

TILTING CORRESPONDENCES OF PERFECTOID RINGS

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Abstract

In this article, we present an alternate proof of a vanishing result of étale cohomology on perfectoid rings due to Česnavičius and more recently proved by a different approach by Bhatt and Scholze. To establish that, we prove a tilting equivalence of étale cohomology of perfectoid rings taking values in commutative, finite étale group schemes. On the way, we algebraically establish an analogue of the tilting correspondences of Scholze, between the category of finite étale schemes over a perfectoid ring and that over its tilt, without using tools from almost ring theory or adic spaces.

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1 Introduction

Our goal in this article is to simplify the proof of the following result of Česnavičius. This theorem plays a central role in the proof of the purity of the Brauer group, notably by demonstrating that the p -primary torsion part of the Brauer group $H_{\text{ét}}^2(A[\frac{1}{p}], \mathbb{G}_m)[p^\infty]$ for a perfectoid ring A vanishes.

Theorem 1.1 ([Čes19, Thm. 4.10], cf. [BS19, Thm. 11.1], see Corollary 3.8). *Let p be a prime and let A be a \mathbb{Z}_p -algebra such that it is a perfectoid ring with an element $\varpi \in A$ such that $\varpi^p \mid p$ and that A is ϖ -adically complete. Then, for a commutative, finite étale $A[\frac{1}{\varpi}]$ -group scheme G of p -power order, we have, for all $i \geq 2$,*

$$H_{\text{ét}}^i(A[\frac{1}{\varpi}], G) = 0.$$

We follow the definition of a perfectoid ring from [BMS18] (see Definition 3.1). We remark that $H_{\text{ét}}^1(A[\frac{1}{\varpi}], G)$ can be computed using the prismatic Dieudonné module of G (see [ČS21, Thm. 4.1.8]). The statement of Theorem 1.1 is a mild generalisation of [Čes19, Thm. 4.10]. Indeed, the beginning of §2 implies that we can find $\pi \in A$ such that A is π -adically complete and that $\pi^p = p$; consequently, $A[\frac{1}{\pi}] = A[\frac{1}{p}]$ and for all $i \geq 2$,

$$H_{\text{ét}}^i(A[\frac{1}{p}], G) = 0.$$

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This slight improvement is possible since our results do not depend on the almost purity theorem of [KL15]. The proof given in [Čes19] uses a non-noetherian version of a result of Huber from [Hub96] to compare the étale cohomology of $A[\frac{1}{p}]$ with coefficients in G with the étale cohomology of the associated adic space $\mathrm{Spa}(A[\frac{1}{p}], A)$ with coefficients in the group associated to G , thus transferring the problem to the world of perfectoid spaces studied by Scholze in [Sch12] and Kedlaya and Liu in [KL15]. In this article, we propose a proof of Theorem 1.1 by using a similar strategy inspired from the work of Česnavičius and Scholze in [ČS21], by replacing the almost purity theorem with ‘algebraic’ tilting results from op. cit. By bootstrapping from the arguments in [ČS21, Thm. 2.2.7], we deduce a non-constant coefficient version of the tilting result of étale cohomology of perfectoid rings (see Theorem 1.3). The details of the proof of Theorem 1.1 using this non-constant coefficient version of the tilting result are given in Corollary 3.8.

Theorem 1.1 is obtained as a consequence of the isomorphism (Theorem 1.3) between the étale cohomology group appearing in Theorem 1.1 and the étale cohomology group of the corresponding perfect \mathbb{F}_p -algebra ‘tilt’ taking values in a commutative, finite étale group scheme G^\flat of p -power order. Indeed, we may apply the vanishing of the étale cohomology of the \mathbb{F}_p -algebra. More precisely, by [SGA4 III, Ex. X Thm. 5.1], the p -cohomological dimension of an affine noetherian \mathbb{F}_p -algebra is ≤ 1 , and consequently, by limit arguments, the étale cohomology of degree $i \geq 2$ of any commutative, finite étale group scheme of p -power order over an \mathbb{F}_p -algebra vanishes. The existence of the ‘tilt’ G^\flat of G will be shown by applying Theorem 1.2, which can be seen as an algebraic analogue of tilting results on perfectoid Banach K -algebras over a perfectoid field K as in [Sch12] or that on perfectoid Banach \mathbb{Q}_p -algebras as in [KL15].

Theorem 1.2. *Let A/\mathbb{Z}_p be a perfectoid ring with an element $\varpi \in A$ such that $\varpi^p \mid p$ and that A is ϖ -adically complete, and let A^\flat be its tilt and $\varpi^\flat \in A^\flat$ be such that $\varpi^{\flat\sharp}$ is a unit multiple of ϖ in A . Then, there is an equivalence, functorial in A , between the categories of finite étale algebras*

$$\mathbf{f\acute{e}t}/A[\frac{1}{\varpi}] \cong \mathbf{f\acute{e}t}/A^\flat[\frac{1}{\varpi^\flat}]. \quad (*)$$

For a ring R , the notion of a (co-commutative) Hopf R -algebra is dual to the notion of a (commutative) affine R -group scheme, i.e., they are R -algebras so that their spectra have the structure of (commutative) affine R -group schemes. Starting with a commutative, finite étale group $A[\frac{1}{\varpi}]$ -scheme G , we observe that, by $(*)$, its coordinate ring $\mathcal{O}(G)$, which is a co-commutative, finite étale Hopf $A[\frac{1}{\varpi}]$ -algebra, has a tilt $\mathcal{O}(G)^\flat$, which a priori is a finite étale $A^\flat[\frac{1}{\varpi^\flat}]$ -algebra. However, the structure of a (co-commutative) Hopf algebra, expressible in terms of diagrams involving $A[\frac{1}{\varpi}]$, $\mathcal{O}(G)$ and $\mathcal{O}(G) \otimes_A \mathcal{O}(G)$, gets transferred by $(*)$ to $\mathcal{O}(G)^\flat$, whence we get the required commutative, finite étale $A^\flat[\frac{1}{\varpi^\flat}]$ -group scheme $G^\flat := \mathrm{Spec} \mathcal{O}(G)^\flat$, as shown in the following theorem.

