Emergent Behavior in Flocks

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***Abstract*—We provide a model (for both continuous and discrete time) describing the evolution of a flock. Our model is parameterized by a constant β capturing the rate of decay—which in our model is polynomial—of the influence between birds in the flock as they separate in space. Our main result shows that when $\beta < 1/2$ convergence of the flock to a common velocity is guaranteed, while for $\beta \geq 1/2$ convergence is guaranteed under some condition on the initial positions and velocities of the birds only.**

***Index Terms*—Consensus reaching problem, emergence, flocking.**

I. Introduction

A GENERAL theme underlying the ideas in this paper is the reaching of consensus without a central direction. A common example of this situation is the emergence of a common belief in a price system when activity takes place in a given market. Another example is the emergence of common languages in primitive societies, or the dawn of vowel systems. As a motivating example in this paper we consider a population, say of birds or fish, whose members are moving in $\mathbb{R}^3$. It has been observed that under some initial conditions, for example on their positions and velocities, the state of the flock converges to one in which all birds fly with the same velocity. A goal of this paper is to provide some justification of this observation. To do so, we will postulate a model for the evolution of the flock and exhibit conditions on the initial state under which a convergence as above is established. In case these conditions are not satisfied, dispersion of the flock may occur. Several parameters give flexibility to our model. A remarkable feature is the existence of critical values for some of these parameters below which convergence is guaranteed. While we focus on this example, our treatment will be abstract enough to provide general insight in other situations.

There has been a large amount of literature on flocking, herding and schooling. Much of it is descriptive, most of the remaining proposes models, which are then studied via computer simulations, e.g., [1] and [2]. A starting point for this paper is the model proposed in the latter of these references which, for convenience, we will call Vicsek's model. Its analytic behavior was subsequently studied in [3] (but convergence could be simply deduced from previous work [4], [5, Lemma 2.1]) and this paper, brought to our attention by Ali Jadbabaie, has been helpful for us. Other work related to ours is [6]–[9]. We note, however, and we will return to this in Remark 1, that convergence results in these references rely on an assumption on the infinite time-sequence of states. In contrast with the above, our convergence results depend on conditions on the initial state only. That is a main virtue of our work. On the other hand, our hypothesis implies that each bird influences all of the other through the adjacency matrix, no matter what the configuration of the birds. Of course, we are making idealizations in this hypothesis. The literature suggests many interpretations for our set up. For example, the distance funtion could be the usual distance in Euclidean space and over large distances the influences could become negligible as in gravity. In another interpretation, the distance function could be interpreted as a visual distance and the euclidean space could be interpreted so that as the norm of x goes to infinity, the ability to communicate visually goes to zero. The methods in this paper can be extended to cover flocking situations where the complete weighted graph is no longer assumed and where symmetry is relaxed. A manuscript is being prepared.

Our model postulates the following behavior: Every bird adjusts its velocity by adding to it a weighted average of the differences of its velocity with those of the other birds. That is, at time $t \in \mathbb{N}$, and for bird i

$$v_i(t+1) - v_i(t) = \sum_{j=1}^{k} a_{ij} \left(v_j(t) - v_i(t) \right). \tag{1}$$

Here, the weights $\{a_{ij}\}$ quantify the way the birds influence each other. It is reasonable to assume that this influence is a function of the distance between birds. We give form to this assumption via a nonincreasing function $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ such that the *adjacency matrix* A_x has entries

$$a_{ij} = \eta \left(\|x_i - x_j\|^2 \right). \tag{2}$$

In this paper we will take, for some fixed K, $\sigma > 0$ and $\beta \geq 0$,

$$\eta(y) = \frac{K}{(\sigma^2 + y)^\beta}. \tag{3}$$

We can write the set of equalities (1) in a more concise form. Let A_x be the $k \times k$ matrix with entries a_{ij}, D_x be the $k \times k$

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diagonal matrix whose ith diagonal entry is $d_i = \sum_{j\le k} a_{ij}$ and $L_x = D_x - A_x$. Then

$$\begin{aligned} v_i(t+1) - v_i(t) &= -\sum_{j=1}^{n} a_{ij}\left(v_i(t) - v_j(t)\right) \\ &= -\left(\sum_{j=1}^{n} a_{ij}\right) v_i(t) + \sum_{j=1}^{n} a_{ij} v_j(t) \\ &= -\left[D_x v(t)\right]_i + \left[A_x v(t)\right]_i \\ &= -\left[L_x v(t)\right]_i . \end{aligned}$$

Note that the matrix notation $A_x v(t)$ does not have the usual meaning of a $k \times k$ matrix acting on $\mathbb{R}^k$. Instead, the matrix A_x is acting on $(\mathbb{R}^3)^k$ by mapping $(v_1, \ldots, v_k)$ to $(a_{i1}v_1 + \ldots + a_{ik}v_k)_{i\le k}$. The same applies to L_x.

Adding a natural equation for the change of positions we obtain the system

$$\begin{aligned} x(t+1) &= x(t) + \Delta t v(t) \\ v(t+1) &= (\mathrm{Id} - L_x) v(t). \end{aligned} \tag{4}$$

We also consider evolution for continuous time. The corresponding model can be given by the system of differential equations

$$\begin{aligned} x' &= v \\ v' &= -L_x v. \end{aligned} \tag{5}$$

Our two main results give conditions to ensure that the birds' velocities converge to a common one and the distance between birds remain bounded, for both continuous and discrete time. They can be stated as follows (more precise statements are in Theorems 2 and 3).

Theorem 1: Let $(x(t), v(t))$ be a solution of (4) with initial conditions $x(0) = x_0$ and $v(0) = v_0$. Assume that $K < (\sigma^{2\beta}/(k-1)\sqrt{k})$. If $\beta < 1/2$ then, when $t \to \infty$ the velocities $v_i(t)$ tend to a common limit $\widehat{v} \in \mathbb{R}^3$ and the vectors $x_i - x_j$ tend to a limit vector $\widehat{x_{ij}}$, for all $i, j \le k$. The same happens if $\beta \ge 1/2$ provided the initial values x_0 and v_0 satisfy a given, explicit, relation.

The same holds for a solution of (5) (but in this case the assumption on K is not necessary).

Remark 1: Although our model (4) is related to Vicsek's, there are some differences which stand out. Vicsek's model is motivated by the idea that bird i has a velocity with constant magnitude, adjusts its heading (or angular velocity) towards the average of its neighbors' headings, and uses a different way of averaging. Actually, Vicsek's model supposes that the heading θ is updated according with the law

$$\theta_i(t+1) = \frac{1}{n_i(t)} \sum_{j\in\mathcal{N}_i(t)} \theta_j(t) \tag{6}$$

where $\mathcal{N}_i(t) = \{j \le k \,|\, \|x_i(t) - x_j(t)\| \le r\}$ and $n_i(t) = \#\mathcal{N}_i(t)$ for some $r > 0$. That is, the updated heading of a bird is the average of the headings of those birds at a distance at most r.

System (6) can be written in a form similar to (4). To do so, consider again the adjacency matrix A_x with $a_{ij} = \eta(\|x_i - x_j\|^2)$ but where now

$$\eta(y) = \begin{cases} 1, & \text{if } y \le r^2 \\ 0, & \text{otherwise.} \end{cases} \tag{7}$$

It is not difficult to check that (6) takes the matrix form

$$\theta(t+1) = \left(\mathrm{Id} - D_x^{-1} L_x\right) \theta(t). \tag{8}$$

Note that, in contrast with the abrupt behavior of the function in (7), the function in (3) decreases continuously with y and the rate of decay is given by $\beta > 0$.

This contrast is at the heart of one of the main differences between Vicsek's model and ours. The adjacency matrix associated to Vicsek's model corresponds to a simple graph. Convergence to a common heading will thus depend on connectivity properties of the successive configurations of the birds and proofs of convergence make assumptions on the infinite time-sequence of these configurations. The adjacency matrix associated to our model corresponds, instead, to a complete weighted graph, with weights decreasing to zero as birds separate. A key feature is now that if the decay of η is polynomial but moderately fast (i.e., if β is at least 0.5) convergence is guaranteed under some condition on the *initial* values x_0 and v_0 only. We believe this is a distinct feature of our analysis as compared with the literature on flocking.

