Tilting Correspondences for Perfectoid Rings

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Introduction

Scholze's work on perfectoid.

Theorem (Scholze'12)

For a perfectoid field K with tilit $K^\flat$ there is an equivalence of categories

$$\text{fét}\,/K \xrightarrow{\sim} \text{fét}\,/K^\flat.$$

- **Scholze'12** for perfectoid Banach K-algebras
- **Kedlaya-Liu'15** Tilting for perfectoid Banach $\mathbb{Q}_p$-algebras
- **Gabber-Ramero'18** Tilting for perfectoid rings
- **Bhatt-Morrow-Scholze'18** Perfectoid rings

Perfectoid Ring I

Bhatt-Morrow-Scholze'18

Definition

A ring A is called *perfectoid* if

1. A is ϖ-adically complete with $\varpi^p | p$
2. $A/\varpi \to A/\varpi^p$ given by $x \mapsto x^p$ is surjective and,
3. $\ker(\theta\colon W(A^\flat) \to A)$ is principal.

Here the tilt is defined as $A^\flat = \lim_{x \mapsto x^p} A/p$ is a perfect ring.

Example

1. Perfectoid field K°
2. Perfectoid Banach K-algebra R°
3. (ϖ-adically) complete Valuation ring V of rank 1 with $\mathrm{Frac}(V)$ alg. clos.

Perfectoid Ring II

Remark

1. The tilt is also the limit $A^\flat = \lim_{x \mapsto x^p} A/\varpi'$; for any $\varpi''|p$.
2. There is an isomorphism of multiplicative monoids

$$A^\flat \cong \lim_{x \mapsto x^p} A.$$

   Define $\sharp \colon A^\flat \to A$.
3. There exists $\varpi^\flat$ such that $({\varpi^\flat}^\sharp) = (\varpi)$. Defn. does not change.

$$\begin{tikzcd}
A^\flat = \lim_{x \mapsto x^p} A/\varpi = \lim_{x \mapsto x^n} A \arrow[r, "\sharp"] \arrow[d, "\text{modulo } \varpi^\flat"] & A \arrow[d, "\text{modulo } \varpi"] \\
A^\flat/\varpi^\flat \arrow[r, "\sim"] & A/\varpi
\end{tikzcd}$$

Algebraic Tilting

Theorem (Česnavičius-Scholze'19)

Let A be a perfectoid ring with tilt $A^\flat$. There is an equivalence of categories

$$\{\text{Perfectoid Algebras}/A\} \longrightarrow \{\text{Perfectoid Algebras}/A^\flat\}$$

$$B \longmapsto B^\flat$$

Moreover if B is ϖ-adically complete iff $B^\flat$ is $\varpi^\flat$-adically complete. If B is a valuation ring (of rank $\leqslant 1$ and algebraically closed fraction field) iff $B^\flat$ is the same.

BC-pair I

We fix a perfectoid ring A which is ϖ-adically complete with $\varpi^p|p$. Let $\varpi^\flat \in A^\flat$ such that $({\varpi^\flat}^\sharp) = (\varpi)$.

Definition

Let $U \subset \mathrm{Spec}(A)$ and $U^\flat \subset \mathrm{Spec}(A^\flat)$ be two opens such that

$$\mathrm{Spec}\left(A\left[\tfrac{1}{\varpi}\right]\right) \subset U \subset \mathrm{Spec}(A) \qquad \mathrm{Spec}\left(A^\flat\left[\tfrac{1}{\varpi^\flat}\right]\right) \subset U^\flat \subset \mathrm{Spec}(A^\flat);$$

$$Z := \mathrm{Spec}(A)\backslash U \qquad Z^\flat := \mathrm{Spec}(A^\flat)\backslash U^\flat$$

We assume that $Z \cong Z^\flat$ under the homeomorphism induced by

$$A/\varpi \cong A^\flat/\varpi^\flat.$$

BC-pair II

Remark

Inspired from the tilting functor of adic spaces.