Theorem 1.3 (see Theorem 3.6). *We assume notations of Theorem 1.2. For commutative group schemes $G \in \mathbf{f\acute{e}t}/A[\frac{1}{\varpi}]$ and $G^\flat \in \mathbf{f\acute{e}t}/A^\flat[\frac{1}{\varpi^\flat}]$ that are identified under the tilting correspondence $(*)$, we have an identification, functorial in A and G ,*

$$R\Gamma_{\acute{\mathrm{e}t}}(A[\frac{1}{\varpi}], G) \cong R\Gamma_{\acute{\mathrm{e}t}}(A^\flat[\frac{1}{\varpi^\flat}], G^\flat). \quad (**)$$

We have the following corollary to Theorem 1.2.

Corollary 1.4. *Let R be a perfectoid Banach K -algebra as in [Sch12, Defn. 5.1], where K is a perfectoid field (resp. a perfectoid Banach \mathbb{Q}_p -algebra as in [KL15, Defn. 3.6.1]). Choose an element $\varpi \in R^\circ$ such that $\varpi^p \mid p$ and that R° is ϖ -adically complete, and choose $\varpi^\flat \in R^{\circ\flat}$ such that $(\varpi^{\flat\sharp}) = (\varpi)$. Then, there is a functorial in R equivalence of categories,*

$$\mathbf{f\acute{e}t}/R^\circ[\frac{1}{\varpi}] \cong \mathbf{f\acute{e}t}/R^{\circ\flat}[\frac{1}{\varpi^\flat}].$$

A key ingredient in the proof of Theorem 1.3 is the theory of the arc topology (Definition 2.5) developed by Bhatt and Mathew in [BM21]. As shown by them, the cohomology

of an étale sheaf of torsion abelian groups satisfies (hyper)descent in the arc topology (see Definition 2.3 for the definition of hyperdescent and Proposition 2.7 for a precise statement). Moreover, they proved that, given an étale sheaf \mathcal{G} of torsion abelian groups, the functor $R' \mapsto \mathrm{R}\Gamma_{\mathrm{\acute{e}t}}(\hat{R}'[\frac{1}{\varpi}], \mathcal{G})$ satisfies ϖ -complete arc hyperdescent (the ϖ -complete arc topology defined in Definition 2.9 is a slight refinement of the arc topology and is better suited to the study of perfectoid rings). This result (Proposition 2.12) and the fact that any perfectoid ring has a ϖ -complete arc hypercover given by a special class of perfectoid rings (see Lemma 3.3) reduces the proof to showing the equivalence for this special class of perfectoid rings, where it is possible to give a direct proof. It must be remarked, however, that except for the aforementioned class of special perfectoid rings, we do not know any direct morphism that establishes an identification (**). This failure is related to the fact that the ‘tilting functor’ (cf. Proposition 3.2), which localises in the analytic topology in the adic case, does not localise in the Zariski topology in the algebraic case.

Notations and Conventions

We fix a prime integer p . The term *rank* shall denote the Krull dimension of a valuation ring. Given a ring A with an element $\varpi \in A$ and an $n \geq 1$, we define $A\langle\varpi^n\rangle$ to be the kernel of multiplication by ϖ^n map in A ; these kernels form an increasing system with union $A\langle\varpi^\infty\rangle$. Given a ring R , the category of *schemes over R* will be denoted by Sch_R and its subcategory of *quasi-compact and quasi-separated R -schemes* will be denoted by $\mathrm{Sch}_R^{\mathrm{qcqs}}$. Given a simplicial object X_\bullet we shall denote the n -th component by X_n and the n -truncation of the object by $X_{\leq n}$. For any category \mathcal{C} and an object $X \in \mathcal{C}$, the *slice category over X* will be denoted by \mathcal{C}/X . The 2-category of small 1-categories will be denoted Cat .

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2 ϖ -complete Arc (hyper)sheaves $\mathrm{f\acute{e}t}$ and $\mathrm{R}\Gamma_{\mathrm{\acute{e}t}}(-, \mathcal{G})$

Our goal of this section is to deduce Proposition 2.12 from Proposition 2.7 (following [BM21]). It is essential in this article to use ∞ -categorical tools, and therefore, we refer the reader to [HTT].

Definition 2.1 ([Sta20, Tag 049J]). For a ring R and an ∞ -category \mathcal{C} with all colimits, we say that a functor $\mathcal{F}: \mathrm{Sch}_R^{\mathrm{op}} \rightarrow \mathcal{C}$ is *locally of finite presentation* (‘finitary’ in [BM21]), if whenever $\{S_\alpha, f_{\alpha\beta}\}_{\alpha, \beta \in I}$ is an inverse system of quasi-compact and quasi-separated R -schemes indexed by a cofiltered partially ordered set I , such that the transition map $f_{\alpha\beta}: S_\alpha \rightarrow S_\beta$ is affine for each $\alpha, \beta \in I$, the map $\mathrm{colim}_\alpha (\mathcal{F}(S_\alpha)) \xrightarrow{\sim} \mathcal{F}(\mathrm{lim}_\alpha S_\alpha)$ is an equivalence in \mathcal{C} .

Definition 2.2 ([Sta20, Tag 01G5], [HTT, Defn. 6.5.3.2]). Given a category \mathcal{X} with finite limits, an element $X \in \mathcal{X}$, and a Grothendieck topology τ on \mathcal{X} , a simplicial object X_\bullet in \mathcal{X}/X is a *hypercover* of X if

1. the morphism $X_0 \rightarrow X$ is a τ -covering, and

2. for every $n \geq 1$, the morphism $X_n \rightarrow (\text{cosk}_n(\text{sk}_{n-1}(X_\bullet)))_n$ is a τ -cover.

