DCDN Cost Analysis by Lester Kim

Introduction

In order to calculate the streaming costs for a potential partner, we need to know how much data B (in bytes) they need to deliver to consumers per unit of time T (in seconds). Let us have N groups of uploader nodes where $N \in \mathbb{N}$. $\forall n \in \{1, ..., N\}$, the average bandwidth of group n is b_n (in bytes/second per node). Let q_n be the quantity of group n nodes. Let $\mathbf{b} = [b_1 ... b_N]^\intercal$ and $\mathbf{q} = [q_1 ... q_N]^\intercal$. Thus, the amount of bytes delivered per second constrained by $\frac{B}{T}$ is

$$g(\mathbf{q}) = \mathbf{b} \cdot \mathbf{q} = \frac{B}{T}. \tag{1}$$

We want to find the optimal $\mathbf{q}^*$ to minimize the cost of distribution $C(\mathbf{q})$. If $\mathbf{p} = [p_1 ... p_N]^\intercal$ where p_n is the price per node for group n, we have

$$C(\mathbf{q}) = \mathbf{p} \cdot \mathbf{q}. \tag{2}$$

Uploader's Profit Maximization

To determine $\mathbf{p}$, let us visit the behavior of a profit maximizing firm. Let f be the production function with energy input E (in kWh) and output q (in nodes). We model this production function as

$$f(E) = AE^\alpha \tag{3}$$

where A is the factor of production (nodes/kWh$^\alpha$) and $\alpha \in [0, 1]$ is the elasticity of production (percent increase in output over percent increase in input) [1].

Let P (in kWh/s) be the power increase when a node starts uploading data. This includes sending data with its network interface controller (NIC) but can also include the machine turning on (from either being off or in standby mode). If each node has power P, then for some E, a single node can run for $\frac{E}{P}$ seconds. However, given a time constraint T to complete the work, there must be $\frac{E}{PT}$ nodes. Thus,

$$A = \frac{D}{(PT)^\alpha} \tag{4}$$

where D is the total factor productivity [2] (in nodes).

Let p be the price of a node and p_E the price of energy (per kWh). The firm's profit function π is

$$\pi(q, E) = pq - p_E E. \tag{5}$$

Bandwidth cost is ignored because it is a fixed cost in the short-term when looking at seconds as opposed to months[1].

We want to maximize the firm's profit given an output requirement at least q;

$$\max_{q,E} \ \pi(q, E) \quad \text{s.t. } f(E) \geq q. \tag{6}$$

To solve this, we take our Lagrangian to be

$$\mathcal{L}(q, E, \lambda) = pq - p_E E - \lambda(AE^{\alpha} - q). \tag{7}$$

Taking partial derivatives and setting them to zero give

$$\frac{\partial \mathcal{L}}{\partial q} = p + \lambda = 0 \tag{8}$$

$$\frac{\partial \mathcal{L}}{\partial E} = -p_E - \lambda A \alpha E^{\alpha - 1} = 0 \tag{9}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = q - AE^{\alpha} = 0. \tag{10}$$

Solving these first-order conditions gives

$$q^* = \left(\frac{\alpha A^{\frac{1}{\alpha}} p}{p_E} \right)^{\frac{\alpha}{1-\alpha}}. \tag{11}$$

Rewriting this (dropping the asterisk from q) gives

$$p = \frac{p_E}{\alpha} \left(\frac{q^{1-\alpha}}{A} \right)^{\frac{1}{\alpha}}. \tag{12}$$

[1]Even with bandwidth included, its cost per second has the same order of magnitude as that of energy. In NYC, 50 MBps costs \$3 x 10^{-5}/second [3].

This formula lets the creator know what p should be to get the desired number of nodes.

From (12), we find that the optimal revenue in terms of q is

$$pq = \frac{p_E}{\alpha}\left(\frac{q}{A}\right)^{\frac{1}{\alpha}}. \tag{13}$$

Distributor's Cost Minimization

Since the revenue for the firm is spending for the consumer (the creator), we can write (2) as

$$C(\mathbf{q}) = \frac{p_E}{\alpha}\sum_{n=1}^{N}\left(\frac{q_n}{A_n}\right)^{\frac{1}{\alpha}}. \tag{14}$$

The creator's cost minimization problem is

$$\min_{\mathbf{q}} \; C(\mathbf{q}) \quad \text{s.t. } g(\mathbf{q}) \geq \frac{B}{T}. \tag{15}$$

The Lagrangian is

$$\mathcal{L}(\mathbf{q}, \lambda) = C(\mathbf{q}) - \lambda(g(\mathbf{q}) - \frac{B}{T}). \tag{16}$$

The first-order conditions are

$$\frac{\partial \mathcal{L}}{\partial \mathbf{q}} = \frac{\partial C}{\partial \mathbf{q}} - \lambda\frac{\partial g}{\partial \mathbf{q}} = 0 \tag{17}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = \frac{B}{T} - g(\mathbf{q}) = 0. \tag{18}$$

