

GERSTEN’S INJECTIVITY FOR SMOOTH ALGEBRAS OVER VALUATION RINGS

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Abstract

Gersten’s injectivity conjecture for a functor F of “motivic type”, predicts that given a semilocal, “non-singular”, integral domain R with a fraction field K , the restriction morphism induces an injection of $F(R)$ inside $F(K)$. We prove two new cases of this conjecture for smooth algebras over valuation rings. Namely, we show that the higher algebraic K -groups of a semilocal, integral domain that is an essentially smooth algebra over an equicharacteristic valuation ring inject inside the same of its fraction field. Secondly, we show that Gersten’s injectivity is true for smooth algebras over, possibly of mixed-characteristic, valuation rings in the case of torsors under tori and also in the case of the Brauer group.

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1 Gersten’s Injectivity Conjecture

Let \mathcal{C} be an additive category, for example, the category of Abelian groups, and let $\mathcal{S} \subseteq \text{CRings}$ be a full subcategory of the category of commutative, unital rings. Given a functor $\mathcal{F}: \text{CRings} \rightarrow \mathcal{C}$, we say that \mathcal{F} *satisfies Gersten’s injectivity for \mathcal{S}* , if for any semilocal, integral domain $R \in \mathcal{S}$ with a fraction field K , we have an injection

$$\mathcal{F}(R) \hookrightarrow \mathcal{F}(K). \quad (1.0.1)$$

Gersten’s injectivity conjecture imprecisely stated predicts the following.

Conjecture 1.1. *A functor \mathcal{F} of ‘motivic type’ satisfies Gersten’s injectivity for a subcategory \mathcal{S} of ‘non-singular’ rings.*

Traditionally, for the subcategory Reg of regular rings, we expect Conjecture 1.1 to be true for the functor of algebraic K -groups, the Milnor K -groups, the Hermitian Witt groups, the de Rham cohomology groups, the étale cohomology groups with coefficients in the ℓ -th roots of unity, the motivic cohomology groups, etc. Validity of a certain ‘effacement theorem’ (in the style of [CHK97, Theorem 2.2.7]) is at the foundation of Conjecture 1.1 for each of above mentioned functors. Knowing that a functor \mathcal{F} satisfies Gersten’s injectivity has diverse practical benefits, in particular, the behaviour of such a functor \mathcal{F} is influenced by its values on fields. To provide background for our discussion, let’s briefly delve into Gersten’s conjecture.

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Digression: Gersten's conjecture

Let $H^i(-)$ denote one of the cohomology theories mentioned above. Gersten's injectivity forms a valuable part of Gersten's conjecture, which, in essence, predicts that for a regular, Noetherian scheme X , the group $H^n(X)$ can be read off by calculating $H^{n-i}(\kappa(x))$, where $0 \leq i \leq n$ and $\kappa(x)$ is the residue field of X at a point $x \in X$ of codimension i . Gersten's conjecture has been extensively studied in the literature thanks to its far reaching consequences and wide range of applications. Let us note some of them below.

- In the case of algebraic K -theory, Gersten's conjecture illuminates the connection of the algebraic cycles with algebraic K -theory (see Bloch's formula proved in [Qui73, §7, Theorem 5.19]). Using Bloch's formula and the cup product structure on algebraic K -groups, one can readily define intersection products of algebraic cycles in smooth algebraic varieties (see [Gra78]).
- In the case of Milnor K -theory, Gersten's conjecture paves the way to prove Levine's generalised Bloch–Kato conjecture for semi-local, equicharacteristic rings. Furthermore, one can deduce Beilinson's conjecture from Gersten's conjecture (see [Ker09], which simultaneously proves Gersten's conjecture in the equicharacteristic case).
- In the case of de Rham cohomology, Gersten's conjecture can be used to verify Washnitzer's conjecture relating the filtration by coniveau with the same by hypercohomology (see [BO74, Corollary 6.9]).

Therefore, unsurprisingly, Gersten's conjecture enjoys a pivotal role in the study of each of these functors.

Warmed up by the above digression, we turn our attention back to Conjecture 1.1. Given the significance of the conjecture, it seems completely justified to question its purview. Hence, we may pose the following.

Question. *Is there a larger category $\mathcal{S} \supset \text{Reg}$ of 'non-singular' rings for which we expect Gersten's injectivity to be satisfied by a functor \mathcal{F} of 'motivic type'?*

As a matter of fact, the answer is yes. Indeed, thanks to Zariski's local uniformisation conjecture (see, for example, [Kun23, Conjecture 2.1.1]), which predicts that any valuation ring is ind-regular, and a limit argument, any functor that commutes with filtered colimits and satisfies Conjecture 1.1 for Reg , also does the same for the category \mathcal{S}_{Val} of essentially smooth algebras over valuation rings. However, as mentioned in loc. cit., although expected to be true, Zariski's conjecture is widely open. This leaves the possibility for a challenge to prove Conjecture 1.1 unconditionally for \mathcal{S}_{Val} in the case of functors \mathcal{F} for which Conjecture 1.1 is known to be true for a subcategory of Reg .

The purpose of this article is twofold. We prove Conjecture 1.1 in the case of two different classes of functors, namely

- the higher algebraic K -groups, and
- the functor that classifies T -torsors and the functor that classifies T -gerbes, where T is some fixed torus.

We write the precise statements of the two theorems below before having separate discussions on each of them later. Let V be a valuation ring and let R be a semilocalisation of a smooth V -algebra with a fraction field F .

Theorem 1.2 (Corollary 3.8). *If V contains a field, then we have an injection*

$$K_i(R) \hookrightarrow K_i(F), \quad \text{for all } i.$$

Theorem 1.3 (Corollary 4.7). *Given an R -torus T , we have injections*

$$H_{\text{ét}}^1(R, T) \hookrightarrow H_{\text{ét}}^1(F, T) \quad \text{and} \quad H_{\text{ét}}^2(R, T) \hookrightarrow H_{\text{ét}}^2(F, T).$$

Plugging $T = \mathbb{G}_m$ in the displayed formula on the right, we get Gersten's injectivity for \mathcal{S}_{Val} in the case of Brauer groups.

Background and Remarks on Theorem 1.2

Let us start this subsection with a review of the known cases of Gersten's conjecture (see [Qui73, §7, Conjecture 5.10]) in the case of algebraic K -theory.

- Gersten in [Ger73, Theorem 1.3] used representation theoretic techniques to prove Gersten’s conjecture for discrete valuation rings with a finite residue field. Sherman in [She82] generalised Gersten’s method to prove the same for discrete valuation rings whose residue field is an algebraic extension of a finite field.
- Quillen in [Qui73, §7, Theorem 5.11] proved Gersten’s conjecture for essentially smooth algebras over a field. The key input is a certain ‘geometric presentation lemma’, which will appear in our discussion later.
- Gillet and Levine in [GL87, Corollary 6] subsequently generalised Quillen’s method to the mixed-characteristic. They showed that Gersten’s conjecture is true for all essentially smooth algebras over discrete valuation rings if the same holds for all discrete valuation rings.
- Kelly and Morrow in [KM21, Theorem 3.1] recently proved Gersten’s injectivity for equicharacteristic valuation rings. Their method is to pass to special classes of valuation rings that we already know are ind-regular, so as to reduce to [Qui73, §7, Theorem 5.11].

In §3, our goal is to establish Theorem 1.2. The technique of the proof is based on the same of [GL87, Theorem]. More precisely, we prove the following generalisation of loc. cit. Let V be a valuation ring and let R be a semilocalisation of a smooth V -algebra. Suppose that $\mathcal{P} \subset \text{Spec}(R)$ is the subset that corresponds to the generic points of the V -special fibre of $\text{Spec}(R)$. Let us denote the semilocalisation of R at the primes in \mathcal{P} by $R_{\mathcal{P}}$.

Theorem 1.4 (Theorem 3.6). *The canonical restriction induces an injection*

$$K_i(R) \hookrightarrow K_i(R_{\mathcal{P}}), \text{ for all } i.$$

Thanks to Lemma 2.13, the semilocal ring $R_{\mathcal{P}}$ is a Prüfer domain (i.e., it’s an integral domain whose local rings are valuation rings). As a consequence, Theorem 1.4 reduces the verification of Gersten’s injectivity for S_{val} to that of semilocal Prüfer domains. Finally, we deduce Theorem 1.2 from [KM21, Theorem 3.1].

It is worth noting that, unlike in Theorem 1.2, the ring V in Theorem 1.4 is not assumed to be equicharacteristic. The main technical component of the proof of this theorem is Presentation Lemma 3.2, which is a “geometric presentation lemma” in the style of Quillen in [Qui73, §7, Lemma 5.12] (see [Čes22_{Surv}, beginning of §4.1]). Gillet and Levine in [GL87, Lemma 1] proved a version of the presentation lemma over mixed-characteristic, discrete valuation rings (cf. [Lüd22, Lemma 2.12] and [Čes22, Variant 3.7]). Employing techniques from the proofs of [Čes22, Variant 3.7] and [Kun23, Proposition 6.4], we construct our version of the presentation lemma (Presentation Lemma 3.2) over valuation rings of finite Krull dimension, generalising [GL87, Lemma 1].

Background and Remarks on Theorem 1.3

Before delving into the discussion on Theorem 1.3, let us contextualise it. Conjecture 1.1 in the case of the functors $H^1(-, \mathbb{G}_m) = \text{Pic}(-)$ and $H^2(-, \mathbb{G}_m)$ for regular rings was explored by Grothendieck in his seminal paper [Gro68a]. He presented an étale cohomological interpretation of the well-known Brauer groups, ultimately establishing Gersten’s injectivity by analysing the long exact sequence associated to a certain ‘divisor short exact sequence’. Subsequently, Colliot-Thélène and Sansuc [CS87] generalised these results to the functors defined by non-split tori, establishing Gersten’s injectivity for them. Their key contribution lies in the introduction of the concept of a particular class of isotrivial tori termed ‘flasque tori’, characterised by simple Galois theoretic data. Remarkably, they demonstrated that any isotrivial tori can be resolved by flasque tori. Recently, employing similar techniques, Guo [Guo22] proved Gersten’s injectivity for valuation rings.

Inspired by the established techniques, in §4, our objective is to prove Theorem 1.3. The strategy is roughly based on the proof of [CS87, Theorem 2.2]. Considering an R -torus T , the key steps are as follows:

- Firstly, leveraging the fact that T has a flasque resolution, denoted by E^- , (refer to §4.5), an analysis of the long exact sequence of cohomology associated to E^- reduces to show Theorem 1.3 in the case when T is flasque.

- Thereafter, writing down the long exact sequence of cohomology with supports in a closed subscheme and subsequently, applying the coniveau spectral sequence associated to the filtration by supports, we further reduce to show the following vanishing statement of local cohomology in low degrees.

Theorem 1.5. *Let $\mathfrak{p} \subset R$ be a prime ideal. For a flasque R -torus T , there is vanishing*

$$H_{\mathfrak{p}}^q(R, T) = 0, \quad \text{for } q \leq 2.$$

- Finally, we establish purity for torsors under tori (see Proposition 4.3), which we derive as a consequence of a weak version of the Auslander–Buchsbaum formula for smooth algebras over valuation rings (see Lemma 4.2).

The results of this section appeared in the author’s thesis [Kun23, Chapter 4] and they were simultaneously and independently obtained by Guo and Liu in [GL24].

Notations and conventions

Let I be an ideal in a ring A .

- The vanishing locus of I is denoted by $V(I) \subseteq \operatorname{Spec} A$. If I is principal with generator, say t , then $V(t)$ denotes $V(I)$.
- The maximal spectrum of A is denoted by $\operatorname{MaxSpec}(A) \subseteq \operatorname{Spec} A$.
- If I is prime, then the localisation of A at I shall be denoted by A_I .
- If A is an integral domain, then the fraction field of A is denoted by $\operatorname{Frac}(A)$.
- An A -algebra is called *essentially smooth* if it can be obtained as the semilocalisation at finitely many primes of a smooth A -algebra.
- Given an A -scheme X and an algebra $a: A \rightarrow A'$, the base change of X along a is denoted by $X_{A'}$.
- If I is prime, the residue field of A at I is denoted by $\kappa(I)$.

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2 Generalities on Prüfer Domains

In this section, we establish several foundational concepts regarding Prüfer domains. These rings play a pivotal role in this article; for instance, they naturally emerge in the proof of the crucial technical foundation of §3, namely, Presentation Lemma 3.2. To establish this lemma, we rely on cut-and-paste type techniques, aided by Lemmas 2.6 and 2.8 (which let us glue Prüfer domains at non-generic points), along with an approximation result, Lemma 2.11. Additionally, the application of Lemma 2.13, which demonstrates the local structure of smooth schemes over Prüfer domains at the generic points of their fibres, recurs throughout the article (see Theorem 3.6 and Theorem 4.6). Therefore, it is one of the most important results of this section.

We start with the definition of Prüfer domains.

Definition 2.1 ([Gil92, §22]). A *Prüfer domain* is an integral domain whose localisation at every prime ideal is a valuation ring.

For equivalent definitions of Prüfer domains, see [Gil92, Theorem 22.1] and [Sta22, Tag 092S].

We state some permanence properties of Prüfer domains below.

Lemma 2.2. *Given a Prüfer domain R , a prime ideal $\mathfrak{p} \subset R$ and a multiplicative subset $S \subset R$, the following are Prüfer domains:*

- (a) *the localisation $S^{-1}R$, and*
- (b) *the quotient R/\mathfrak{p} .*

In Lemma 2.5, we will demonstrate that the concept of semilocal Prüfer domains coincides with another significant class of rings known as the semilocal Krull domains (see Definition 2.3 below). We will take advantage of this fact in Lemmas 2.6, 2.8 and 2.11.

Definition 2.3 (cf. [Mat86, Chapter 4, §12]). An integral domain R with a fraction field K is called a *semilocal Krull domain* if there exist a nonempty finite set Λ and valuation rings $\{R_\lambda\}_{\lambda \in \Lambda}$ in K such that $R = \bigcap_{\lambda \in \Lambda} R_\lambda$, where the intersection is taken in K .

Remarks 2.4.