For example, given a τ -cover $Y \rightarrow X$, we have the Čech nerve hypercover

$$(\cdots \rightrightarrows Y \times_X Y \rightrightarrows Y) \text{ of } X.$$

Definition 2.3 (cf. [HTT, §6.5.2]). Given a ring R , a Grothendieck topology τ on Sch_R , and an ∞ -category \mathcal{C} with all limits, a functor $\mathcal{F}: \text{Sch}_R^{\text{op}} \rightarrow \mathcal{C}$ is said to be a τ -sheaf or to satisfy τ -descent, if \mathcal{F} carries arbitrary coproducts of schemes to products in \mathcal{C} and for every τ -cover $Y \rightarrow X$, the map

$$\mathcal{F}(X) \xrightarrow{\sim} \lim(\mathcal{F}(Y) \rightrightarrows \mathcal{F}(Y \times_X Y) \rightrightarrows \cdots) \text{ is an equivalence.}$$

A τ -sheaf \mathcal{F} is said to be a τ -hypersheaf or to satisfy τ -hyperdescent, if for every hypercover $X_\bullet := (\cdots \rightrightarrows X_1 \rightrightarrows X_0)$ of X in the τ -topology, the map

$$\mathcal{F}(X) \xrightarrow{\sim} \lim(\mathcal{F}(X_0) \rightrightarrows \mathcal{F}(X_1) \rightrightarrows \cdots) \text{ is an equivalence.}$$

Given an ∞ -category \mathcal{C} and two objects $C, D \in \mathcal{C}$, [HTT, Defn. 1.2.2.1] associates a Kan complex $\text{Map}_{\mathcal{C}}(C, D)$, called the ‘mapping space’ from C to D . This is a generalisation of the set of morphisms from C to D in a 1-category, obtained by admitting homotopies between morphisms. For $n \geq -1$, an object $C \in \mathcal{C}$ is said to be n -truncated if $\text{Map}_{\mathcal{C}}(D, C)$ is n -truncated, for all objects $D \in \mathcal{C}$ (see [HTT, Defn. 5.5.6.1]), i.e., if for all $k > n$ the homotopy groups $\pi_k(\text{Map}_{\mathcal{C}}(D, C))$ vanish; dually C is n -cotruncated (resp., cotruncated) if it is n -truncated (resp., m -cotruncated for some $m \geq -1$) in \mathcal{C}^{op} . A cocomplete ∞ -category \mathcal{C} is said to be *generated under colimits by cotruncated objects* if any compact object in \mathcal{C} is cotruncated and any object $C \in \mathcal{C}$ can be written as a colimit of compact objects (cf. [EHIK21, Defn. 3.1.4]). This automatically guarantees that the class of compact objects forms a set. Examples include ‘nice’ cocomplete n -categories (in which every object is n -truncated), where ‘nice’ refers to some set theoretic finiteness condition to make the category small enough, and the derived ∞ -category $\mathcal{D}(\mathbb{Z})^{\geq 0}$ of bounded below by 0 complexes of abelian groups (in which the perfect complexes form the set cotruncated objects by [HA, Warning 1.2.1.9] and [Sta20, Tag 07VI]; also cf. [BM21, Eg. 3.6(1)]).

Proposition 2.4 ([EHIK21, Lem. 3.1.7]). *For a ring R , a Grothendieck topology τ on Sch_R , and an ∞ -category \mathcal{C} generated under colimits by cotruncated objects, any sheaf $\mathcal{F}: \text{Sch}_R^{\text{op}} \rightarrow \mathcal{C}$ is automatically a hypersheaf.*

Following notations of op. cit., we denote the ∞ -category of Kan complexes as \mathcal{S} and for each $n \geq -1$, its full ∞ -subcategory of n -truncated Kan complexes as $\mathcal{S}_{\leq n}$. In loc. cit. the source of \mathcal{F} is assumed to be an ‘ ∞ -topos’, however, the same proof works in the case of an ordinary Grothendieck site, which is sufficient for our purposes. We now present a sketch of the proof of Proposition 2.4 following arguments from op. cit. and [HTT].

Proof. The property of a sheaf (or a hypersheaf) can be tested by applying the Yoneda embedding; whence, since \mathcal{C} is generated under colimits by cotruncated objects, \mathcal{F} has the property if and only if for every cotruncated object $C \in \mathcal{C}$, the functor $\text{Map}_{\mathcal{C}}(C, \mathcal{F}(-)): \text{Sch}_R^{\text{op}} \rightarrow \mathcal{S}_{\leq n}$ has the same property. The latter is a truncated object in the category of sheaves on Sch_R taking values in \mathcal{S} , and therefore, by [HTT, Lem. 6.5.2.9], it is a hypersheaf. \square

We have inherently used [HTT, Cor. 6.5.3.13], which shows that the notion of a hypersheaf coincides with the notion of ‘hypercompleteness’ (see [HTT, §6.5.2] for the definition).

The following topology was defined by Bhatt and Mathew in [BM21], where they proved that the étale cohomology satisfies arc descent (see Proposition 2.7).

Definition 2.5 ([BM21, Defn. 1.2]). A morphism $X' \rightarrow X$ of schemes is an *arc cover* if for every rank ≤ 1 valuation ring V and every morphism $\text{Spec } V \rightarrow X$, there are a faithfully flat

extension $V \rightarrow V'$ of valuation rings and a morphism $\mathrm{Spec} V' \rightarrow X'$ that lifts the composition $\mathrm{Spec} V' \rightarrow \mathrm{Spec} V \rightarrow X$ to a commutative square

$$\begin{array}{ccc} \mathrm{Spec} V' & \longrightarrow & X' \\ \downarrow & & \downarrow \\ \mathrm{Spec} V & \longrightarrow & X. \end{array} \quad (2A)$$

If we remove the rank ≤ 1 hypothesis from Definition 2.5 and consider any valuation ring V , we obtain the v -topology of [BM21, Defn. 1.1] or the ‘universally subtrusive topology’ of [Ryd10, Defn. 2.2]. The ‘universally submersive topology’ of [Ryd10, pg. 187] is contained in between the arc and the v -topology ([HS21]). In the proof of the following we shall need to use a result [Ryd10, Thm. 3.12 and Rem. 3.13] of Rydh which says that any finitely presented v -cover of schemes has a refinement given by a composition of a Zariski open covering and a proper surjective morphism of finite presentation, and therefore, a Zariski sheaf locally of finite presentation that takes values in an ∞ -category generated under colimits by cotruncated objects and which satisfies descent for proper surjective morphisms of finite presentation is a v -sheaf.