In the original model proposed by Vicsek the magnitude of the bird's velocities is constrained to be constant. That is, the model is nonholonomic, and the control is in changing the angular velocity. In our model, each agent has inertia and the system is fully actuated. In other words, Vicsek's model is kinematic whereas our is dynamic.

II. Some Preliminaries

Given a nonnegative, symmetric, $k \times k$ matrix A the *Laplacian* L of A is defined to be

$$L = D - A$$

where $D = \mathsf{diag}(d_1, \ldots, d_k)$ and $d_\ell = \sum_{j=1}^{k} a_{\ell j}$. Some features of L are immediate. It is symmetric and it does not depend on the diagonal entries of A. The Laplacian as just defined has its origins in graph theory where the matrix A is the adjacency matrix of a graph G and many of the properties of G can be read out from L (see [10]).

The matrix L_x in (4) and (5) is thus the Laplacian of A_x. Denote 3-dimensional Euclidean space by $\mathbb{E}^3$ and let $(\mathbb{E}^3)^k$ be its k-fold product endowed with the induced inner product structure. Then L_x acts on $(\mathbb{E}^3)^k$ and satisfies the following.

a) For all $v \in \mathbb{E}^3$, $L_x(v, \ldots, v) = 0$.
b) If $\lambda_1, \ldots, \lambda_k$ are the eigenvalues of L_x, then

$$0 = l_1 \le l_2 \le \ldots \le l_k = \|L_x\|.$$

c) For all $v \in (\mathbb{E}^3)^k$,

$$\langle L_x v, v\rangle = \frac{1}{2}\sum_{i,j=1}^{k} a_{ij}\|v_i - v_j\|^2.$$

A proof for c) can be found in [11]. The other two properties are easy to prove. Note that b) implies L_x is positive semidefinite.

The quantity $E_x(v) = \sum_{i,j=1}^{k} a_{ij}\|v_i - v_j\|^2$ is the *energy* of the flock (at a position $x \in (\mathbb{E}^3)^k$ and a velocity $v \in (\mathbb{E}^3)^k$). Note that $E_x(v) = 0$ when all birds are flying with the same velocity. That is, they fly with the same heading and at the same speed.

The matrix $\mathrm{Id} - L_x$ in (4) acts on $(\mathbb{E}^3)^k$. The fixed points for this action are easily characterized.

Proposition 1: Let $v \in (\mathbb{E}^3)^k$. The following are equivalent.

1) v is a fixed point (i.e., $(\mathrm{Id} - L_x)v = v$).
2) $L_x(v) = 0$.
3) $E_x(v) = 0$.

Proof: The equivalence between 1) and 2) is obvious. The implication $(2) \implies (3)$ is trivial. Finally, note that 3) implies that $v_i = v_j$ for all $i \neq j$ and this, together with a) above, implies 2). ■

The second eigenvalue λ_2 of L_x is called the *Fiedler number* of A_x. We denote the Fiedler number of A_x by ϕ_x.

We end these preliminaries introducing some concepts which will be useful in this paper.

Let Δ be the diagonal of $(\mathbb{E}^3)^k$, i.e.,

$$\Delta = \left\{(v, v, \ldots, v) \mid v \in \mathbb{E}^3\right\}$$

and $\Delta^\perp$ be the orthogonal complement of Δ in $(\mathbb{E}^3)^k$. Then, every point $x \in (\mathbb{E}^3)^k$ decomposes in a unique way as $x = x_\Delta + x_\perp$ with $x_\Delta \in \Delta$ and $x_\perp \in \Delta^\perp$. Note that if $x(t+1) = x(t) + \Delta t v(t)$ then $x(t+1)_\perp = x(t)_\perp + \Delta t v(t)_\perp$. Similarly, if $v(t+1) = -(\mathrm{Id} - L_x)v(t)$ then

$$v(t+1)_\perp = -(\mathrm{Id} - L_x)v(t)_\perp$$

since $L_x(\Delta) = 0$ and $L_x(\Delta^\perp) \subseteq \Delta^\perp$. Finally, note that for all $x \in (\mathbb{E}^3)^k$ the matrices A_x and $A_{x_\perp}$ are equal. It follows that the projections over $\Delta^\perp$ of the solutions of (4) are the solutions of the restriction of (4) to $\Delta^\perp$. A similar remark holds for (5).

These projections over $\Delta^\perp$ are of the essence since we are interested on the differences $x_i - x_j$ and $v_i - v_j$, for $i \neq j$, rather than on the x_i or v_i themselves.

We denote $\Gamma = (1/2)\sum_{i\neq j}\|x_i - x_j\|^2$ and $\Lambda = (1/2)\sum_{i\neq j}\|v_i - v_j\|^2$. To better deal with these functions consider $Q : (\mathbb{E}^3)^k \times (\mathbb{E}^3)^k \to \mathbb{R}$ defined by

$$Q(u, v) = \frac{1}{2}\sum_{i,j=1}^{k}\langle u_i - u_j, v_i - v_j\rangle.$$

Then, Q is bilinear, symmetric, and, when restricted to $\Delta^\perp \times \Delta^\perp$, positive definite. It follows that it defines an inner product $\langle , \rangle_Q$ on $(\mathbb{E}^3)^k/\Delta \simeq \Delta^\perp$. Now note that $\Lambda = \|v\|_Q^2$ and $\Gamma = \|x\|_Q^2$ and that $\Gamma(x) = \Gamma(x_\perp)$ and $\Lambda(v) = \Lambda(v_\perp)$.

Let $\nu, \overline{\nu} > 0$ be such that, restricted to $\Delta^\perp$,

$$\nu\| \;\|^2 \leq \| \;\|_Q^2 \leq \overline{\nu}\| \;\|^2.$$

Note that $\nu, \overline{\nu}$ depend only on k. We now show bounds for them in terms of k.

Lemma 1: For all $k \geq 2$, $\nu(k) \geq (1/3k)$ and $\overline{\nu}(k) \leq 2k(k-1)$.

Proof: By definition, $\overline{\nu} \leq \max_{\|x\|=1}\|x\|_Q^2$. Since $\|x\| = 1$, $\|x_i\| \leq 1$ for $i = 1, \ldots, k$ and, therefore, $\|x_i - x_j\|^2 \leq 4$ for all $i \neq j$. This implies

$$\|x\|_Q^2 \leq \frac{1}{2}k(k-1)4 = 2k(k-1).$$

Also by definition, $(1/\nu) \leq \max_{\|x\|_Q=1}\|x\|^2$. Let $x \in \Delta^\perp$ such that $\|x\|_Q = 1$. We claim that, for all $i \leq k$ and $\ell \leq 3$, $|x_{i\ell}| < 1$. Assume the contrary. Then there exists i_0 and ℓ such that $|x_{i_0\ell}| \geq 1$. Without loss of generality, $x_{i_0\ell} \geq 1$. Since $\sum x_i = 0$, there exists i_1 such that $x_{i_1\ell} < 0$. However, then

$$\|x\|_Q^2 = \frac{1}{2}\sum_{i\neq j}\|x_i - x_j\|^2 \geq \|x_{i_0} - x_{i_1}\|^2 \geq (x_{i_0\ell} - x_{i_1\ell})^2 > 1$$

contradicting $\|x\|_Q^2 = 1$. So, the claim is proved. Finally

$$\|x\|^2 = \sum_{i=1}^{k}\sum_{\ell=1}^{3} x_{i\ell}^2 \leq 3k$$

which shows $(1/\nu) \leq 3k$. ■

Remark 2: The condition "the velocities $v_i(t)$ tend to a common limit $\widehat{v} \in \mathbb{E}^3$" in Theorem 1 is equivalent to the condition "$v_\perp(t) \to 0$." Also, the condition "the vectors $x_i - x_j$ tend to a limit vector $\widehat{x_{ij}}$, for all $i, j \leq k$" is equivalent to "$x_\perp(t)$ tend to a limit vector $\widehat{x}$ in $\Delta^\perp$." This suggests that we are actually interested on the solutions of the systems induced by (4) and (5), respectively, on the space $\Delta^\perp \times \Delta^\perp$. Since, as we mentioned, these induced systems have the same form as (4) and (5), we will keep referring to them but we will consider them on $\Delta^\perp \times \Delta^\perp$. Actually, we will consider positions in

$$X := (\mathbb{E}^3)^k/\Delta \simeq \Delta^\perp$$

and velocities in

$$V := (\mathbb{E}^3)^k/\Delta \simeq \Delta^\perp.$$

III. Convergence in Continuous Time

In the following, we fix a solution (x, v) of (5). At a time $t \in \mathbb{R}_+$, $x(t)$ and $v(t)$ are elements in X and V, respectively. In particular, $x(t)$ determines an adjacency matrix $A_{x(t)}$. For notational simplicity, we will denote this matrix by A_t and its