From (14), (4), and (1),

$$\frac{\partial C}{\partial \mathbf{q}} = \frac{p_E T\mathbf{P}}{\alpha^2} \circ (\mathbf{q}^{\circ(1-\alpha)} \oslash \mathbf{D})^{\circ\frac{1}{\alpha}} \tag{19}$$

$$\frac{\partial g}{\partial \mathbf{q}} = \mathbf{b} \tag{20}$$

where $(\mathbf{D},\mathbf{P}) = ([D_1...D_N]^\intercal, [P_1...P_N]^\intercal)$. "$\circ$", "$^\circ$", "$\oslash$" are the Hadamard (entrywise) product, power, and division, respectively [4]. So $\forall m, n \in \{1, ..., N\}$,

$$\frac{P_m^\alpha q_m^{1-\alpha}}{b_m^\alpha D_m} = \frac{P_n^\alpha q_n^{1-\alpha}}{b_n^\alpha D_n}. \tag{21}$$

Thus,

$$b_m q_m = \left(\frac{b_m D_m P_n^\alpha}{P_m^\alpha b_n D_n}\right)^{\frac{1}{1-\alpha}} b_n q_n. \tag{22}$$

Combining (22) with (18) gives

$$\boxed{\mathbf{q}^* = \frac{B\mathbf{b}^{\circ -1}}{T\kappa} \circ (\mathbf{b} \circ \mathbf{D} \oslash \mathbf{P}^{\circ\alpha})^{\circ \frac{1}{1-\alpha}}} \tag{23}$$

$$\boxed{C^* = \frac{p_E T}{\alpha}\left(\frac{B}{T\kappa^{1-\alpha}}\right)^{\frac{1}{\alpha}}} \tag{24}$$

where

$$\kappa \equiv \sum_{m=1}^{N} \left(\frac{b_m D_m}{P_m^\alpha}\right)^{\frac{1}{1-\alpha}}. \tag{25}$$

Case: $\alpha = 1$

When $\alpha = 1$, (23) and (24) become

$$q_n^* = \begin{cases} \frac{B}{|\Upsilon| T b_n} & n \in \Upsilon \\ 0 & n \notin \Upsilon \end{cases} \tag{26}$$

$$C^* = \frac{p_E B P_n}{b_n D_n} \qquad \text{any } n \in \Upsilon \tag{27}$$

where

$$\Upsilon \equiv \left\{ n \in \{1, ..., N\} \;\middle|\; n = \underset{1 \le m \le N}{\arg\max} \frac{b_m D_m}{P_m} \right\}. \tag{28}$$

$\forall n \in \Upsilon$, each node in group n would deliver $\frac{B}{|\Upsilon| q_n^*} (= b_n T)$ of data and receive at least $\frac{p_E P_n T}{D_n}$ in compensation. However, there are multiple solutions to $\mathbf{q}^*$. For example, for any $n \in \Upsilon$, group n can take on all the work by employing $\frac{B}{T b_n}$ nodes.

Example

Let's work out an example in NYC where

$$\alpha = 1 \tag{29}$$
$$B = 1 \text{ GB} \tag{30}$$
$$N = 2 \tag{31}$$
$$p_E = \$0.2321/\text{kWh } [5] \tag{32}$$
$$T = 1 \text{ s} \tag{33}$$
$$\mathbf{b} = \begin{bmatrix} 100 \text{ MB/s} \\ 1 \text{ MB/s} \end{bmatrix} [6] \tag{34}$$
$$\mathbf{D} = \begin{bmatrix} 1 \text{ node} \\ 1 \text{ node} \end{bmatrix} \tag{35}$$
$$\mathbf{P} = \begin{bmatrix} 200 \text{ W} \\ 2 \text{ W} \end{bmatrix} [7][8] \tag{36}$$

to find examples of $\mathbf{q}^*$ and C^* for a creator. Then,

$$\mathbf{q}^* = \begin{bmatrix} 5 \text{ nodes} \\ 500 \text{ nodes} \end{bmatrix} \tag{37}$$
$$C^* \approx \$1.29 \text{ x } 10^{-4}. \tag{38}$$

This is 155 to 659 times (99.35% - 99.85%) cheaper than AWS Cloudfront's on-demand pricing ($0.020/GB - $0.085/GB) [9]. Each group 1 node would handle 100 MB whereas each group 2 node would handle 1 MB. Each node in group 1 and 2 would need more than $1.29 x 10^{-5} and $1.29 x 10^{-7}, respectively.

To put this in perspective, assume Netflix is a potential partner. In 2017, Netflix averaged more than 140 million hours of content watched per day [10]. On average, a Netflix video is one GB/hour [11]. On the Ara platform, the 51.1-exabyte [12] annual spend would only be $6.6 million/year ($0.2106/second). If we estimate Netflix's streaming cost to be $0.03/GB [13], we get $1.5 billion/year

(\$46.61/second). Using the Ara network would nearly quadruple Netflix's 2017 net income of \$558.9 million [14]. (Manhattan has 1.66 million people [15] with 287,008 Netflix users[2] streaming 321.45 TB/day, 3.72 GB/s. That requires 3,721 group 2 nodes at a rate of \$41.47/day).

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