Exercise 1: Unconstrained Optimization via Penalty Method and DFP Algorithm

Method Used:

To numerically solve the constrained optimization problem:

$$\min_{x_1,x_2,x_3} \quad x_1^2 + x_2^2 + x_3^2 \quad \text{subject to} \quad x_1 + 3x_2 + 2x_3 - 12 = 0$$

we applied the Penalty Method to transform it into an unconstrained optimization problem. The penalized objective function was defined as:

$$F(x) = x^\top x + \mu \cdot (x_1 + 3x_2 + 2x_3 - 12)^2$$

where μ is the penalty coefficient (set to 1000.0).To minimize this new objective, we implemented the **Davidon–Fletcher–Powell (DFP) method combined with Line search using the Armijo** rule to determine the appropriate step size at each iteration.

Results:

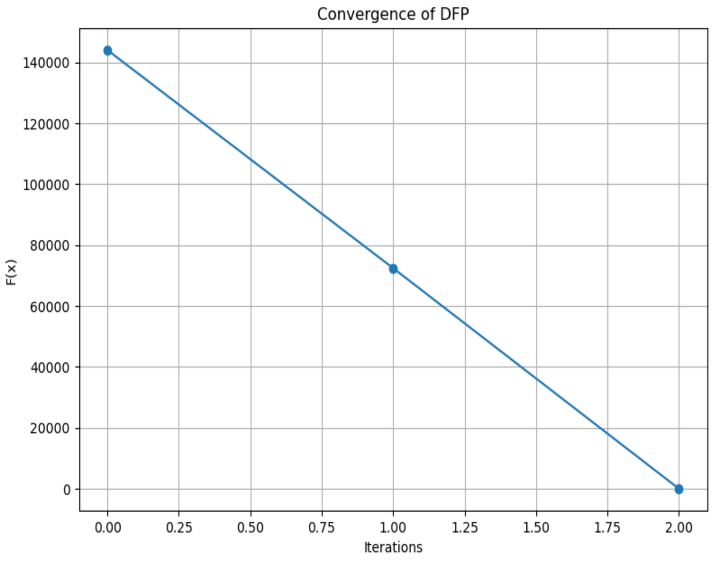
From the terminal output, the algorithm converged quickly in 3 iterations, showing the efficiency of the DFP method in this quadratic setting. The final results were:

- Optimal solution:

$$x^* = [0.8571, 2.5712, 1.7142]$$

- Final objective value:

$$F(x^*) = 10.285$$



Exercise 2 Report: Muscle Force Distribution Using the Augmented Lagrangian Method

Methodology

To solve the muscle force distribution problem presented in Exercise 2, we applied the **Augmented Lagrangian Method (ALM)**. This method transforms a constrained optimization problem into a series of unconstrained ones by adding a penalty term and Lagrange multipliers to the objective function.

The original objective function to minimize was:

$$Z(F) = \sum_{i=1}^{9} \left(\frac{F_i}{A_i} \right)^2$$

- Subject to the equality constraints:

$$g_1(F) = d_1 F_1 - d_2 F_2 - d_{3a} F_3 - M_1 = 0$$
$$g_2(F) = -d_{3k} F_3 + d_4 F_4 + d_{5k} F_5 - d_6 F_6 - d_{7k} F_7 - M_2 = 0$$
$$g_3(F) = d_{5h} F_5 - d_{7h} F_7 + d_8 F_8 - d_9 F_9 - M_3 = 0$$

- With positivity constraint:

$$F_i \geq 0 \quad \text{for all } i = 1, \ldots, 9$$

Before numerical optimization, the following were derived manually:

- **The penalized objective function:**

$$L(F, \lambda, \rho)$$

- The gradient of $\boldsymbol{Z(F)}$ with respect to each $\boldsymbol{Fi}$ **.**
- the partial derivatives of each constraint $\boldsymbol{gj(F)}$ with respect to each $\boldsymbol{Fi}$**.**
- **Gradient of the penalized function:**

$$\nabla L(F)$$

- **Minimize the objective function:**

$$Z(F) = \sum_{i=1}^{9} \left(\frac{F_i}{A_i}\right)^2$$

- **Subject to the equality constraints:**

$$g_1(F) = d_1F_1 - d_2F_2 - d_{3a}F_3 - M_1 = 0$$
$$g_2(F) = -d_{3k}F_3 + d_4F_4 + d_{5k}F_5 - d_6F_6 - d_{7k}F_7 - M_2 = 0$$
$$g_3(F) = d_{5h}F_5 - d_{7h}F_7 + d_8F_8 - d_9F_9 - M_3 = 0$$

- **With positivity constraint:**

$$F_i \geq 0 \quad \text{for all } i = 1, \ldots, 9$$

- ALM function:

$$L(F,\lambda,P) = Z(F) + \sum_{j=1}^{3} \left(\lambda_j g_j(F) + \frac{P}{2} g_j(F)^2\right) \qquad (1)$$

where:
- λ_j Lagrange multipliers
- $P>0$ Penalization parameter

$$L(F,\lambda,P) = \underbrace{\sum_{i=1}^{9}\left(\frac{F_i}{A_i}\right)^2}_{Z(F)} + \underbrace{\lambda_1(g_1(F)) + \lambda_2(g_2(F)) + \lambda_3(g_3(F))}_{\text{Lagrange multipliers x constraints}} + \underbrace{\frac{P}{2}\left((g_1(F))^2 + (g_2(F))^2 + (g_3(F))^2\right)}_{\text{Squared penalization}} \qquad (1.2)$$

