Some technical stuff

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1 Derived completion

Let $K \in \mathbf{D}(A)$ and $f \in A$. We denote $T(K, f)$ the derived limit of the system

$$\cdots \to K \xrightarrow{f} K \xrightarrow{f} K$$

Lemma 1.1. *The following are equivalent*

1. $\mathrm{Ext}^n_A(A_f, K) = 0$ *for all* n,
2. $\mathrm{Hom}_{\boldsymbol{D}(A)}(E, K) = 0$ *for all* E *in* $\boldsymbol{D}(A_f)$,
3. $T(K, f) = 0$,
4. *for every* $p \in \mathbb{Z}$ *we have* $T(H^p(K), f) = 0$,
5. *for every* $p \in \mathbb{Z}$ *we have* $\mathrm{Hom}_A(A_f, H^p(K)) = 0$ *and* $\mathrm{Ext}^1_A(A_f, H^p(K)) = 0$,
6. $R\,\mathrm{Hom}_A(A_f, K) = 0$,
7. *the map* $\prod_{n \geq 0} K \to \prod_{n \geq 0} K$ *sending* $(x_0, x_1, \dots)$ *to* $(x_0 - fx_1, x_1 - fx_2, \dots)$ *is an isomorphism in* $\boldsymbol{D}(A)$.

Proof. Since $\mathrm{Ext}^n_A(A_f, K) = \mathrm{Hom}_{\mathbf{D}(A)}(A_f, K[n]) = \mathrm{Hom}_{\mathbf{D}(A)}(A_f[-n], K)$, we clearly have that 2) implies 1). And since $\mathrm{Ext}^n_A(A_f, K) = H^n(R\,\mathrm{Hom}_A(A_f, K))$, the equivalence between 1) and 6) is clear. Assume condition 1) and let $I^\bullet$ be a K-injective complex of A-modules representing K. By definition of $R\,\mathrm{Hom}$, condition 6) $\Leftrightarrow$ 1) signifies that the complex $\mathrm{Hom}_A(A_f, I^\bullet)$ is acyclic. Now for an element $E \in \mathbf{D}(A_f)$, let $M^\bullet$ be a complex of A_f-modules representing E. Then

$$\mathrm{Hom}_{\mathbf{D}(A)}(E, K) = \mathrm{Hom}_{\mathbf{K}(A)}(M^\bullet, I^\bullet) = \mathrm{Hom}_{\mathbf{K}(A_f)}(M^\bullet, \mathrm{Hom}_A(A_f, I^\bullet))$$

As $\mathrm{Hom}_A(A_f, I^\bullet))$ is a K-injective complex of A_f-modules (the above equation proves that), the fact that it is acyclic implies that it is homotopy equivalent to 0. Therefore $\mathrm{Hom}_{\mathbf{K}(A_f)}(M^\bullet, \mathrm{Hom}_A(A_f, I^\bullet)) = 0$ which proves 2).

Consider the following free resolution of A_f as an A-module

$$0 \to \bigoplus_{n \in \mathbb{N}} A \to \bigoplus_{n \in \mathbb{N}} A \to A_f \to 0$$

where the first map sends $(a_0, a_1, a_2 \dots)$ to $(a_0, a_1 - fa_0, a_2 - fa_1, \dots)$ and the second map sends $(a_0, a_1, a_2, \dots)$ to $a_0 + a_1/f + a_2/f^2 + \dots$. Applying $\mathrm{Hom}_A(-, I^\bullet)$, we get

$$0 \to \mathrm{Hom}_A(A_f, I^\bullet) \to \prod I^\bullet \to \prod I^\bullet \to 0$$

Since $\prod I^\bullet$ represents $\prod_{n\in\mathbb{N}} K$ this proves the equivalence of 1) and 7). Moreover, by definition of the derived limit, the above exact sequence shows that $T(K,f)$ is a representative of $R\operatorname{Hom}_A(A_f,K)$ in $\mathbf{D}(A)$. This gives the equivalence of 1) and 3).