Laplacian and Fiedler number by L_t and ϕ_t, respectively. Similarly, we will write $\Lambda(t)$ and $\Gamma(t)$ for the values of Λ and Γ, respectively, at $(v(t), x(t))$. Finally, we will write Γ_0 for $\Gamma(0)$ and similarly for Λ_0.

The main result in this section is the following.

Theorem 2: Assume that, for some constants $K, \sigma > 0$ and $\beta \geq 0$

$$a_{ij} = \frac{K}{(\sigma^2 + \|x_i - x_j\|^2)^\beta}.$$

Assume also that one of the three following hypothesis hold.

i) $\beta < 1/2$.

ii) $\beta = 1/2$ and $\Lambda_0 < ((\nu K)^2/8)$.

iii) $\beta > 1/2$ and

$$\left[\left(\frac{1}{2\beta}\right)^{\frac{1}{2\beta-1}} - \left(\frac{1}{2\beta}\right)^{\frac{2\beta}{2\beta-1}}\right]\left(\frac{(\nu K)^2}{8\Lambda_0}\right)^{\frac{1}{2\beta-1}} > 2\Gamma_0 + \sigma^2.$$

Then there exists a constant B_0 (independent of t, made explicit in the proof of each of the three cases) such that $\Gamma(t) \leq B_0$ for all $t \in \mathbb{R}_+$. In addition, $\Lambda(t) \to 0$ when $t \to \infty$. Finally, there exists $\widehat{x} \in X$ such that $x(t) \to \widehat{x}$ when $t \to \infty$.

We next prove some stepping stones towards the proof of Theorem 2. Denote $\Phi_t = \min_{\tau \in [0,t]} \phi_\tau$.

Proposition 2: For all $t \geq 0$

$$\Lambda(t) \leq \Lambda_0 e^{-2t\Phi_t}.$$

Proof: Let $\tau \in [0, t]$. Then

$$\begin{aligned}\Lambda'(\tau) &= \frac{d}{d\tau}\langle v(\tau), v(\tau)\rangle_Q \\ &= 2\langle v'(\tau), v(\tau)\rangle_Q \\ &= -2\langle L_\tau v(\tau), v(\tau)\rangle_Q \\ &\leq -2\phi_{x(\tau)}\Lambda(\tau).\end{aligned}$$

Here, we have used that L_τ is symmetric positive definite on V. Using this inequality

$$\ln(\Lambda(\tau))|_0^t = \int_0^t \frac{\Lambda'(\tau)}{\Lambda(\tau)} d\tau \leq \int_0^t -2\phi_\tau d\tau \leq -2t\Phi_t$$

i.e.,

$$\ln(\Lambda(t)) - \ln(\Lambda_0) \leq -2t\Phi_t$$

from which the statement follows. ■

Proposition 3: For $T > 0$

$$\Gamma(T) \leq 2\left(\Gamma_0 + \frac{\Lambda_0}{\Phi_T^2}\right).$$

Proof: We have $|\Gamma'(t)| = |2\langle v(t), x(t)\rangle_Q| \leq 2\|v(t)\|_Q\|x(t)\|_Q$. But $\|x(t)\|_Q = \Gamma(t)^{1/2}$ and $\|v(t)\|_Q^2 = \Lambda(t) \leq \Lambda_0 e^{-2t\Phi_t}$, by Proposition 2. Therefore

$$\Gamma'(t) \leq |\Gamma'(t)| \leq 2(\Lambda_0 e^{-2t\Phi_t})^{1/2}\Gamma(t)^{1/2} \tag{9}$$

and, using that $t \mapsto \Phi_t$ is nonincreasing

$$\begin{aligned}\int_0^T \frac{\Gamma'(t)}{\Gamma(t)^{1/2}} dt &\leq 2\int_0^T (\Lambda_0 e^{-2t\Phi_t})^{1/2} dt \\ &\leq 2\int_0^T \Lambda_0^{1/2} e^{-t\Phi_T} dt \\ &= 2\Lambda_0^{1/2}\left(-\frac{1}{\Phi_T}\right)e^{-t\Phi(T)}\Big|_0^T \leq \frac{2\Lambda_0^{1/2}}{\Phi_T}\end{aligned}$$

which implies

$$\Gamma(t)^{1/2}\Big|_0^T = \frac{1}{2}\int_0^T \frac{\Gamma'(t)}{\Gamma(t)^{1/2}} dt \leq \frac{\Lambda_0^{1/2}}{\Phi_T}$$

from which it follows that

$$\Gamma(T) \leq \left(\Gamma_0^{1/2} + \frac{\Lambda_0^{1/2}}{\Phi_T}\right)^2.$$

The statement now follows from the elementary inequality $(\alpha + \beta)^2 \leq 2(\alpha^2 + \beta^2)$. ■

Proposition 4: Let A be a nonnegative, symmetric matrix, $L = D - A$ its Laplacian, ϕ its Fiedler number, and $\mu = \min_{i \neq j} a_{ij}$. Then, $\phi \geq (1/2)\nu\mu$. In particular, if $a_{ij} = \eta(\|x_i - x_j\|^2)$ then

$$\phi \geq \frac{1}{2}\nu\eta(\Gamma(x)).$$

Proof: For all $v \in V$

$$\begin{aligned}\|Lv\|\|v\| \geq \langle Lv, v\rangle &= \frac{1}{2}\sum_{i,j=1}^k a_{ij}\|v_i - v_j\|^2 \\ &\geq \frac{1}{2}\mu\|v\|_Q^2 \geq \frac{1}{2}\nu\mu\|v\|^2.\end{aligned}$$

It follows that $\|Lv\| \geq (1/2)\nu\mu\|v\|$ and thus the statement. ■

A proof of the following lemma is in [12, Lemma 7].

