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Nonlinear Dynamical Systems: ACMS 403630 — 01
Thursday, January 30th 2014
Homework 1

Question 1

Question: Consider a differential equation $\dot{x} = f(x) = ax - x^3$ for some $a \in \mathbb{R}$. Determine fixed points and their stability for cases where a is positive, negative, and zero.

Answer: In order to determine the fixed points, we must set $f(\bar{x}) = 0$ and solve for \bar{x} . This yields:

$$f(\bar{x}) = 0 = a\bar{x} - \bar{x}^3 = \bar{x}(a - \bar{x}^2)$$

$$\bar{x} = 0, \pm\sqrt{a}$$

Thus, our fixed points are going to be located at $\bar{x} = 0$ and $\pm\sqrt{a}$. In order to determine the stability at each of these points for the three cases of a , we will graph them using $a = 4, a = 0$ and $a = -4$ as representative cases.

Our first case $a = 4$, represents the case where $a > 0$. Seen in the graphs below, this scenario yields stable fixed points at $\pm\sqrt{a}$ and an unstable node at $a = 0$. For $a = 0$, we have one stable node at the origin. Finally, for the $a < 0$ case, we use $a = -4$ as a proxy. This graph reveals a similar one to the $a = 0$ case in that there is a stable node at the origin.

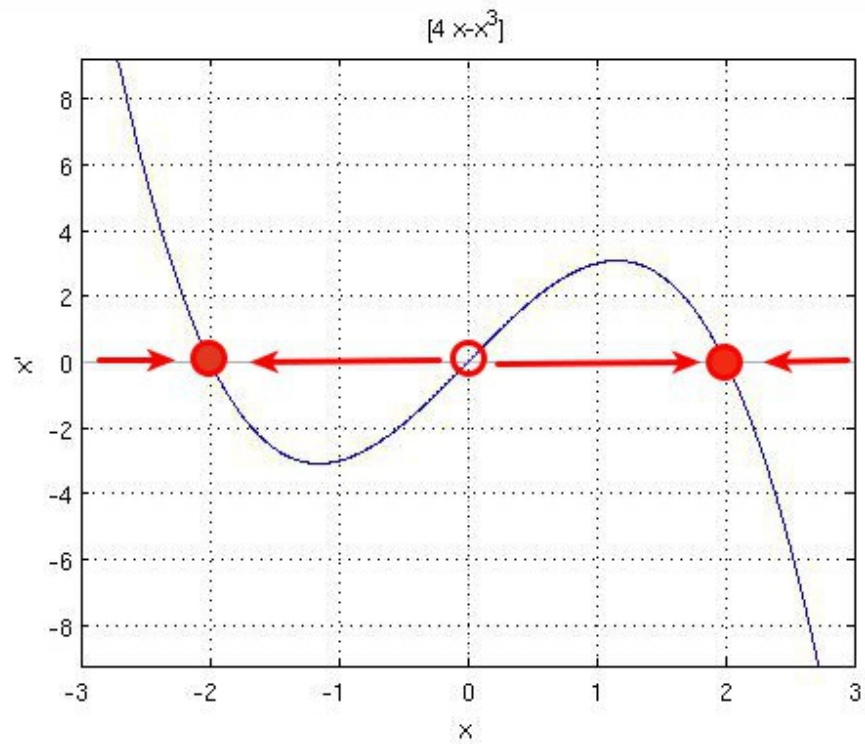


Figure 1: $a > 0$ represented by $a = 4$

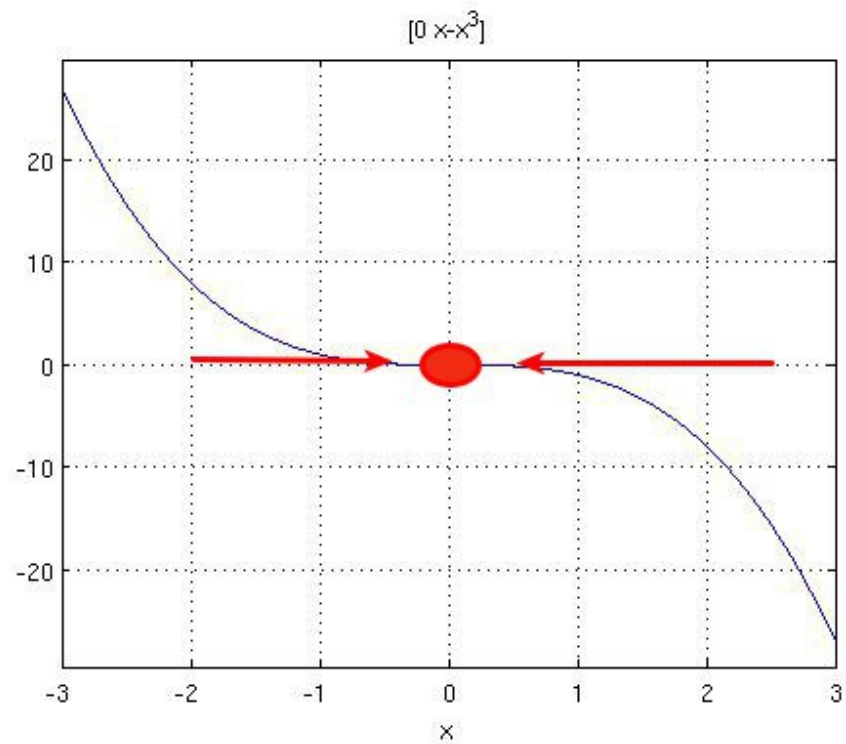


Figure 2: $a = 0$

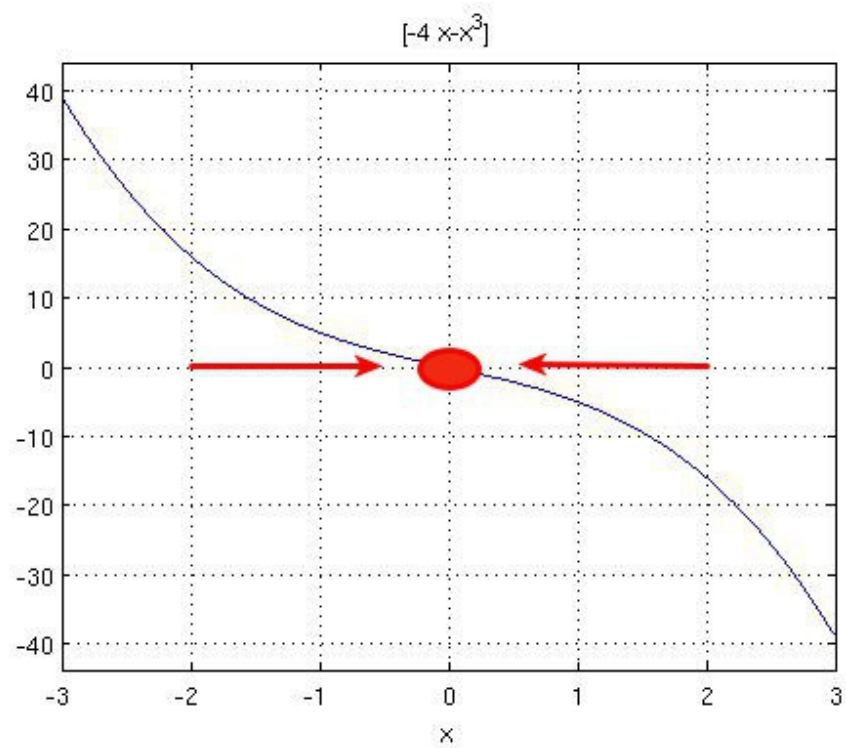


Figure 3: $a = 0$

Question 2

Question: Find fixed points and sketch nullclines, vector field and plausible phase portrait of the system:

$$\begin{aligned}\dot{x} &= x(x - y) = f(x, y) \\ \dot{y} &= y(2x - y) = g(x, y)\end{aligned}$$

Answer: In order to find the fixed points, we need to find (\bar{x}, \bar{y}) such that $f(\bar{x}, \bar{y}) = 0$ and $g(\bar{x}, \bar{y}) = 0$. Thus, the only fixed point occurs at $(0, 0)$.