Remark 2.6. Although the arc topology (Definition 2.5) is defined for all schemes, for all the rest we shall specialise to quasi-compact and quasi-separated schemes.

Proposition 2.7 ([BM21, Thm. 5.6(2), Thm. 5.4]). *Given a ring R , the functor \mathbf{fet} that associates to a quasi-compact and quasi-separated R -scheme X the category of finite étale X -schemes satisfies arc hyperdescent; moreover, the functor given by $X \mapsto \mathrm{R}\Gamma_{\mathrm{\acute{e}t}}(X, \mathcal{G})$ satisfies arc hyperdescent.*

We need the following lemma to prove the above proposition.

Lemma 2.8 ([Sta20, Tag 09ZL], [Gab94, Thm. 1]). *Given a ring A with an ideal $I \subset A$ such that A is I -henselian, there is an equivalence $\mathbf{fet}/A \xrightarrow{\sim} \mathbf{fet}/(A/I)$; moreover, given an étale sheaf \mathcal{G} on A of torsion abelian groups, the morphism $\mathrm{R}\Gamma_{\mathrm{\acute{e}t}}(A, \mathcal{G}) \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{\acute{e}t}}(A/I, \mathcal{G})$ is an isomorphism*

Proof of Proposition 2.7. This proof is the same as the proofs in [BM21]. We shall prove that \mathbf{fet} satisfies arc descent, since then it automatically satisfies arc hyperdescent by Proposition 2.4. Indeed, \mathcal{C} can taken to be the essentially small 2-category generated under colimits by the categories of finite étale X -schemes, where X is a finite type R -scheme.

By [BM21, Thm. 4.1], since \mathbf{fet} is locally of finite presentation, it is enough to prove that \mathbf{fet} is a v -sheaf, and that for any valuation ring with an algebraically closed fraction field and any prime ideal $\mathfrak{p} \subset V$, the square

$$\begin{array}{ccc} \mathbf{fet}/V & \longrightarrow & \mathbf{fet}/(V/\mathfrak{p}) \\ \downarrow & & \downarrow \\ \mathbf{fet}/V_{\mathfrak{p}} & \longrightarrow & \mathbf{fet}/\kappa(\mathfrak{p}) \end{array} \quad (2B)$$

is cartesian. Indeed, (2B) is cartesian because each ring among V/\mathfrak{p} , $V_{\mathfrak{p}}$ and $\kappa(\mathfrak{p})$ is a valuation ring with an algebraically closed fraction field, implying that each is a strictly henselian local ring. Since $V \rightarrow V/\mathfrak{p}$ is a morphism between henselian local rings with the same residue field, the top horizontal morphism is an identification by Lemma 2.8, and similarly, the bottom horizontal morphism is an isomorphism.

Thus, it reduces to prove that \mathbf{fet} is a v -sheaf: by [Ryd10, Thm. 3.12 and Rem. 3.13], it suffices to show that it satisfies descent for proper surjective morphisms of finite presentation, which follows from [SGA1, Exp. IX Thm. 4.12].

The proof for the second functor has the same structure as the proof for the first. The proper base change theorem [SGA4 III, Exp. XII Thm. 5.1] and [BM21, Lem. 5.1] show that the functor satisfies descent for proper surjective morphisms of finite presentation (see the proof of [BM21, Thm. 5.4] for details), and hence, by [Ryd10, Thm. 3.12 and Rem. 3.13],

it satisfies v -descent. The square analogous to (2B) is cartesian because the horizontal morphisms are isomorphism by Lemma 2.8; the lemma being applicable as the corresponding horizontal morphisms are between henselian rings with the same residue fields. The result [Sta20, Tag 09YQ] proves that the functor is locally of finite presentation, consequently, the functor is an arc sheaf thanks to [BM21, Thm. 4.1], and finally by Proposition 2.4, where we take $\mathcal{C} = \mathcal{D}(\mathbb{Z})^{\geq 0}$, an arc hypersheaf. \square

Definition 2.9 ([BM21, Defn. 6.14], [ČS21, §2.2.1]). Given a ring A with an element $\varpi \in A$, a morphism $A \rightarrow A'$ of rings is a ϖ -complete arc cover if for any rank ≤ 1 and ϖ -adically complete valuation ring V and any morphism $A \rightarrow V$, there are a faithfully flat extension $V \rightarrow V'$ of valuation rings and a morphism $\text{Spec } V' \rightarrow X' := \text{Spec } A'$ lifting the composition $\text{Spec } V' \rightarrow \text{Spec } V \rightarrow X := \text{Spec } A$ to a commutative square as in (2A).

Remark 2.10. For a ring A , an element $\varpi \in A$, and a ϖ -complete arc cover $A \rightarrow A'$, the reduction $A/\varpi \rightarrow A'/\varpi$ is an arc cover. Conversely, an arc cover $A \rightarrow A'$ is a ϖ -complete arc cover (see [Sta20, Tag 090T]).

Remark 2.11. Given a ring A with an element $\varpi \in A$ such that A is ϖ -adically complete, the functor fet on the category of ϖ -adically complete A -algebras taking such an A -algebra A' to the category of finite étale A' -algebras is a ϖ -complete arc sheaf by Remark 2.10 and Lemma 2.8 (and hence a ϖ -complete arc hypersheaf by Proposition 2.4), similarly, given an étale sheaf \mathcal{G} on A of torsion abelian groups, the functor $A' \mapsto \text{R}\Gamma_{\text{ét}}(A', \mathcal{G})$ on the same category is a ϖ -complete arc hypersheaf.