Lemma 2: Let $c_1, c_2 > 0$ and $s > q > 0$. Then the equation

$$F(z) = z^s - c_1 z^q - c_2 = 0$$

has a unique positive zero z_*. In addition

$$z_* \leq \max\left\{(2c_1)^{\frac{1}{s-q}}, (2c_2)^{\frac{1}{s}}\right\}$$

and $F(z) \leq 0$ for $0 \leq z \leq z^*$. ■

Proof of Theorem 2: By Proposition 4, for all $x \in X$,

$$\phi_x \geq \frac{\nu K}{2(\sigma^2 + \max_{i\neq j}\|x_i - x_j\|^2)^\beta} \geq \frac{\nu K}{2(\sigma^2 + \Gamma_x)^\beta}.$$

Let $t^* \in [0,t]$ be the point maximizing Γ in $[0,t]$. Then

$$\Phi_t = \min_{\tau\in[0,t]} \phi_\tau \geq \min_{\tau\in[0,t]} \frac{\nu K}{2\left(\sigma^2+\Gamma(\tau)\right)^\beta} \geq \frac{\nu K}{2\left(\sigma^2+\Gamma(t^*)\right)^\beta}.$$

By Proposition 3

$$\Gamma(t) \leq 2\Gamma_0 + 8\Lambda_0 \frac{\left(\sigma^2+\Gamma(t^*)\right)^{2\beta}}{(\nu K)^2}. \tag{10}$$

Since t^* maximizes Γ in $[0,t]$ it also does so in $[0,t^*]$. Thus, for $t=t^*$, (10) takes the form

$$\left(\sigma^2+\Gamma(t^*)\right) - 8\Lambda_0 \frac{\left(\sigma^2+\Gamma(t^*)\right)^{2\beta}}{(\nu K)^2} - (2\Gamma_0+\sigma^2) \leq 0. \tag{11}$$

Let $z=\Gamma(t^*)+\sigma^2$,

$$\mathbf{a} = \frac{8\Lambda_0}{(\nu K)^2}, \quad \text{and} \quad \mathbf{b} = 2\Gamma_0+\sigma^2.$$

Then (11) can be rewritten as $F(z)\leq 0$ with

$$F(z) = z - \mathbf{a}z^{2\beta} - \mathbf{b}.$$

i) Assume $\beta < 1/2$. By Lemma 2, $F(z)\leq 0$ implies that $z = (\sigma^2+\Gamma(t^*)) \leq U_0$ with

$$U_0 = \max\left\{ \left(\frac{16\Lambda_0}{(\nu K)^2}\right)^{\frac{1}{1-2\beta}}, 2(2\Gamma_0+\sigma^2)\right\}.$$

That is $\Gamma(t^*) \leq B_0 := U_0 - \sigma^2$. Since B_0 is independent of t, we deduce that, for all $t\in\mathbb{R}_+$, $\Gamma(t)\leq B_0$. However, this implies that $\phi_t \geq (\nu K/2(\sigma^2+B_0)^\beta)$ for all $t\in\mathbb{R}_+$ and, therefore, the same bound holds for Φ_t. By Proposition 2

$$\Lambda(t) \leq \Lambda_0 e^{-\frac{\nu K}{(\sigma^2+B_0)^\beta}t} \tag{12}$$

which shows that $\Lambda(t)\to 0$ when $t\to\infty$. Finally, for all $T>t$

$$\begin{aligned}
\|x(T)-x(t)\| &= \left\|\int_t^T v\right\| \leq \int_t^T \|v\| \leq \int_t^T \frac{1}{\nu}\Lambda^{1/2} \\
&\leq \int_t^T \frac{1}{\nu}\Lambda_0^{1/2} e^{-\frac{\nu K}{2(\sigma^2+B_0)^\beta}s} ds \\
&= \frac{1}{\nu}\Lambda_0^{1/2}\left(-\frac{2(\sigma^2+B_0)^\beta}{\nu K} e^{-\frac{\nu K}{2(\sigma^2+B_0)^\beta}s}\right)\Bigg|_t^T \\
&= \frac{2\Lambda_0^{1/2}(\sigma^2+B_0)^\beta}{\nu^2 K} \\
&\quad\times\left(e^{-\frac{\nu K}{2(\sigma^2+B_0)^\beta}t} - e^{-\frac{\nu K}{2(\sigma^2+B_0)^\beta}T}\right) \\
&\leq \frac{2\Lambda_0^{1/2}(\sigma^2+B_0)^\beta}{\nu^2 K} e^{-\frac{\nu K}{2(\sigma^2+B_0)^\beta}t}.
\end{aligned}$$

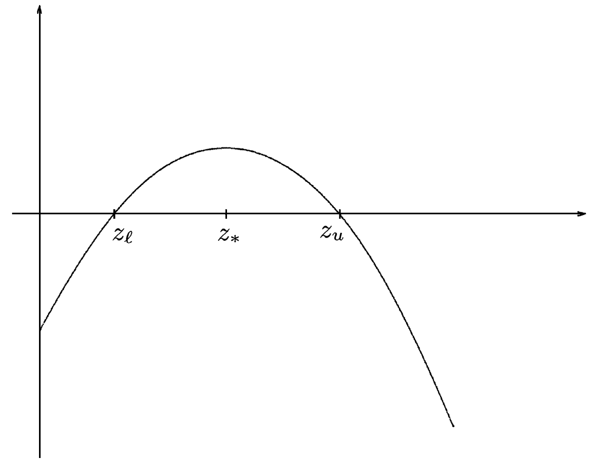

Fig. 1. Shape of F.

Since the last tends to zero with t and is independent of T we deduce that there exists $\widehat{x}\in X$ such that, $x\to\widehat{x}$.

ii) Assume now $\beta=1/2$. Then (11) takes the form

$$\left(\sigma^2+\Gamma(t^*)\right)\left(1-\frac{8\Lambda_0}{(\nu K)^2}\right) - (2\Gamma_0+\sigma^2) \leq 0$$

which implies that

$$\Gamma(t^*) \leq B_0 := \frac{2\Gamma_0+\sigma^2}{1-\frac{8\Lambda_0}{(\nu K)^2}} - \sigma^2.$$

Note that $B_0>0$ since $\Lambda_0 < ((\nu K)^2/8)$. We now proceed as in case i).

iii) Assume finally $\beta > 1/2$ and let $\alpha = 2\beta$ so that $F(z) = z - \mathbf{a}z^\alpha - \mathbf{b}$. The derivative $F'(z) = 1-\alpha\mathbf{a}z^{\alpha-1}$ has a unique zero at $z_* = (1/\alpha\mathbf{a})^{1/(\alpha-1)}$ and

$$\begin{aligned}
F(z_*) &= \left(\frac{1}{\alpha\mathbf{a}}\right)^{\frac{1}{\alpha-1}} - \mathbf{a}\left(\frac{1}{\alpha\mathbf{a}}\right)^{\frac{\alpha}{\alpha-1}} - \mathbf{b} \\
&= \left(\frac{1}{\alpha}\right)^{\frac{1}{\alpha-1}}\left(\frac{1}{\mathbf{a}}\right)^{\frac{1}{\alpha-1}} - \left(\frac{1}{\alpha}\right)^{\frac{\alpha}{\alpha-1}}\left(\frac{1}{\mathbf{a}}\right)^{\frac{1}{\alpha-1}} - \mathbf{b} \\
&= \left(\frac{1}{\mathbf{a}}\right)^{\frac{1}{\alpha-1}}\left[\left(\frac{1}{\alpha}\right)^{\frac{1}{\alpha-1}} - \left(\frac{1}{\alpha}\right)^{\frac{\alpha}{\alpha-1}}\right] - \mathbf{b} \\
&\geq 0
\end{aligned}$$

the last by our hypothesis. Since $F(0) = -\mathbf{b} < 0$ and $F(z)\to-\infty$ when $z\to\infty$ we deduce that the shape of F is as shown in Fig. 1.

