

Linear Independence

Geometric Algorithms

Lecture 6

Practice Problem

Do these three vectors span all of \mathbb{R}^3 ?

$$\mathbf{v}_1 = \begin{bmatrix} -4 \\ 4 \\ 2 \end{bmatrix}$$

$$\mathbf{v}_2 = \begin{bmatrix} -3 \\ 6 \\ -3 \end{bmatrix}$$

$$\mathbf{v}_3 = \begin{bmatrix} -5 \\ 8 \\ -2 \end{bmatrix}$$

Answer

$$\mathbf{v}_1 = \begin{bmatrix} -4 \\ 4 \\ 2 \end{bmatrix}$$

$$\mathbf{v}_2 = \begin{bmatrix} -3 \\ 6 \\ -3 \end{bmatrix}$$

$$\mathbf{v}_3 = \begin{bmatrix} -5 \\ 8 \\ -2 \end{bmatrix}$$

Outline

- » Motivate and define **linear independence**
- » See several perspectives on linear independence
- » If there's time: see an application of linear systems to **network flows**

Keywords

linear independence

linear dependence

homogenous systems of linear equations

trivial and nontrivial solutions

Homogeneous Linear Systems

Recall: The Zero Vector

$$0 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Recall: The Zero Vector

$$\mathbf{v} + \mathbf{0} = \mathbf{0} + \mathbf{v} = \mathbf{v}$$

$$c\mathbf{0} = \mathbf{0}$$

$$\mathbf{u} + -\mathbf{u} = \mathbf{0}$$

$$\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Recall: The Zero Vector

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$$c\mathbf{0} = \mathbf{0}$$

$$\mathbf{u} + -\mathbf{u} = \mathbf{0}$$

$$\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_n$$

the
dimension is
implicit in
the notation

Homogenous Linear Systems

Definition. A system of linear equations is called *homogeneous* if it can be expressed as

$$A\mathbf{x} = \mathbf{0}$$

Homogenous Linear Systems

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$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \dots + x_n\mathbf{a}_n = \mathbf{0}$$

Homogenous Linear Systems

Definition. A system of linear equations is called *homogeneous* if it can be expressed as

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = 0$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = 0$$

⋮

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = 0$$

Trivial Solutions

Definition. For the matrix equation $Ax = 0$ the solution $x = 0$ is called the *trivial solution*

Any other solution is called *nontrivial*

Trivial Solutions

Definition. For the vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \dots + x_n\mathbf{a}_n = \mathbf{0}$$

the solution $\mathbf{x} = \mathbf{0}$ is called the *trivial solution*

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Trivial Solutions

Definition. For the system of linear equations

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = 0$$

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⋮

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = 0$$

the solution $\mathbf{x} = \mathbf{0}$ is called the *trivial solution*

Any other solution is called *nontrivial*

Questions about Homogeneous Systems

When does $Ax = 0$ have only the **trivial solution**?

When does $Ax = 0$ have **nontrivial solutions**?

What does it mean *geometrically* in each case?

An Important Feature of Homogenous Systems

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 1 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \end{bmatrix} \quad \boxed{}$$

An Important Feature of Homogenous Systems

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 1 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \end{bmatrix} \quad \boxed{}$$

What do we know about the covered column?

An Important Feature of Homogenous Systems

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 1 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -3 & -6 & 0 \end{bmatrix}$$

What do we know about the covered column?

It has to be all zeros

Linear Independence

Linear Independence

Definition. A set of vectors $\{v_1, v_2, \dots, v_n\}$ is *linearly independent* if the vectors equation

$$x_1v_1 + x_2v_2 + \dots + x_nv_n = \mathbf{0}$$

has exactly one solution (the trivial solution)

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has exactly one solution (the trivial solution)

The columns of A are linearly independent
if $A\mathbf{x} = \mathbf{0}$ has exactly one solution

Linear Dependence

Definition. A set of vectors $\{v_1, v_2, \dots, v_n\}$ is *linearly dependent* if the vectors equation

$$x_1v_1 + x_2v_2 + \dots + x_nv_n = 0$$

has a *nontrivial* solution

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$$x_1v_1 + x_2v_2 + \dots + x_nv_n = 0$$

has a *nontrivial* solution

A set of vectors is linearly dependent if there is a nontrivial linear combination of the vectors which equals 0

Linear Dependence (Alternative)

Definition. A set of vectors is **linearly dependent** if it is not linearly independent

Linear Dependence (Alternative)

Definition. A set of vectors is **linearly dependent** if it is not linearly independent

