δ-Rings

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1 Motivation

For this section we roughly follow the set of notes [Bor]. Fix a prime p. The letter R shall always denote a commutative ring with identity.

Definition 1.1. A ring homomophism $\phi\colon R \to R$ is called a Frobenius lift if descends to $\bar{\phi}\colon R/p \to R/p$, i.e. for every $x \in R$ there is a $x' \in R$ such that $\phi(x) = x^p + px'$.

Examples 1.2.

1. $R = \mathbb{Z}$ and $\phi(x) = x$ for every $x \in Z$.
2. $R = \mathbb{Z}[x]$ and $\phi(x) = x^p + pf(x)$ where f is any polynomial.

We want to study the category of rings with Frobenius lifts, i.e. the category FrobLifts with objects with objects as the pairs (R, ϕ) of rings with a Frobenius lift and morphisms $f : (R, \phi_R) \to (S, \phi_S)$ which commute the following diagram

$$\begin{array}{ccc} R & \xrightarrow{\phi_R} & R \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} \\ S & \xrightarrow{\phi_S} & S. \end{array} \tag{1.1}$$

But we have the porblem that that in this category we do not have existence of limits, for example we do not have equalisers. Therefore to solve this problem we can introduce the $x' = \frac{\phi(x) - x^p}{p}$ in the definition.

Definition 1.3 (p-derivation)**.** A set function $\delta : R \to R$ is said to be a p-derivation if the following holds

- $\delta(0) = \delta(1) = 0$,
- $\delta(x + y) = \delta(x) + \delta(y) + \frac{x^p + y^p - (x+y)^p}{p}$,
- $\delta(xy) = x^p\delta(y) + y^p\delta(x) + p\delta(x)\delta(y)$.

A delta ring is a pair (R, δ) of a ring R with a p-derivation δ.

Exercise 1.4.

1. Given an $\mathbb{F}_p$-algebra R it admits a p-derivation if and only if $R = \{0\}$.
2. Same for $\mathbb{Z}/p^n\mathbb{Z}$-algebra R.

Remark 1.5. Given a p-derivation δ we can define a lift of Frobenius by $\phi(x) = x^p + p\delta(x)$. The converse is true for p-torsion free rings.

Given the category of delta rings δ-Rings with objects as δ-rings (R, δ) with morphisms $f\colon (R, \delta_R) \to (S, \delta_S)$ which commute a square like 1.1. We have a forgetful functor $F\colon \delta{-}Rings \to$ Rings given by $(R, \delta) \mapsto R$. This functor has both left and right adjoints

$$\begin{tikzcd} \delta\text{-Rings} \arrow[d, "F"] \\ \text{Rings} \arrow[u, bend left, start anchor=west, end anchor=west] \arrow[u, bend right, start anchor=east, end anchor=east, "W"'] \end{tikzcd} \tag{1.2}$$

1.1 Rings with derivations

Definition 1.6. A derivation $d\colon R \to R$ is an additive map such that $d(xy) = xd(y) + yd(x)$.

We have a category diff-Rings of differential rings with objects (R, d) where R is a ring with a derivation d and morphisms $(R, d_R) \to (S, d_S)$ that commute a square like 1.1. Then we have a forgetful functor with two adjoints.

$$\begin{tikzcd} \text{diff-Rings} \arrow[d, "F"] \\ \text{Rings.} \arrow[u, bend left, start anchor=west, end anchor=west] \arrow[u, bend right, start anchor=east, end anchor=east, "W^{\text{diff}}"'] \end{tikzcd} \tag{1.3}$$

Where $W^{\text{diff}}(A) = \{\sum a_n \frac{t^n}{n!} | a_n \in A\}$ is the divided power algebra over A. This ring has a natural differential structure given by $d\left(\frac{x^n}{n!}\right) = \frac{x^{n-1}}{(n-1)!}$ for $n \geq 1$. There is a natural projection map of rings $\pi\colon W^{\text{diff}}(A) \to A$ given by $\sum a_n \frac{t^n}{n!} \mapsto a_0$. Then from the right adjointness we have the following universal property of $W^{\text{diff}}(A)$: given a differential ring (R, d_R) and a morphism of rings $g\colon R \to A$ there is a unique morphism of differential rings $\tilde{g}\colon (R, d_R) \to W^{\text{diff}}(A)$

$$\begin{tikzcd} R \arrow[rr, dashed, "\tilde{g}"] \arrow[dr, "g"'] & & W^{\text{diff}}(A) \arrow[dl, "\sum a_n \frac{t^n}{n!} \mapsto a_0"] \\ & A & \end{tikzcd} \tag{1.4}$$