Even though t^* is not continuous as a function of t, the mapping $t\mapsto\Gamma(t^*)+\sigma^2$ is continuous and, therefore, so is the mapping $t\mapsto F(\Gamma(t^*)+\sigma^2)$. This fact, together with (11), shows that, for all $t \geq 0$, $F(\Gamma(t^*) + \sigma^2) \leq 0$. In addition, when $t = 0$ we have $t^* = 0$ as well and

$$\begin{aligned}\Gamma_0 + \sigma^2 &\leq 2\Gamma_0 + \sigma^2 = \mathbf{b} \\ &< \left(\frac{1}{\mathbf{a}}\right)^{\frac{1}{\alpha-1}} \left[\left(\frac{1}{\alpha}\right)^{\frac{1}{\alpha-1}} - \left(\frac{1}{\alpha}\right)^{\frac{\alpha}{\alpha-1}}\right] \\ &< \left(\frac{1}{\mathbf{a}}\right)^{\frac{1}{\alpha-1}} \left(\frac{1}{\alpha}\right)^{\frac{1}{\alpha-1}} \\ &= z_*.\end{aligned}$$

This implies that $\Gamma_0 + \sigma^2 < z_\ell$ (the latter being the smallest zero of F on $\mathbb{R}_+$, see the previous figure) and the continuity of the map $t \mapsto \Gamma(t^*) + \sigma^2$ implies that, for all $t \geq 0$,

$$\Gamma(t^*) + \sigma^2 \leq z_\ell \leq z_*.$$

Therefore

$$\Gamma(t^*) \leq B_0 := \left(\frac{1}{\alpha\mathbf{a}}\right)^{\frac{1}{\alpha-1}} - \sigma^2 = \left(\frac{(\nu K)^2}{8\alpha\Lambda_0}\right)^{\frac{1}{\alpha-1}} - \sigma^2.$$

We now proceed as in case i). ∎

Remark 3:

i) In Theorem 2, the condition that $a_{ij} = K/(\sigma^2 + \|x_i - x_j\|^2)^\beta$ may be relaxed to $a_{ij} \geq (K/(\sigma^2 + \|x_i - x_j\|^2)^\beta)$.

ii) The bound $\beta < 1/2$ for unconditional convergence in Theorem 2 is essentially sharp. We will indicate this in Remark 4 by studying the special case of a flock with two birds flying on a line.

IV. A Flock of Two Birds

We give here a more detailed analysis of the case of two birds flying on a line (i.e., we take $\mathbb{R}$ instead of $\mathbb{E}^3$ for both positions and velocities).

We define $\mathbf{x} = x_1 - x_2$ and $\mathbf{v} = v_1 - v_2$ and assume that the state $(\mathbf{x}, \mathbf{v})$ of the pair satisfies the system of ODE's

$$\begin{aligned}\mathbf{x}' &= \mathbf{v} \\ \mathbf{v}' &= -\frac{\mathbf{v}}{(1+\mathbf{x})^\alpha}.\end{aligned} \tag{13}$$

This is not exactly (5) but it is easier to dealt with and, we will see below, it is close to this system.

The arguments used in the preceding section show that when $\alpha < 1$, for all initial $\mathbf{x}_0$ and $\mathbf{v}_0$, we have that $\mathbf{x}$ is bounded and $\mathbf{v} \to 0$ when $t \to \infty$. The next proposition gives conditions on $\mathbf{x}_0$ and $\mathbf{v}_0$ for such a convergence to hold when $\alpha > 1$.

Proposition 5: Let $\alpha > 1$. Assume that $\mathbf{x}_0 > 0$ and $\mathbf{v}_0 > 0$ and that

$$\mathbf{x}_0 < \widehat{\mathbf{x}_0} := \left(\frac{\alpha - 1}{\mathbf{v}_0}\right)^{\frac{1}{\alpha-1}} + 1.$$

Then, $\mathbf{x}$ is bounded and increasing. In addition, when $t \to \infty$, $\mathbf{v}(t) \to 0$ and

$$\mathbf{x}(t) \to \left(\frac{\alpha - 1}{\mathbf{v}_0 - \frac{\alpha-1}{(1+\mathbf{x}_0)^{\alpha-1}}}\right)^{\frac{1}{\alpha-1}} + 1.$$

Proof: It follows from the system (13) that, for all $t \geq 0$,

$$\int_0^t \mathbf{v}' = \int_0^t \frac{\mathbf{x}'}{(1+\mathbf{x})^\alpha}$$

and therefore, integrating both sides between 0 and t, that

$$\mathbf{v}(t) - \mathbf{v}_0 = \frac{\alpha - 1}{(1+\mathbf{x}(t))^{\alpha-1}} - \frac{\alpha - 1}{(1+\mathbf{x}_0)^{\alpha-1}}$$

or yet, that

$$\mathbf{v}(t) = \frac{\alpha - 1}{(1+\mathbf{x}(t))^{\alpha-1}} - \gamma_0 \tag{14}$$

where $\gamma_0 > 0$ since $\mathbf{x}_0 < \widehat{\mathbf{x}_0}$.

If, for some t_*, $\mathbf{v}(t_*) = 0$ then $\mathbf{v}'(t_*) = 0$. But then the pair $(\widetilde{x}, \widetilde{v})$ defined by $\widetilde{x}_1(t) = x_1(t_*)$, $\widetilde{x}_2(t) = x_2(t_*)$ and $\widetilde{v}(t) = 0$, for all $t \geq 0$, is a solution of (13) satisfying the conditions $\widetilde{\mathbf{x}}(t_*) = \mathbf{x}(t_*)$ and $\widetilde{\mathbf{v}} = 0$. By the unicity of the solutions of (13) it follows that $\widetilde{\mathbf{v}} = \mathbf{v}$ and hence that $\widetilde{\mathbf{v}}_0 = 0$ in contradiction with our assumptions. We conclude that $\mathbf{v}(t) > 0$ for all $t \geq 0$. However, then

$$0 < \mathbf{v}(t) = \frac{\alpha - 1}{(1+\mathbf{x}(t))^{\alpha-1}} - \gamma_0$$

implies that

$$\mathbf{x}(t) < \left(\frac{\alpha - 1}{\gamma_0}\right)^{\frac{1}{\alpha-1}} + 1.$$

Thus, $\mathbf{x}$ remains bounded on $\mathbb{R}_+$. Furthermore $\mathbf{x}$ is increasing since $\mathbf{v} > 0$. This implies that there exists $\mathbf{x}_* > 0$ such that $\mathbf{x}(t) \to \mathbf{x}_*$ and $\mathbf{x}'(t) \to 0$ when $t \to \infty$. It follows from $\mathbf{x}' = \mathbf{v}$ and (14) that $\mathbf{x}_*$ is as claimed. ∎

Remark 4: It follows from the proof of Proposition 5 that, for all $\alpha > 1$, $\mathbf{v}$ fails to converge if $\mathbf{x} \geq \widehat{\mathbf{x}_0}$. Also, for $\beta = \alpha/2$ and since

$$\frac{1}{(1+\mathbf{x})^\alpha} \leq \frac{1}{(1+\mathbf{x}^2)^\beta} \leq 2\frac{\sqrt{2}}{(1+\mathbf{x})^\alpha}$$

the system (5) is tightly bounded in between two versions of (13) differing only by a constant factor. This indicates that convergence may fail as well in (5) for $\beta > 1/2$.

V. Convergence in Discrete Time

We now focus on discrete time. The model is thus (4). A motivation to consider discrete time is that we want to derive (possibly a small variation of) our model from a mechanism based on exchanges of signals. The techniques to do so, learning theory, are better adapted to discrete time. Also, we want our model to include noisy environments and this issue becomes more technically involved in continuous time.

We assume as before that there are constants $K, \sigma > 0$ and $\beta \geq 0$ such that

$$a_{ij} = \frac{K}{(\sigma^2 + \|x_i - x_j\|^2)^\beta}.$$

Note that, by Proposition 4, this implies that $\phi_x > 0$ for all $x \in X$. This, in turn, shows that L_x is a self-adjoint, positive definite linear map, whose smallest eigenvalue is ϕ_x.

Lemma 3: For all $x \in X$

$$\|L_x\| \leq \mathcal{L} := \frac{2(k-1)\sqrt{k}K}{\sigma^{2\beta}}.$$

In particular, if $K < (\sigma^{2\beta}/(k-1)\sqrt{k})$, then $\|L_x\| < 2$.

Proof: For all $i, j \leq k$, $a_{ij} \leq (K/\sigma^{2\beta})$. Therefore

$$\|L_x\|_\infty = \max_{i \leq k} \sum_{j=1}^{k} |(L_x)_{ij}| \leq 2(k-1)\frac{K}{\sigma^{2\beta}}.$$

Now use that $\|L_x\| \leq \sqrt{k}\|L_x\|_\infty$ [13, Table 6.2] to deduce the result. ■

In what follows, we assume that $K < (\sigma^{2\beta}/(k-1)\sqrt{k})$ and, therefore, that $\mathcal{L} < 2$.