1. Definition 2.3 extends the one in loc. cit. because we allow non-discrete valuations.
2. Contrary to loc. cit., we restrict Definition 2.3 to the semilocal setting for simplicity. This particular case will be adequate for our purposes.

Lemma 2.5. *A Prüfer domain R is the intersection in $\text{Frac}(R)$ of the valuation rings obtained by the localisations at the maximal ideals. Additionally, if R is semilocal,*

- (a) *for a subfield $K \subseteq \text{Frac}(R)$, the intersection $R_K := R \cap K$ is a semilocal Prüfer domain with $\text{Frac}(R_K) = K$ such that the canonical morphism $\text{MaxSpec}(R) \twoheadrightarrow \text{MaxSpec}(R_K)$ is a surjection, and*
- (b) *the ring R is the increasing union of its subrings that are semilocal Prüfer domain of finite Krull dimension.*

Proof. The first claim is a consequence of the fact that any integral domain is the intersection of its localisations at the maximal ideals.

(a): This is a consequence of [Mat86, Theorem 12.2] (or [BouCA, Chapter VI, §7, No. 1, Proposition 2]). Indeed, by [Sta22, Tag 0AAV], the intersection $R_{\mathfrak{m}} \cap K$ is a valuation ring with fraction field K , for each maximal ideal $\mathfrak{m} \subset R$. Therefore,

$$R_K = \bigcap_{\mathfrak{m} \in \text{MaxSpec}(R)} (R_{\mathfrak{m}} \cap K)$$

is a semilocal Krull domain with fraction field K , namely, there are a surjection $\text{MaxSpec}(R) \twoheadrightarrow \text{MaxSpec}(R_K)$ given by $\mathfrak{m} \mapsto \mathfrak{m}_K := \mathfrak{m} \cap R_K$ and an isomorphism $(R_K)_{\mathfrak{m}_K} \cong R_{\mathfrak{m}} \cap K$ for each maximal ideal.

(b): The fraction field K of R can be written as $K = \bigcup K'$, the increasing union being taken over subfields $K' \subset K$ that are finitely generated extensions over the prime subfield $\mathbb{F} \subset K$; as a consequence, $R = \bigcup (R \cap K')$. Thanks to (a), the semilocal Prüfer domain $R' := R \cap K'$ has fraction field K' ; therefore, the fact that $\text{tr.deg}_{\mathbb{F}}(K') < \infty$ ensures that R' has finite Krull dimension ([BouCA, Chapter VI, Section 10, Number 3, Corollary 1]). Therefore, we have exhibited R as an increasing union of subrings R' that are semilocal Prüfer domain of finite Krull dimension. Hence, we are done. \square

By Lemma 2.5, the notion of semilocal Krull domains coincides with the same of semilocal Prüfer domains. Consequently, as a result of Lemma 2.2, the category of semilocal Krull domains is closed under localisations and under quotients by prime ideals.

Lemmas 2.6-2.8 demonstrate that the category of semilocal Prüfer domains is also closed under gluing, possibly, at non-generic points.

Lemma 2.6. *Let R_1 be a semilocal Prüfer domain with a maximal ideal \mathfrak{m} and a residue field k and let R_2 be a semilocal Prüfer domain whose fraction field is k . The subring*

$$R := R_1 \times_k R_2 \subseteq R_1$$

is a semilocal Prüfer domain whose residue field at $\mathfrak{p} := \mathfrak{m} \cap R$ is k . Moreover, the semilocalisation of R at \mathfrak{p} and the maximal ideals not containing \mathfrak{p} is R_1 and $R/\mathfrak{p} \cong R_2$.

Proof. We begin by establishing that R is a semilocal Prüfer domain. According to Lemma 2.5, this is equivalent to demonstrating that it is a semilocal Krull domain. The same lemma shows that the rings R_1 and R_2 are semilocal Krull domains. To prove that R itself is a semilocal Krull domain, we can, without loss of generality, reduce to the case where R_1 and R_2 are valuation rings. Indeed, this reduction is possible thanks to the fact that intersections of rings commute with products of rings. The claim when R_1 and R_2 are valuation rings follows from [FK18, Chapter 0, Proposition 6.4.1(1)] (cf. [Sta22, Tag 088Z]).

Therefore, it suffices to demonstrate that $\mathfrak{p} \subset R$ satisfies the remaining claims. Since, by [Sta22, Tag 0B7J], we have that $\text{Spec}(R) = \text{Spec}(R_1) \sqcup_{\text{Spec}(k)} \text{Spec}(R_2)$, the claim that $R/\mathfrak{p} \cong R_2$ follows. Furthermore, the same coproduct description of $\text{Spec}(R)$ demonstrates that the semilocalisation of R at \mathfrak{p} and the maximal ideals not containing \mathfrak{p} is R_1 . Thus, the proof is complete. \square

Remark 2.7. Given a ring R , we may produce a graph with vertices in $\text{Spec}(R)$, where two vertices are joined by an edge if either is an immediate specialisation of the other. For example, the graph of a valuation ring R is a path with two endpoints, which corresponds to the zero and the maximal ideal of R respectively. Consequently, by the compatibility of forming this graph with localisation of rings, it is clear that the graph of a Prüfer domain is a connected, acyclic graph. Conversely, an application of Lemma 2.6 demonstrates that any finite, connected, acyclic graph may be recursively realised as the graph of a semilocal Prüfer domain of finite Krull dimension.

Lemma 2.8. *Given a semilocal Prüfer domain R of finite Krull dimension, a prime ideal $\mathfrak{p} \subset R$ and an element $a \in R$ such that $V(a) = \{\mathfrak{q} \in \text{Spec}(R) \mid \mathfrak{q} \supsetneq \mathfrak{p}\}$, the canonical morphism*

$$R \xrightarrow{\sim} (R[\frac{1}{a}]) \times_{(R[\frac{1}{a}]/(\mathfrak{p}R[\frac{1}{a}]))} (R/\mathfrak{p}) \quad \text{is an isomorphism.} \quad (2.8.1)$$

Remarks 2.9.

1. Since R has finitely many prime ideals, such an a always exist thanks to prime avoidance [Sta22, Tag 00DS].
2. The element a is chosen such that $R[\frac{1}{a}]$ is the semilocalisation of R at \mathfrak{p} and the maximal ideals not containing \mathfrak{p} . In particular, $\mathfrak{p}R[\frac{1}{a}] \subset R[\frac{1}{a}]$ is a maximal ideal.

Proof of Lemma 2.8. The claim (2.8.1) when R is a valuation ring is the content of [FK18, Chapter 0, Proposition 6.4.1(1)]. Therefore, without loss of generality, we may assume that R is not a valuation ring. Since R is an integral domain, we have that $R \subseteq R[\frac{1}{a}]$; consequently, the canonical morphism in claim (2.8.1) is an injection. We need to show that this morphism is a bijection. Letting $R' := (R[\frac{1}{a}]) \times_{(\kappa(\mathfrak{p}))} (R/\mathfrak{p})$, by [Sta22, Tag 0B7J], in similar vein as the proof of Lemma 2.6, we have that

$$\text{Spec}(R') = \text{Spec}(R[\frac{1}{a}]) \sqcup_{\text{Spec}(\kappa(\mathfrak{p}))} \text{Spec}(R/\mathfrak{p}).$$

However, the latter topological space is homeomorphic to $\text{Spec}(R)$. Therefore, the maximal ideals (in reality, all the prime ideals) of R' and R correspond to each other. Moreover, by Lemma 2.2, each multiplicative factor of $R' = (R[\frac{1}{a}]) \times_{(\kappa(\mathfrak{p}))} (R/\mathfrak{p})$ is a semilocal Prüfer domain. As a result, by Lemma 2.6, we get that R' itself is a semilocal Prüfer domain. Since the maximal ideals of R' and R correspond and each ring is a semilocal Krull domain (Lemma 2.5), to verify that the morphism in claim (2.8.1) is an isomorphism, it suffices to ensure that their localisations at each maximal ideal are equal ([Mat86, Theorem 12.2] or [BouCA, Chpater 6, Section 7, Number 1, Proposition 2]). We verify this claim below. Let $\mathfrak{m} \subseteq R$ be a maximal ideal. If $\mathfrak{m} \not\supseteq \mathfrak{p}$, we get that

$$(R')_{(\mathfrak{m}R')} = (R[\frac{1}{a}])_{(\mathfrak{m}R[\frac{1}{a}])} = R_{\mathfrak{m}}.$$

Therefore, it remains to verify the case when $\mathfrak{m} \supseteq \mathfrak{p}$. In this case,

$$(R')_{(\mathfrak{m}R')} = (R[\frac{1}{a}])_{(\mathfrak{m}R[\frac{1}{a}])} \times_{(\kappa(\mathfrak{p}))} (R/\mathfrak{p})_{(\mathfrak{m}/\mathfrak{p})} = R_{\mathfrak{p}} \times_{(\kappa(\mathfrak{p}))} (R_{\mathfrak{m}}/(\mathfrak{p}R_{\mathfrak{m}})).$$

Given that, by hypothesis, $R_{\mathfrak{m}}$ is a valuation of finite rank (whose spectrum forms a finite linear chain), the second equality follows from the fact that $(R[\frac{1}{a}])_{(\mathfrak{m}R[\frac{1}{a}])} = (R_{\mathfrak{m}})[\frac{1}{a}] = R_{\mathfrak{p}}$. On the other hand, the valuation

ring case of the claim (2.8.1) produces the equality

$$R_{\mathfrak{p}} \times_{(\kappa(\mathfrak{p}))} (R_{\mathfrak{m}}/(\mathfrak{p}R_{\mathfrak{m}})) = R_{\mathfrak{m}}.$$

As a result, we get that $R_{\mathfrak{m}} = (R')_{\mathfrak{m}R'}$, as claimed. Thus, we are done. \square

The following is an input in the proof of Lemma 2.11 below.

Lemma 2.10. *Let R be an integral domain, let $\mathfrak{m} \subset R$ be a maximal ideal with a residue field k , and let $\ell \subseteq k$ be a subfield. Suppose that $\varphi: R \rightarrow k$ is the canonical surjection. The subring*

$$R_{\ell} := \{r \in R \mid \varphi(r) \in \ell\} \text{ is an integral subdomain such that } R_{\ell} = \text{Frac}(R_{\ell}) \cap R.$$

Proof. Let $R'_{\ell} := \text{Frac}(R_{\ell}) \cap R$ and $\ell' := \varphi(R'_{\ell})$. To demonstrate that $R_{\ell} = R'_{\ell}$, it suffices to prove that $R'_{\ell} \subseteq R_{\ell}$. Applying φ , it is equivalent to show that $\ell' \subseteq \ell$. This follows since ℓ is a field and since $R'_{\ell} \subseteq \text{Frac}(R_{\ell})$. \square

We are now ready to prove a generalisation of Lemma 2.5(b), which can be recovered by setting $\mathfrak{p} = (0)$ in Lemma 2.11(ii). We introduce some terminology below for convenience.

Given a semilocal Prüfer domain R , a set of prime ideals $\mathcal{P} = \{\mathfrak{p}_r \subset R\}$ and a subring $R' \subseteq R$, we say that R' is *finitely generated at \mathcal{P}* if the residue field of R' at $\mathfrak{p}_r \cap R'$ is finitely generated over its prime subfield, for each $\mathfrak{p}_r \in \mathcal{P}$.

Lemma 2.11. *Let R be a semilocal Prüfer domain of finite Krull dimension with a fraction field K and let $\mathcal{P} := \{\mathfrak{p}_r \subset R\}$ be a set of prime ideals.*

- (i) *Given a subfield $K' \subseteq K$, if R is finitely generated at \mathcal{P} , then same is true for the subring $R' := R \cap K'$.*
- (ii) *The ring R is a filtered union of semilocal Prüfer domains R_{α} of finite Krull dimension each of which is finitely generated at \mathcal{P} .*
- (iii) *The field K can be written as a filtered union of subfields $K_{\alpha} \subseteq K$ so that each $R_{\alpha} := R \cap K_{\alpha}$ is finitely generated at \mathcal{P} .*

Remark 2.12. [BouCA, Chapter VI, § 10, No. 3, Corollary 1] demonstrates that the transcendence degrees of $\kappa(\mathfrak{p}_r)$ over their respective prime subfields are bounded, for all r , as soon as K is finitely generated over its prime subfield. However, ensuring that $\kappa(\mathfrak{p})$ are also finitely generated is more delicate.

Proof of Lemma 2.11. First, we observe that (ii) is implied by (iii). Indeed, by Lemma 2.5(a), such subrings $R_{\alpha} := R \cap K_{\alpha}$ are automatically semilocal Prüfer domains of finite Krull dimension. Therefore, it suffices to demonstrate only (i) and (iii). We shall prove them by inducting on r .

The claims are trivial in the base case, i.e., when $r = 0$. Therefore, we may assume that $r \geq 1$. As inductive hypothesis, we suppose that both (i) and (iii) are true for all such R when $r = n$. As inductive step, we shall prove both the claims when $r = n + 1$. Let $\mathfrak{p} \in \mathcal{P}$ be a maximal element ordered by inclusion and let $a \in R$ be such that $V(a) = \{\mathfrak{q} \in \text{Spec}(R) \mid \mathfrak{q} \supsetneq \mathfrak{p}\}$. Since R has only finitely many prime ideals, the element a exists by prime avoidance [Sta22, Tag 00DS] (cf. Remarks 2.9(1)). Thanks to Lemma 2.8,

$$R \xrightarrow{\sim} (R[\frac{1}{a}]) \times_{(\kappa(\mathfrak{p}))} (R/\mathfrak{p}) \quad \text{is an isomorphism.} \quad (2.12.1)$$

Furthermore, $R[\frac{1}{a}]$ is a semilocal Prüfer domain of finite Krull dimension whose residue field at the maximal ideal $\mathfrak{p}R[\frac{1}{a}]$ is $\kappa(\mathfrak{p})$ and R/\mathfrak{p} is a semilocal Prüfer domain of finite Krull dimension whose fraction field is $\kappa(\mathfrak{p})$ (Lemma 2.2). Again, considering the fact that R has only finitely many primes, it follows that $\text{Spec}(R_{\mathfrak{p}}) \subseteq \text{Spec}(R)$ is a Zariski open. As a consequence, since both the claims are Zariski local in R , by localising R at \mathfrak{p} , we may, without loss of generality, assume that R is a valuation ring with a maximal ideal $\mathfrak{p} \subset R$. First, we prove (i).