The Jacobian matrix at a general point in this system is given as:

$$J = \begin{Bmatrix} 2x - y & -x \\ 2y & -2y + 2x \end{Bmatrix}$$

Evaluating the Jacobian at the fixed point $(0, 0)$ yields:

$$J = \begin{Bmatrix} 0 & 0 \\ 0 & 0 \end{Bmatrix}, \quad \Lambda^2 = 0, \quad \Lambda = 0, \quad \det = 0, \quad \text{trace} = 0$$

This analysis tells us that linearization of our system predicts a center at $(0, 0)$ and this is our fixed point, as can be seen by the intersection of the nullclines in the graphs below.

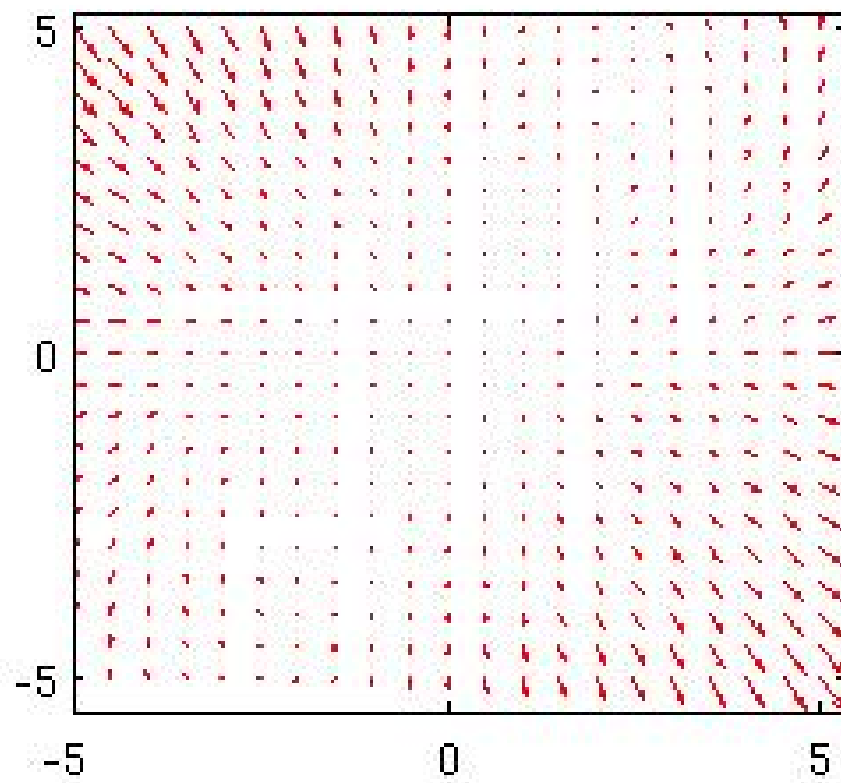


Figure 4: Vector Field and Nullclines

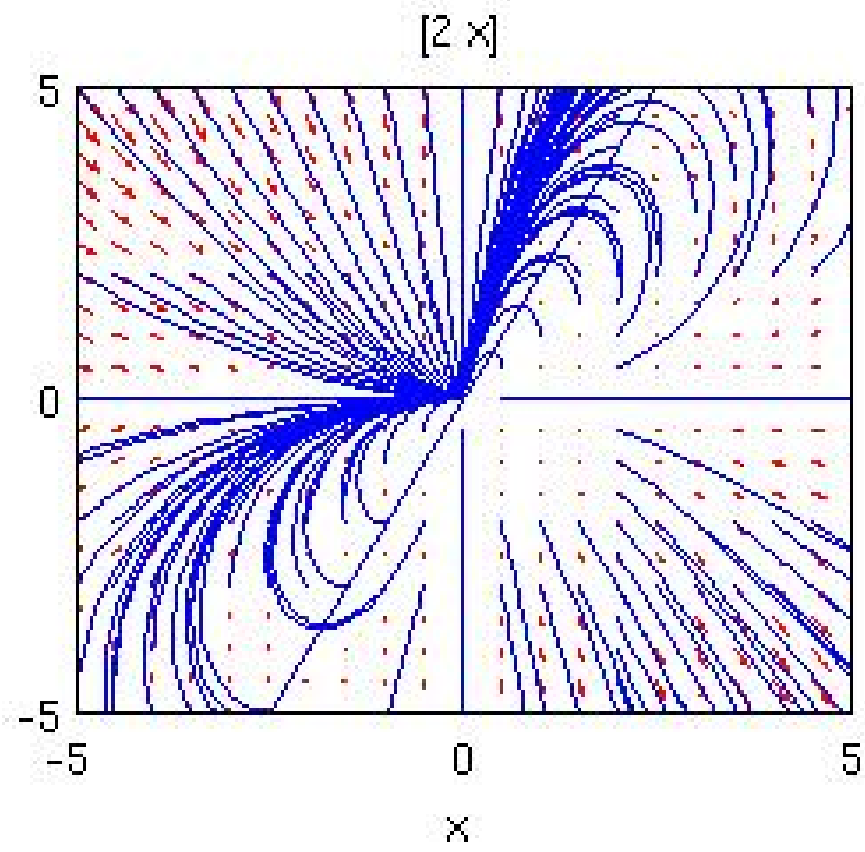


Figure 5: Phase Portrait

Question 3

Question: Show that the following system is reversible:

$$\begin{aligned}\dot{x} &= y(1 - x^2) = f(x, y) \\ \dot{y} &= 1 - y^2 = g(x, y)\end{aligned}$$

Answer: We can show that this system is reversible by showing that $f(x, -y) = -f(x, y)$ and $g(x, -y) = g(x, y)$, as seen below.

$$\begin{aligned}f(x, -y) &= -y(1 - x^2) = -f(x, y) \\ g(x, -y) &= 1 - (-y)^2 = 1 - y^2 = g(x, y)\end{aligned}$$

Question 4

Question: Is the origin a nonlinear center of the system?

$$\begin{aligned}\dot{x} &= -y - x^2 = f(x, y) \\ \dot{y} &= x = g(x, y)\end{aligned}$$