Example

1. $U = \mathrm{Spec}(A)$ and $U^\flat = \mathrm{Spec}(A^\flat)$,
2. $U = \mathrm{Spec}(A\left[\frac{1}{\varpi}\right])$ and $U^\flat = \mathrm{Spec}(A^\flat\left[\frac{1}{\varpi^\flat}\right])$.

Main Theorems

Theorem (1)

We have a natural equivalence

$$\text{fét}\,/U \xrightarrow{\ \sim\ } \text{fét}\,/U^\flat.$$

Let G be a commuative affine group with G/A finite étale. Then by tilting we have that $G^\flat$ is a commutative affine group with $G^\flat/A^\flat$ finite étale.

Theorem (2)

There is a natural equivalence

$$R\Gamma_{\text{ét}}(U, G) \xrightarrow{\ \sim\ } R\Gamma_{\text{ét}}(U^\flat, G^\flat).$$

Known results

Theorem (?)

$$\text{fét}\,/A\left[\tfrac{1}{\varpi}\right] \to \text{fét}\,/A^\flat\left[\tfrac{1}{\varpi^\flat}\right].$$

Theorem (Česnavičius'19)

$$R\Gamma_{\text{ét}}(A\left[\tfrac{1}{\varpi}\right], G) \xrightarrow{\sim} R\Gamma_{\text{ét}}(A^\flat\left[\tfrac{1}{\varpi^\flat}\right], G^\flat).$$

Theorem (Česnavičius-Scholze'19)

Let $\mathcal{F}$ be a constant torsion étale sheaf over U. Then there is a natural equivalence

$$R\Gamma_{\text{ét}}(U, \mathcal{F}) \xrightarrow{\sim} R\Gamma_{\text{ét}}(U^\flat, \mathcal{F}).$$

Strategy

Use the method from Česnavičius-Scholze'19.

- Reduce to the case of perfectoid rings.
- Prove that there is arc-descent for fét and $R\Gamma_{\text{ét}}$.
- Using 2. reduce to the case of "special" perfectoid rings.

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arc-topology(defn)

Bhatt-Mathew'18

Definition (arc-topology)

A morphism of schemes $X' \to X$ is an *arc-cover* if for every valuation ring V of rank $\leqslant 1$ and a morphism $\mathrm{Spec}(V) \to X$ there exists a valuation V' which is a faithfully flat extension $V \to V'$ such that the following diagram

$$\begin{tikzcd} \mathrm{Spec}(V') \arrow[r] \arrow[d] & X' \arrow[d] \\ \mathrm{Spec}(V) \arrow[r] & X \end{tikzcd}$$

is commuatative.

Remark

In the definition we can assume V' is of rank $\leqslant 1$ and $\mathrm{Frac}(V)$ as well as $\mathrm{Frac}(V')$ is algebraically closed.

ϖ-complete arc-topology

A faithfully flat map of valuation rings $V \to V'$ is equivalently a local injective morphism.

Definition (ϖ-complete arc topology)

Let $X = \mathrm{Spec}(A)$ and $X' = \mathrm{Spec}(A')$ and $\varpi \in A$. Then $X' \to X$ is said to be a *ϖ-complete arc-cover* if for every ϖ-adically complete valauation ring V of rank $\leqslant 1$ and a morphism $\mathrm{Spec}(V) \to X$ there exists a valuation V' which is a faithfully flat extension $V \to V'$ such that the following diagram

$$\begin{tikzcd} \mathrm{Spec}(V') \arrow[r] \arrow[d] & X' \arrow[d] \\ \mathrm{Spec}(V) \arrow[r] & X \end{tikzcd}$$

is commuatative.

Remark

We can assume that V' is ϖ-adically complete.

Example

Let V be a valuation ring of rank 2 and let $\mathfrak{p}$ be the prime of height 1. Then $V \to V_{\mathfrak{p}} \times V/\mathfrak{p}$ is an arc-cover but not a v-cover.

1. Faithfully flat extensions are arc-covers.
2. arc-covers are ϖ-complete arc-covers.
3. ϖ-complete arc-cover are ϖ'-complete arc-covers for $\varpi|\varpi'$.