(i): Let $K' \subseteq K$ be subfield. As a first case, we assume that $\mathcal{P} = \{\mathfrak{p}\}$. Since $R' \subseteq R$, the residue field k' of R' at $\mathfrak{p} \cap R'$ is a subfield of $\kappa(\mathfrak{p})$, which is finitely generated over its prime subfield. Consequently, by [Bou03, Chapter V, §14, No. 7, Corollary 3], the same is true for k' . Thus, in this case, we are done.

Otherwise, the set $\mathcal{P}' := (\mathcal{P} \setminus \{\mathfrak{p}\})$ is nonempty. Let \mathfrak{q} be the maximal element in \mathcal{P}' (the prime \mathfrak{q} is unique because $\text{Spec}(R)$ is totally ordered). Similar to (2.12.1), we have an isomorphism

$$R \xrightarrow[\text{(2.8.1)}]{\sim} R_{\mathfrak{q}} \times_{(\kappa(\mathfrak{q}))} (R/\mathfrak{q}). \quad (2.12.2)$$

Letting $\mathfrak{q}' := \mathfrak{q} \cap R'$, since $R'_{\mathfrak{q}'}$ is a valuation ring that is dominated by $R_{\mathfrak{q}} \cap K'$, we obtain the equality $R'_{\mathfrak{q}'} = R_{\mathfrak{q}} \cap K'$. Let k' be the residue field of R' at \mathfrak{q}' . Since R is finitely generated at \mathcal{P} , it follows that $R_{\mathfrak{q}}$ (respectively, R/\mathfrak{q}) is finitely generated at \mathcal{P}' (respectively, at $\{(\mathfrak{p}/\mathfrak{q})\}$). Consequently, by the induction hypothesis (respectively, by the $r = 1$ case proved above), the ring $R'_{\mathfrak{q}'}$ (respectively, the ring R'/\mathfrak{q}') is finitely generated at \mathcal{P}' (respectively, at $\{(\mathfrak{p}/\mathfrak{q})\}$). Therefore, the ring

$$R' \xrightarrow[\text{(2.8.1)}]{\sim} R'_{\mathfrak{q}'} \times_{k'} (R'/\mathfrak{q}')$$

is finitely generated at \mathcal{P} , as required. The induction is thus complete and (i) is proven.

Finally, we prove (iii).

(iii): In a similar vein as the proof of (i), as a first case, we assume that $\mathcal{P} = \{\mathfrak{p}\}$. Letting $\mathbb{F} \subseteq \kappa(\mathfrak{p})$ be the prime subfield, we can write $\kappa(\mathfrak{p})$ as a filtered unions of subfields $k_{\beta} \subseteq \kappa(\mathfrak{p})$ which are finitely generated over \mathbb{F} . For each β , we let

$$R_{\beta} := \{r \in R \mid r \bmod \mathfrak{p} \in k_{\beta}\} \quad \text{and} \quad K_{\beta} := \text{Frac}(R_{\beta}).$$

By Lemmas 2.10 and 2.5(b), the ring R_{β} is a semilocal Prüfer domain of finite Krull dimension whose residue field at the maximal ideal $\mathfrak{m}_{\beta} := \mathfrak{p} \cap R_{\beta}$ is k_{β} , for each β . Furthermore, the same lemmas prove the equality $R_{\beta} = R \cap K_{\beta}$, for each β . As a consequence, it suffices to check that $\bigcup K_{\beta} = K$, which is true because $\bigcup k_{\beta} = k$. Thus, in this case, we are done.

Otherwise, the set $\mathcal{P}' := \mathcal{P} \setminus \{\mathfrak{p}\}$ is nonempty. Similar to the proof of (i), let \mathfrak{q} be the maximal element in \mathcal{P}' . By the induction hypothesis (respectively, by the $r = 1$ case proved above), the field K can be written as a filtered union of subfields $K_{\gamma} \subseteq K$ (respectively, subfields $K_{\beta} \subseteq K$) such that $R_{\gamma} := R \cap K_{\gamma}$ (respectively, $R_{\beta} := R \cap K_{\beta}$) is finitely generated at \mathcal{P}' (respectively, at $\{\mathfrak{p}\}$), for all γ (respectively, for all β). Possibly by reindexing, we consider a mono-indexed cofinal collection $\{K_{\alpha}\}$ in the bi-indexed collection $\{K_{\beta} \cap K_{\gamma}\}$ of subfields of K and we let

$$R_{\alpha} := R \cap K_{\alpha}, \text{ for each } \alpha.$$

Since $\{K_{\alpha}\}$ is a cofinal collection of $\{K_{\beta} \cap K_{\gamma}\}$, letting α be an index, there exists some β and some γ (we fix one such pair) so that $K_{\alpha} = K_{\beta} \cap K_{\gamma}$. Therefore, by definition, we get

$$R_{\alpha} = R_{\beta} \cap K_{\gamma} = R_{\gamma} \cap K_{\beta}.$$

Applying (i) to R_{γ} (respectively, to R_{β}), we obtain that R_{α} is finitely generated at \mathcal{P}' (respectively, at $\{\mathfrak{p}\}$). Combining the statements, it follows that R_{α} is finitely generated at $\mathcal{P} = \mathcal{P}' \cup \{\mathfrak{p}\}$, as required. Thus, the induction step is complete, and consequently, the claim is proven. Hence, we are done. \square

Below, in Lemma 2.13(1), we recall from [Kun23, Lemma 3.10] (cf. [Mor22, théorème A]) an intriguing property of smooth algebras A over valuation rings V . Specifically, we demonstrate that the local rings of A at the generic points of its R -fibres are also valuation rings. We will leverage this to ultimately reduce Theorem 1.2 for A to a Gersten's injectivity claim for the local ring of A at a generic point of its R -special fibre (see Corollary 3.8).

Before stating Lemma 2.13, we introduce some notations for clarity. Given a ring R , we denote its Krull dimension by $\dim(R)$. Given a finite set of primes $\mathcal{P} \subseteq \text{Spec}(A)$ of a ring A , we denote the semilocalisation of A at the primes in \mathcal{P} by $A_{\mathcal{P}}$. Given an extension $e: R \rightarrow S$ of rings and a subset $\mathcal{P} \subset \text{Spec}(S)$, let

$$\text{Flat}_{\mathcal{P}}(S/R) := \{f \in S \mid V(f) \cap \mathcal{P} = \emptyset \text{ and } R \rightarrow S/fS \text{ is flat}\}.$$

In terms of the previously introduced notations, for any extension e and subset \mathcal{P} , it is noteworthy that $\text{Flat}_{\mathcal{P}}(S/R)$ is a filtered set.

Lemma 2.13.

- (1) Given a valuation ring V , an integral domain A that is a smooth V -algebra, and a subset \mathcal{P} of the set $\tilde{\mathcal{P}}$ of generic points of the V -special fibre of $\mathrm{Spec}(A)$, the morphism $V \rightarrow A_{\mathcal{P}}$ is faithfully flat extension of semilocal Prüfer domains.
- (2) Moreover, if $\dim(V) < \infty$, then there exists an affine open neighbourhood $\mathrm{Spec}(A') \subseteq \mathrm{Spec}(A)$ of \mathcal{P} such that we have

$$\mathrm{colim}_{f \in \mathrm{Flat}_{\mathcal{P}}(A'/V)} A'[\frac{1}{f}] = A'_{\mathcal{P}}.$$

Remark 2.14.

1. Although, our lemma is stated for a general \mathcal{P} , the most interesting cases are when \mathcal{P} is a singleton set and when $\mathcal{P} = \tilde{\mathcal{P}}$. In the former case, the proof of (1) below shows that, in fact, $V \rightarrow A_{\mathcal{P}}$ is an extension of valuation rings which induces an isomorphism between the respective value groups.
2. It is worth noting that the R -special fibre of $\mathrm{Spec}(A)$ need not be connected, even when R is a discrete valuation ring and $R \rightarrow \mathcal{A}$ is étale. For example, when \mathcal{A} is the integral closure of R in an unramified extension $L \supseteq \mathrm{Frac}(R)$ ([Sta22, Tag 09E9]), the maximal ideals of \mathcal{A} , which lie in the R -special fibre of $\mathrm{Spec}(A)$, correspond to the extensions of valuations $R \subset L$ centred on \mathcal{A} ([Sta22, Tag 09E8]). In particular, if R is not Henselian, such extensions of valuations need not be unique.

Proof of Lemma 2.13. We prove the claims simultaneously. We shall reduce, by the local structure of smooth morphisms, to establishing the claims when A is étale, and when A is a polynomial algebra, in which we do an explicit computation.

By the semilocal structure of smooth morphisms [SGA 1, Exposé II, théorème 4.10(ii)]² (cf. [Sta22, Tag 052E]), there exist an integer $n \geq 0$, a connected, affine, open neighbourhood $\mathrm{Spec}(B) \subseteq \mathrm{Spec}(A)$ of \mathcal{P} and an étale morphism $j: \mathrm{Spec}(B) \rightarrow \mathrm{Spec}(V[x_1, \dots, x_n])$. Since the claims are of Zariski semilocal in nature, without loss of generality, we may assume that $A = B$. If $V \rightarrow A$ is étale, then $n = 0$, and, by [Sta22, Tag 0ASJ], claim (1) is true. Furthermore, if $\dim(V) < \infty$, the ring A is a semilocal Prüfer domain of finite Krull dimension. Therefore, by prime avoidance [Sta22, Tag 00DS], $\mathrm{Spec}(A_{\mathcal{P}}) \subseteq \mathrm{Spec}(A)$ is an open subset. Consequently, to demonstrate claim (2), it suffices to take $A' = A_{\mathcal{P}}$. Thus, the claims are proven in the case when $n = 0$.

Therefore, without loss of generality, we may assume that $n \geq 1$. Since étale morphisms are of relative dimension 0, it follows that $j(\mathfrak{p})$ is the generic point η of the V -special fibre of $\mathrm{Spec}(V[x_1, \dots, x_n])$, for any $\mathfrak{p} \in \mathcal{P}$. As a consequence, letting $V' := V[x_1, \dots, x_n]$ and $\mathfrak{p}' \subset V'$ be the prime corresponding to η , we have a factorisation

$$V \xrightarrow{g} V'_{\mathfrak{p}'} \xrightarrow{h} A_{\mathcal{P}}.$$

Thus, it is sufficient to show the claims are true for the morphism g . Indeed, supposing that the claims are true for g , the proven $n = 0$ case applied to h implies that $h \circ g$ is a faithfully flat extension of semilocal Prüfer domains, as required to show (1). Similarly, to prove claim (2), we note that there is an inclusion $\mathrm{Flat}_{\{\mathfrak{p}\}}(V'/V) \subseteq \mathrm{Flat}_{\mathcal{P}}(A/V)$ and base change the equality obtained by (2) for g along $V' \rightarrow A$ to obtain an equality

$$A_{\mathfrak{p}'} = \mathrm{colim}_{f \in \mathrm{Flat}_{\{\mathfrak{p}'\}}(V'/V)} A[\frac{1}{f}] = \mathrm{colim}_{f \in \mathrm{Flat}_{\mathcal{P}}(A/V)} A[\frac{1}{f}]. \quad (2.14.1)$$

Finally, base changing the equality (2.14.1) along the open embedding $\mathrm{Spec}(A_{\mathcal{P}}) \subset \mathrm{Spec}(A)$, we obtain the required equality which shows (2). Therefore, without loss of generality, it suffices to establish the claims for g .

In consequence, we may assume that $A = V[x_1, \dots, x_n]$. Let $\mathfrak{m} \subset V$ be the maximal ideal and let $\mathfrak{p} := \mathfrak{m}[x_1, \dots, x_n]$. Given the assumption, we have that $\mathcal{P} = \{\mathfrak{p}\}$.

(1): In fact, we shall demonstrate that $V \rightarrow A_{\mathfrak{p}}$ is a faithfully flat extension of valuation rings that induces an isomorphism an isomorphism between the respective value groups. First, we prove that $A_{\mathfrak{p}}$ is a

²Even though [SGA 1, Exposé II, théorème 4.10(ii)] concerns only the Zariski local structure of smooth morphisms, we can establish their Zariski semilocal structure with a slight modification of the proof in loc. cit.

valuation ring. To do so, it suffices to verify that for any

$$t \in \text{Frac}(A_{\mathfrak{p}}) = \text{Frac}(V[x_1, \dots, x_n]), \text{ either } t \in A_{\mathfrak{p}} \text{ or } 1/t \in A_{\mathfrak{p}}.$$

Let $t = f/g \in \text{Frac}(A_{\mathfrak{p}})$, where $f, g \in A$. Using the valuation on V , we define $\text{val}(f) \in V$ (resp., $\text{val}(g) \in V$) to be the element, which is well defined up to a unit in V , such that $f/\text{val}(f) \in A \setminus \mathfrak{p}$ (resp., $g/\text{val}(g) \in A \setminus \mathfrak{p}$). If $\text{val}(g) \mid \text{val}(f)$, then $t \in A_{\mathfrak{p}}$, otherwise, $1/t \in A_{\mathfrak{p}}$, and we are done. It remains to show that the morphism

$$\varphi: \text{Frac}(V)^{\times}/V^{\times} \rightarrow \text{Frac}(A)^{\times}/A_{\mathfrak{p}}^{\times} \text{ of value groups is an isomorphism.}$$

Since φ is injective, it suffices to show that φ is surjective. In a similar vein as the previous arguments, given $t \in \text{Frac}(A)^{\times}$, there exists $u \in \text{Frac}(V)^{\times}$ such that $t/u \in A_{\mathfrak{p}}^{\times}$. Thus, (1) is proven.