We can think of $W^{\text{diff}}(A)$ as an infinite product $A^{\mathbb{N}}$ with each $\sum a_n \frac{t^n}{n!}$ being represented by $(a_0, \ldots)$. The differential operator can be thought of as the shifting operator $d(a_0, \ldots) = (a_1, \ldots)$. In this way the product can be represented by $(a_0, \ldots)(b_0, \ldots) = (a_0 b_0, \ldots)$. Then the n-th term is $\sum_i \binom{n}{i} a_i b_{n-i} + b_i a_{n-1}$. **Compare Leibnitz rule!**

Similarly we define the Witt vectors to be $W(A) = A^{\mathbb{N}}$ with the ring structure given the Leibniz rules with respect to $+$ and $*$, that is

$$(a_0, a_1, \ldots) + (b_0, b_1, \ldots) = (a_0 + b_0, a_1 + b_1 - \sum \frac{1}{p}\binom{p}{i} a_0^i b_0^{p-i}, \ldots) \tag{1.5}$$

$$(a_0, a_1, \ldots) * (b_0, b_1, \ldots) = (a_0 b_0, a_0^p b_1 + a_1 b_0^p + p a_1 b_1, \ldots)$$

. There is a natural projection map of rings $\pi \colon W(A) \to A$ given by $(a_0, a_1, \ldots) \mapsto a_0$. Then from the right adjointness we have the following universal property of $W(A)$: given a δ-ring (R, δ_R) and a morphism of rings $g \colon R \to A$ there is a unique morphism of $\delta-$rings $\tilde{g} \colon (R, \delta_R) \to W(A)$

$$\begin{tikzcd} R \arrow[rr, dashed, "\tilde{g}"] \arrow[dr, "g"'] & & W(A) \arrow[dl, "(a_0,a_1,\ldots)\mapsto a_0"] \\ & A & \end{tikzcd} \tag{1.6}$$

Theorem 1.7 (Theorem II.6.8 [Ser79])**.** *Let $A/\mathbb{F}_p$ be a perfect algebra. Then $W(A)$ is a p-adically complete p-torsion free ring.*

Remark 1.8. If we take the Frobenius associated to the δ-map, then it is an isomorphism. This follows from the universal property of the Witt vectors.

Theorem 1.9 (Corollary and Proposition II.5.10[Ser79])**.** *Let A, A' be two p-adically complete p-torsion free rings with perfect residue rings. Then given any homomorphism $\phi : A/p \to A'/p$ there exists a unique homomorphism $g : A \to A'$ making the diagram commute*

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d] & A' \arrow[d] \\ A/p \arrow[r, "f"] & A'/p. \end{tikzcd} \tag{1.7}$$

In particular any two such rings with isomorphic residue rings are isormorphic.

2 δ-Rings

In this section we roughly follow the chapter 2 of the paper [BS19]. A friendlier reference is the set of notes [Bha].

Examples 2.1.

1. The ring $\mathbb{Z}$ has a unique δ structure $\delta(x) = \frac{x-x^p}{p}$. This is the initial object in the category of δ-Rings.

2. Let $A/\mathbb{F}_p$ be a perfect algebra. Then $W(A)$ has a δ-strucrture induced by the Frobenius lift. 1.5.

3. There is a unique δ-structure on $\mathbb{Z}[x]/[x^p, px]$ with $\delta(x) = 0$.

4. (Free δ-ring) Let (A, δ) be a delta ring. Then the infinite polynomial algebra $A[x_0, x_1, \ldots]$ has a δ-structure $\delta(x_i) = x_{i+1}$ for $i \geq 0$ extending the one on A. This rings shall be denoted as $A_\delta\{x\}$ or simply $A\{x\}$.

We notice a calculation $\delta(xy) = (y^p\delta(x)+p\delta(y)\delta(x))+x^p\delta(y) = \phi(y)\delta(x)+x^p\delta(y)$.