(Hyper)descent I

- Let $\mathcal{C}$ be an ∞-category with all limits. Let $F : \mathrm{Sch}_R \to \mathcal{C}$ be a contravariant functor, where R is a ring.
- Can assume $\mathcal{C} = Cat$, 2-category of categories.
- τ be a Grothendieck topology on a subcategory of Sch_R.

(Hyper)descent II

Definition

Let $f : X' \to X$ be a morphism in Sch_R. Then F is said to satisfy *f-descent* if there is an equivalence

$$F(X) \xrightarrow{\sim} \lim(F(X') \rightrightarrows F(X' \times_X X') \substack{\rightarrow\\\rightarrow\\\rightarrow} \cdots).$$

Then $\mathcal{F}$ is said to satisfy *f-hyperdescent* if there is an equivalence

$$F(X) \xrightarrow{\sim} \lim(F(X_0) \rightrightarrows F(X_1) \substack{\rightarrow\\\rightarrow\\\rightarrow} \cdots)$$

for any hypercover

$$\cdots \substack{\rightarrow\\\rightarrow\\\rightarrow} X_1 \rightrightarrows X_0 = X' \longrightarrow X.$$

(Hyper)descent III

Definition

Let τ be a Grothendieck topology on a subcategory of Sch_R. Then F is said to satisfy τ-(hyper)descent if $\mathcal{F}$ satisfies f-descent for every τ-(hyper)covering $f \colon X' \to X$.

Remark

If $\mathcal{C} = \mathrm{Cat}$, then the above can be written as

$$F(X) \xrightarrow{\sim} \lim(F(X') \rightrightarrows F(X' \times_X X') \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\; F(X' \times_X X' \times_X X')).$$

$$F(X) \xrightarrow{\sim} \lim(F(X_0) \rightrightarrows F(X_1) \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\; F(X_2))$$

This is a generalisation of descent for fibered categories.

arc-Hyperdescent I

Theorem (Bhatt-Mathew'18)

Let Qcqs_R *be the subcategory of qcqs schemes over* R. *Then* fét *satisfies arc (hyper)descent in* Qcqs_R.

Definition

We consider the ∞-derived category $\mathscr{D}(\Lambda)^{\geqslant 0}$ of bounded below chain complexes over a ring Λ. Let $\mathcal{F}$ be a sheaf of Λ-modules over Sch_R for the étale topology. Then there is a functor $R\Gamma_{\text{ét}}(-,\mathcal{F}) : \mathsf{Sch}_R \to \mathscr{D}(\Gamma)^{\geqslant 0}$ given by $X \mapsto R\Gamma_{\text{ét}}(X,\mathcal{F})$.

Theorem (Bhatt-Mathew'18)

Let $\mathcal{F}$ *be a torsion étale sheaf on* R. *There is arc (hyper)descent for* $R\Gamma_{\text{ét}}(-,\mathcal{F})$ *in* Qcqs_R.

Let A be a ϖ-adically henselian ring(for example A is ϖ-adically complete for $\varpi \in A$). Let $\varpi\text{-Henselain}$ be the category of ϖ-adically henselian rings over A.

arc-Hyperdescent II

Theorem

There is ϖ-complete arc-descent for $\mathsf{f\acute{e}t}$ *in* ϖ-Henselian.

Theorem (Česnavičius-Scholze'19)

Let $\mathcal{F}$ be a torsion sheaf over Sch_R. There is ϖ-complete arc-hyperdescent for $R\Gamma_{\acute{e}t}(-,\mathcal{F})$ in ϖ-Henselian

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Strategy

Use the method from Česnavičius-Scholze'19.

- Reduce to the case when either $U = \mathrm{Spec}(A), U^\flat = \mathrm{Spec}(A)$ or $U = \mathrm{Spec}(A), U^\flat = \mathrm{Spec}(A^\flat)$.
- Prove that there is arc-descent for fét and $R\Gamma_{\text{ét}}$.
- Using 2. reduce to the case of "special" perfectoid rings.