(2): Since, by definition, $A_{\mathfrak{p}} = \text{colim}_{f \in A \setminus \{\mathfrak{p}\}} A[\frac{1}{f}]$, it is enough to show that $A \setminus \{\mathfrak{p}\} \subseteq \text{Flat}_{\{\mathfrak{p}\}}(A/V)$. In other words, given $f \in A \setminus \{\mathfrak{p}\}$, we need to demonstrate that $V \rightarrow A/fA$ is flat. As V is a valuation ring, it is equivalent to prove that such an f satisfies that A/fA is V -torsion free. Letting $f \in A \setminus \{\mathfrak{p}\}$, suppose that $v \in V$ and $a, b \in A$ be any elements such that $va = bf$. In order to establish that A/fA is V -torsion free, we need to show that either $v = 0$ or $a \in fA$. Therefore, further assuming that $v \neq 0$, it is sufficient to prove that $a \in fA$. Adopting a notation of the above paragraph and keeping in mind that $f \notin \mathfrak{p}$, we conclude that v divides $\text{val}(bf) = \text{val}(b)$. As a consequence, $b' := b/v \in A$, whence we deduce that $a = b'f$, as required. Thus, (2) is also proven, and hence, we are done. \square

3 Gersten's Injectivity of G -theory

In this section, our goal is to prove Theorem 3.6. Although the anatomy of its proof is based on the proof of [GL87, Theorem], considerable amount of customisation is needed to upgrade the arguments to the non-Noetherian setting. Already, since the spectra of valuation rings contain more points than their discrete counterparts, the construction of Presentation Lemma 3.2, which is the principal technical ingredient in this section, is surprisingly more delicate than we expect (cf. [GL87, Lemma 1] and [Lüd22, Lemma 2.12]). We assemble our version of the presentation lemma by synthesising the proofs of [Čes22, Variant 3.7] and [Kun23, Proposition 6.4]. In conclusion of this section, and as an application of Theorem 1.4, we extend the result in [GL87, Theorem] to demonstrate that the algebraic K -theory functor satisfies Gersten's injectivity for semilocal, essentially smooth algebras over equicharacteristic valuation rings (Corollary 3.8).

In preparation for Presentation Lemma 3.2, let us state a straightforward consequence of [Čes22, Proposition 3.6].

Lemma 3.1. *Given a field k , an affine, smooth k -scheme X of pure relative dimension $d > 0$, points $x_1, \dots, x_n \in X$ and a nowhere dense, closed subscheme $Y \hookrightarrow X$, there exist affine opens $x_1, \dots, x_n \in U \subseteq X$ and $S \subseteq \mathbb{A}_k^{d-1}$ and a smooth k -morphism $\pi: U \rightarrow S$ of relative dimension 1 such that $\pi|_{Y \cap U}$ is finite.*

Proof. We shall deduce the claim via a direct application of loc. cit. At the cost of shrinking X to an open, affine neighbourhood of x_1, \dots, x_n , we may assume that there is a closed embedding $X \hookrightarrow \mathbb{A}_k^m$, and thus, by composing with a chosen open embedding $\mathbb{A}_k^m \hookrightarrow \mathbb{P}_k^m$, we obtain an embedding $\iota: X \hookrightarrow \mathbb{P}_k^m$. We define \overline{X} (respectively, \overline{Y}) to be the schematic closure of ι (respectively, the same of $Y \subset \overline{X}$). The fact that Y is dense in \overline{Y} ensures that $\text{codim}_{\overline{X}}(\overline{Y}) = \text{codim}_X(Y) \geq 1$. Moreover, the same fact implies that $\overline{Y} \setminus Y$ does not contain any generic point of \overline{Y} . As a consequence, it follows that $\text{codim}_{\overline{X}}(\overline{Y} \setminus Y) \geq 2$. Therefore, the claim follows by putting $(\overline{X}, X, x_1, \dots, x_n \in X, \overline{Y})$ in loc. cit. \square

Presentation Lemma 3.2. Given

- a semilocal Prüfer domain R of finite Krull dimension,
- an affine, smooth R -scheme X of pure relative dimension $d > 0$,
- points $x_1, \dots, x_n \in X$ and
- an R -flat, closed subscheme $Y \hookrightarrow X$ that does not contain any component of the R -fibres of X ,

there exist affine opens $x_1, \dots, x_n \in U \subseteq X$ and $S \subseteq \mathbb{A}_R^{d-1}$ and a smooth R -morphism $\pi: U \rightarrow S$ of relative dimension 1 such that $\pi|_{Y \cap U}$ is quasi-finite.

For brevity, such a morphism π (along with the data of U and S) will be called a *presentation of X over R* .

Proof. Let $C \subseteq \text{Spec } R$ be the closed subscheme of closed points and let X_C be the fibre of X over C . Since Lemma 3.1 takes care of the case when R is a field, without loss of generality, we may assume that $C \subsetneq \text{Spec } R$. Since the claim is semilocal, we are allowed to replace any x_i with one its specialisations.

Case 1: We suppose further that each x_i specialises to a point in X_C . Because of this assumption, we may replace x_i with one of its specialisations that are closed points in X_C (we use the fact that X_C is a Jacobson scheme in order to specialise each x_i to a closed point in X_C) to assume further that each $x_i \in X_C$ is closed. Thanks to Lemma 3.1, there exist affine opens

$$x_1, \dots, x_n \in U' \subset X_C \text{ and } S' \subset \mathbb{A}_C^{d-1} \text{ and a smooth } C\text{-morphism } \pi': U' \rightarrow S'$$

such that $\pi'|_{Y_C \cap U'}$ is quasi-finite (in fact, we can ensure that the latter is finite, but we will not require it for the proof of the claim). Let \mathcal{J} be the ideal of vanishing of $X_C \hookrightarrow X$ and let $\tilde{U} \subset X$ be an open subset such that $\tilde{U} \cap X_C = U'$. By lifting sections along $\mathcal{O}_{\tilde{U}} \rightarrow \mathcal{O}_{\tilde{U}}/(\mathcal{J}|_{\tilde{U}})$, we define a morphism $\tilde{\pi}: \tilde{U} \rightarrow \mathbb{A}_R^{d-1}$. Since \tilde{U} is R -finite type and R -flat, it is R -finitely presented (by [RG71, Première partie, théorème 3.4.6] and by the fact that R is an integral domain). Thanks to the fibrewise criterion of flatness [Sta22, Tag 039C], $\tilde{\pi}$ is flat at each x_i , whence thanks to the fibrewise criterion of smoothness [Sta22, Tag 01V8], $\tilde{\pi}$ is smooth at each x_i . Therefore, after shrinking \tilde{U} , we may assume that $\tilde{\pi}$ is smooth. The openness of the quasi-finite locus [Sta22, Tag 01T1] implies that there exists an open subset $U_1 \subset Y$ containing $Y \cap \pi^{-1}(\pi(x_i))$, for all i , such that $\pi|_{U_1}$ is quasi-finite. We choose affine opens $\pi(x_1), \dots, \pi(x_n) \in S \subseteq \pi(\tilde{U} \setminus (Y \setminus U_1))$ and $U \subseteq \pi^{-1}(S)$.

Case 2: In general, some of the points among x_1, \dots, x_n might not specialise to a point of X_C , say y_1, \dots, y_m . Let $\mathcal{P} \subseteq \text{Spec}(R)$ be the images of y_1, \dots, y_m . We shall tailor X in such a way that each y_i specialises to X_C , effectively replacing the original X with this customised version. For this purpose, we use a trick due to Česnavičius in the proof of [Čes22, Variant 3.7]. In this regard, thanks to Lemma 2.11(ii) and a limit argument, without loss of generality, we may assume that the residue field $\kappa(\mathfrak{p})$ of R at each $\mathfrak{p} \in \mathcal{P}$ is finitely generated over its prime subfield. By, for example, [Kun23, Lemma 6.1], each field $\kappa(\mathfrak{p})$ is a fraction field of a regular domain $A_{\mathfrak{p}}$ that is smooth over \mathbb{F}_p or \mathbb{Z} . Moreover, each $A_{\mathfrak{p}}$ is of positive Krull dimension, since otherwise K is a finite field, in which case, it contradicts our assumption that R is not a field. By localising $A_{\mathfrak{p}}$ and possibly by selecting a local R -projective embedding of X , we may assume that

1. the scheme $X_{\kappa(\mathfrak{p})}$ spreads out to a smooth $A_{\mathfrak{p}}$ -scheme $X_{\mathfrak{p}}$ that is fibrewise of pure dimension d (see [Sta22, Tag 01V8] and [EGA IV₃, théorème 12.1.1(iv)]),
2. each point y_i lying over \mathfrak{p} spreads out to an $A_{\mathfrak{p}}$ -finite, closed subscheme in $X_{\mathfrak{p}}$, and
3. the closed subscheme $Y_{\kappa(\mathfrak{p})}$ spreads out to an $A_{\mathfrak{p}}$ -flat, closed subscheme $Y_{\mathfrak{p}}$ such that $Y_{\mathfrak{p}}$ is $A_{\mathfrak{p}}$ -fibrewise of codimension ≥ 1 in $X_{\mathfrak{p}}$ (see [Sta22, Tag 039C] and [EGA IV₃, théorème 12.1.1(v)]).

Now that we have constructed the objects above for each $\mathfrak{p} \in \mathcal{P}$, let us proceed with the customization of X .

Method: Simultaneously,

- we shall iteratively glue R with a discrete valuation ring $A'_{\mathfrak{p}}$ whose fraction field is $\kappa(\mathfrak{p})$ at each $\mathfrak{p} \in \mathcal{P}$ to ultimately produce a semilocal Prüfer domain \tilde{R} of finite Krull dimension \tilde{R} , and
- we shall iterative glue X with an $A'_{\mathfrak{p}}$ -scheme $X_{\mathfrak{p}}$ whose generic fibre is $X_{\kappa(\mathfrak{p})}$ at each $\mathfrak{p} \in \mathcal{P}$ to ultimately produce an \tilde{R} -scheme \tilde{X} which has the property that each y_i specialises to a point of the \tilde{R} -special fibre of \tilde{X} .

Once this is done, we apply the already proven Case 1 to \tilde{R} and \tilde{X} (Case 1 applies thanks to our construction of \tilde{R} and \tilde{X}) to obtain a presentation $\tilde{\pi}$ of \tilde{X} over \tilde{R} . Given that a presentation of X over R is Zariski semilocal around x_1, \dots, x_n , considering the open embedding $j: \text{Spec}(R) \hookrightarrow \text{Spec}(\tilde{R})$, we can perform a base change of $\tilde{\pi}$ along j to obtain a presentation π of X over R , as required.

Consequently, it suffices to construct such \tilde{R} and such \tilde{X} . We proceed iteratively, incrementally considering primes $\mathfrak{p} \in \mathcal{P}$ ordered by their height. We fix a prime $\mathfrak{p} \in \mathcal{P}$ of height n . By abuse of notation, let R be the semilocal Prüfer domain (respectively, the R -scheme X) constructed by applying the procedure called Method to all $\mathfrak{q} \in \mathcal{P}$ of height $< n$. Following Method, we shall construct \tilde{R} by gluing R with a discrete valuation ring $A_{\mathfrak{p}}$ at \mathfrak{p} and \tilde{X} by gluing X with an $A_{\mathfrak{p}}$ -scheme $X_{\mathfrak{p}}$ whose generic fibre is $X_{\kappa(\mathfrak{p})}$. Let $a \in R$ be a element whose vanishing set is $\{\mathfrak{q} \in \text{Spec}(R) \mid \mathfrak{q} \supsetneq \mathfrak{p}\}$. As observed in Lemma 2.8, thanks to prime avoidance [Sta22, Tag 00DS], such an a always exists because R has finitely many prime ideals. By (2.8.1), we may write

$$R \xrightarrow{\sim} R[\frac{1}{a}] \times_{\kappa(\mathfrak{p})} (R/\mathfrak{p}).$$

Given that $A_{\mathfrak{p}}$ is of positive Krull dimension, it has infinitely many primes of height 1, permitting us to choose such a prime $\mathfrak{r} \subset A$ so that the localisation $A'_{\mathfrak{p}}$, which is necessarily a discrete valuation ring, of $A_{\mathfrak{p}}$ at \mathfrak{r} is different from each of the localisations of R/\mathfrak{p} . Choosing such a prime $\mathfrak{r} \subset A$, we substitute $A_{\mathfrak{p}}$ with $A'_{\mathfrak{p}}$ and consider $R' := (R/\mathfrak{p}) \cap A_{\mathfrak{p}}$, where the intersection is taken in $\kappa(\mathfrak{p})$. Thanks to Lemma 2.5, the ring R' is a semilocal Prüfer domain of finite Krull dimension with fraction field $\kappa(\mathfrak{p})$. We define \tilde{R} by the following diagram

$$\tilde{R} \xrightarrow{\sim} R[\frac{1}{a}] \times_{\kappa(\mathfrak{p})} R'. \quad (3.2.1)$$

Lemma 2.6 ensures that \tilde{R} is a semilocal Prüfer domain of finite Krull dimension. It remains to construct \tilde{X} , which we do below. Over the open cover $\text{Spec}(R/\mathfrak{p})$ and $\text{Spec}(A_{\mathfrak{p}})$ of $\text{Spec}(R')$

- we glue $X_{R/\mathfrak{p}}$ and $X_{\mathfrak{p}}$ along $X_{\kappa(\mathfrak{p})}$ to obtain a smooth R' -scheme X' that is fibrewise of pure dimension d , and
- we glue $Y_{R/\mathfrak{p}}$ and $Y_{\mathfrak{p}}$ along $Y_{\kappa(\mathfrak{p})}$ to obtain an R' -flat, closed subscheme $Y' \subset X'$ which is R' -fibrewise of codimension ≥ 1 .

By construction, each y_i lying over \mathfrak{p} specialises to a point in an R' -special fibre of X' . Finally, we define \tilde{X} . Thanks to [Sta22, Tag 0B7J] and [Sta22, Tag 0D2I](1),

- we glue $X_{R[\frac{1}{a}]}$ and X' along $X_{\kappa(\mathfrak{p})}$ to obtain an \tilde{R} -flat scheme \tilde{X} that is fibrewise of pure dimension d , and
- we glue $Y_{R[\frac{1}{a}]}$ and Y' along $Y_{\kappa(\mathfrak{p})}$ to obtain an \tilde{R} -flat, closed subscheme $\tilde{Y} \subset \tilde{X}$ which is \tilde{R} -fibrewise of codimension ≥ 1 .