We have a spectral sequence (I think this is just the spectral sequence that computes the cohomology of the Hom-bicomplex)

$$E_2^{p,q} = \operatorname{Ext}^q_A(A_f, H^p(K)) \Rightarrow \operatorname{Ext}^{p+q}_A(A_f, K)$$

It degenerates at E_2 since A_f has a projective resolution of length 1 (the above free resolution) and so there are only two non-zero rows ($q = 0,1$), which gives us the exact sequence

$$0 \to \operatorname{Ext}^1_A(A_f, H^{p-1}(K)) \to \operatorname{Ext}^p_A(A_f, K) \to \operatorname{Hom}_A(A_f, H^p(K)) \to 0$$

This shows that 4) and 5) are equivalent to 1). $\square$

Lemma 1.2. *Let $I \subset A$ be an ideal and M be an A-module.*

1. *If M is I-adically complete, then $T(M,f) = 0$ for every $f \in I$*
2. *If $T(M,f) = 0$ for every $f \in I$, and I is finitely generated, then the map $M \to \varprojlim M/I^nM$ is surjective*

Proof. 1) assume that M is p-adically complete. By 5. of lemma 1.1, it suffices to prove that $\operatorname{Ext}_A(A_f, M) = 0$ and $\operatorname{Hom}_A(A_f, M) = 0$. But

$$\operatorname{Hom}_A(A_f, M) = \operatorname{Hom}_A(A_f, \varprojlim M/I^nM) = \varprojlim \operatorname{Hom}_A(A_f, M/I^nM) = 0$$

since for every $n \geq 1$, $\operatorname{Hom}_A(A_f, M/I^nM) = 0$.
Now since Ext^1 characterises extensions, we need to show that every extension

$$0 \to M \to E \to A_f \to 0$$

is split. So for each $n \geq 1$, select a $e_n \in E$ mapping to $1/f^n$, and set $\delta_n = fe_{n+1} - e_n \in M$. So the element

$$e'_n = e_n + \delta_n + f\delta_{n+1} + f^2\delta_{n+2} + \dots$$

exists since M is f-adically complete and maps to $1/f^n$. Since $e'_n = fe^{n+1}$, we can define a splitting sending $1/f^n$ to e'_n.

2) Assume that $I = (f_1, \dots, f_r)$ and that $T(M, f_i) = 0$ for $i = 1, \dots, r$. One easily shows that if $M \to \varprojlim M/f_i^nM$ is surjective for every f_i, then $M \to \varprojlim M/I^nM$ is surjective. So we can assume that $I = (f)$ and that $T(M,f) = 0$.
Consider some $x_n \in M$ for $n \geq 0$ and the extension

$$0 \to M \to E \to A_f \to 0$$

where $E = (M \oplus \bigoplus Ae_n)/\langle x_n - fe_{n+1} + e_n\rangle$. Again by 5. of lemma 1.1, this extension is split, so we obtain an element that we can write $x + e_0$ ($x \in M$) that generates a copy of A_f in E $x + e_0 = x - x_0 + fe_1 = x - x_0 - fx_1 + f^2e_2 = \dots$ By the snake lemma, we have $M/f^nM = E/f^nE$ and since $x + e_0 \in f^nE$, we get that $x = x_0 + fx_1 + \dots + f^{n-1}x_{n-1} \mod f^nM$. Which shows the surjectivity of the desired map. $\square$

Definition 1.3. Let I be an ideal of A and $K \in \mathbf{D}(A)$. We say that K is derived complete with respect to I if for every $f \in I$ we have $T(K, f) = 0$. We denote by $\mathbf{D}_{comp}(A) = \mathbf{D}_{comp}(A, I)$ the full subcategory of $\mathbf{D}(A)$ consisting of derived complete objects with respect to I.
If M is an A-module, we say that M is derived complete with respect to I if $M[0] \in \mathbf{D}(A)$ is derived complete with respect to I.