Reduce to the case of perfectoid rings

- Can use Zariski coverings of U and $U^\flat$ and reduce to quasi-compact opens.
- Reduce to the case when $U = \mathrm{Spec}(A\left[\frac{1}{\varpi}\right]) \cup \mathrm{Spec}(A\left[\frac{1}{f}\right])$ and $U^\flat = \mathrm{Spec}(A^\flat\left[\frac{1}{\varpi^\flat}\right]) \cup \mathrm{Spec}(A^\flat\left[\frac{1}{f^\flat}\right])$ and then use Beauville–Laszlo gluing.
- Reduce to proving for the case $(U = \mathrm{Spec}(A), U^\flat = \mathrm{Spec}(A^\flat))$ and $(U = \mathrm{Spec}(A\left[\frac{1}{\varpi}\right]), U^\flat = \mathrm{Spec}(A^\flat\left[\frac{1}{\varpi^\flat}\right]))$.

Reduction to special perfectoid rings I

Let A be a perfectoid ring which is ϖ-adically complete $\varpi \in A$ such that $\varpi^p | p$. We need to show that there is a natural equivalence

$$\text{fét}\,/A \longrightarrow \text{fét}\,/A^\flat,$$

$$\text{fét}\,/A\left[\frac{1}{\varpi}\right] \longrightarrow \text{fét}\,/A^\flat\left[\frac{1}{\varpi^\flat}\right].$$

Proposition (CS)

For any ring A with $\varpi \in A$ there is a ϖ-complete arc-hypercover

$$A \longrightarrow A_0 \rightrightarrows A_1 \Rrightarrow \cdots$$

such that $A_r = \prod_{s \in I_r} V_s$, where V_s are ϖ-adically complete valuation rings of rank $\leqslant 1$ with algebraically closed fraction fields.

Reduction to special perfectoid rings II

Proposition (CS)

For any perfectoid ring A $\varpi \in A$-adically complete with $\varpi^p | p$ the following is a ϖ-complete arc-hypercover

$$A \longrightarrow A_0 \rightrightarrows A_1 \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow} \cdots$$

iff the following is a $\varpi^\flat$-complete arc hypercover

$$A^\flat \longrightarrow A_0^\flat \rightrightarrows A_1^\flat \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow} \cdots$$

$$\text{fét}\,/A \longrightarrow \text{fét}\,/A_0 \rightrightarrows \text{fét}\,/A_1 \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow} \text{fét}\,/A_2$$

$$\text{fét}\,/A^\flat \longrightarrow \text{fét}\,/A_0^\flat \rightrightarrows \text{fét}\,/A_1^\flat \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow} \text{fét}\,/A_2^\flat$$

Reduction to special perfectoid rings III

$$\text{f\'et}\,/A\left[\tfrac{1}{\varpi}\right] \longrightarrow \text{f\'et}\,/A_0\left[\tfrac{1}{\varpi}\right] \rightrightarrows \text{f\'et}\,/A_1\left[\tfrac{1}{\varpi}\right] \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} \text{f\'et}\,/A_2\left[\tfrac{1}{\varpi}\right]$$

$$\text{f\'et}\,/A^\flat\left[\tfrac{1}{\varpi}\right] \longrightarrow \text{f\'et}\,/A_0^\flat\left[\tfrac{1}{\varpi}\right] \rightrightarrows \text{f\'et}\,/A_1^\flat\left[\tfrac{1}{\varpi}\right] \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} \text{f\'et}\,/A_2^\flat\left[\tfrac{1}{\varpi}\right]$$

Let $V' = \prod_{i\in I} V_i$ be ϖ-adically complete valuation rings of rank $\leqslant 1$ with algebraically closed fraction fields.

$$\text{f\'et}\,/V' \longrightarrow \text{f\'et}\,/V'^\flat,$$

$$\text{f\'et}\,/V'\left[\tfrac{1}{\varpi}\right] \longrightarrow \text{f\'et}\,/V'^\flat\left[\tfrac{1}{\varpi^\flat}\right].$$