Thanks to the fibrewise criterion of smoothness [Sta22, Tag 01V8], it follows that \tilde{X} is \tilde{R} -smooth. By construction, the property that each y_i lying over \mathfrak{p} specialises to a point in an \tilde{R} -special fibre of \tilde{X} continues to be true. Therefore, we may apply Case 1 to $(\tilde{R}; \tilde{X}; x_1, \dots, x_m; \tilde{Y})$ to obtain a presentation of \tilde{X} over \tilde{R} , as required. This concludes the proof. \square

Remarks 3.3.

- (1) Even though for the sake of demonstrating Theorem 3.6, we are only interested in a special case of Presentation Lemma 3.2, namely, when R is a valuation ring, the reason to state Presentation Lemma 3.2 in its current generality is not merely curiosity. As a matter of fact, our proof, more precisely, Case 2 of our proof forces us to modify R via cute-and-glue tailoring (Lemmas 2.6-2.8). As a result, Prüfer domains naturally appear there during this process.
- (2) There are significant differences between the statements of [Kun23, Proposition 6.4] and Presentation Lemma 3.2. Although our assumption that Y is R -flat, as opposed to a fibrewise smoothness condition imposed on Y in loc. cit., makes our proof easier than there, a drawback is that we can only ensure that $\pi|_{Y \cap U}$ is quasi-finite, instead of finite. Quasi-finiteness is sufficient for the proof of Theorem 3.6 thanks to the ‘linearity of G -theory’, which greatly simplifies the arguments. Indeed, following the clever idea in the proof of [GL87, Theorem], after base changing along π (cf. (3.7.1)), we employ Zariski’s main theorem [Sta22, Tag 00Q9] to enlarge our scheme, so that essentially $\pi|_{Y \cap U}$ may be assumed to be finite (cf. (3.7.2)), at least in some vague sense.
- (3) In the proof of Presentation Lemma 3.2, one can also argue that we can always reduce to Case 1, i.e., the case when x_1, \dots, x_n are assumed to lie in the R -special fibre. Indeed, as in the beginning of the proof of [ČF23, Lemma 2.1], we can localise at the images of x_1, \dots, x_n in $\text{Spec}(R)$ and spread out.

While working with non-Noetherian rings, we constantly need to make the distinction between finite type objects and finitely presented ones over them. This is vital, for example, especially while using Noetherian approximation techniques, where we require finitely presented, and not just finite type. Coherent rings, which we introduce below, form an important class of rings where this disparity between finite type and finite presentation reduces. Thankfully, rings in \mathcal{S}_{val} are coherent, which permits us to adapt some of the techniques from [GL87].

3.4. Coherence. Given a scheme X , an \mathcal{O}_X -module \mathcal{F} is called *coherent* if it is of finite type and for every open $U \subseteq X$ and every finite collection $s_i \in \mathcal{F}(U)$, $i = 1, \dots, n$, the kernel of the associated morphism $\bigoplus_{i=1, \dots, n} \mathcal{O}_U \rightarrow \mathcal{F}$ is of finite type ([Sta22, Tag 01BV]). A coherent \mathcal{O}_X -module is finitely presented, and therefore, quasi-coherent ([Sta22, Tag 01BW]). A scheme X is called *locally coherent* if \mathcal{O}_X is a coherent module over itself ([GR18, Definition 8.1.54]). A ring A is called *coherent* if any finitely generated ideal of A is finitely presented ([Sta22, Tag 05CV]).

A scheme X that is locally of finite presentation over a Prüfer domain R is locally coherent. Indeed, since the property of being locally coherent is Zariski local, it suffices to check that any ring A that is a finitely presented R -algebra is coherent. Let $f: A' := R[x_1, \dots, x_n] \twoheadrightarrow A$ be a presentation of A such that $\ker(f) \subset A'$ is a finitely generated ideal. Since $\ker(f) \subset A'$ is finitely generated, it is enough to show that the ring A' is coherent. Letting $I \subset A'$ be a finitely generated ideal, we shall show that I is a finitely presented A' -module. Putting $X = \text{Spec } A'$, $S = \text{Spec } R$ and $\mathcal{M} = \tilde{I}$ in [RG71, Première partie, théorème 3.4.6] (by [BouCA, Chapter I, §2.4, Proposition 3(ii)], [Sta22, Tag 090Q] and the fact that flatness is a local property [Sta22, Tag 0250], the R -torsion-free module I is flat), we obtain that I is a finitely presented A' -module, showing that A' is coherent.

In Theorem 3.6 below, following ideas due to Quillen in [Qui73] and Gillet and Levine in [GL87], we show that the algebraic K -groups of a semilocal ring $R \in \mathcal{S}_{\text{val}}$ (where \mathcal{S}_{val} is the category of rings which are essentially smooth algebras over valuation rings) inject inside the respective algebraic K -groups of the semilocalisation of R at the generic points of the special fibre of $\text{Spec}(R)$. As a consequence, thanks to [KM21], we will demonstrate that the algebraic K -theory functor satisfies Gersten’s injectivity for such a ring R which contain a field. Let us start by defining the functors that play a role in Theorem 3.6.

3.5. G -theory. Given a small exact category \mathcal{C} , let $K(\mathcal{C})$ denote its ‘Quillen K -theory’ spectrum³ defined in [Qui73, §2]. Given a ring R , let $K(R)$, termed its *algebraic K -theory*, denote the Quillen K -theory of the category $\text{Proj}(R)$ of finite type, projective R -modules. On the other hand, given a coherent ring R , let $G(R)$, termed its *G -theory*, denote the Quillen K -theory of the category $\text{Coh}(R)$ of R -finitely presented modules.

By Quillen’s dévissage theorem [Qui73, §5, Theorem 4], it follows that the canonical inclusion $\text{Proj}(R) \hookrightarrow \text{Coh}(R)$ induces an isomorphism

$$K(R) \cong G(R) \text{ when } R \in \mathcal{S}_{\text{val}},$$

i.e., when R is a smooth algebra over a valuation ring. Indeed, for such a ring R , finitely presented R -modules admits a finite length resolution by finite type, projective R -modules. From the displayed isomorphism above, it follows that

G -theory satisfies localisation for \mathcal{S}_{val} (see [AMM22, Proposition 2.5]).

The rationale behind naming this section “Gersten’s Injectivity of G -theory” instead of “Gersten’s Injectivity of K -theory” becomes evident upon examining the proof of Theorem 3.6. It is apparent that $\text{Coh}(R)$, rather than $\text{Proj}(R)$, plays the key role in the proof.

Let \mathcal{S} be the stable ∞ -category of spectra (see [HA, §1.4.3]) endowed with the t -structure as in loc. cit., and let $\pi_i(-)$ be the homotopy groups induced by this t -structure. For each i , we have that $\pi_i(-)$ commutes

³The algebraic K -theory functor was originally defined Quillen [Qui73, §2] as a covariant functor from the category of small exact categories to spaces using the Q -construction. Waldhausen [Wal85, 1.5 and Appendix 1.9] generalised this definition to further incorporate Waldhausen categories and demonstrated that the algebraic K -theory functor actually take values in “infinite loop spaces” (in the sense of Adams [Ada78, §1.4]). Consequently, according to op. cit. §1.7, the algebraic K -theory functor can be regarded as a functor that takes values in spectra (cf. [TT90, 1.5.2 and Theorem 1.11.2]). Blumberg, Gepner, and Tabuada in [BGT13] demonstrated that the source category of the algebraic K -theory functor (called connective algebraic K -theory in op. cit.) can be enlarged to include the ∞ -category of small stable ∞ -categories.

with filtered colimits because it is corepresented by an suspension of the ‘sphere spectrum’ \mathbb{S} , which is a compact object (see [HA, Corollary 1.4.4.6]). We define $K_i(-) := \pi_i \circ K(-)$, for each i .

We are now ready to state and prove the main theorem of this section.

Theorem 3.6. *Given*

- *a valuation ring R ,*
 - *an integral domain \mathcal{A} that is a smooth R -algebra,*
 - *a faithful R -algebra A that is the semilocalisation of \mathcal{A} at finitely many primes, and*
 - *the subset $\mathcal{P} \subset \mathrm{Spec}(A)$ of primes that correspond to generic points of the R -special fibre of $\mathrm{Spec}(A)$,*
- the pullback morphism induces an injection $K_i(A) \hookrightarrow K_i(A_{\mathcal{P}})$, for all i .*

Remarks 3.7.

- (1) Modulo the technicalities, the proof follows the arguments in [GL87], which are, in itself, mixed-characteristic adaptations of Quillen’s ([Qui73, §7, Theorem 5.11]).
- (2) Using the property that the algebraic K -theory commutes with colimits of rings and that it satisfies localisation (in our generality, this is proved in [AMM22, Proposition 2.5]), we reduce to show that $K(-)$ applied to the pushforward along $j: Y \hookrightarrow X$ is the zero morphism, where $X := \mathrm{Spec}(\mathcal{A})$ and Y is any R -relative Cartier divisor. We then use Presentation Lemma 3.2 to further reduce to the case when j has a retraction. Finally, in this case, we utilise [GL87, Lemma 2] (can be thought of as some sort of additivity property) to conclude.
- (3) The statement of [GL87, Lemma 2] is specific to the algebraic K -theory, in particular, it does not hold for an arbitrary localising invariant (in the sense of [BGT13, §1.1]). In order to elaborate, let us provide a sketch of the proof of [GL87, Lemma 2]. Assuming the notations of loc. cit., in order to show that the natural morphism $K(A'') \rightarrow K(B)$ is trivial, it suffices to show that the induced morphism $f: K(A')[1] \rightarrow K(B)$ is itself trivial. We can verify that $[\theta] \in K_1(\mathrm{Hom}(A', B))$ induces f thanks to the following commutative diagram

$$\begin{array}{ccc} K(\mathrm{Hom}(A', B)) \otimes K(A') & \longrightarrow & K(\mathrm{Hom}(A', B) \times A') \\ & \searrow & \downarrow \\ & & K(B). \end{array}$$

However, by the hypothesis of loc. cit., θ is the identity endomorphism of $G|_{A'}$, and as a consequence, $[\theta] = 0$. This shows that $f = 0$, as claimed.

- (4) It is worth noting that the \mathbb{A}^1 -homotopy invariance of the algebraic K -theory, recently proved in the generality of ‘weakly regular stably coherent’ rings (which contains all essentially smooth algebras over valuation rings) in [AMM22, Propostion 2.4], does not seem to play a role in the proof.
- (5) We recall that the R -special fibre of $\mathrm{Spec}(A)$ need not be connected, even when R is a discrete valuation ring and $R \rightarrow \mathcal{A}$ is étale (see Remark 2.14(2)).

Proof of Theorem 3.6. Firstly, we shall reduce to the case when R is of finite Krull dimension. Thanks to Lemma 2.5(b), the ring R can be written as filtered colimit of valuation rings R_α of finite Krull dimension. Since \mathcal{A} is V -smooth, it is V -flat and V -finite type. Consequently, \mathcal{A} is R -finitely presented ([RG71, Première partie, théorème 3.4.6]). Therefore, possibly by reindexing, the smooth algebra $R \rightarrow \mathcal{A}$ descends to a smooth algebra $R_\alpha \rightarrow \mathcal{A}_\alpha$ ([Sta22, Tag 0C0B]), for all α . We suppose that x_1, \dots, x_n are the set of points corresponding to the semilocalisation $\mathcal{A} \rightarrow A$. Let A_α be the semilocalisation of \mathcal{A}_α at the images of the points x_1, \dots, x_n under the morphism $\mathrm{Spec}(\mathcal{A}) \rightarrow \mathrm{Spec}(\mathcal{A}_\alpha)$. Let $\mathfrak{p}_\alpha \subset A_\alpha$ be a prime that corresponds to the generic point of the R_α -special fibre of $\mathrm{Spec}(A_\alpha)$. Since the filtered colimit of A_α (respectively, $(A_\alpha)_{\mathfrak{p}_\alpha}$) is A (respectively, $A_{\mathfrak{p}}$) and $K_i(-)$ commutes with filtered colimits of rings⁴, for all i , it suffices to show that the

⁴We can show that there is an isomorphism of functors $K_i(-) \cong \pi_i \circ K \circ D^{\mathrm{perf}}(-)$. The (connective) algebraic K -theory commutes with filtered colimits of small stable ∞ -categories (see [BGT13, §1.1]). By [Sta22, Tag 09RF], the last functor commutes with filtered colimits of rings.

pullback morphism induces an injection $K_i(A_\alpha) \hookrightarrow K_i((A_\alpha)_{\mathfrak{p}_\alpha})$, for each α and for all i . As a consequence, without loss of generality, we may assume that $R = R_\alpha$. In particular,

we may assume that R is of finite Krull dimension.

Secondly, if \mathcal{A} is of R -relative dimension 0, then it is R -étale. In this case, \mathcal{A} is a semilocal Prüfer domain and $A = A_{\mathcal{P}}$. Thus, the claim is trivial. Therefore, we may assume that \mathcal{A} is of R -relative dimension $d > 0$.