Answer: Maybe, the origin might be a nonlinear center to this system. Reversibility would show that this is a nonlinear center, but this system does not seem to be reversible in the y direction (despite the answer in the back of the book). The phase portrait in the graph below confirm the suspicion that this is not a nonlinear center, as there is no robust center at $(0, 0)$

The linearization process consists in verifying that $(0, 0)$ is a fixed point and then evaluating the Jacobian of this system at the $(0, 0)$ and calculating the resulting eigenvalues.

The Jacobian matrix at a general point in this system is given as:

$$J = \begin{Bmatrix} -2x & -1 \\ 1 & 0 \end{Bmatrix}$$

Evaluating the Jacobian at the fixed point $(0, 0)$ yields:

$$J = \begin{Bmatrix} 0 & -1 \\ 1 & 0 \end{Bmatrix}, \quad \Lambda^2 = -1, \quad \Lambda = \pm i, \quad \det = 1, \quad \text{trace} = 0$$

Since the $\det > 0$ and the $\text{trace} = 0$, by linearization the fixed point $(0, 0)$ would be classified as a center. In order to verify this, we overlay the phase portrait on the vector field. With initial conditions starting at $(.1, .1)$ the system returns a closed orbit around the fixed point, which is expected behavior for a fixed point. This closed orbit verifies that the fixed point at $(0, 0)$ is indeed a center.