Proposition 2.12 (cf. [Mat20, Thm. 5.19], [BM21, Cor. 6.17]). *Given a ring R with an element $\varpi \in R$, the functor taking an R -algebra R' , with ϖ -adic completion \hat{R}' , to the category of finite étale $\hat{R}'[\frac{1}{\varpi}]$ -algebras satisfies ϖ -complete arc hyperdescent. Moreover, given an étale sheaf \mathcal{G} on R of torsion abelian groups, the functor $R' \mapsto \text{R}\Gamma_{\text{ét}}(\hat{R}'[\frac{1}{\varpi}], \mathcal{G})$ satisfies ϖ -complete arc hyperdescent.*

See [Mat20, Thm. 5.19] for the case of finite étale algebras and [BM21, Cor. 6.17] for the case of the étale cohomology. In either case, it reduces to prove arc descent for the respective functors. Notably, the case of the étale cohomology is easier, given that there is a cartesian square

$$\begin{array}{ccc} \text{R}\Gamma_{\text{ét}}(R, \mathcal{G}) & \longrightarrow & \text{R}\Gamma_{\text{ét}}(R[\frac{1}{\varpi}], \mathcal{G}) \\ \downarrow & & \downarrow \\ \text{R}\Gamma_{\text{ét}}(\hat{R}, \mathcal{G}) & \longrightarrow & \text{R}\Gamma_{\text{ét}}(\hat{R}[\frac{1}{\varpi}], \mathcal{G}), \end{array} \quad (2C)$$

which, since $\mathcal{D}(\mathbb{Z})^{\geq 0}$ is a stable ∞ -category ([HA, Defn. 1.1.1.9]), is a co-cartesian square (see [HA, Prop. 1.1.3.4]). Consequently, it suffices to show that each of the other three functors appearing in (2C) are arc sheaves, which follows from Proposition 2.7 and Lemma 2.8. But in the case of finite étale schemes, loc. cit. does not apply and thus, the above proof can not be naively adapted to work.

3 Tilting for fet and $\text{R}\Gamma_{\text{ét}}(-, \mathcal{G})$ over Perfectoid Rings

In the first part of §3 we recall the basics of tilting of perfectoid rings (following [ČS21]) which form the base of the proof of Theorem 3.6. As a corollary, we have the generalisation of [Čes19, Thm. 4.10] stated in Corollary 3.8.

Given a \mathbb{Z}_p -algebra A , we define the *tilt* $A^b := \lim_{x \mapsto x^p} A/(p)$. Given a ring A and an element $\varpi \in A$ such that $\varpi \mid p$ and A is ϖ -adically complete, [BMS18, Lem. 3.2(i)] implies that there is a multiplicative monoidal isomorphism

$$\lim_{x \mapsto x^p} A \xrightarrow{\sim} A^b. \quad (3A)$$

The *untilt* morphism is the projection onto the first factor $\sharp: A^b \rightarrow A$. It can be written as a composition of the Teichmüller map $a \mapsto [a]$ and the canonical morphism θ from the p -typical Witt vectors of A^b to A defined by $[a] \mapsto a^\sharp$.

Definition 3.1 ([BMS18, Defn. 3.5]). A ring A with tilt A^b is called *perfectoid* (also referred to as ‘integral perfectoid’ by some authors) if there exists a $\varpi \in A$ such that $\varpi^p \mid p$ and such that A is ϖ -adically complete and

1. the p -power morphism $x \mapsto x^p$ in $A/(\varpi) \rightarrow A/(\varpi^p)$ is surjective, and
2. the kernel of the map $\theta: W(A^b) \rightarrow A$ (defined above) is principal.

Any perfect \mathbb{F}_p -algebra is perfectoid because we may take $\varpi = 0$.

Let A be a perfectoid ring with an element $\varpi \in A$ such that $\varpi^p \mid p$ and that A is ϖ -adically complete. We recall from [ČS21, §2.1.3] that A has bounded ϖ -torsion; more precisely, we have $A\langle\varpi\rangle = A\langle\varpi^\infty\rangle$. Additionally, by [BMS18, Lem. 3.9], there exists a unit $u \in A$ such that the element $u\varpi$ admits compatible p -power roots in A , that is, there exists an element ϖ^b with untild $u\varpi$. Then, by the proof of [BMS18, Lem. 3.10], there is an isomorphism

$$A/(\varpi) \cong A^b/(\varpi^b). \quad (3B)$$

Moreover, since A^b is perfect, arguing as in the proof of [Bha17, Cor. 3.2.3], the above isomorphism implies that A^b is ϖ^b -adically complete. The isomorphism (3A) of multiplicative monoids extends (thanks to [ČS21, §2.1.7]) to an isomorphism

$$A^b[\frac{1}{\varpi^b}] \xrightarrow{\sim} \lim_{x \mapsto x^p} A[\frac{1}{\varpi}]. \quad (3C)$$

Proposition 3.2 ([ČS21, Prop. 2.1.9]). *Given a perfectoid ring A and an element $\varpi \in A$ such that $\varpi^p \mid p$ and that A is ϖ -adically complete, and its tilt A^b with an element $\varpi^b \in A^b$ such that $\varpi^{b\sharp}$ is a unit multiple of ϖ in A ; we have an equivalence between the categories of ϖ -adically complete A -algebras which are valuation rings of rank ≤ 1 with algebraically closed fraction fields and the category of ϖ^b -adically complete A^b -algebras which are valuation rings of rank ≤ 1 with algebraically closed fraction fields.*

In fact, op. cit. proves a stronger result, by showing an equivalence between the categories of ϖ -adically complete perfectoid rings over A and that of ϖ^b -adically complete perfect rings over A^b (cf. [Sch12, Thm. 5.2] and [KL15, Thm. 3.6.5]). It is to be noted, however, for the sake of clarity, that ϖ -adically complete valuation rings over A of rank ≤ 1 with algebraically closed fraction fields are perfectoid rings (resp., valuation rings of rank ≤ 1 with algebraically closed fraction fields of characteristic p are perfect rings) by [ČS21, §2.1.1].