Thanks to Lemma 2.13(2), we have an equality

$$A_{\mathcal{P}} = \operatorname{colim}_{f \in \operatorname{Flat}_{\mathcal{P}}(A/R)} A[\frac{1}{f}].$$

Similar to above, since $K_i(-)$ commutes with colimits of rings, for any i , it suffices to show that $K_i(A) \hookrightarrow K_i(A[\frac{1}{f}])$, for all i and for each $f \in \operatorname{Flat}_{\mathcal{P}}(A/R)$. Since we have a natural fibre sequence $G(A/(f)) \rightarrow K(A) \rightarrow K(A[\frac{1}{f}])$ (see [AMM22, Proposition 2.5]), it suffices to show that the morphism induced by applying $K(-)$ to the pushforward morphism $\operatorname{Coh}(A/(f)) \rightarrow \operatorname{Coh}(A)$ is zero. Let $x_1, \dots, x_n \in \operatorname{Spec}(\mathcal{A})$ be the points corresponding to the semilocalisation $\mathcal{A} \rightarrow A$. Possibly by shrinking $\operatorname{Spec}(\mathcal{A})$ to an affine neighbourhood of x_1, \dots, x_n and by spreading out, we may assume that $f \in \mathcal{A}$, whence the openness of the flat locus [Sta22, Tag 0399] implies that, by further shrinking $\operatorname{Spec}(\mathcal{A})$, we may suppose that $R \rightarrow \mathcal{A}/(f)$ is flat. Since, by assumption, f does not vanish at the generic points of the R -special fibre of $\operatorname{Spec}(\mathcal{A})$, the closed subscheme $i: Y := \operatorname{Spec}(\mathcal{A}/(f)) \hookrightarrow X := \operatorname{Spec} \mathcal{A}$ does not contain any R -fibre of the target scheme ([Sta22, Tag 0D4H]). Thus, applying Presentation Lemma 3.2, at the cost of shrinking X further, we get a smooth R -morphism $\pi: X \rightarrow \mathbb{A}_R^{d-1}$ of relative dimension 1 such that $\pi|_Y$ is quasi-finite. Following the proof of [Qui73, §7, Theorem 5.11], we define $C := Y \times_{\mathbb{A}_R^{d-1}} X$, i.e., we define C via the pullback diagram given below.

$$\begin{array}{ccc} C & \xrightarrow{q} & X \\ \downarrow c & \nearrow i & \downarrow \pi \\ Y & \xrightarrow{\pi|_Y} & \mathbb{A}_R^{d-1} \end{array} \quad \text{with a dashed arrow } s: Y \rightarrow C \text{ from } Y \text{ to } C \text{ along } c \text{ and } i. \quad (3.7.1)$$

The section s , which is induced by i , is a closed immersion as c is affine. However, by definition, q is only quasi-finite, and not finite. This is the price we pay in mixed characteristic, as opposed to Quillen's presentation lemma [Qui73, §7, Lemma 5.12], where $\pi|_Y$ can even be arranged to be finite. To deal with this shortcoming, for the rest of the proof we follow [GL87, §2, proof of Theorem]. Thanks to [SGA 1, corollaire 4.17], the section s of the smooth morphism c is a regular immersion. Consequently, at the cost of shrinking X , we may assume that the ideal of vanishing of $s(Y) \subset C$ is principal, say with a generator g . By Zariski's Main Theorem [Sta22, Tag 00QB], there exists an open immersion $u: C \hookrightarrow \bar{C}$ with an extension of q , i.e., a finite morphism $\bar{q}: \bar{C} \rightarrow X$. We define $\bar{s} := u \circ s$. The diagram is depicted below.

$$\begin{array}{ccccc} & & \bar{C} & & \\ & \nearrow \bar{s} & \uparrow u & \nwarrow \bar{q} & \\ Y & \xrightarrow{c} & C & \xrightarrow{q} & X \\ & \searrow s & \downarrow i & & \end{array} \quad (3.7.2)$$

We observe that \bar{s} is a monomorphism that is also proper. Indeed, as \bar{s} is a morphism from a proper X -scheme to a separated X -scheme, it is proper. Thus, it follows that \bar{s} is a closed immersion ([Sta22, Tag 04XV]). We need to show that $K(-)$ applied to $i_*: \operatorname{Coh}(Y) \rightarrow \operatorname{Coh}(X)$ is the zero functor. Thanks to the fact that both \bar{s} (since it's a closed immersion) and \bar{q} are finite, it follows that each gives rise to a pushforward morphism between categories of coherent modules. Consequently, the equality $i = \bar{q} \circ \bar{s}$ induces an isomorphism of

functors $i_* \cong \bar{q}_* \circ \bar{s}_*$. Therefore, it is enough to show that $K(\bar{s}_*) \cong 0$. On the other hand, the equality $c \circ s = \text{id}_Y$ furnishes the isomorphism $s^* \circ c^* \cong \text{id}_{\text{Coh}(Y)}$. As a consequence, it suffices to show that $K(-)$ applied to $\bar{s}_* \circ s^*: \text{Coh}(C) \rightarrow \text{Coh}(\bar{C})$ is the zero functor. Let $Z := (\bar{C} \setminus C)$ be endowed with a closed subscheme structure such that its ideal \mathcal{I} of vanishing is finitely generated (we prove the existence⁵ of such a closed subscheme structure below in Claim 1). Since $\bar{s}(Y) \subset C$, we have that $\bar{s}(Y) \cap Z = \emptyset$. Thanks to [AMM22, Proposition 2.5]), since \mathcal{I} is finitely generated, there is a natural fibre sequence

$$G(Z) \longrightarrow K(\bar{C}) \xrightarrow{K(u^*)} K(C). \quad (3.7.3)$$

We can further assume that the ideal I of vanishing of $\bar{s}(Y) \subset \bar{C}$ is principal (we also prove this below in Claim 1), say with generator \tilde{g} . Moreover, since $\bar{s}(Y) \cap Z = \emptyset$, by an application of the Chinese remainder theorem, we may assume that $\tilde{g} \equiv 1 \pmod{\mathcal{I}}$ (see Claim 1 for more details). Since $\tilde{g} \in \mathcal{R} := \mathcal{O}_{\bar{C}}(\bar{C})$ is a nonzerodivisor, it induces an isomorphism $\mathcal{R} \cong I$ of \mathcal{R} -modules. The claim now follows by [GL87, Lemma 2]. Indeed, assuming \bar{C} is coherent (we prove this in Claim 2), it follows that the exact sequence $0 \rightarrow I \rightarrow \mathcal{R} \rightarrow \mathcal{A}/(f) \rightarrow 0$ induces a short exact sequence of functors $\text{Coh}(\bar{C}) \rightarrow \text{Coh}(\bar{C})$ given by

$$0 \longrightarrow \text{id}_{\text{Coh}(\bar{C})} \xrightarrow{\tilde{g}} \text{id}_{\text{Coh}(\bar{C})} \longrightarrow F \circ u^* \longrightarrow 0, \quad (3.7.4)$$

where we have invoked the isomorphism $\mathcal{R} \cong I$ to write the left morphism. The diagrams (3.7.3) and (3.7.4) together satisfy the condition of loc. cit., more precisely, we apply loc. cit. to $A' = \text{Coh}(Z)$, $A = \text{Coh}(\bar{C})$, $A'' = \text{Coh}(C)$, $B = \text{Coh}(\bar{C})$, $F = \bar{s}_* \circ s^*$, $G = \text{id}_{\text{Coh}(\bar{C})}$ and $\theta = \tilde{g}$ (this θ satisfies the hypothesis of loc. cit. because $\tilde{g} \equiv 1 \pmod{\mathcal{I}}$) to obtain that $K(\bar{s}_* \circ s^*) = 0$, as required. Thus, we are done modulo the claims made above, which we prove below.

Let us introduce some notations for convenience before proving the first claim. Specialising each x_i to a closed point in X , we may, without loss of generality, assume that each x_i is a closed point. Let $S := \{x_1, \dots, x_n\}$, let $T := q^{-1}(S)$ and let $\bar{T} := \bar{q}^{-1}(S)$. Since $C \subset \bar{C}$, by definition, we have that $T \subset \bar{T}$. On the other hand, as \bar{q} is finite, we conclude that \bar{T} is a finite set.

Claim 1: Without loss of generality, we may assume that both \mathcal{I} and I are principal. Furthermore, we may pick a generator \tilde{g} of I such that $\tilde{g} \equiv 1 \pmod{\mathcal{I}}$.

Proof: Firstly, we establish that \mathcal{I} can be assumed to be principal. We semilocalise \bar{C} at \bar{T} (resp., C at T) to obtain $\mathcal{R}_{\bar{T}}$ (resp., \mathcal{R}_T). Notably, $\text{Spec}(\mathcal{R}_T) \subseteq \text{Spec}(\mathcal{R}_{\bar{T}})$ is an open subscheme. By prime avoidance [Sta22, Tag 00DS], we select an element $z \in \mathcal{I}\mathcal{R}_{\bar{T}}$ such that z does not vanish at y_i , for each $y_i \in T$. Consequently, according to the definition, we have

$$V(z) = Z \cap \text{Spec}(\mathcal{R}_{\bar{T}}).$$

Hence, by replacing $\mathcal{I}\mathcal{R}_{\bar{T}}$ with $z\mathcal{R}_{\bar{T}}$, we may assume that $\mathcal{I}\mathcal{R}_{\bar{T}}$ is principal. To show that \mathcal{I} itself may be assumed to be principal, it is enough to spread out z to a element of \mathcal{R} . By spreading out, there exist an open neighbourhood $U \subseteq \bar{C}$ of \bar{T} and an element

$$\tilde{z} \in \mathcal{O}_{\bar{C}}(U) \text{ such that } V(\tilde{z}) = Z \cap U.$$

We define $W := \bar{q}(\bar{C} \setminus U)$. We observe that $W \subseteq X$ is a closed subset since \bar{q} is finite, and thus, a closed morphism. Then, we select a nonempty, affine open subset $X' \subseteq (X \setminus W)$. Finally, we replace diagram (3.7.2) with its base change along $X' \hookrightarrow X$ (and by abuse of notation, refer to the objects with their respective names) to obtain an element $\tilde{z} \in \mathcal{R}$ whose vanishing locus is Z , as required. This completes the proof.

It remains to show that I can be assumed to be principal with a generator \tilde{g} satisfying $\tilde{g} \equiv 1 \pmod{\mathcal{I}}$. Let $T_1 := \bar{s}(Y) \cap \bar{T}$. The Chinese remainder theorem [Sta22, Tag 00DT] furnishes an element $g \in I\mathcal{R}_{\bar{T}}$ such that $g \equiv 1 \pmod{\mathfrak{m}}$, for each maximal ideal \mathfrak{m} corresponding to a point in $\bar{T} \setminus T_1$. In particular, by definition, we have

$$V(g) = Y \cap \text{Spec}(\mathcal{R}_{\bar{T}}).$$

⁵Since we are working with non-Noetherian rings, it does not automatically follow that \mathcal{I} is finitely generated.

Therefore, by substituting $I\mathcal{R}_{\bar{T}}$ with $g\mathcal{R}_{\bar{T}}$, we may assume that $I\mathcal{R}_{\bar{T}}$ is principal. Moreover, since, by construction, $V(g-1) \supseteq V(\mathcal{I}\mathcal{R}_{\bar{T}})$, possibly by replacing g with a polynomial in g , we may also assume that $g \equiv 1 \pmod{\mathcal{I}\mathcal{R}_{\bar{T}}}$. In a similar vein as above, a spreading out argument yields a nonempty, affine open subset $X' \subseteq X$. Upon base changing diagram (3.7.2) along $X' \hookrightarrow X$ (and by abuse of notation, referring to the objects with their respective names), we obtain an element $\tilde{g} \in \mathcal{R}$ whose vanishing locus is $\bar{s}(Y)$ and which satisfies that $\tilde{g} \equiv 1 \pmod{\mathcal{I}}$, as required. Thus, the claim is proven.

We prove the second claim below.

Claim 2: The scheme \bar{C} is coherent.

Proof: It is enough to show that \bar{C} is R -finitely presented. By [RG71, Première partie, théorème 3.4.6], it suffices to verify that it is R -flat. Since R is a valuation ring, it is equivalent to verify that \bar{C} is R -torsion free. However, by the finer version of Zariski's Main Theorem [Sta22, Tag 00QB], we know that $\mathcal{R} \subseteq \mathcal{O}_C(C)$, where the larger ring is R -torsion free because C is R -flat. Thus, the claim is proven. \square

We are finally ready to establish the promised Gersten's injectivity in the case of K -theory (Theorem 1.2). By applying Theorem 3.6 and employing standard reductions, the proof of the corollary below reduces to [KM21, Theorem 3.1].

Corollary 3.8. *Let R be a semilocal Prüfer domain and let A be an integral domain that is the semilocalisation of a smooth R -algebra at finitely many primes. If R contains a field, then the pullback morphism induces an injection*

$$K_i(A) \hookrightarrow K_i(\text{Frac}(A)), \text{ for all } i. \quad (3.8.1)$$

Proof. Similar to the beginning of the proof of Theorem 3.6, by a limit argument, we may, without loss of generality, assume that R is of finite Krull dimension. Let $F := \text{Frac}(R)$ and let $A_F := A \otimes_R F$. Thanks to [Qui73, §7, Theorem 5.11], the restriction map induces an injection $K_i(A_F) \hookrightarrow K_i(\text{Frac}(A))$, for each i . Consequently, it suffices to show that the restriction map induces an injection $K_i(A) \hookrightarrow K_i(A_F)$, for each i .

As a first case, we suppose that (3.8.1) is true when R is a valuation ring. With this assumption, we shall show that (3.8.1) is true in general. To do so, we induct on

$$d(R) := \sum_{\mathfrak{m} \in \text{MaxSpec}(R)} \dim(R_{\mathfrak{m}}).$$

Let $\mathfrak{m} \subset R$ be a maximal ideal and suppose that $a \in R$ is an element such that $V(a) = \{\mathfrak{m}\}$ (such an a exists thanks to prime avoidance [Sta22, Tag 00DS]). Therefore, by definition, $\text{Spec}(R) \setminus \{\mathfrak{m}\} = \text{Spec}(R[\frac{1}{a}])$. Proceeding inductively, it suffices to show that $K_i(A) \hookrightarrow K_i(A[\frac{1}{a}])$, for each i . Thanks to localisation [AMM22, Proposition 2.5], we obtain a morphism of exact triangles

$$\begin{array}{ccccc} G(A/(aA)) & \longrightarrow & K(A) & \longrightarrow & K(A[\frac{1}{a}]) \\ \downarrow & & \downarrow & & \downarrow \\ G(A_{\mathfrak{m}}/(aA_{\mathfrak{m}})) & \longrightarrow & K(A_{\mathfrak{m}}) & \longrightarrow & K(A_{\mathfrak{m}}[\frac{1}{a}]). \end{array}$$

However, the canonical morphism induces an isomorphism $A/(aA) \cong A_{\mathfrak{m}}/(aA_{\mathfrak{m}})$, demonstrating that the right square is Cartesian. As $R_{\mathfrak{m}}$ is a valuation ring, since we have assumed that (3.8.1) is true when $R = R_{\mathfrak{m}}$ (i.e., a valuation ring), the lower left horizontal morphism is zero. Consequently, it follows that the upper left horizontal morphism is also zero, establishing the required injection. Thus, in this case, the proof is complete.