This system can explicitly be shown to have a *nonlinear center* by showing that this system is reversible under the transformations $x \mapsto -x$ and $y \mapsto -y$.

$$\begin{aligned} f(-x, y) &= -y - (-x)^2 = -y - (x)^2 = f(x, y) \\ g(-x, y) &= (-x) = -g(x, y) \end{aligned}$$

And (the back of the book agrees that this system is reversible. I cannot figure out how $f(x, -y) = -f(x, y)$, however.)

$$\begin{aligned} f(x, -y) &= -(-y) - x^2 =? -f(x, y) = -(-y - x^2) = y + x^2 \\ g(x, -y) &= x = g(x, y) \end{aligned}$$

If this system is indeed reversible, then we can use Thm 6.61 (Strogatz) to prove that this is a robust *nonlinear center*.

However, I am still not convinced that this system is reversible in the y direction. I cannot figure it out explicitly and the phase portrait below also fails to show $(0, 0)$ as a stable center.

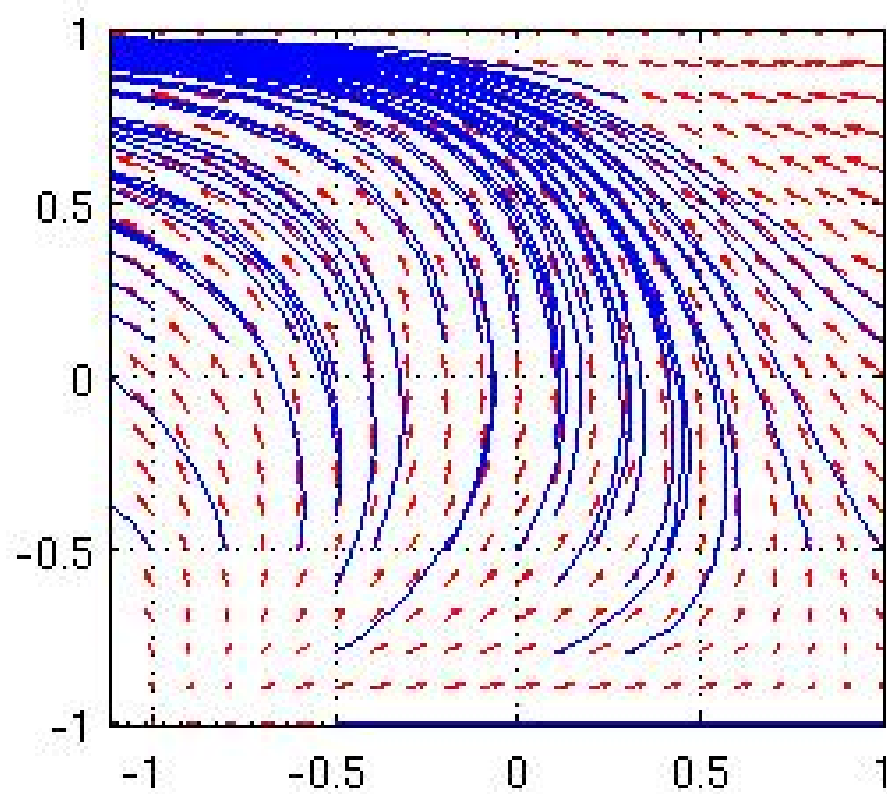


Figure 6: Phase Portrait

Question 5

Question: Study the Lotka-Volterra predator-prey model given by the system of equations:

$$\begin{aligned}\dot{R} &= aR - bRF \\ \dot{F} &= -cF + dRF\end{aligned}$$

a) Show that the model can be recast in a non-dimensional form:

$$\begin{aligned}\dot{x} &= x(1 - y) \\ \dot{y} &= \mu y(x - 1)\end{aligned}$$

b) Plot the vector field of the system. c) Discuss the biological relevance of solutions of the system. **Answer:** a)