We also fix a solution (x, v) of (4). At a time $t \in \mathbb{N}$, $x(t)$ and $v(t)$ are elements in X and V, respectively. The meaning of expressions like ϕ_t, L_t, $\Lambda(t)$, or $\Gamma(t)$ is as described in Section III.

Proposition 6: For all $t \in \mathbb{N}$, $\|v(t+1)\| \leq (1-\mu(t))\|v(t)\|$ with

$$\mu(t) = \min\{2 - \mathcal{L}, \phi_t\}.$$

In particular, $\|v\|$ is decreasing as a function of t.

Proof: The linear map $\mathrm{Id} - L_t$ is self-adjoint and its eigenvalues are in the interval $(-1, 1)$. Its largest (in absolute value) eigenvalue is $1 - \mu(t)$. Therefore

$$\begin{aligned}\|v(t+1)\| &= \|(\mathrm{Id} - L_t)v(t)\| \\ &\leq \|\mathrm{Id} - L_t\| \, \|v(t)\| = (1 - \mu(t)) \, \|v(t)\|.\end{aligned}$$

■

Corollary 1: For all $t \in \mathbb{N}$, $\|v(t)\| \leq \|v(0)\| \prod_{i=0}^{t-1}(1 - \mu(i))$. ■

Theorem 3: Assume that, for some constants $\sigma > 0$, $\beta \geq 0$, and $K < (\sigma^{2\beta}/(k-1)\sqrt{k})$,

$$a_{ij} = \frac{K}{(\sigma^2 + \|x_i - x_j\|^2)^\beta}.$$

Assume also that one of the three following hypothesis hold.

i) $\beta < 1/2$.
ii) $\beta = 1/2$ and $\|v(0)\| \leq (\nu K / 2\overline{\nu}^{1/2}\Delta t)$.
iii) $\beta > 1/2$ and

$$\left(\frac{1}{\mathbf{a}}\right)^{\frac{2}{\alpha-1}} \left[\left(\frac{1}{\alpha}\right)^{\frac{2}{\alpha-1}} - \left(\frac{1}{\alpha}\right)^{\frac{\alpha+1}{\alpha-1}}\right] > \overline{\nu}\left(V_0^2 + 2V_0\left((\alpha\mathbf{a})^{-\frac{2}{\alpha-1}} - \sigma^2\right)\overline{\nu}^{-1/2}\right) + \mathbf{b}.$$

Here, $\alpha = 2\beta$, $V_0 := \Delta t \|v(0)\|$,

$$\mathbf{a} = \frac{2\overline{\nu}^{1/2}}{\nu K}V_0, \quad \text{and} \quad \mathbf{b} = \overline{\nu}^{1/2}\|x(0)\| + \sigma.$$

Then there exists a constant B_0 (independent of t, made explicit in the proof of each of the three cases) such that $\|x(t)\| \leq B_0$ for all $t \in \mathbb{N}$. In addition, $\|v(t)\| \to 0$ when $t \to \infty$. Finally, there exists $\widehat{x} \in X$ such that $x(t) \to \widehat{x}$ when $t \to \infty$.

Proof: We divide the proof in two cases.

Case I. There exists $t \in \mathbb{N}$ such that $\mu(t) = \phi_t$.

Let $t_\dagger$ be the first $t \in \mathbb{N}$ for which $\phi_t \leq 2 - \mathcal{L}$. For $t \in \mathbb{N}$, $t \geq t_\dagger$, let t^* be the point maximizing $\|x\|$ in $\{0, 1, \ldots, t\}$. Then, by Proposition 4 for $i \in \{0, 1, \ldots, t\}$,

$$\begin{aligned}\phi_i &\geq \frac{1}{2}\nu\eta(\Gamma(i)) \geq \frac{\nu K}{2\left(\sigma^2 + \overline{\nu}\|x(i)\|^2\right)^\beta} \\ &\geq H(t^*) := \frac{\nu K}{2\left(\sigma^2 + \overline{\nu}\|x(t^*)\|^2\right)^\beta}.\end{aligned}$$

In particular, $\phi_{t_\dagger} \geq H(t^*)$. Therefore

$$\mu(i) = \min\{2 - \mathcal{L}, \phi_i\} \geq \min\{2 - \mathcal{L}, H(t^*)\} = H(t^*)$$

the last since $H(t^*) \leq \phi_{t_\dagger} \leq 2 - \mathcal{L}$. Using Corollary 1 we obtain, for all $\tau \leq t$,

$$\begin{aligned}\|x(\tau)\| &\leq \|x(0)\| + \sum_{j=0}^{\tau-1}\|x(j+1) - x(j)\| \\ &\leq \|x(0)\| + \Delta t \sum_{j=0}^{\tau-1}\|v(j)\| \\ &\leq \|x(0)\| + \Delta t\left(\|v(0)\| + \sum_{j=1}^{\tau-1}\|v(j)\|\right) \\ &\leq \|x(0)\| + \Delta t\|v(0)\|\left(1 + \sum_{j=1}^{\tau-1}\prod_{i=1}^{j}(1 - \mu(i))\right) \\ &\leq \|x(0)\| + \Delta t\|v(0)\|\sum_{j=0}^{\tau-1}(1 - H(t^*))^j \\ &\leq \|x(0)\| + \Delta t\frac{1}{H(t^*)}\|v(0)\| \\ &= \|x(0)\| + \Delta t\frac{2\left(\sigma^2 + \overline{\nu}\|x(t^*)\|^2\right)^\beta}{\nu K}\|v(0)\|.\end{aligned}$$

For $\tau = t^*$, the previous inequality takes then the following equivalent form:

$$\sigma + \overline{\nu}^{1/2} \|x(t^*)\| \le \left(\overline{\nu}^{1/2} \|x(0)\| + \sigma\right) + \overline{\nu}^{1/2} \Delta t \frac{2\left(\sigma^2 + \overline{\nu}\|x(t^*)\|^2\right)^{\beta}}{\nu K} \|v(0)\|$$

which implies

$$\left(\sigma^2 + \overline{\nu}\|x(t^*)\|^2\right)^{1/2} \le \left(\overline{\nu}^{1/2} \|x(0)\| + \sigma\right) + \overline{\nu}^{1/2} \Delta t \frac{2\left(\sigma^2 + \overline{\nu}\|x(t^*)\|^2\right)^{\beta}}{\nu K} \|v(0)\|. \quad (15)$$

Let $z = (\sigma^2 + \overline{\nu}\|x(t^*)\|^2)^{1/2}$

$$\mathbf{a} = \frac{2\overline{\nu}^{1/2}\Delta t}{\nu K} \|v(0)\|, \quad \text{and} \quad \mathbf{b} = \overline{\nu}^{1/2} \|x(0)\| + \sigma.$$

Then, (15) can be rewritten as $F(z) \le 0$ with

$$F(z) = z - \mathbf{a} z^{2\beta} - \mathbf{b}.$$

i) Assume $\beta < 1/2$. By Lemma 2, $F(z) \le 0$ implies that $(\sigma^2 + \overline{\nu}\|x(t^*)\|^2) \le U_0^2$ with

$$U_0 = \max\left\{ \left(\frac{4\overline{\nu}^{1/2}\Delta t}{\nu K} \|v(0)\|\right)^{\frac{1}{1-2\beta}}, 2\left(\overline{\nu}^{1/2} \|x(0)\| + \sigma\right) \right\}.$$

Since U_0 is independent of t we deduce that, for all $t \ge t_\dagger$

$$\|x(t)\|^2 \le B_0^2 := \frac{U_0^2 - \sigma^2}{\overline{\nu}}$$

and, therefore

$$\phi_t \ge F_0 := \frac{\nu K}{2\left(\sigma^2 + \overline{\nu} B_0^2\right)^{\beta}}.$$

