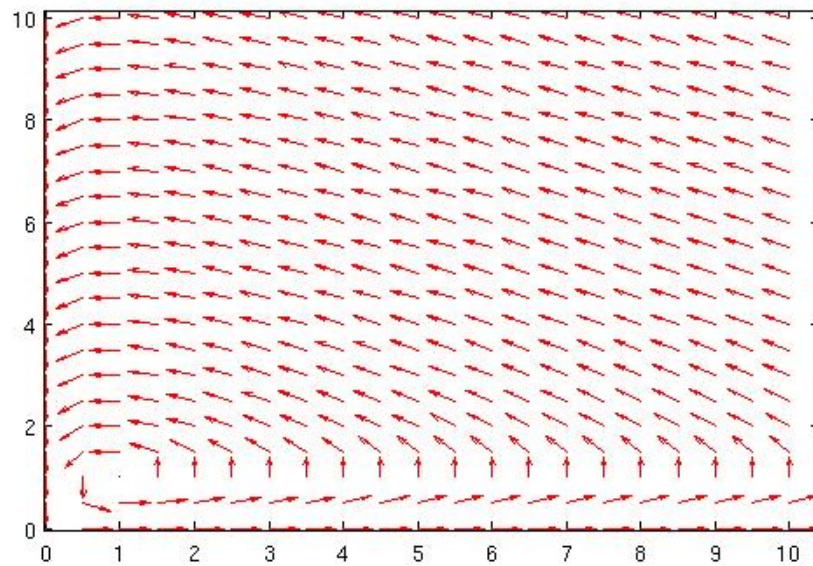
Figure 7: Vector Field