Reduction to special perfectoid rings IV

Remark

V ϖ-adically complete valuation rings of rank $\leqslant 1$ with algebraically closed fraction fields

- perfectoid
- strict henselian

Consequently $V' = \prod_{i \in I} V_i$

- (CS) perfectoid,
- every étale cover has a section,
- every finite étale cover is a finite disjoint union of an open closed subscheme

Thus finite étale schemes corresponds to idempotents. Thus it reduces to checking

$$\mathrm{Idem}(V') \longrightarrow \mathrm{Idem}(V'^{\flat}),$$

$$\mathrm{Idem}(V'\left[\tfrac{1}{\varpi}\right]) \longrightarrow \mathrm{Idem}(V'^{\flat}\left[\tfrac{1}{\varpi^{\flat}}\right]).$$

Reduction to special perfectoid rings V

This can be done because of the isomorphisms

$$V'^{\flat} \cong \lim_{x \mapsto x^p} V',$$

$$V'^{\flat}\left[\frac{1}{\varpi}\right] \cong \lim_{x \mapsto x^p} V'\left[\frac{1}{\varpi^{\flat}}\right].$$

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Applications I

Corollary

Given A be a perfectoid ring which is ϖ-adically complete, where $\varpi \in A$ is such that $\varpi^p | p$, there is an equivalence

$$\text{fét}\,/A\left[\tfrac{1}{\varpi}\right] \xrightarrow{\sim} \text{fét}\,/A^\flat\left[\tfrac{1}{\varpi^\flat}\right]. \tag{4.1}$$

In particular, for any Perfectoid Banach K-algebra R(à la Scholze) or $\mathbb{Q}_p$-algebra (à la Kedlaya-Liu) with an element $\varpi \in R^\circ$ such that there exists $\varpi^\flat \in R^{\circ\flat}$ such that $({\varpi^\flat}^\sharp) = (\varpi)$

$$\text{fét}\,/R^\circ\left[\tfrac{1}{\varpi}\right] \longrightarrow \text{fét}\,/R^{\circ\flat}\left[\tfrac{1}{\varpi^\flat}\right].$$

Applications II

Corollary

Given a perfectoid ring A which is ϖ-adically complete where $\varpi \in A$ is such that $\varpi^p \mid p$, and commutative group schemes $G \in \mathrm{f\acute{e}t}\,/A\left[\frac{1}{\varpi}\right]$ and $G^\flat \in \mathrm{f\acute{e}t}\,/A^\flat\left[\frac{1}{\varpi^\flat}\right]$ that are identified under the isomorphism of 4.1, there is an equivalence

$$R\Gamma_{\text{ét}}(A\left[\tfrac{1}{\varpi}\right], G) \xrightarrow{\sim} R\Gamma_{\text{ét}}(A^\flat\left[\tfrac{1}{\varpi^\flat}\right], G^\flat).$$

Corollary (Česnavičius'19)

For a perfectoid ring A and a commutative, finite, étale $A[\frac{1}{p}]$-group scheme G of p-power order, $H^i_{\text{ét}}(A[\frac{1}{p}], G) = 0$ for all $i \geqslant 2$

Applications III

Proof.

We have an isomorphism

$$R\Gamma_{\text{ét}}(A[\tfrac{1}{p}], G) \xrightarrow{\sim} R\Gamma_{\text{ét}}(A^\flat\left[\tfrac{1}{\varpi^\flat}\right], G^\flat).$$

Then the argument follows from SGA4 which implies in this case that for all $i \geqslant 2$ we have $H^i_{\text{ét}}(A^\flat\left[\frac{1}{\varpi^\flat}\right], G^\flat) = 0$. □

Remark

The original proof rests on a non-noetherian version of a comparison result Huber for étale cohomology of a noetherian henselian adic ring with the étale cohomology of its associated affinoid space. Therefore the proof above can be seen as a simplification.

Thank you for your attention!