Lemma 2.2. *If $u \in \mathbb{Z}^{\times}{}_{(p)}$ and $m \geq 1$ then $\delta(p^m u) = p^{m-1}v$ for some $v \in \mathbb{Z}_{(p)}^{\times}$.*

Proof. When $u = 1$ then $\delta(p^m) = \frac{\phi(p^m)-p^{mp}}{p} = p^{m-1}(1-p^{(m-1)p}) = p^{m-1}w$ for some $w \in \mathbb{Z}_{(p)}^{\times}$. This is true because $\mathbb{Z}$ is the initial object and $\phi(x) = x$ in $\mathbb{Z}$. Now in the general case $\delta(p^m u) = \phi(p^m)\delta(u) + u^p\delta(p^m) = p^m\delta(u) + u^p p^{m-1}w = p^{m-1}(u^p w + p\delta(u)) = p^{m-1}w$. But here $v = u^p w + p\delta(u)$ is a unit because u and w are units and $p \in \operatorname{rad}\mathbb{Z}_{(p)}$ the Jacobson radical of $\mathbb{Z}_{(p)}$. □

Definition 2.3. An element $x \in A$ has rank 1 is $\delta(x) = 0$ i.e. $\phi(x) = x^p$.

Lemma 2.4. *The category of δ-rings gas all limits and colimits. The forgetful functor F commutes with these limits and colimits.*

Proof. Since it has both adjoints..... □

Example 2.5. Using the fact that the category has all colimits we can produce new δ-rings with the ones that we know. Consider the pushout diagram

$$\begin{tikzcd}
\mathbb{Z}\{z\} \arrow[r] \arrow[d] & \mathbb{Z}\{x,y\} \arrow[d] \\
\mathbb{Z} \arrow[r] & \mathbb{Z}\{x,y\}/(x^3+y^3+xy)_\delta
\end{tikzcd} \tag{2.1}$$

where the upper horizontal arrow $\mathbb{Z}\{z\} \to \mathbb{Z}\{x,y\}$ is the map $z \mapsto x^3+y^3+xy$ of δ-rings where $\mathbb{Z}\{z\}$ is a free δ-ring with the variable z and $\mathbb{Z}\{x,y\}$ is a free δ-ring with two variables x, y. The left vertical map $\mathbb{Z}\{z\} \to \mathbb{Z}$ is the map of δ-rings $z \mapsto 0$. The colimit $\mathbb{Z}\{x,y\}/(x^3+y^3+xy)_\delta$ is a quotient of the ring $\mathbb{Z}\{x,y\}$ along the ideal generated by the δ-iterates of x^3+y^3+xy.

From this point on we fix a δ-ring (A, δ).

Lemma 2.6 (Localisation, [BS19] Lemma 2.15)**.** *Let $S \subset A$ be a multiplicative set such that $\phi(S) \subset S$. Then there is a unique δ-structure on $S^{-1}A$ such that the natural map $A \to S^{-1}A$ is a map of δ-rings.*

Lemma 2.7 (Quotient, [BS19] Lemma 2.9)**.** *Let $I \subset A$ be an ideal such that $\delta(I) \subset I$. Then A/I admits a unique structure of a δ-ring such that the natural quotient map $A \to A/I$ is a map of δ-rings.*

Lemma 2.8 (Completion, [BS19] Lemma 2.17)**.** *Let $p \in I \subset A$ be a finitely generated ideal. Then $\delta\colon A \to A$ is an I-adically continuous map. Moreover, the classical I-adic completion of A along I admits a unique δ-structure such that the natural map $A \to \hat{A}$ is a map of δ-rings.*

2.1 Perfect δ-rings

Definition 2.9. A δ-ring A is called perfect if the Frobenius lift ϕ is an isomophism of rings.

Remark 2.10. The inclusion functor $i\colon$ Perfect δ-Rings $\to$ δ-Rings admits both adjoints. The left adjoint is given by the functor $R \mapsto R_{\mathrm{Perf}} := \operatorname{colim}_{\phi} R$ and the right adjoint is given by $R \mapsto R^{\mathrm{Perf}} := \lim_\phi R$.

Lemma 2.11 (p-torsion in δ-rings, [BS19] Lemma 2.28)**.** *The ring A is p-torsion free is one of the following holds*

1. *$\phi\colon A \to A$ is an injection,*
2. *A is reduced.*

Example 2.12.

1. 1 holds but 2 does not hold: $A = \mathbb{Z}_p[x]/(x^2)$ where $\phi(x) = px$. Then we may check that $\phi(a+bx) = a+bpx$ and therefore it is injective but A is not reduced.
2. 1 does not hold but 2 holds: $A = \mathbb{Z}_p[x]/(x^p-1)$ where $\phi(x) = x^p = 1$. Then we may check that $\phi(x-1) = 0$ and therefore it is not injective but A is reduced.