Lemma 3.3 ([ČS21, Lem. 2.2.2, Lem. 2.2.3]). *Let A be a perfectoid ring with an element $\varpi \in A$ such that $\varpi^p \mid p$ and that A is ϖ -adically complete, and let A^b be its tilt and $\varpi^b \in A^b$ be such that $\varpi^{b\sharp}$ is a unit multiple of ϖ in A . Then, there exists a ϖ -complete arc cover $A \rightarrow A'$ such that $A' = \prod_{i \in I} V_i$, where I is an indexing set, and V_i is a ϖ -adically complete valuation ring over A of rank ≤ 1 with an algebraically closed fraction field for each $i \in I$, and $A^b \rightarrow A'^b$ is a ϖ^b -complete arc cover.*

Lemma 3.4 ([ČS21, Lem. 2.2.4]). *Given a collection of valuation rings $\{V_i\}_{i \in I}$ with algebraically closed fraction fields, any étale cover of a quasi-compact open $U \subset \mathrm{Spec}(\prod_{i \in I} V_i)$ has a section, and in particular, any finite étale U -scheme is a finite disjoint union of subsets $T \subset U$ which are both open and closed.*

Proposition 3.5 (Beauville–Laszlo). *Let R be a ring with an element $\varpi \in R$ such that R has bounded ϖ -torsion (i.e., there exists an $n \geq 1$ such that $R\langle\varpi^\infty\rangle = R\langle\varpi^n\rangle$) and S be a flat, ϖ -henselian R -algebra with ϖ -adic completion \hat{S} . Given a quasi-compact open $\mathrm{Spec}(R[\frac{1}{\varpi}]) \subset U \subset \mathrm{Spec}(R)$ for which $\mathrm{Spec}(R[\frac{1}{\varpi}] \times S)$ is an fpqc cover, and an arc sheaf \mathcal{F} locally of finite presentation on R such that for every R -henselian pair (A, I) , the morphism*

$\mathcal{F}(A) \xrightarrow{\sim} \mathcal{F}(A/I)$ is an isomorphism, we have a cartesian diagram

$$\begin{array}{ccc} \mathcal{F}(U) & \longrightarrow & \mathcal{F}(R[\frac{1}{\varpi}]) \\ \downarrow & & \downarrow \\ \mathcal{F}(\hat{S}) & \longrightarrow & \mathcal{F}(\hat{S}[\frac{1}{\varpi}]). \end{array} \quad (3D)$$

In particular, the above holds for the functor of idempotents and the functor of finite étale algebras, and given a torsion étale sheaf \mathcal{G} on R , for the functor $\mathcal{F} = \mathrm{R}\Gamma_{\text{ét}}(-, \mathcal{G})$.

Proof. By fpqc descent, we have an equivalence

$$\mathcal{F}(U) \longrightarrow \lim(\mathcal{F}(R[\frac{1}{\varpi}]) \times \mathcal{F}(S) \rightrightarrows \mathcal{F}(S[\frac{1}{\varpi}])).$$

Thanks to [BC20, Thm. 2.1.15], the hypothesis implies that $\mathcal{F}(S[\frac{1}{\varpi}]) \cong \mathcal{F}(\hat{S}[\frac{1}{\varpi}])$. It remains to show that there is an equivalence

$$\mathcal{F}(S) \longrightarrow \lim(\mathcal{F}(S[\frac{1}{\varpi}]) \times \mathcal{F}(\hat{S}) \rightrightarrows \mathcal{F}(\hat{S}[\frac{1}{\varpi}])).$$

The ring S , being flat over R , has bounded ϖ -torsion, consequently, so does \hat{S} and, in fact, $\hat{S}\langle\varpi^\infty\rangle = S\langle\varpi^\infty\rangle$. This implies that Beauville–Laszlo gluing condition is satisfied ($S \rightarrow \hat{S}$ is a ‘gluable pair’ as in [BM21, Thm. 6.4]), proving the required equivalence.

For the final assertion of the claim, the three functors in question are arc sheaves locally of finite presentation (Proposition 2.7) and Lemma 2.8 shows that there is an isomorphism $\mathcal{F}(A) \cong \mathcal{F}(A/I)$ for any henselian pair (A, I) . □

Theorem 3.6. *Let A be a perfectoid ring with an element $\varpi \in A$ such that $\varpi^p \mid p$ and that A is ϖ -adically complete, and let A^\flat be its tilt and $\varpi^\flat \in A^\flat$ be such that ϖ^{\flat^\sharp} is a unit multiple of ϖ in A . Suppose $U \subset \mathrm{Spec} A$ (resp. $U^\flat \subset \mathrm{Spec} A^\flat$) is an open containing $\mathrm{Spec}(A[\frac{1}{\varpi}])$, (resp. containing $\mathrm{Spec}(A^\flat[\frac{1}{\varpi^\flat}])$), such that the closed subsets $Z := \mathrm{Spec} A \setminus U$ and $Z^\flat := \mathrm{Spec} A^\flat \setminus U^\flat$ agree under the isomorphism induced by (3B). Then there are compatible equivalences, functorial in A and U and compatible with orthogonality relation on idempotents,*

$$\mathrm{Idem}(U) \cong \mathrm{Idem}(U^\flat), \quad \text{and} \quad (3E)$$

$$\mathrm{fet}/U \cong \mathrm{fet}/U^\flat. \quad (3F)$$

Moreover, given a commutative, finite étale U -group scheme G , there are a commutative, finite étale U^\flat -group scheme G^\flat obtained by (3F) and a functorial in A , U and G equivalence

$$\mathrm{R}\Gamma_{\text{ét}}(U, G) \cong \mathrm{R}\Gamma_{\text{ét}}(U^\flat, G^\flat). \quad (3G)$$

We note that the inspiration for defining the ‘tilt’ U^\flat of U comes from the tilting functor in the adic theory of perfectoid spaces. In [CS21, Thm. 2.2.7], the authors showed the equivalence (3E) and (3G), the latter in the case of constant coefficients, that is, by replacing both G and G^\flat by an abstract abelian group. The proof given below is an adaptation of the proof of loc. cit.