Corollary 1.4. *If the ideal $I \subset A$ is finitely generated, and M is an A-module, then the following are equivalent*

1. *M is I-adically complete,*
2. *M is derived complete with respect to I and I-adically separated*

Proof. Direct consequence of 1.2. □

Proposition 1.5. *Let I be a finitely generated ideal of a ring A. The inclusion functor $\boldsymbol{D}_{comp}(A, I) \to \boldsymbol{D}(A)$ has a left adjoint, i.e, there exist a map sending any object K of $\boldsymbol{D}(A)$ to a derived complete object $K^\wedge$ of $\boldsymbol{D}(A)$ such that the map*

$$\operatorname{Hom}_{\boldsymbol{D}(A)}(K^\wedge, E) \to \operatorname{Hom}_{\boldsymbol{D}(A)}(K, E)$$

is a bijection whenever E is derived complete. In fact, if A is generated by $f_1, \ldots, f_r \in A$, we have

$$K^\wedge = R\operatorname{Hom}\left((A \to \prod_{i_0} A_{f_{i_0}} \to \prod_{i_0<i_1} A_{f_{i_0}, f_{i_1}} \to \cdots \to A_{f_1,\ldots,f_r}), K\right)$$

Proof. Let $K^\wedge$ be defined as above. Then the map of complexes

$$(A \to \prod_{i_0} A_{f_{i_0}} \to \prod_{i_0<i_1} A_{f_{i_0}, f_{i_1}} \to \cdots \to A_{f_1,\ldots,f_r}) \to A$$

induces a map $K \to K^\wedge$. It suffices to show that $K^\wedge$ is derived complete and that $K \to K^\wedge$ is an isomorphism if K is derived complete.
Let $f \in A$ We have

$$\begin{aligned} R\operatorname{Hom}_A(A_f, K^\wedge) &= R\operatorname{Hom}\left(A_f, R\operatorname{Hom}\left((A \to \prod_{i_0} A_{f_{i_0}} \to \prod_{i_0<i_1} A_{f_{i_0}, f_{i_1}} \to \cdots \to A_{f_1,\ldots,f_r}), K\right)\right) \\ &= R\operatorname{Hom}\left(A_f \otimes_A^{\mathbb{L}} (A \to \prod_{i_0} A_{f_{i_0}} \to \prod_{i_0<i_1} A_{f_{i_0}, f_{i_1}} \to \cdots \to A_{f_1,\ldots,f_r}), K\right) \\ &= R\operatorname{Hom}\left((A_f \to \prod_{i_0} A_{f f_{i_0}} \to \prod_{i_0<i_1} A_{f f_{i_0}, f_{i_1}} \to \cdots \to A_{f f_1,\ldots,f_r}), K\right) \end{aligned}$$

The last equality is true by looking at the definition of the derived tensor product and noticing that the complex $(A \to \prod_{i_0} A_{f_{i_0}} \to \prod_{i_0<i_1} A_{f_{i_0}, f_{i_1}} \to \cdots \to A_{f_1,\ldots,f_r})$ is K-flat (every element of the complex is a flat A-module).
Now for $f \in I$, the complex

$$(A_f \to \prod_{i_0} A_{f f_{i_0}} \to \prod_{i_0<i_1} A_{f f_{i_0}, f_{i_1}} \to \cdots \to A_{f f_1,\ldots,f_r})$$

is 0 in $\mathbf{D}(A)$ by corollary 3.4. Hence $R\operatorname{Hom}_A(A_f, K^\wedge) = 0$, so $K^\wedge$ is derived complete by lemma 1.1.
Conversely, by the same lemma 1.1, we have $R\operatorname{Hom}_A(A_f, K) = 0$ for each $f = f_{i_0} \cdots f_{i_p}$, hence $K \to K^\wedge$ is an isomorphism in $D(A)$. □

Lemma 1.6. *Let $I \subset A$ be an ideal and let (K_n) be an inverse system of objects of $\boldsymbol{D}(A)$ such that for all $f \in I$, there exist $e = e(n, f)$ such that f^e is zero on K_n. Then for $K \in \boldsymbol{D}(A)$, the object $K' = K \otimes^{\mathbb{L}}_A K_n$ is derived complete with respect to I.*

Proof. The category of derived complete objects being preserved under $R\lim$, it suffices to show that each $K \otimes^{\mathbb{L}}_A K_n$ is derived complete. But by assumption, for all $f \in I$, there exist e such that f^e is zero in $K \otimes^{\mathbb{L}}_A K_n$. Hence $T(K \otimes^{\mathbb{L}}_A K_n, f)$=0. $\square$

1.1 Some useful facts in the principal case

In this subsection, we assume that $I = (f)$ for some $f \in A$. One can prove -I am definitely not doing that here but it is just technical- that in this case, we have

$$K^{\wedge} = R\lim \left(K \otimes^{\mathbb{L}}_A (A \xrightarrow{f^n} A)\right)$$

At least one can see directly from lemma 1.6 that this object is derived complete.