$A\mathbf{x} = \mathbf{0}$ has a nontrivial solution

≡

$A\mathbf{x} = \mathbf{0}$ does not have only the trivial solution

Examples

$$\left\{ \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Examples

$$\left\{ \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ 6 \\ -2 \end{bmatrix} \right\}$$

Examples

$$\left\{ \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} \right\}$$

Examples

{ }

Another Interpretation of Linear Dependence

demo
(from ILA)

Three Vectors in \mathbb{R}^3

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It's possible for three vectors in \mathbb{R}^3 to span all of \mathbb{R}^3 , but it's not guaranteed

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There may be vectors which lies in the plane spanned by two other vectors

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It's possible for three vectors in \mathbb{R}^3 to span all of \mathbb{R}^3 , but it's not guaranteed

There may be vectors which lies in the plane spanned by two other vectors

Or even two vectors which lie in the span of one of the others

Fundamental Concern

How do we classify when a set of vectors does not span as much as it possibly can? When it is "smaller" than it could be?

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How do we classify when a set of vectors does not span as much as it possibly can? When it is "smaller" than it could be?

This is the role of linear dependence

Linear Dependence (Another Alternative)

Linear Dependence (Another Alternative)

Definition. A set of vectors $\{v_1, v_2, \dots, v_n\}$ is *linearly dependent* if it is nonempty and one of its vectors can be written as a linear combination of the others (not including itself)

The Linear Combination Perspective

Suppose we have four vectors such that

$$\mathbf{v}_3 = 2\mathbf{v}_1 + 3\mathbf{v}_2 + 5\mathbf{v}_4$$

what do we know about the equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 + x_4\mathbf{v}_4 = \mathbf{0}$$

The Linear Combination Perspective

$$\mathbf{v}_3 = 2\mathbf{v}_1 + 3\mathbf{v}_2 + \dots + 5\mathbf{v}_4$$

Implies $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 + x_4\mathbf{v}_4 = \mathbf{0}$ has a nontrivial solution:

$$(2, 3, -1, 5)$$

The Vector Equation Perspective

Suppose $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 + x_4\mathbf{v}_4 = \mathbf{0}$ has a nontrivial solution $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ where, say, $\alpha_2 \neq 0$

The Vector Equation Perspective

Suppose $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 + x_4\mathbf{v}_4 = \mathbf{0}$ has a nontrivial solution $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ where, say, $\alpha_2 \neq 0$

We can turn this into a linear combination

The Vector Equation Perspective

$$\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \alpha_3 \mathbf{v}_3 + \alpha_4 \mathbf{v}_4 = \mathbf{0}$$

Suppose $x_1 \mathbf{v}_1 + x_2 \mathbf{v}_2 + x_3 \mathbf{v}_3 + x_4 \mathbf{v}_4 = \mathbf{0}$ has a nontrivial solution $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ where, say, $\alpha_2 \neq 0$

In All

Theorem. A set of vectors is linearly dependent if and only if it is nonempty and *at least* one of its vectors can be written as a linear combination of the others

P if and only if Q means
P implies Q and Q implies P

Linear Dependence Relation

Definition. If $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ are linearly dependent, then a *linear dependence relation* is an equation of the form

$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \dots + \alpha_n\mathbf{v}_n = \mathbf{0}$$

A linear dependence relation
witnesses the linear dependence

How To: Linear Dependence Relation

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Question. Write down a linear dependence relation for the vectors v_1, v_2, \dots, v_n

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Solution. Find a nontrivial solution to the equation

$$[v_1 \ v_2 \ \dots \ v_n] x = 0$$

How To: Linear Dependence Relation

Question. Write down a linear dependence relation for the vectors v_1, v_2, \dots, v_n

Solution. Find a nontrivial solution to the equation

$$[v_1 \ v_2 \ \dots \ v_n] x = 0$$

(there will be a free variable you can choose to be nonzero)

Example

Write down the linear dependence relation for the following vectors

$$\mathbf{v}_1 = \begin{bmatrix} -4 \\ 4 \\ 2 \end{bmatrix}$$

$$\mathbf{v}_2 = \begin{bmatrix} -3 \\ 6 \\ -3 \end{bmatrix}$$

$$\mathbf{v}_3 = \begin{bmatrix} -5 \\ 8 \\ -2 \end{bmatrix}$$

Answer

$$\begin{bmatrix} -4 & -3 & -5 \\ 0 & 3 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

Simple Cases

The Empty Set

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{ } is linearly independent

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$\{\}$ is linearly independent

There is no nontrivial linear combination of the vectors equaling 0

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$\{\}$ is linearly independent

There is no nontrivial linear combination of the vectors equaling 0

There are none at all...