Proof of Theorem 3.6. For any scheme X , the commutative, finite étale X -group schemes are commutative group objects in the category of finite étale X -schemes. Thus, given a commutative, finite étale U -group scheme G , the identification (3F), functorially in A and U , identifies G with a commutative, finite étale U^\flat -group scheme.

Our strategy is to find a suitable open covering of U which has a corresponding open covering of U^b : precisely, we would like each open V in the covering of U to have a corresponding open V^b (i.e., such that $\mathrm{Spec} A[\frac{1}{\varpi}] \subset V$ and $\mathrm{Spec} A^b[\frac{1}{\varpi^b}] \subset V^b$, and $\mathrm{Spec} A \setminus V$ identifies with $\mathrm{Spec} A^b \setminus V^b$ under (3B)) in the covering of U^b ; and, therefore, by Zariski descent of idempotents (resp., of \mathbf{fet} , resp., of étale cohomology), we will reduce to producing the functorial equivalence (3E) (resp., (3F), resp., (3G)) for each of the open V in covering of U . In fact, it is possible to obtain each open V such that it is covered by $\mathrm{Spec} A[\frac{1}{f}]$ and an open $\mathrm{Spec} B \subset \mathrm{Spec} A$, and such that V^b is covered by $\mathrm{Spec} A^b[\frac{1}{f^b}]$ and an open $\mathrm{Spec} B' \subset \mathrm{Spec} A^b$ in the following way: covering Z by principal open subsets $\{\mathrm{Spec} A[\frac{1}{f}] \subset U\}$, we can find elements $f^b \in A^b$ (guaranteed by Definition 3.1(2) and (3A)), for each f , such that $f^{b\#} \equiv f \pmod{\varpi}$, and such that the opens $\{\mathrm{Spec} A^b[\frac{1}{f^b}] \subset U^b\}$ cover Z^b . Indeed, fixing one such cover of Z , for each open $\mathrm{Spec} A[\frac{1}{f}]$ in the cover, we may take $B = A[\frac{1}{f}]$ and $B' = A^b[\frac{1}{f^b}]$, and

$$V = \mathrm{Spec} A[\frac{1}{\varpi}] \cup \mathrm{Spec} B \quad \text{and}$$

$$V^b = \mathrm{Spec} A^b[\frac{1}{\varpi^b}] \cup \mathrm{Spec} B'.$$

Henceforth, we assume that U is the union of $\mathrm{Spec} A[\frac{1}{\varpi}]$ and $\mathrm{Spec} B$ (resp., U^b is the union of $\mathrm{Spec} A^b[\frac{1}{\varpi^b}]$ and $\mathrm{Spec} B'$) for some open $\mathrm{Spec} B \subset \mathrm{Spec} A$ (resp., $\mathrm{Spec} B' \subset \mathrm{Spec} A^b$). Our construction implies that $B/\varpi \cong B'/\varpi^b$, and therefore, the ϖ -adic completion \hat{B} of B , which is perfectoid thanks to [ČS21, Cor. 2.1.6], has tilt \hat{B}' , the ϖ^b -adic completion of B' (which is also perfectoid thanks to loc. cit.). The Beauville–Laszlo gluing (Proposition 3.5) applies by using $R = A$ and $S = \varpi$ -henselisation of B (resp., by using $R = A^b$ and $S = \varpi^b$ -henselisation of B'), reducing the task of producing a functorial equivalence (3E) (resp., (3F), resp., (3G)) to the following two cases, namely, when $U = \mathrm{Spec} A[\frac{1}{\varpi}]$ and $U^b = \mathrm{Spec} A^b[\frac{1}{\varpi^b}]$ and when $U = \mathrm{Spec} A$ and $U^b = \mathrm{Spec} A^b$ (therefore, in particular, implying the case when $U = \mathrm{Spec} \hat{B}$ and $U^b = \mathrm{Spec} \hat{B}'$). The latter case is easier, because we can use (3B) and Lemma 2.8 (this is applicable since A and A^b are ϖ and ϖ^b -henselian respectively). In the case of idempotents, the first case follows from (3A).

To deal with the cases of \mathbf{fet} and the étale cohomology, we will need to use the descent results from §2. Let

$$A \longrightarrow A_0 \rightrightarrows A_1 \rightrightarrows \cdots$$

be a ϖ -complete arc hypercover supplied by Lemma 3.3 and

$$A^b \longrightarrow A_0^b \rightrightarrows A_1^b \rightrightarrows \cdots$$

be the tilt ϖ^b -complete arc hypercover. By Proposition 2.12, the functor on ϖ -complete A -algebras taking such an A -algebra \tilde{A} to the category of finite étale $\tilde{A}[\frac{1}{\varpi}]$ -algebras (resp., to $\mathrm{R}\Gamma_{\mathrm{ét}}(\tilde{A}[\frac{1}{\varpi}], G)$) satisfies ϖ -complete arc hyperdescent and the functor on ϖ^b -complete A^b -algebras taking such an A^b -algebra \tilde{A} to the category of finite étale $\tilde{A}[\frac{1}{\varpi^b}]$ -algebras (resp., to $\mathrm{R}\Gamma_{\mathrm{ét}}(\tilde{A}[\frac{1}{\varpi^b}], G^b)$) satisfies ϖ^b -complete arc hyperdescent. Hence, to show that there is a functorial equivalence $\mathbf{fet}/A[\frac{1}{\varpi}] \cong \mathbf{fet}/A^b[\frac{1}{\varpi^b}]$ (resp., $\mathrm{R}\Gamma_{\mathrm{ét}}(A[\frac{1}{\varpi}], G) \cong \mathrm{R}\Gamma_{\mathrm{ét}}(A^b[\frac{1}{\varpi^b}], G^b)$), it is enough to exhibit functorial equivalences for all i ,

$$\mathbf{fet}/A_i[\frac{1}{\varpi}] \cong \mathbf{fet}/A_i^b[\frac{1}{\varpi^b}]$$

$$(\text{resp., } \mathrm{R}\Gamma_{\mathrm{ét}}(A_i[\frac{1}{\varpi}], G) \cong \mathrm{R}\Gamma_{\mathrm{ét}}(A_i^b[\frac{1}{\varpi^b}], G^b)).$$