Proposition 2.13. *We define the following categories.*

1. *Perf p-complete δ-Rings be the full sub-category of δ-Rings with objects (R, δ) where R is a perfect p-adically complete δ-ring, and*
2. *Perf p-complete Rings be the full sub-category of Rings with objects R where R is a p-torsion free and p-adically complete ring with R/p perfect, and*
3. *Perf the category of perfect $\mathbb{F}_p$-algebras.*

Then they are equivalent, more precisely we have a diagram

$$\begin{tikzcd}
\mathit{Perf\ p\text{-}complete\ \delta\text{-}Rings} \arrow[rr, "\sim", "Forgetful"'] & & \mathit{Perf\ p\text{-}complete\ Rings} \arrow[dl, "\sim", "A\mapsto A/p"'] \\
& \mathit{Perf} \arrow[ul, "\sim"', "R\mapsto W(R)"] &
\end{tikzcd} \tag{2.2}$$

The forgetful functor is well defined because perfect δ-rings are p-torsion free due to 2.11.

Proof. 1 $\iff$ 3 We check from the universal property of Witt vectors that given a perfect $\mathbb{F}_p$-algebra A the ring of Witt vectors $W(A)$ is a perfect δ-ring. The Witt vector functor is fully faithful due to 1.9 and the universal property of Witt vectors, namely we show that the induced map g in the theorem is a map of δ-rings. It is essentially surjective because of the same theorem. Hence these two categories are equivalent. 2 $\iff$ 3 Follows the same line of argument except that in this case it is easier. Therefore all the categories are equivalent. $\square$

Lemma 2.14 ([BS19] Lemma 2.32)**.** *Let $x \in A$. Then $\delta(x^{p^n}) \in p^n A$ for all $n \geq 0$. In particular if A is p-adically separated and y admits p^n-th power roots for all $n \geq 0$ then $\delta(y) = 0$, i.e. y has rank* 1.

2.2 Relation to divided power algebra

We assume in the section that A is a p-torsion free δ-ring. In this case we have a natural inclusion $A \in A\left[\frac{1}{p}\right]$. Then we can naturally identify elements $\gamma_n(x) = x^n/n!$ inside $A\left[\frac{1}{p}\right]$, where $\gamma_n(x)$ is the n-th divided power of $x \in A$.

Lemma 2.15 ([BS19] Lemma 2.35)**.** *Fix $z \in A$ with $\gamma_p(z) \in A$. Then $\gamma_n(z) \in A$ for all $n \geq 0$.*

Lemma 2.16 ([BS19] Lemma 2.36)**.** *Define $C = Z_{(p)}\{x, \frac{\phi(x)}{p}\}$ identifies with the pd envelope $D := D_{\mathbb{Z}_{(p)}\{x\}}(x)$ of the ring $\mathbb{Z}_{(p)}\{x\}$ along the ideal generated by x.*

2.3 Distinguished Elements

Definition 2.17. An element $d \in A$ is called a distinguished element if $\delta(d) \in A^\times$.

We can check easily that a morphism of δ-rings preserves distinguished elements.

Examples 2.18.

(Crystalline Cohomology) $A = \mathbb{Z}_p$. Here $d = p$ is a distinguished element.

(q-de Rham Cohomology) $A = \mathbb{Z}_p[q]$. Here $d = \frac{q^p-1}{q-1}$ is a distinguished element. This follows because $\delta(q) = q^p$.

Lemma 2.19 ([BS19] Lemma 2.23)**.** *Fix $d \in A$ distinguished element and $u \in A^\times$. If $d, p \in \mathrm{rad}(A)$ then ud is distinguished.*

Lemma 2.20 ([BS19] Lemma 2.24)**.** *Fix $d \in A$ such that $d, p \in \mathrm{rad}(A)$. Then d is a distinguished element if and only if $p \in (d, \phi(d))$. In particular 'd is distinguished' depends only on the ideal (d).*

Lemma 2.21 ([BS19] Lemma 2.25)**.** *Let A be a perfect p-complete δ-ring and $d \in A$. Then d is distinguished if and only if the coefficient of p in the Teichmüller expansion of d is a unit.*

[Bha] [Bor] [BS19]
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References

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