Arbeitsgemeinshaft: Grothendieck–Serre Conjecture

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Venue:	TBA University of Toronto, St. George Campus
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1. THE CONJECTURE

The subject of this working group is the study of torsors under reductive group schemes. One of the central problems in this subject is the conjecture of Grothendieck and Serre stated below. It appeared in the Chevalley seminar papers of Serre in [Ser58, page 31, remarque] and Grothendieck in [Gro58, pages 26–27, remarques 3]. We present the statement of the conjecture below.

Conjecture 1.1 (Grothendieck–Serre)**.** *For a regular local ring R with a fraction field K and a reductive R-group scheme G, a generically trivial G-torsor E on* $\operatorname{Spec} R$ *is trivial, i.e.,*

$$\ker(H^1(R,G) \to H^1(K,G)) = \{*\}.$$

Equivalently, given a Noetherian, regular scheme X and a reductive X-group scheme G, a generically trivial G-torsor E on X is Zariski locally trivial.

The useful references for learning about this conjecture is [Čes22] and [$\text{Čes22}_{\text{Surv}}$, §3.1]. We shall follow these references during Talks 5-8. The thesis [Kun23] could also serve as a complementary reference, and, in fact, this introduction is based on the same of op. cit.

To get a feeling for Conjecture 1.1, we could test the hypothesis by replacing G by certain reductive groups for which the torsors have been well studied. The best examples of reductive groups whose torsors are thoroughly studied in the literature are the groups GL_n and SL_n. Torsors under GL_n (resp., SL_n) correspond to vector bundles of rank n (resp., vector bundles of rank n whose determinant is trivial). We test out the hypothesis of Conjecture 1.1 below.

1.2. The GL_n-case : In the case when $G = \mathrm{GL}_n$ or when $G = \mathrm{SL}_n$, the Hilbert theorem 90 yields that on local rings $H^1(R,G) = H^1(K,G) = \{*\}$. Indeed, GL_n-torsors (resp., SL_n-torsors) over $\operatorname{Spec} R$ correspond to locally free sheaves of rank n (resp., locally free sheaves of rank n whose determinant is trivial) on $\operatorname{Spec} R$. Since R is local, locally free sheaves are free, whence the claim follows. We shall study this case in Lecture 2.

Another example of a reductive group whose torsors are well studied in the literature is PGL_n, whose torsors were studied by Grothendieck in [Gro68a, §I.1] (see also [CS21, §3.1])) for their connection with the Brauer group. Since PGL_n is the automorphism group of the matrix algebra Mat_n, its torsors correspond to forms of Mat_n, i.e., Azumaya algebras of rank n. By taking the long exact sequence of cohomology associated to the short exact sequence $1 \to \mathbb{G}_m \to \mathrm{GL}_n \to \mathrm{PGL}_n \to 1$ and using the fact that $H^1(R, \mathrm{GL}_n) = \{*\}$, Grothendieck showed that

$$H^1(R, \mathrm{PGL}_n) \hookrightarrow H^2(R, \mathbb{G}_m). \tag{1.1.1}$$

The case when $G = \mathrm{PGL}_n$ was one of the main motivations for Conjecture 1.1 (see [Gro68b, remarques 1.11 a]).

There is a rich history surrounding the proofs of several instances of Conjecture 1.1, starting from the early work of Nisnevich in the 70's. For more details on the known cases, the reader is invited to read [Česa22$_{\mathrm{Surv}}$, §3.1.4] or [Kun23, Chapter 2].

Known Cases of Conjecture 1.1

(1) The case when R is a discrete valuation ring was proved by Nisnevich in his thesis [Nis82]. The idea is to use Harder-type approximation results to reduce the proof of Conjecture 1.1 over R to its counterpart over the completion $\hat{R}$. Subsequently, Bruhat–Tits theory is exploited to settle the conjecture when R is a complete discrete valuation ring. We shall discuss this case in Talks 3-4.

(2) The case when G is a torus was proved by Colliot-Thélène and Sansuc in [CS87].

(3) The equicharacteristic case, i.e., when R contains a field, was settled by Fedorov and Panin in [FP15]. Combining the ideas underlying Artin's results on good neighbourhoods from [SGA 4$_{\mathrm{III}}$, Exposé XI] with Voevodsky's "standard triples" from [MVW06, Definition 11.5], they define the notion of "nice triples" (which are smooth relative curves over R equipped with a section and an R-finite closed subscheme). This enables them to pass via Nisnevich-type gluing to the study of torsors over the relative affine line $\mathbb{A}^1_R$. They conclude via Horrocks-style results (see Lectures 4-6) to show that such torsors pull-back to the trivial one. Our goal is to give a proof of Panin–Fedorov result. However, we shall follow the strategy of Česnavičius in [Čes22], [Čes22$_{\mathrm{Surv}}$] and [Čes$_{\mathrm{Preprint}}$].