Lemma 1.7. *Let $f \in A$. If there exist an integer $c \geq 1$ such that $A[f^c] = A[f^{c+1}] = \dots$, then for all $n \geq 1$, there exist maps*

$$(A \xrightarrow{f^n} A) \to A/f^n, \qquad \text{and} \qquad A/(f^{n+c}) \to (A \xrightarrow{f^n} A)$$

in $\boldsymbol{D}(A)$ inducing an isomorphism of the pro-objects $\{A/f^nA\}$ and $\{(A \xrightarrow{f^n} A)\}$ in $\boldsymbol{D}(A)$.

Proof. The first map is given by the following commutative diagram

$$\begin{array}{ccccccccccc}
\dots & \longrightarrow & 0 & \longrightarrow & A & \xrightarrow{f^n} & A & \longrightarrow & 0 & \longrightarrow & \dots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\dots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & A & \longrightarrow & 0 & \longrightarrow & \dots
\end{array}$$

For the second arrow, first we define a map

$$\begin{array}{ccccccccccc}
\dots & \longrightarrow & 0 & \longrightarrow & A/A[f^c] & \xrightarrow{f^{n+c}} & A & \longrightarrow & 0 & \longrightarrow & \dots \\
 & & \downarrow & & \downarrow{\scriptstyle f^c} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow & & \\
\dots & \longrightarrow & 0 & \longrightarrow & A & \xrightarrow{f^n} & A & \longrightarrow & 0 & \longrightarrow & \dots
\end{array}$$

But since the arrow $A/A[f^c] \xrightarrow{f^{n+c}} A$ is injective, the first row is quasi-isomorphic to $A/f^{n+c}A$ which gives the second map. $\square$

Lemma 1.8. *Let A be a ring and $f \in \mathbb{A}$. We have the naive derived completion $K \mapsto K' = R\lim(K \otimes^{\mathbb{L}}_A A/f^nA)$ and $K \mapsto K^{\wedge} = R\lim\left(K \otimes^{\mathbb{L}}_A (A \xrightarrow{f^n} A)\right)$. The natural transformation $K^{\wedge} \mapsto K'$ is an isomorphism if and only if the f-power torsion of A is bounded.*

Proof. We won't need the only if part, so we will only prove the if part. But by lemma 1.7, the pro-objects $\{A/f^nA\}$ and $\{(A \xrightarrow{f^n} A)\}$ are isomorphic. The result follows from lemma 091B (Stack project). $\square$

2 p-complete flatness

Definition 2.1. Given $a, b \in \mathbb{Z} \cup \{\infty\}$, we say that $M \in \mathbf{D}(A)$ has Tor amplitude $[a, b]$ if for any A-module N, we have $M \otimes^{\mathbb{L}}_A N \in \mathbf{D}^{[a,b]}(A)$. If $a = b$, we say that M has Tor amplitude concentrated in degree a.

Definition 2.2. Fix $M \in \mathbf{D}(A)$ and $a, b \in \mathbb{Z} \cup \{\infty\}$.

- We say that M has p-complete Tor amplitude $\in [a, b]$ if $M \otimes^{\mathbb{L}}_A A/pA \in \mathbf{D}(A/pA)$ has Tor amplitude concentrated in $[a, b]$. If $a = b$, we say that $M \in \mathbf{D}(A)$ has p-complete Tor amplitude concentrated in degree a.
- We say that M is p-completely (faithfully) flat if $M \otimes^{\mathbb{L}}_A A/pA \in \mathbf{D}(A/pA)$ is concentrated in degree 0 and is a (faithfully) flat A/pA-module.