- Now we should for the gradient of $\nabla L(F)$:

→ Solve for $Z(F)$ gradient:

$$\frac{\partial}{\partial F_i}\left(\frac{F_i}{A_i}\right)^2$$

- Let $u(F_i) = \frac{F_i}{A_i} \Rightarrow (u(F_i))^2 = \left(\frac{F_i}{A_i}\right)^2$

by chain rule:

$$\frac{d}{dF_i}(u(F_i))^2 = \frac{d}{du}(u(F_i))^2 \cdot \frac{d}{dF_i}u(F_i)$$
$$= 2(u(F_i)) \cdot \frac{1}{A_i}$$
$$= 2\left(\frac{F_i}{A_i}\right) \cdot \frac{1}{A_i}$$
$$= \boxed{\frac{2F_i}{A_i^2}} \qquad (2.1)$$

$\rightarrow \lambda_j g_j(F)$:

$$\frac{d g_i}{dF_i}\left(\lambda_j g_j(F)\right) = \boxed{\lambda_j \frac{d g_j(F)}{dF_i}} \quad (2.2)$$

$\rightarrow \frac{P}{2} g_j(F)^2$ let $u = g_j(F)$

$$= \frac{P}{2}\frac{d}{du}u^2 \cdot \frac{d}{dF_i}u$$

$$= \frac{P}{2} 2u \cdot \frac{d g_j(F)}{dF_i}$$

$$= \boxed{P g_j(F) \frac{d g_j(F)}{dF_i}} \quad (2.3)$$

→ By putting eqs. 2.1, 2.2 and 2.3 together we have:

$$\nabla L(F) = \frac{2F_i}{A_i^2} + \sum_{j=1}^{3}\left(\lambda_j \frac{d g_j(F)}{dF_i} + P g_j(F)\frac{d g_j(F)}{dF_i}\right)$$

$$= \boxed{\underbrace{\frac{2F_i}{A_i^2}}_{\nabla Z(F)} + \sum_{j=1}^{3}\left(\lambda_j + P g_j(F)\right)\underbrace{\frac{d g_j(F)}{dF_i}}_{\nabla g_j(F)}} \quad (2.4)$$

- So far we have the objective function $Z(F)$, The penalized objective function for ALM $L(F,\lambda,P)$, and the gradient of the penalized ALM function $\nabla L(F)$, the next step should be apply a numerical method to optimize forces F_i, in my case i used Gradient descent (GD) where $F_{i_{new}} = F_i - \alpha \nabla L(F)$, however we need to solve for $\frac{d g_j(F)}{dF_i}$, by calculating partial derivatives of all $g_j(F)$ with respect to F_i.

• Solve for $\frac{dg_j(F)}{dF_i}$:

→ In g_1 we have → F_1 , F_2 and F_3 and they depend on $d_1, -d_2, -d_{39}$ respectively, so:

$$\frac{\partial g_1(F)}{\partial F_1} = d_1 \quad , \quad \frac{\partial g_1(F)}{\partial F_2} = -d_2 \quad , \quad \frac{\partial g_1(F)}{\partial F_3} = -d_{39}$$ ✓

→ In g_2 we have → F_3 , F_4 , F_5 , F_6 , F_7 , and they depend on $-d_{3k}$, d_4, d_{5k} , $-d_6$, $-d_{7k}$; respectively, so we say:

$$\frac{\partial g_2(F)}{\partial F_3} = -d_{3k} \quad , \quad \frac{\partial g_2(F)}{\partial F_4} = d_4 \quad , \quad \frac{\partial g_2(F)}{\partial F_5} = d_{5k} \quad , \quad \frac{\partial g_2(F)}{\partial F_6} = -d_6 \quad ,$$ ✓

$$\frac{\partial g_2(F)}{\partial F_7} = -d_{7k}$$

→ In g_3 we have → F_5 , F_7 , F_8 , F_9 , that depend on d_{5h} , $-d_{7h}$, d_8 , $-d_9$, respectively, so by solvin partial derivatives we have:

$$\frac{\partial g_3(F)}{\partial F_5} = d_{5h} \quad , \quad \frac{\partial g_3(F)}{\partial F_7} = -d_{7h} \quad , \quad \frac{\partial g_3(F)}{\partial F_8} = d_8 \quad , \quad \frac{\partial g_3(F)}{\partial F_9} = -d_9$$

Numerical Optimization:

- The optimization of the penalized objective was carried out using **Gradient Descent (GD)**.
- The constraint violations were monitored calculating:

and convergence was determined once this norm dropped below a small threshold.

Results:

After **12** outer iterations, the algorithm converged. The final optimized muscle forces were: $\|g(F)\|$.

F1 = 283.00
F2 = 10.75
F3 = 10.46
F4 = 74.91
F5 = 88.36
F6 = 3.36
F7 = 5.72
F8 = 18.45
F9 = 5.32

The final value of the objective function was:

$$\boldsymbol{Z(F)} = 626.27$$