(4) In mixed characteristic, the case when R is unramified (a local ring R with a maximal ideal $\mathfrak{m} \subset R$ is *unramified* if it contains a field or if $\mathrm{char}(R/\mathfrak{m}) \notin \mathfrak{m}^2$) and G has a Borel R-subgroup was settled by Česnavičius in [Čes22]. Streamlining Panin–Fedorov's strategy, he replaced Artin's good neighbourhoods by a presentation lemma over discrete valuation rings in the style of Gabber (see Lecture 7). As mentioned above, we shall follow Česnavičius' strategy to prove Conjecture 1.1. Extra mileage from his method was used to prove the quasi-split case of Conjecture 1.1 in the case of smooth algebras over valuation rings of rank 1 (see [Kun23]).

There are numerous applications of Conjecture 1.1 that are found in the literature. For an account, the reader is invited to have a look at [Čes22], [Čes22$_{\mathrm{Surv}}$, §3.1.4] and [Kun23, Chapter 2].

2. DESCRIPTION OF THE SEMINAR TALKS

Each talk is presented with a title which indicates its purpose or the takeaway. It is followed by a list of topics that the speaker is expected to cover. The talk marked $(*)$ requires you to be able to read Mathematical articles written in French.

Lecture 1. The conjecture of Grothendieck and Serre. State Conjecture 1.1 and discuss its known cases. Mention some of its applications that are found in the literature from [Kun23, Chpater 2]. Give a brief description of the talks that will follow.

Lecture 2. Hilbert's Theorem 90. Discuss the notion of sheaves on a Grothendieck site from [Mil80, Chapter II, §1] (cf. [Mil13, Chapters 5-6]). Discuss examples of sheaves on the étale site. Explain that representable presheaves are sheaves in this site. Discuss [Mil80, Examples 2.18]. Discuss the Leray spectral sequence [Mil80, Chapter III, Theorem 1.18(a)]. Discuss the notion of the first Čech cohomology set and the notion of torsors from [Mil80, Chapter III, §4] (cf. [Mil13, Chapter 11]). State the fact that short exact sequences of groups induce long exact sequences of Čech cohomology sets [Mil80, Chapter III, Proposition 4.5]. Show that the torsors are classified by their Čech cocycles [Mil80, Chapter III, Proposition 4.6]. Prove Hilbert's Theorem 90 [Mil80, Chapter III, Proposition 4.9].

Lecture 3. Torsors on Dedekind schemes. State [Guo20, Theorem 3.1] and mention a few words about its proof. State and prove [Guo20, Proposition 4.5]. State and prove [Guo20, Proposition 4.6]. State and prove [Guo20, Proposition 4.7]. State and start the proof of [Guo20, Theorem 5.1].

Lecture 4. Torsors on the Relative Affine Line over a Field$(*)$. Discuss the relevant material on Bruhat–Tits buildings from [$\mathrm{BT}_{\mathrm{II}}$, §4.6] (also, briefly discuss the concepts on which they are founded). Hence, finish the proof of [Guo20, Theorem 5.1]. State [Gil02, théorème 3.4]. State [Gil02, théorème 3.5] and prove the surjectivity of the isomorphism discussed in loc. cit. State and prove [Gil02, corollaire 3.10(a)].

Lecture 5. Torsors on the Relative Affine Line over a Local Ring : Totally Isotropic Groups. State [$\text{Čes22}_{\text{Surv}}$, Conjecture 3.5.1]. Prove the conjecture in the case when G is semi-simple and simply connected following [$\text{Čes22}_{\text{Surv}}$, §3.5]. State [$\text{Čes22}_{\text{Surv}}$, Lemma 3.5.3] and sketch its proof from [Čes22, Lemma 8.3]. State and prove [$\text{Čes22}_{\text{Surv}}$, Lemma 3.5.4]. State and give a sketch of [$\text{Čes22}_{\text{Surv}}$, Lemma 3.5.5]. Hence, conclude a proof of [$\text{Čes22}_{\text{Surv}}$, Conjecture 3.5.1] in the semi-simple and simply connected case.

Lecture 6. Torsors on the Relative Affine Line over a Local Ring : General Case. State and give a sketch of [$\text{Čes22}_{\text{Surv}}$, Lemma 5.3.5]. State [$\text{Čes22}_{\text{Surv}}$, Proposition 5.3.6] and give a list of techniques from the analysis of the geometry of the affine Grassmannian [$\text{Čes22}_{\text{Surv}}$, §5.3] that are required in its proof. Discuss the geometric properties of the affine Grassmannian from [$\text{Čes22}_{\text{Surv}}$, §5.3] that are relevant to the proof of [$\text{Čes22}_{\text{Surv}}$, Proposition 5.3.6]. State the general case of [$\text{Čes}_{\text{Preprint}}$, Theorem 4.5] and give a sketch of its proof in the special case when $C = \mathbb{A}^1_A$ and $B = A$.

Lecture 7. The Presentation Lemma. State [Čes22, Lemma 3.1] and [Čes22, Lemma 3.2] in detail. Introduce weighted projective spaces and weighted blowups. Prove the presentation lemma [Čes22, Proposition 3.6] in the case when k is perfect. If time permits, mention a few words about the case when k is imperfect. This is a key step in the proof of the Grothendieck–Serre conjecture. There has been other version of the presentation lemma, for instance, see [GS88].

Lecture 8. The Simplified Proof of Conjecture 1.1 in the equal-characteristic. Finish the proof of Conjecture 1.1.

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