As a consequence, it remains to establish the claim when R is a valuation ring. Let $\mathcal{P} \subset \text{Spec}(A)$ be the subset of primes that correspond to the generic points of the R -special fibre of $\text{Spec}(A)$. By Theorem 3.6, we have $K_i(A) \hookrightarrow K_i(A_{\mathcal{P}})$, for each i . Since $\text{Frac}(A_{\mathcal{P}}) = \text{Frac}(A)$, without loss of generality, we can assume that $A = A_{\mathcal{P}}$. Moreover, under this assumption, thanks to Lemma 2.13(1), the ring A is a Prüfer domain. Using an induction argument similar to the one above, we reduce the problem to showing the claim when A is a valuation ring. In this case, the required injectivity is the content of [KM21, Theorem 3.1]. \square

Remark 3.9. In effect, the proof of Corollary 3.8 demonstrates that if Gersten’s injectivity holds for all valuation rings, it extends to all smooth algebras over valuation rings. However, the mixed-characteristic counterpart of [KM21, Theorem 3.1] remains largely unexplored. Nevertheless, there are some preliminary results in the case of perfectoid valuation rings (see [AMM22, Theorem 1.2]).

4 Toral Case of Gersten’s Injectivity

In this section, unless otherwise stated, all the cohomology groups that appear are étale. Let X be a ‘non-singular’ scheme and let $j: U \hookrightarrow X$ be an open subscheme whose complement is of depth ≥ 2 . Consider an isotrivial X -torus T . The primary objective of this section is to establish Theorem 4.6. Our key technical tool is a version of purity of torsors under tori (see (4.3.3)), which essentially provides a necessary and sufficient condition for a T -torsor on U to extend globally over X . We prove this purity via an application of a weak version of Auslander–Buchsbaum formula (4.3.1), which demonstrates that j_* preserves reflexive sheaves.

To start, we recall the notion of reflexive sheaves.

4.1. Reflexive sheaves. Let X be a scheme. The dual \mathcal{F}^\vee of an \mathcal{O}_X -module \mathcal{F} is defined to be the \mathcal{O}_X -module $\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$. A coherent \mathcal{O}_X -module \mathcal{F} is called *reflexive* if for every $x \in X$, there is a neighbourhood $x \in U \subset X$ such that the canonical morphism

$$\beta_{\mathcal{F}|_U}: \mathcal{F}|_U \xrightarrow{\sim} \mathcal{F}|_U^{\vee\vee} \quad \text{is an isomorphism.}$$

Given a coherent \mathcal{O}_X -module \mathcal{F} and a presentation

$$\mathcal{O}_X^{\oplus m} \longrightarrow \mathcal{O}_X^{\oplus n} \longrightarrow \mathcal{F} \longrightarrow 0, \quad (4.1.1)$$

we can dualise to obtain a short exact sequence

$$0 \longrightarrow \mathcal{F}^\vee \longrightarrow \mathcal{O}_X^{\oplus n} \longrightarrow \mathcal{O}_X^{\oplus m}. \quad (4.1.2)$$

Therefore, by [Sta22, Tag 01BY], if X is a locally coherent scheme, then for a coherent sheaf (in particular, reflexive) \mathcal{F} , the dual \mathcal{F}^\vee is coherent.

In preparation for Proposition 4.3, we prove the following lemma, drawing inspiration from [GR18, Proposition 11.3.8] and [CS79, Lemma 2.1].

Lemma 4.2. *For a locally coherent scheme X , a quasi-compact open $j: U \hookrightarrow X$ such that at each point $z \in Z := X \setminus U$, we have⁶ $\mathrm{depth}(\mathcal{O}_{X,z}) \geq 2$, and a reflexive \mathcal{O}_X -module \mathcal{F} , the restriction induces an isomorphism*

$$\mathcal{F} \xrightarrow{\sim} j_* j^* \mathcal{F}. \quad (4.2.1)$$

Moreover, if X is reduced, for a reflexive \mathcal{O}_U -module \mathcal{G} ,

$$\text{the pushforward } j_* \mathcal{G} \text{ is a reflexive } \mathcal{O}_X\text{-module.} \quad (4.2.2)$$

Proof. (4.2.1): Thanks to [ČS21, Lemma 7.2.7(b)], the restriction induces an isomorphism

$$\mathcal{O}_X \xrightarrow{\sim} j_* \mathcal{O}_U. \quad (4.2.3)$$

We shall reduce to the special case $\mathcal{F} = \mathcal{O}_X$. Since it is enough to show (4.2.1) locally, given a reflexive \mathcal{O}_X -module \mathcal{F} , we may assume that there is a presentation (4.1.1) of \mathcal{F}^\vee , which can be dualised to obtain a short exact sequence like (4.1.2). Since j_* is left exact, this gives us a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{O}_X^{\oplus n} & \longrightarrow & \mathcal{O}_X^{\oplus m} \\ & & \downarrow & & \downarrow (4.2.3) & & \downarrow (4.2.3) \\ 0 & \longrightarrow & j_* j^* \mathcal{F} & \longrightarrow & \mathcal{O}_X^{\oplus n} & \longrightarrow & \mathcal{O}_X^{\oplus m}, \end{array}$$

⁶A module M over a local ring (A, \mathfrak{m}) has $\mathrm{depth}_A(M) \geq d$, if there is an M -regular sequence $x_1, \dots, x_d \in \mathfrak{m}$; the depth of A is $\mathrm{depth}_A(A)$ (see [EGA IV₁, Chapitre 0, Définition 15.1.7 and §15.2.2]). There is no condition on the quotients being nonzero.

from which we are reduced to the case when $\mathcal{F} = \mathcal{O}_X$, and we are done.

(4.2.2): In view of (4.2.1), it is enough to show that there exists a reflexive \mathcal{O}_X -module \mathcal{F} such that $\mathcal{F}|_U = \mathcal{G}$. By [Sta22, Tag 0G41], there is a finitely presented \mathcal{O}_X -module \mathcal{F}' such that $\mathcal{F}'|_U = \mathcal{G}$. Thanks to [Sta22, Tag 01BZ], keeping in mind that X is locally coherent, the \mathcal{O}_X -module \mathcal{F}' is automatically coherent. Taking $\mathcal{F} = \mathcal{F}'^{\vee\vee}$, we note that $\mathcal{F}|_U = \mathcal{F}'^{\vee\vee}|_U = \mathcal{G}^{\vee\vee} = \mathcal{G}$ (using the fact that \mathcal{G} is reflexive). It remains to check that \mathcal{F} is reflexive, for which we follow the proof of [Sta22, Tag 0AY4]. Since the result is local, it can be assumed that $X = \text{Spec } A$ is affine. Choosing a presentation

$$A^{\oplus m} \rightarrow A^{\oplus n} \rightarrow \Gamma(\text{Spec } A, \mathcal{F}') \rightarrow 0,$$

and dualising it, in order to conclude, it is sufficient to show the following claim.

Claim: Given an exact sequence

$$0 \rightarrow M \rightarrow M' \rightarrow M''$$

of finitely presented A -modules, the module M is reflexive if M' and M'' are reflexive.

Proof: We suppose that M' and M'' are reflexive. Proceeding as in the proof of [Sta22, Tag 0EB8], we shall show that M is reflexive. Double dualising the displayed short exact sequence in the claim and writing down canonical morphisms, we get the following morphism of complexes

$$\begin{array}{ccccc} M & \longrightarrow & M' & \longrightarrow & M'' \\ \downarrow & & \downarrow & & \downarrow \\ \text{Hom}_A(\text{Hom}_A(M, A), A) & \xrightarrow{\alpha} & \text{Hom}_A(\text{Hom}_A(M', A), A) & \longrightarrow & \text{Hom}_A(\text{Hom}_A(M'', A), A). \end{array}$$

By the assumption, the middle and the right vertical arrows are isomorphisms. We need to show that the left vertical arrow is an isomorphism. It suffices to show that α is injective. We consider module Q defined by the exact sequence $\text{Hom}_A(M', A) \rightarrow \text{Hom}_A(M, A) \rightarrow Q \rightarrow 0$. Letting K be the total ring of fractions of A (see [Sta22, Tag 00EW]), by the finite presentation property [Sta22, Tag 0583], tensoring the exact sequence with K , we obtain the exact sequence

$$\text{Hom}_K(M' \otimes_A K, K) \rightarrow \text{Hom}_K(M \otimes_A K, K) \rightarrow Q \otimes_A K \rightarrow 0.$$

However, since K is a product of fields, the injection $M \otimes_A K \hookrightarrow M' \otimes_A K$ is split, consequently, $Q \otimes_A K = 0$, implying that Q is a torsion A -module. In that case, $\text{Hom}_A(Q, A) = 0$, ensuring that α is injective. \square

Given a flat (resp., smooth) group scheme G over a scheme S and an S -scheme S' , we let $\mathbf{BG}(S')$ denote the category of fppf locally (resp., étale locally) trivial G -torsors on S' . Likewise, let \mathbf{BG} be the presheaf on the category of S -schemes defined by $S' \mapsto \mathbf{BG}(S')$ (see [Sta22, Tag 0048]).

We are now ready to prove our version of the Auslander–Buchsbaum formula (4.3.1). As a consequence, we derive a categorical variant of the purity of torsors under tori (4.3.3).

Proposition 4.3. *Let R be a Prüfer domain, let X be a smooth, integral R -scheme and let $j: U \hookrightarrow X$ be a quasi-compact open such that at each point $x \in Z := X \setminus U$ with $f(x) = y$, we have $\dim(\mathcal{O}_{X_y, x}) + \min(1, \dim(R_y)) \geq 2$. Then, for a locally free \mathcal{O}_U -module \mathcal{L} of rank 1,*

$$\text{the pushforward } j_*\mathcal{L} \text{ is a locally free } \mathcal{O}_X\text{-module of rank 1,} \quad (4.3.1)$$

in particular, for any étale X -scheme X' , the restriction induces an equivalence of categories

$$\mathbf{BG}_m(X') \xrightarrow{\sim} \mathbf{BG}_m(X' \times_X U). \quad (4.3.2)$$

More generally, for an X -torus T and for any étale X -scheme X' , the restriction induces an equivalence of categories

$$\mathbf{BT}(X') \xrightarrow{\sim} \mathbf{BT}(X' \times_X U), \quad (4.3.3)$$

in particular,

$$H^q(X', T) \cong H^q(X' \times_X U, T), \quad \text{for } q \leq 1. \quad (4.3.4)$$

Remark 4.4. The seemingly unusual inequality $\dim(\mathcal{O}_{X_y, x}) + \min(1, \dim(R_y)) \geq 2$ serves a specific purpose: ensuring that $\text{depth}(\mathcal{O}_{X, x}) \geq 2$ at each point $x \in Z$. Relatedly, Prüfer domains are far from being Cohen–Macaulay. In effect, as previously noted in §2, by prime avoidance [Sta22, Tag 00DS], for any valuation ring V of finite Krull dimension, there exists an element $a \in V$ so that $V(a) = \{\mathfrak{m}\}$, where $\mathfrak{m} \subset V$ is the maximal ideal.

Proof of Proposition 4.3. We show that $\text{depth}(\mathcal{O}_{X, z}) \geq 2$, at any point $z \in Z$. Let $f: X \rightarrow \text{Spec } R$. Thanks to [EGA IV₃, Théorème 11.3.8] (specifically, (c) \implies (a)), it suffices to argue that $\text{depth}(\mathcal{O}_{X_{f(z)}, z}) + \min(1, \dim(R_{f(z)})) \geq 2$. It follows from the hypothesis and the equality $\text{depth}(\mathcal{O}_{X_{f(z)}, z}) = \dim(\mathcal{O}_{X_{f(z)}, z})$, which is true because f is smooth.

We show the key claim, i.e., (4.3.1), below. The claim (4.3.2) is a consequence of (4.2.1) and (4.3.1). Since the torus T trivialises étale locally on X , the claims (4.3.3) and (4.3.4) reduce to (4.3.2) by an étale descent argument.

(4.3.1): We follow the proof of [GR18, Proposition 11.4.1(iv)]. A complex of sheaves on X of abelian groups that is concentrated in cohomological degree 0 with the 0-th term \mathcal{A} shall be denoted by $\mathcal{A}[0]$. Thanks to (4.2.2), the pushforward $\mathcal{M} := j_*\mathcal{L}$ is a reflexive \mathcal{O}_X -module, and also, torsion-free and hence, flat over R . Assuming that $\mathcal{M}[0]$ is a perfect \mathcal{O}_X -complex, the result follows. Indeed, letting $\det(\mathcal{M}[0])$ be the determinant line bundle of $\mathcal{M}[0]$ (see [KM76, Theorem 1]), there is a sequence of isomorphisms

$$\mathcal{M} \xrightarrow{\sim} j_*\mathcal{L} \xrightarrow{\sim} j_*\det(\mathcal{L}[0]) \xrightarrow{\sim} j_*j^*(\det(\mathcal{M}[0])) \xleftarrow[\sim]{(4.2.1)} \det(\mathcal{M}[0]),$$

from which the result follows. We verify that $\mathcal{M}[0]$ is a perfect \mathcal{O}_X -complex. It suffices to assume that $X = \text{Spec } A$ is affine and to show that $\mathcal{M}[0]$ is quasi-isomorphic to a bounded complex of finite free A -modules ([Sta22, Tag 0BCJ]). Let $M := \Gamma(A, \mathcal{M})$. By [Sta22, Tag 0G9A], it suffices to show that

$$\text{Ext}_A^q(M, N) = 0 \text{ for any finitely presented } A\text{-module } N \text{ and any } q \gg 0 \quad (4.4.1)$$

(the complex $M[0]$ is pseudo-coherent because M is a coherent A -module). Using the finitely presented property of the variable N in (4.4.1), it is enough to show that there exists an integer n such that $\text{proj.dim}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}) \leq n$ for any prime $\mathfrak{p} \subset A$ (see [Wei94, Lemma 3.3.8]). This follows from [GR18, Proposition 11.4.1(ii)] (or [Guo22, Lemma 7.2(i)]), which shows that taking $n = \dim_R(A)$ suffices. Hence, we are done.