Note that $M \in \mathbf{D}(A)$ having Tor amplitude concentrated in degree 0 just means that M is concentrated in degree 0 and is a flat A-module.
Therefore $M \in \mathbf{D}(A)$ is p-completely flat if and only if it has p-complete Tor amplitude concentrated in degree 0.

Remark 2.3. One can replace in the definition A/pA by A/p^nA for every $n \geq 1$ without changing its meaning.
Indeed, suppose that we have an extension of rings $R \to S$ with $S = R/I$ for an ideal I such that $I^2 = 0$ (I is canonically an S-module). Then $M \in \mathbf{D}(R)$ has tor amplitude in $[a, b]$ if and only if $M \otimes^{\mathbb{L}}_R S \in \mathbf{D}(S)$ has tor amplitude in $[a, b]$.
The only if part, is just a consequence of the stability of the tor amplitude under base change. And for the if part, consider the exact triangle $I \to R \to S$. Applying $M \otimes^{\mathbb{L}}_R -$ gives an exact triangle

$$(M \otimes^{\mathbb{L}}_R S) \otimes^{\mathbb{L}}_S I \to M \to M \otimes^{\mathbb{L}}_R S$$

The leftmost term is in $\mathbf{D}^{[a,b]}(R)$ so tensoring with an R-module N we get an object of $\mathbf{D}^{[a,b]}(R)$. Also by hypothesis we have $(M \otimes^{\mathbb{L}}_R S) \otimes^{\mathbb{L}}_R N = (M \otimes^{\mathbb{L}}_R S) \otimes^{\mathbb{L}}_S (N \otimes^{\mathbb{L}}_R S) \in \mathbf{D}^{[a,b]}(R)$. Therefore $M \otimes^{\mathbb{N}}_R \in \mathbf{D}^{[a,b]}(R)$.

Lemma 2.4. *Fix $M \in \boldsymbol{D}(A)$ and $a, b \in \mathbb{Z} \cup \{\infty\}$. Let $\widehat{M} \in \boldsymbol{D}(A)$ be the derived p-completion of M. The following are equivalent*

1. *M has p-complete* Tor *amplitude in $[a, b]$ (resp. is p-completely (faithfully) flat)*
2. *$\widehat{M}$ has p-complete* Tor *amplitude in $[a, b]$ (resp. is p-completely (faithfully) flat)*

Proof. The map $M \mapsto \widehat{M}$ induces an isomorphism $M \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z} \cong \widehat{M} \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z}$. Indeed, we have for every $N \in \mathbf{D}(A)$,

$$\begin{aligned} \mathrm{Hom}_{\mathbf{D}(A)}(\widehat{M} \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z}, N) &\cong \mathrm{Hom}_{\mathbf{D}(A)}(\widehat{M} \otimes^{\mathbb{L}}_A (\mathbb{Z}/p\mathbb{Z} \otimes^{\mathbb{L}}_{\mathbb{Z}} A), N) \\ &\cong \mathrm{Hom}_{\mathbf{D}(A)}(\widehat{M}, R\,\mathrm{Hom}_A(\mathbb{Z}/p\mathbb{Z} \otimes^{\mathbb{L}}_{\mathbb{Z}} A, N)) \end{aligned}$$

$R\,\mathrm{Hom}_A(\mathbb{Z}/p\mathbb{Z} \otimes^{\mathbb{L}}_{\mathbb{Z}} A, N)$ can easily be seen to be derived p-complete (use 2. of lemma 1.1). Hence by proposition 1.5, we get that

$$\begin{aligned} Hom_{\mathbf{D}(A)}(M \otimes^{\mathbb{L}}_A \mathbb{Z}/p\mathbb{Z}, N) &\cong \mathrm{Hom}_{\mathbf{D}(A)}(\widehat{M}, R\,\mathrm{Hom}_A(\mathbb{Z}/p\mathbb{Z} \otimes^{\mathbb{L}}_{\mathbb{Z}} A, N)) \\ &\cong \mathrm{Hom}_{\mathbf{D}(A)}(M \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z}, N) \end{aligned}$$