Because of the nature of the rings A_i , it is enough to establish functorial equivalences (3F) and (3G) in the case when $R = \prod_i V_i$, where V_i are ϖ -adically complete valuation rings of rank ≤ 1 with algebraically closed fraction fields. By, for example [ČS21, Prop. 2.1.8], V_i

are perfectoid and by Proposition 3.2, V_i^b are ϖ^b -adically complete valuation rings of rank ≤ 1 with algebraically closed fraction fields; consequently, R is perfectoid with tilt R^b due to op. cit. Prop. 2.1.11(d). By Lemma 3.4, the finite étale schemes over $R[\frac{1}{\varpi}]$ (resp., over $R^b[\frac{1}{\varpi^b}]$) correspond to disjoint unions of subsets of $\mathrm{Spec} R[\frac{1}{\varpi}]$ (resp., of $\mathrm{Spec} R^b[\frac{1}{\varpi^b}]$) which are both open and closed, which, by [Sta20, Tag 00EE], correspond to finite collections of idempotents of $R[\frac{1}{\varpi}]$ (resp., of $R^b[\frac{1}{\varpi^b}]$); whence, the functorial equivalence (3E) implies (3F). Again, by Lemma 3.4, the ring $R[\frac{1}{\varpi}]$ (resp., the ring $R^b[\frac{1}{\varpi^b}]$) has no non-split étale covers, and therefore $\mathrm{R}\Gamma_{\mathrm{\acute{e}t}}(R[\frac{1}{\varpi}], G)$ (resp., $\mathrm{R}\Gamma_{\mathrm{\acute{e}t}}(R^b[\frac{1}{\varpi^b}], G^b)$) is concentrated in degree 0; thus, the cohomology $\mathrm{R}\Gamma_{\mathrm{\acute{e}t}}(R[\frac{1}{\varpi}], G) \cong H^0(R[\frac{1}{\varpi}], G)$ (resp., the cohomology $\mathrm{R}\Gamma_{\mathrm{\acute{e}t}}(R^b[\frac{1}{\varpi^b}], G^b) \cong H^0(R^b[\frac{1}{\varpi^b}], G^b)$) can be identified with the group of sections $R[\frac{1}{\varpi}] \rightarrow G$ (resp., sections $R^b[\frac{1}{\varpi^b}] \rightarrow G^b$). Consequently, the full faithfulness of the functorial equivalence (3F) implies (3G). \square

In the above proof, we could have replaced the ϖ -complete arc hyperdescent with the ‘ ϖ -complete v ’-hyperdescent. Indeed, the proof of [ČS21, Lem. 2.2.3] can be tailored to produce a ‘ ϖ -complete v ’-cover in Lemma 3.3. However, it does not save us much work if we just prove the ‘ ϖ complete v ’-hyperdescent for the functors in Proposition 2.7 and Proposition 2.12.

Remark 3.7. Let R be a perfectoid Banach K -algebra as in [Sch12, Defn. 5.1], where K is a perfectoid field as in op. cit. Defn. 3.1, (resp., perfectoid Banach \mathbb{Q}_p -algebra as in [KL15, Defn. 3.6.1]). Then, the ring R° of power-bounded elements is a perfectoid ring (cf. [BMS18, Lem. 3.20]). Choosing an element $\varpi \in R^\circ$ such that $\varpi^p \mid p$ and that R° is ϖ -adically complete, and choosing $\varpi^b \in R^{b^\circ}$ such that ϖ^{b^\sharp} is a unit multiple of ϖ , we have, by Theorem 3.6, an equivalence

$$\mathrm{fct}/R^\circ[\frac{1}{\varpi}] \cong \mathrm{fct}/R^{b^\circ}[\frac{1}{\varpi^b}].$$

Corollary 3.8 ([Čes19, Thm. 4.10]). *Let A be a \mathbb{Z}_p -algebra such that it is a perfectoid ring with an element $\varpi \in A$ such that $\varpi^p \mid p$ and that A is ϖ -adically complete. Then, for a commutative, finite étale $A[\frac{1}{\varpi}]$ -group scheme G of p -power order, we have, for all $i \geq 2$,*

$$H_{\mathrm{\acute{e}t}}^i(A[\frac{1}{\varpi}], G) = 0. \quad (3H)$$

In particular, for a commutative, finite étale $A[\frac{1}{p}]$ -group scheme G of p -power order, we have, for all $i \geq 2$,

$$H_{\mathrm{\acute{e}t}}^i(A[\frac{1}{p}], G) = 0.$$

Proof. The second vanishing follows from (3H) because there exists a $\varpi \in A$ such that ϖ^p is a unit multiple of p , and $A[\frac{1}{\varpi}] = A[\frac{1}{p}]$. Indeed, [BMS18, Lem. 3.9] implies that there exists a unit $v \in A$ such that vp admits compatible p -power roots, consequently, we can take $\varpi \in A$ such that $\varpi^p = vp$.

Letting $\varpi^b \in A^b$ be such that $\varpi^{b^\sharp} = vp$, Theorem 3.6 implies that we have an isomorphism

$$\mathrm{R}\Gamma_{\mathrm{\acute{e}t}}(A[\frac{1}{\varpi}], G) \cong \mathrm{R}\Gamma_{\mathrm{\acute{e}t}}(A^b[\frac{1}{\varpi^b}], G^b).$$

It suffices to show the cohomology vanishing of the second complex, which reduces us to prove (3H) for the perfect \mathbb{F}_p -algebra A^b . Due to a limit argument, [SGA4 III, Ex. X Thm. 5.1], which shows that the p -cohomological degree of any noetherian \mathbb{F}_p -algebra is ≤ 1 , shows that for all $i \geq 2$,

$$H_{\mathrm{\acute{e}t}}^i(A^b[\frac{1}{\varpi^b}], G^b) = 0. \quad \square$$

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