(4.3.2): Let $j': U' := X' \times_X U \hookrightarrow X$ be the inclusion. Since $\dim_X(X') = 0$, we obtain that (X', U') satisfies the hypothesis of Proposition 4.3. We shall show that the pushforward $j'_*: \mathbf{BG}_m(U') \rightarrow \mathbf{BG}_m(X')$, which is well defined as a consequence of (4.3.1), is an inverse to the pullback $j'^*: \mathbf{BG}_m(X') \rightarrow \mathbf{BG}_m(U')$. Since the equality $j'^*j'_* = \text{id}$ follows from the definition, it suffices to show that for a line bundle $\mathcal{L} \in \mathbf{BG}_m(X')$, the restriction induces an isomorphism $\mathcal{L} \xrightarrow{\sim} j'_*j'^*\mathcal{L}$. However, this results from (4.2.1).

(4.3.3): Let $U' := X' \times_X U$. Thanks to [Sta22, Tag 04UK], $\mathbf{BT}_{X'}$ satisfies étale descent, consequently, the same holds for the presheaf $j_*\mathbf{BT}_{U'}$. By [SGA 3_{II}, Exposé X, Corollaire 4.5], there exists an étale surjection $\tilde{X} \rightarrow X'$ that splits T . In view of the étale descent property of $\mathbf{BT}_{X'}$ and $j_*\mathbf{BT}_{U'}$, it suffices to show that for any étale \tilde{X} -scheme X'' , the restriction induces an equivalence of categories $\mathbf{BT}_{\tilde{X}}(X'') \xrightarrow{\sim} \mathbf{BT}_{\tilde{X}}(X'' \times_{X'} U')$, which is the content of (4.3.2).

(4.3.4) This follows from (4.3.3). Indeed, $H^0(X', T)$ (resp., $H^0(U', T)$) is the automorphism group of the trivial torsor in $\mathbf{BT}(X')$ (resp., in $\mathbf{BT}(U')$) and $H^1(X', T)$ (resp., $H^1(U', T)$) is the group of isomorphism classes of objects in $\mathbf{BT}(X')$ (resp., in $\mathbf{BT}(U')$). \square

With Proposition 4.3 proven, to state our main theorem, we now need to recall the notion of flasque tori.

4.5. Flasque Torus and Flasque Resolution. We recall some definitions from [CS87, §0.5]. Let G be a finite group. A finitely generated, free \mathbb{Z} -module \mathcal{P} with a linear action of G is called a *permutation module* if \mathcal{P} admits a G -stable \mathbb{Z} -basis. A finitely generated, free \mathbb{Z} -module \mathcal{F} with a linear action of G is called *flasque* if

$$\text{Ext}_{\mathbb{Z}[G]}^1(\mathcal{F}, \mathcal{P}) = 0 \quad \text{or, equivalently,} \quad H^1(G, \text{Hom}_{\mathbb{Z}}(\mathcal{F}, \mathcal{P})) = 0,$$

Let X be a scheme. An X -torus T is called *isotrivial* if it is split by a finite étale surjection $\tilde{X} \rightarrow X$. The *character group* of an X -torus T is the sheaf of abelian groups $T^\vee := \underline{\mathrm{Hom}}_{X\text{-gps}}(T, \mathbb{G}_{m,X})$. An isotrivial X -torus T is called *quasi-trivial* (resp., *flasque*) if for any connected component $Z \subset X$, there exists a connected, Galois, finite étale cover $\tilde{Z} \rightarrow Z$ that splits T such that the induced $\mathbb{Z}[\mathrm{Gal}(\tilde{Z}/Z)]$ -module $T^\vee(\tilde{Z})$ is a permutation module (resp., is flasque) (see [CS87, Definition 1.2]). In fact, by op. cit. Lemma 1.1, any connected, Galois, finite étale cover $\tilde{Z} \rightarrow Z$ that splits T can be chosen in the previous definition, and the choice of the closed subscheme structure on $Z \subset X$ is irrelevant. As we might expect, the notions of flasque and quasi-trivial tori are preserved under base change (cf., op. cit. Proposition 1.3). For a connected scheme X , any quasi-trivial torus Q can be written as a finite product of Weil restrictions $\mathrm{Res}_{X_i/X}(\mathbb{G}_{m,X_i})$, for finite étale covers $X_i \rightarrow X$ (see [Čes22_{Surv}, Lemma A.2.6]).

$$1 \rightarrow F \rightarrow Q \rightarrow T \rightarrow 1, \text{ where } F \text{ is a flasque and } Q \text{ is a quasi-trivial } X\text{-torus, and} \quad (4.5.1)$$

We are now prepared to prove our main theorem. Before stating the theorem, let us outline the steps of its proof. To show Gersten's injectivity for a flasque A -torus F , standard reductions show that it is enough to establish vanishing

of the local cohomology at a point $z \in \operatorname{Spec}(A)$. If z is of codimension 1 or if it is the generic point of an R -fibre of $\operatorname{Spec}(A)$, Lemma 2.13 shows that A_z is a valuation ring. Therefore, in this case, (4.5.3) follows from the results in [Guo22, §2]. Otherwise, our analysis shows that it follows from Proposition 4.3.

- (i) the morphism $H^1(A, F) \rightarrow H^1(U, F)$ is surjective, and
- (ii) the morphism $H^2(A, F) \rightarrow H^2(U, F)$ is injective.

Again, thanks to Lemma 2.5(b), by a limit argument (see [Sta22, Tag 09YQ]), we may assume that R has finite Krull dimension. Letting $Z := \operatorname{Spec} A \setminus U$, an analysis of the long exact sequence of cohomology with supports

indicates that it suffices to establish that $H_Z^2(A, F) = 0$. On the other hand, since the R -fibres of $\mathrm{Spec}(A)$ are Noetherian, the topological space $\mathrm{Spec}(A)$ itself is Noetherian. Therefore, thanks to the coniveau spectral

sequence associated to the filtration by supports inside closed subschemes [ILO14, Exposé XVIII-A, §2.2.1] (see also [Gro68b, Section 10.1])

$$E_1^{p,q}: \bigoplus_{z \in Z \text{ with } \dim A_z = p} H_{\{z\}}^{p+q}(A_z, F) \Rightarrow H_Z^{p+q}(A, F), \quad (4.6.1)$$

we are reduced to establishing that

$$H_{\{z\}}^q(A_z, F) = 0, \text{ for each } z \in Z \text{ and for all } q \leq 2. \quad (4.6.2)$$

We first examine the following two cases, which readily follow from results in the literature.

- If z is a generic point of an R -fibre, then Lemma 2.13 shows that A_z is a valuation ring. Indeed, letting x be the image of z along $\text{Spec } A \rightarrow \text{Spec } R$, the image of z in $\text{Spec } A_x$ is a generic point of the special fibre over the valuation ring R_x . In this case, to prove (4.6.2), we again write a long exact sequence of cohomology with supports. Ultimately, it follows from [Guo22, Lemma 2.3, Proposition 2.4 and Corollary 2.5].
- If z is a point of codimension 1 in the generic fibre, then A_z is an equicharacteristic discrete valuation ring. In a similar vein as above, vanishing (4.6.2) follows.

Consequently, without loss of generality, we may assume that z is neither a generic point of an R -fibre of $\text{Spec}(A)$ nor it is a point of codimension 1 in the R -generic fibre. This assumption is important for our eventual application of Proposition 4.3. Thanks to the local-to-global spectral sequence ([SGA 4_{II}, Exposé V, Proposition 6.5])

$$E_2^{p,q}: H^p(A_z, \mathcal{H}_{\{z\}}^q(A_z, F)) \Rightarrow H_{\{z\}}^{p+q}(A_z, F),$$

it is enough to demonstrate that $\mathcal{H}_{\{z\}}^q(A_z, F) = 0$, for all $q \leq 2$. Letting $A_{\bar{z}}$ be the strict Henselisation of A_z and by taking stalks, it is equivalent to show that $H_{\{\bar{z}\}}^q(A_{\bar{z}}, F) = 0$, for all $q \leq 2$. An inspection of the long exact sequence of cohomology with supports

$$\begin{array}{ccccccc} 0 & \longrightarrow & H_{\{\bar{z}\}}^0(A_{\bar{z}}, F) & \longrightarrow & H^0(A_{\bar{z}}, F) & \longrightarrow & H^0(\text{Spec } A_{\bar{z}} \setminus \{\bar{z}\}, F) \\ & & & & & & \searrow \\ & & & & & & H_{\{\bar{z}\}}^1(A_{\bar{z}}, F) \longrightarrow H^1(A_{\bar{z}}, F) \longrightarrow H^1(\text{Spec } A_{\bar{z}} \setminus \{\bar{z}\}, F) \\ & & & & & & \searrow \\ & & & & & & H_{\{\bar{z}\}}^2(A_{\bar{z}}, F) \longrightarrow H^2(A_{\bar{z}}, F) \longrightarrow \dots \end{array}$$

and the vanishing of the positive degrees of étale cohomology of strictly Henselian rings [Sta22, Tag 03QO] proves that

$$H_{\{\bar{z}\}}^2(A_{\bar{z}}, F) \cong H^1(\text{Spec } A_{\bar{z}} \setminus \{\bar{z}\}, F). \quad (4.6.3)$$

In conclusion, it is left to show that $H^q(\text{Spec } A_{\bar{z}}, F) \cong H^q(\text{Spec } A_{\bar{z}} \setminus \{\bar{z}\}, F)$, for $q = 0$ and $q = 1$. Since étale cohomology commutes with cofiltered limits of schemes, by the definition of strict Henselisation, it remains to apply Proposition 4.3 to $X = \text{Spec } A$, $Z = \{\bar{z}\}$ and $T = F$ (since the topological space $\text{Spec } A$ is Noetherian, any open subset of $\text{Spec } A$ is quasi-compact). \square

Finally, we are ready to establish Gersten's injectivity in the case of $H^1(-, T)$ as well as $H^2(-, T)$, for a torus T over a smooth algebra over a semilocal Prüfer domain. In other words, we prove a generalisation of Theorem 1.3 below. By employing flasque resolutions of isotrivial tori, we derive this result as a consequence of Theorem 4.6.

Corollary 4.7. *For a semilocal Prüfer domain R , a semilocal, integral domain A that is R -essentially smooth and an A -torus T ,*

(i) the morphism $H^1(A, T) \hookrightarrow H^1(K, T)$ is injective, and

(ii) the morphism $H^2(A, T) \hookrightarrow H^2(K, T)$ is injective.

Proof. (i): Letting $1 \rightarrow F \rightarrow Q \rightarrow T \rightarrow 1$ be a flasque resolution (4.5.1), we get the following morphism of long exact sequences

$$\begin{array}{ccccccc} \cdots & \longrightarrow & H^1(A, Q) & \longrightarrow & H^1(A, T) & \longrightarrow & H^2(A, F) \longrightarrow \cdots \\ & & \downarrow & & \downarrow \theta_1^T & & \downarrow \theta_2^F \\ \cdots & \longrightarrow & H^1(K, Q) & \longrightarrow & H^1(K, T) & \longrightarrow & H^2(K, F) \longrightarrow \cdots \end{array} \quad (4.7.1)$$

Since Q is quasi-trivial, there are connected finite étale covers $A \rightarrow A_i$ such that $Q \cong \prod \text{Res}_{A_i/A}(\mathbb{G}_m)$. By the fact that higher direct images vanish along finite morphisms [Sta22, Tag 03QP], the cohomology rewrites itself as $H^1(A, Q) \cong \prod H^1(A_i, \mathbb{G}_m)$, and since all the rings A_i are semilocal (since an étale morphism is quasi-finite), the cohomology vanishes thanks to the Hilbert theorem 90 [Mil80, Chapter III, Section 4, Proposition 4.9]. By a similar argument, the cohomology $H^1(K, Q)$ vanishes, and in view of (4.7.1), to prove (i), it is enough to show that θ_2^F is injective. By using the fact that the étale cohomology commutes with filtered colimits of rings, this is a consequence of Theorem 4.6.

(ii): This time, letting $1 \rightarrow T \rightarrow F' \rightarrow Q' \rightarrow 1$ be a different flasque resolution (4.5.2), we get the following morphism of long exact sequences

$$\begin{array}{ccccccc} \cdots & \longrightarrow & H^1(A, Q') & \longrightarrow & H^2(A, T) & \longrightarrow & H^2(A, F') \longrightarrow \cdots \\ & & \downarrow & & \downarrow \theta_2^T & & \downarrow \theta_2^{F'} \\ \cdots & \longrightarrow & H^1(K, Q') & \longrightarrow & H^1(K, T) & \longrightarrow & H^1(K, F') \longrightarrow \cdots \end{array}$$

In a similar vein as above, since Q' is quasi-trivial, it follows that $H^1(A, Q') = H^1(K, Q') = 0$. Consequently, a similar argument as above shows that $\theta_2^{F'}$ is injective. Therefore, θ_2^T is injective, and we are done. \square

Remarkably, for a flasque torus F , the \mathbb{A}^1 -invariance of $H^1(-, F)$ follows as a direct consequence of Corollary 4.7. This is encapsulated in the corollary below. This conclusion readily follows from the Noetherian case studied in [CS87, Lemma 2.4].

Corollary 4.8. *For a semilocal Prüfer domain R , a semilocal, integral domain A that is R -essentially smooth, a flasque A -torus and a nonempty, quasi-compact open $U \subset \mathbb{A}_A^n$, the composite morphism*

$$H^1(A, F) \rightarrow H^1(\mathbb{A}_A^n, F) \rightarrow H^1(U, F) \quad \text{is surjective.}$$

Proof. Thanks to (i), the second morphism is surjective. Consequently, it reduces us to proving the surjectivity of the first morphism. In fact, we shall show that

$$H^1(A, F) \xrightarrow{\sim} H^1(\mathbb{A}_A^n, F).$$

Assuming that A is normal, by a limit argument, it is enough to show the displayed isomorphism for Noetherian, normal domains, which is the content of loc. cit. Therefore, it remains to show that A is normal. This follows from [Sta22, Tag 00GY, Tag 030A] and [SGA 1, Exposé I, théorème 9.5(i)] (cf. [Mil80, Chapter I, Proposition 3.17(b)]). \square

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