which shows the claim.
Now notice that we have $A \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z} = (\cdots \to 0 \to A \xrightarrow{p} A \to 0 \to \dots)$ is quasi-isomorphic to $(\cdots \to 0 \to A[p] \xrightarrow{0} A/pA \to 0 \to \dots)$. This induces an isomorphism

$$\begin{aligned} M \otimes^{\mathbb{L}}_A A/pA \oplus M[1] \otimes^{\mathbb{L}}_A A[p] &\cong M \otimes^{\mathbb{L}}_A (A \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z}) \\ &\cong M \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z} \\ &\cong \widehat{M} \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z} \\ &\cong \widehat{M} \otimes^{\mathbb{L}}_A A/pA \oplus \widehat{M}[1] \otimes^{\mathbb{L}}_A A[p] \end{aligned}$$

Since the morphism induced from $M \to \widehat{M}$ by $- \otimes^{\mathbb{L}}_A (A \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z}$ respects the summands, we get that $M \otimes^{\mathbb{L}}_A A/pA \cong \widehat{M} \otimes^{\mathbb{L}}_A A/pA$, which gives the result.

$\square$

Lemma 2.5. *Let $A \to B$ be a map of rings, $M \in \boldsymbol{D}(A)$ and $a, b \in \mathbb{Z} \cap \{\infty\}$.*

1. *If $M \in \boldsymbol{D}(A)$ has p-complete* Tor *amplitude in $[a, b]$ (resp. p-completely (faithfully) flat), then the same holds true for $M \otimes^{\mathbb{L}}_A B \in \boldsymbol{D}(B)$.*

2. *If $A \to B$ is p-completely faithfully flat, then the converse of* 1. *holds true.*

Proof. This is immediate from the discrete case. $\square$

Lemma 2.6. *Suppose that A has p^∞-torsion and let $M \in \boldsymbol{D}(A)$ be derived p-complete with p-complete tor amplitude in $[a, b]$, $a, b \in \mathbb{Z} \cup \{\infty\}$. Then $M \in \boldsymbol{D}^{[a,b]}(A)$.*

Proof. By lemma 1.8, M is the derived limit of $M \otimes^{\mathbb{L}}_A A/p^n A$. But by remark 2.3, all $M \otimes^{\mathbb{L}}_A A/p^n A \in \mathbf{D}^{[a,b]}(A/p^n A)$. Looking at the long exact sequence of cohomology from the exact triangle

$$M \to \prod_n M \otimes^{\mathbb{L}}_A A/p^n A \to \prod_n M \otimes^{\mathbb{L}}_A A/p^n A$$

and noticing that the maps on the highest degree $H^b(M \otimes^{\mathbb{L}}_A A/p^n A)$ are surjective, we get that $M \in \mathbf{D}^{[a,b]}(A)$. $\square$

Lemma 2.7. *Suppose that A has bounded p^∞-torsion.*

1. *If $M \in \boldsymbol{D}(A)$ is derived p-complete and p-completely flat then it is a classically p-complete A-module concentrated in degree 0, with bounded p^∞-torsion, such that $M/p^n M$ is flat over $A/p^n A$ for every $n \geq 1$. Moreover, for every $n \geq 1$, the map*
$$M \otimes_A A[p^n] \to M[p^n]$$
is an isomorphism.

2. *Conversely, if N is a classically p-adically complete A-module with bounded p^∞-torsion such that $N/p^n N$ is flat over $A/p^n A$ for all $n \geq 1$, then $N[0] \in \boldsymbol{D}(A)$ is p-completely flat.*

Proof. **1)** Lemma 2.6 implies that M is concentrated in degree 0. The condition that M is p-completely flat implies that $M \otimes^{\mathbb{L}}_A A/p^n A$ is a flat $A/p^n A$-module for all $n \geq 1$. But

$$\begin{aligned} M \otimes^{\mathbb{L}}_A A/p^n A = M \otimes^{\mathbb{L}}_A (\cdots \to A \xrightarrow{p^n} A \to 0 \to \dots) &= (\cdots \to M \xrightarrow{p^n} M \to 0 \to \dots) \\ \cong (\cdots \to M/p^n M \to \dots) \in \mathbf{D}^{[0,0]}(A/p^n A) \end{aligned}$$