Also, since $\phi_t \ge \phi_{t_\dagger}$ for $t < t_\dagger$, we deduce that $\phi_t \ge F_0$ holds for all $t \in \mathbb{N}$ and it readily follows that $\mu(i) \ge F_0$ for all $i \in \mathbb{N}$. By Corollary 1, for $t \in \mathbb{N}$,

$$\|v(t)\| \le \|v(0)\| \prod_{i=0}^{t-1} (1 - \mu(i)) \le (1 - F_0)^t \|v(0)\|$$

and this expression tends to zero when $t \to \infty$.
Finally, for $T > t$, reasoning as before, we have

$$\begin{aligned}
\|x(T) - x(t)\| &\le \sum_{j=t}^{T-1} \|x(j+1) - x(j)\| \\
&\le \Delta t \sum_{j=t}^{T-1} \|v(j)\| \\
&\le \Delta t \sum_{j=t}^{T-1} (1 - F_0)^j \|v(t)\| \\
&\le \Delta t \frac{1}{F_0} \|v(t)\|.
\end{aligned}$$

Since $\|v(t)\|$ tends to zero, we deduce that $\{x(t)\}_{t\in\mathbb{N}}$ is a Cauchy sequence and there exists $\widehat{x} \in X$ such that $x(t) \to \widehat{x}$.

ii) Assume now $\beta = 1/2$. Then (15) takes the form

$$\left(\sigma^2 + \overline{\nu}\|x(t^*)\|^2\right)^{1/2} \left(1 - \frac{2\overline{\nu}^{1/2}\Delta t}{\nu K} \|v(0)\|\right) - \left(\overline{\nu}^{1/2} \|x(0)\| + \sigma\right) \le 0$$

which implies that

$$\|x(t^*)\|^2 \le B_0 := \frac{1}{\overline{\nu}} \left(\left(\frac{\overline{\nu}^{1/2} \|x(0)\| + \sigma}{1 - \frac{2\overline{\nu}^{1/2}\Delta t}{\nu K} \|v(0)\|} \right)^2 - \sigma^2 \right)$$

which is positive since $0 < 1 - (2\overline{\nu}^{1/2}\Delta t/\nu K)\|v(0)\| \le 1$, by hypothesis. We now proceed as in case i).

iii) Assume finally $\beta > 1/2$. Letting $\alpha = 2\beta$ as in the proof of Theorem 2, the arguments therein show that the derivative $F'(z) = 1 - \alpha \mathbf{a} z^{\alpha-1}$ has a unique zero at $z_* = (1/\alpha\mathbf{a})^{1/(\alpha-1)}$ and $F(z_*) = (1/\mathbf{a})^{1/(\alpha-1)}[(1/\alpha)^{1/(\alpha-1)} - (1/\alpha)^{\alpha/(\alpha-1)}] - \mathbf{b}$. Our hypothesis then implies that $F(z_*) \ge 0$. This shows that the graph of F is as in Fig. 1.

For $t \in \mathbb{N}$ let $z(t) = (\sigma^2 + \overline{\nu}\|x(t^*)\|^2)^{1/2}$. When $t = 0$ we have $t^* = 0$ as well and

$$z(0) \le \overline{\nu}^{1/2} \|x(0)\| + \sigma = \mathbf{b} < \left(\frac{1}{\mathbf{a}}\right)^{\frac{1}{\alpha-1}} \left(\frac{1}{\alpha}\right)^{\frac{1}{\alpha-1}} = z_*.$$

This actually implies that $z(0) \le z_\ell$. Assume that there exists $t \in \mathbb{N}$ such that $z(t) \ge z_u$ and let T be the first such t. Then, $T = T^* \ge 1$ and, for all $t < T$

$$\left(\sigma^2 + \overline{\nu}\|x(t)\|^2\right)^{1/2} \le z(T-1) \le z_\ell.$$

This shows that, for all $t < T$

$$\|x(t)\| \le \left(\frac{z_\ell^2 - \sigma^2}{\overline{\nu}}\right)^{1/2} \le B_0 := \left(\frac{z_*^2 - \sigma^2}{\overline{\nu}}\right)^{1/2}.$$

In particular

$$\|x(T-1)\|^2 \le \frac{z_\ell^2 - \sigma^2}{\overline{\nu}}.$$

For T instead, we have

$$\|x(T)\|^2 \ge \frac{z_u^2 - \sigma^2}{\overline{\nu}}.$$

This implies

$$\|x(T)\|^2 - \|x(T-1)\|^2 \ge \frac{z_u^2 - z_\ell^2}{\overline{\nu}} \ge \frac{z_*^2 - z_\ell^2}{\overline{\nu}} \ge \frac{(z_* - z_\ell) z_*}{\overline{\nu}}. \quad (16)$$

From the intermediate value theorem, there is $\xi \in [z_\ell, z_*]$ such that $F(z_*) = F'(\xi)(z_* - z_\ell)$. But $F'(\xi) \geq 0$ and $F'(\xi) = 1 - a\alpha\xi^{\alpha-1} \leq 1$. Therefore,

$$z_* - z_\ell \geq F(z_*)$$

and it follows from (16) that

$$\|x(T)\|^2 - \|x(T-1)\|^2 \geq \frac{z_* F(z_*)}{\overline{\nu}}. \tag{17}$$

However

$$\begin{aligned}\|x(T)\| - \|x(T-1)\| &\leq \|x(T) - x(T-1)\| \\ &= \Delta t \, \|v(T-1)\| \\ &\leq \Delta t \, \|v(0)\|\end{aligned}$$

the last since $\|v\|$ is decreasing. Therefore

$$\begin{aligned}\|x(T)\|^2 - \|x(T-1)\|^2 &\leq (\Delta t)^2 \, \|v(0)\|^2 \\ &\quad + 2\Delta t \, \|v(0)\| \, \|x(T-1)\| \\ &\leq (\Delta t)^2 \, \|v(0)\|^2 + 2\Delta t \, \|v(0)\| \, B_0.\end{aligned}$$

Putting this inequality together with (17) shows that

$$z_* F(z_*) \leq \overline{\nu} \left((\Delta t)^2 \, \|v(0)\|^2 + 2\Delta t \, \|v(0)\| \, B_0 \right)$$

or, equivalently

$$\begin{aligned}&\left(\frac{1}{\mathbf{a}}\right)^{\frac{2}{\alpha-1}} \left[\left(\frac{1}{\alpha}\right)^{\frac{2}{\alpha-1}} - \left(\frac{1}{\alpha}\right)^{\frac{\alpha+1}{\alpha-1}} \right] - \mathbf{b} \\ &\qquad \leq \overline{\nu} \left((\Delta t)^2 \, \|v(0)\|^2 + 2\Delta t \, \|v(0)\| \, B_0 \right)\end{aligned}$$

which contradicts our hypothesis.

We conclude that, for all $t \in \mathbb{N}$, $z(t) \leq z_\ell$ and, hence, $\|x(t)\| \leq B_0$. We now proceed as in case i).

Case II. For all $t \in \mathbb{N}$, $\mu(t) = 2 - \mathcal{L} < \phi_t$.