So $M \otimes^{\mathbb{L}}_A A/p^n A = M/p^n M$ is a flat $A/p^n A$-module for all $n \geq 1$. Moreover, by lemma 1.8, M is the limit of $M \otimes^{\mathbb{L}}_A A/p^n A = M/p^n M$ so it is classically p-complete. $\square$

Corollary 2.8. *Let $A \to B$ be a map of derived p-complete rings.*

1. *If A has bounded p^∞-torsion and $A \to B$ is p-completely flat, then B has bounded p^∞-torsion.*

2. *Conversely, if B has bounded p^∞-torsion and $A \to B$ is p-completely faithfully flat, then A has bounded p^∞-torsion.*

3. *Assume that A and B both have bounded p^∞-torsion. Then the map $A \to B$ is p-completely flat (resp. p-completely faithfully flat) if and only if $A/p^n \to B/p^nB$ is flat (resp. faithfully flat) for all $n \geq 1$.*

3 Appendix

3.1 The Koszul Complex

Definition 3.1. Let R be a ring, E an R-module and $\varphi : E \to R$ an R-module map. We define the Koszul complex $\mathbf{K}_\bullet(\varphi)$ to be the commutative differential graded algebra verifying

1. the underlying graded algebra is the exterior algebra $\wedge(E)$

2. the derivation $d : \mathbf{K}_\bullet(\varphi) \to \mathbf{K}_\bullet(\varphi)$ is the unique derivation such that $d(e) = \varphi(e)$ for all $e \in E = \mathbf{K}_1(\varphi)$

If $e_1 \wedge \cdots \wedge e_n$ is one of the generators of degree n in $\mathbf{K}_\bullet(\varphi)$, then

$$d(e_1 \wedge \cdots \wedge e_n) = \sum_i (-1)^{i+1} \varphi(e_i) e_1 \wedge \cdots \wedge \widehat{e_i} \wedge \cdots \wedge e_n$$

If $f_1, \ldots, f_n \in R$, the Koszul complex on $f_1, \ldots, f_n$, denoted by $\mathbf{K}_\bullet(f_\bullet)$ is the Koszul complex associated to the map $(f_1, \ldots, f_r) : R^r \to R$.

Lemma 3.2. *Let $e \in E$ and $f = \varphi(e) \in R$. Then we have*

$$f = de + ed$$

as endomorphisms of $\boldsymbol{K}_\bullet(\varphi)$.
In particular, multiplication by f_i is homotopic to zero on $\boldsymbol{K}_\bullet(f_\bullet)$. So the homology module $H_i(\boldsymbol{K}_\bullet(f_\bullet))$ are annihilated by $(f_1, \ldots, f_r)$

Proof. We have $d(ea) = d(e)a - ed(a) = fa - ed(a)$. □

Lemma 3.3. *The alternating Cech complex*

$$R \to \prod_{i_0} R_{f_{i_0}} \to \prod_{i_0 < i_1} R_{f_{i_0}, f_{i_1}} \to \cdots \to R_{f_1 \ldots f_r}$$

is the colimit of the Koszul complexes $\boldsymbol{K}_\bullet(f_\bullet^n)$.

Proof. The transition maps $\mathbf{K}_\bullet(f_\bullet^n) \to \mathbf{K}_\bullet(f_\bullet^{n+1})$ send $e_{i_0} \wedge \cdots \wedge e_{i_p}$ to $f_{i_0} \ldots f_{i_p} e_{i_0} \wedge \cdots \wedge e_{i_p}$. Hence by sending each Koszul complex to the complex $R \to \prod_{i_0} R \to \prod_{i_0 < i_1} R \to \cdots \to R$ (the obvious map), we get the result by noticing that $R_g = \mathrm{colim}(\cdots \to R \xrightarrow{g} R \xrightarrow{g} R)$. □

Corollary 3.4. *If $(f_1, \ldots, f_r) = R$ then the alternating Cech complex is acyclic.*

Proof. This combines lemma 3.2 and 3.3. □