Note that in this case $1 < \mathcal{L} < 2$. Moreover, for all $t \in \mathbb{N}$

$$\|v(t)\| \leq \|v(0)\| \prod_{i=0}^{t-1} (1 - \mu(i)) \leq \|v(0)\| \, (\mathcal{L} - 1)^t$$

and

$$\begin{aligned}\|x(t)\| &\leq \|x_0\| + \Delta t \, \|v(o)\| \sum_{j=0}^{t-1} (\mathcal{L} - 1)^j \\ &\leq \|x_0\| + \Delta t \, \|v(0)\| \, \frac{1}{2 - \mathcal{L}}.\end{aligned}$$

The result now trivially follows. ∎

Remark 5: In the proof of Theorem 3, we could have used the bounds for $\overline{\nu}$ and ν exhibited in Lemma 1 and, in case iii) the trivial bound $(\alpha\mathbf{a})^{-(2/(\alpha-1))} - \sigma^2 \leq (\alpha\mathbf{a})^{-(2/(\alpha-1))}$. Recall, $V_0 := \Delta t \|v(0)\|$. Denoting as well $X_0 := \|x(0)\|$ and

$$g(\alpha) := \left[\left(\frac{1}{\alpha}\right)^{\frac{2}{\alpha-1}} - \left(\frac{1}{\alpha}\right)^{\frac{\alpha+1}{\alpha-1}} \right]$$

the sufficiency condition for convergence in case iii) becomes

$$\begin{aligned}g(\alpha) \left(\frac{K}{6\sqrt{2}k^2 V_0} \right)^{\frac{2}{\alpha-1}} &\geq \sqrt{2}k X_0 + \sigma + 2k^2 V_0 \\ &\quad + 2\sqrt{2}k^3 V_0 \left(\frac{K}{6\sqrt{2}\alpha k^2 V_0} \right)^{\frac{2}{\alpha-1}}.\end{aligned}$$

It is apparent from the previous expression that this condition is satisfied when V_0 is sufficiently small. It is also apparent that the larger k is, the smaller V_0 needs to be to satisfy the condition.

We note also that, for $\alpha > 1$, we have $0 < g(\alpha) < 1$ and that $g(\alpha) \to 0$ when $\alpha \to 1$.

We end this section with a short discussion on our assumption on K. We made this assumption to ensure that $\|L_x\| < 2$ for all $x \in X$. Note that we did not need this condition for continuous time. The next example shows that we do need it with discrete time since otherwise convergence may fail.

Example 1: For $\beta > 0$, let $K > \sigma^{2\beta}$ be arbitrary and let

$$\gamma = \frac{K^{1/\beta} - \sigma^2}{2} > 0$$

so that $K/(\sigma^2 + 2\gamma)^\beta = 1$. For $\beta = 0$ we let $K = \sigma^{2\beta} = 1$ and $\gamma > 0$ be arbitrary. Finally, let $\Delta t = 1$.

Now, consider the situation of two birds flying on a line. That is, $k = 2$ and instead of $\mathbb{E}^3$ we take $\mathbb{R}$. Let $x(0) = (-\gamma, \gamma)$ and $v(0) = (2\gamma, -2\gamma)$. Then

$$A_{x(0)} = \begin{pmatrix} \frac{K}{\sigma^{2\beta}} & 1 \\ 1 & \frac{K}{\sigma^{2\beta}} \end{pmatrix} \quad \text{and} \quad L_{x(0)} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

It is easy to check that $\|L_{x(0)}\| = 2$. In addition,

$$x(1) = x(0) + v(0) = \begin{pmatrix} -\gamma \\ \gamma \end{pmatrix} + \begin{pmatrix} 2\gamma \\ -2\gamma \end{pmatrix} = \begin{pmatrix} \gamma \\ -\gamma \end{pmatrix}$$

and

$$v(1) = \left(\mathrm{Id} - L_{x(0)}\right) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2\gamma \\ -2\gamma \end{pmatrix} = \begin{pmatrix} -2\gamma \\ 2\gamma \end{pmatrix}.$$

We see that, for all $r \in \mathbb{N}$, $\|L_{x(r)}\| = 2$, $x(2r) = (-\gamma, \gamma)$ and $v(2r) = (2\gamma, -2\gamma)$ while $x(2r+1) = (\gamma, -\gamma)$ and $v(2r+1) = (-2\gamma, 2\gamma)$. Thus, $\|v(r)\|$ keeps constant (with value 4γ) and does not converge to 0.

VI. Language Evolution

We now consider a linguistic population with k agents evolving with time. At time t, the state of the population is given by $(x(t), f(t)) \in (\mathbb{E}^3)^k \times \mathcal{H}^k$. Here $\mathbb{E}^3$ is interpreted as the space of positions and $\mathcal{H}$ as the space of languages of

[14]. Thus, unlike the development in Section III, the functions x and f do not belong to the same space.

We model the evolution of the population with the system of differential equations

$$\begin{aligned} x' &= -L_f x \\ f' &= -L_x f. \end{aligned}$$

Again, L_x is the Laplacian of the matrix A_x given by $a_{ij} = \eta_X(\|x_i - x_j\|^2)$ for some function $\eta_X : \mathbb{R}_+ \to \mathbb{R}_+$. Similarly with L_f for some function $\eta_{\mathcal{H}} : \mathbb{R}_+ \to \mathbb{R}_+$. The distance between languages in $\mathcal{H}$ is defined as in [14].

A rationale for this model could be the following. Agents tend to move towards other agents using languages close to theirs (and therefore, communicating better). Hence, the first equation. Also, languages evolve by the influence from other agents' languages and this influence decrease with distance (for instance, because of a decrease in the frequency of linguistic encounters). Hence, the second equation.

Theorem 4: Let $\eta_X, \eta_{\mathcal{H}} : \mathbb{R}_+ \to (0, \infty)$ be non-increasing. Then, when $t \to \infty$, the state (x, f) tends to a point in the diagonal of $(\mathbb{E}^3 \times \mathcal{H})^k$.

Proof: We use the ideas and notations from Section III. In particular, we denote $\Lambda(x) = \langle x, x\rangle_Q$ and $\Lambda(f) = \langle f, f\rangle_Q$ and we denote by ϕ_x and ϕ_f the Fiedler numbers of L_x and L_f, respectively. Reasoning as in Proposition 2 we obtain, for all $t > 0$

$$\Lambda'(f(t)) \leq -2\phi_{x(t)}\Lambda(f(t))$$

and

$$\Lambda'(x(t)) \leq -2\phi_{f(t)}\Lambda(x(t)).$$

This shows that both $\Lambda(f(t))$ and $\Lambda(x(t))$ are decreasing functions on t and satisfy

$$\Lambda(f(t)) \leq \Lambda(f(0))\, e^{-2\int_0^t \phi_{x(\tau)} d\tau}$$

and

$$\Lambda(x(t)) \leq \Lambda(x(0))\, e^{-2\int_0^t \phi_{f(\tau)} d\tau}.$$

However, since both η and $\Lambda(x(t))$ are nonincreasing, by Proposition 4

$$\begin{aligned} \phi_{x(\tau)} &\geq \nu\eta\left(\max_{i\neq j}\|x_i(\tau) - x_j(\tau)\|^2\right) \\ &\geq \nu\eta\left(\Lambda_{x(\tau)}\right) \geq \nu\eta\left(\Lambda(x(0))\right). \end{aligned}$$

Thus

$$\int_0^t \phi_{x(\tau)} d\tau \geq t\nu\eta\left(\Lambda(x(0))\right)$$

and $\Lambda(f(t)) \leq \Lambda(f(0))e^{-2t\nu\eta(\Lambda(x(0)))}$. This shows the convergence to 0 of $\Lambda(f(t))$. That of $\Lambda(x(t))$ is similar. ■

Remark 6:

i) We interpret the convergence of $x(t)$ to a fixed $\mathbf{x} \in \Delta_X$ as the formation of a tribe and the convergence of $f(t)$ to a fixed $\mathbf{f} \in \Delta_{\mathcal{H}}$ as the emergence of a common language as in Examples 2 and 3 of [14]. The first such example is taken from [15] were models are proposed (and studied via simulation) for the origins of language. The second, is a modification of it proposed in [14] for the emergence of common vowel sounds.

ii) The assumption of symmetry is plausible in contexts where (unlike the Mother/Baby case discussed in [14, Ex. 4]) there are no leaders in the liguistic population.

iii) Detailed learning mechanisms could be introduced by first deriving a result akin to Proposition 3 for discrete time and then follow [14].

iv) We have not used any argument as those in the proof of Proposition 3. These arguments involved expressions like $\langle x, f\rangle$ which, in the situation at hand, would be meaningless.

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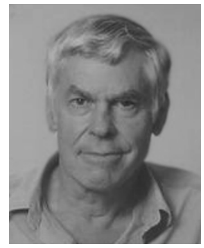

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