The Empty Set

$\{\}$ is linearly independent

There is no nontrivial linear combination of the vectors equaling 0

There are none at all...

0 is in every span, even the span of the empty set

One Vector

A single vector v is linearly independent if and only if $v \neq 0$

(Note that $x_1\mathbf{0} = \mathbf{0}$ has many nontrivial solutions)

The Zero Vector and Linear Dependence

If a set of vectors V contains the 0 , then it is linearly dependent

The Zero Vector and Linear Dependence

If a set of vectors V contains the 0 , then it is linearly dependent

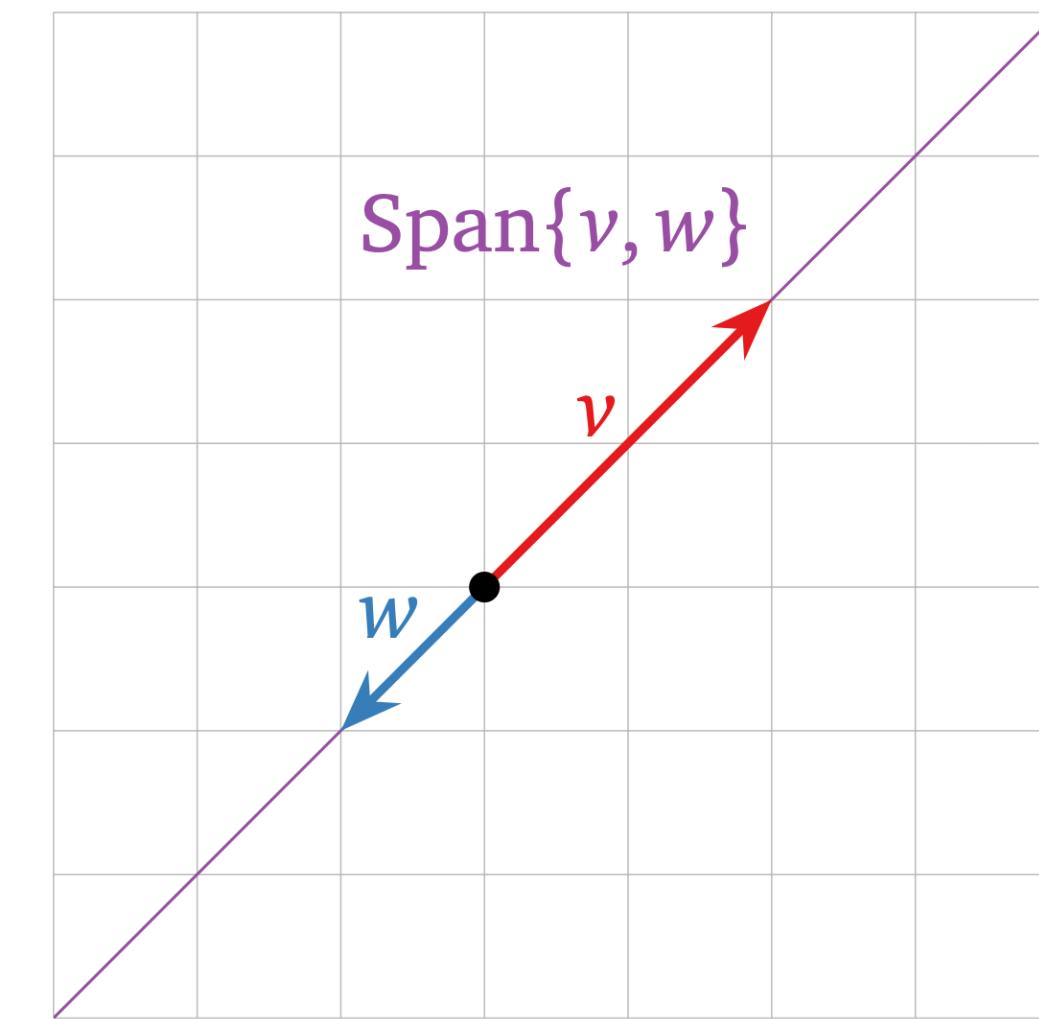
$$(1) 0 + 0\mathbf{v}_2 + 0\mathbf{v}_2 + \dots + 0\mathbf{v}_n = 0$$

There is a very simple nontrivial solution

Two Vectors

Definition. Two vectors are *colinear* if they are scalar multiples of each other

e.g., $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 1.5 \\ 1.5 \\ 3 \end{bmatrix}$ or $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ -1 \end{bmatrix}$

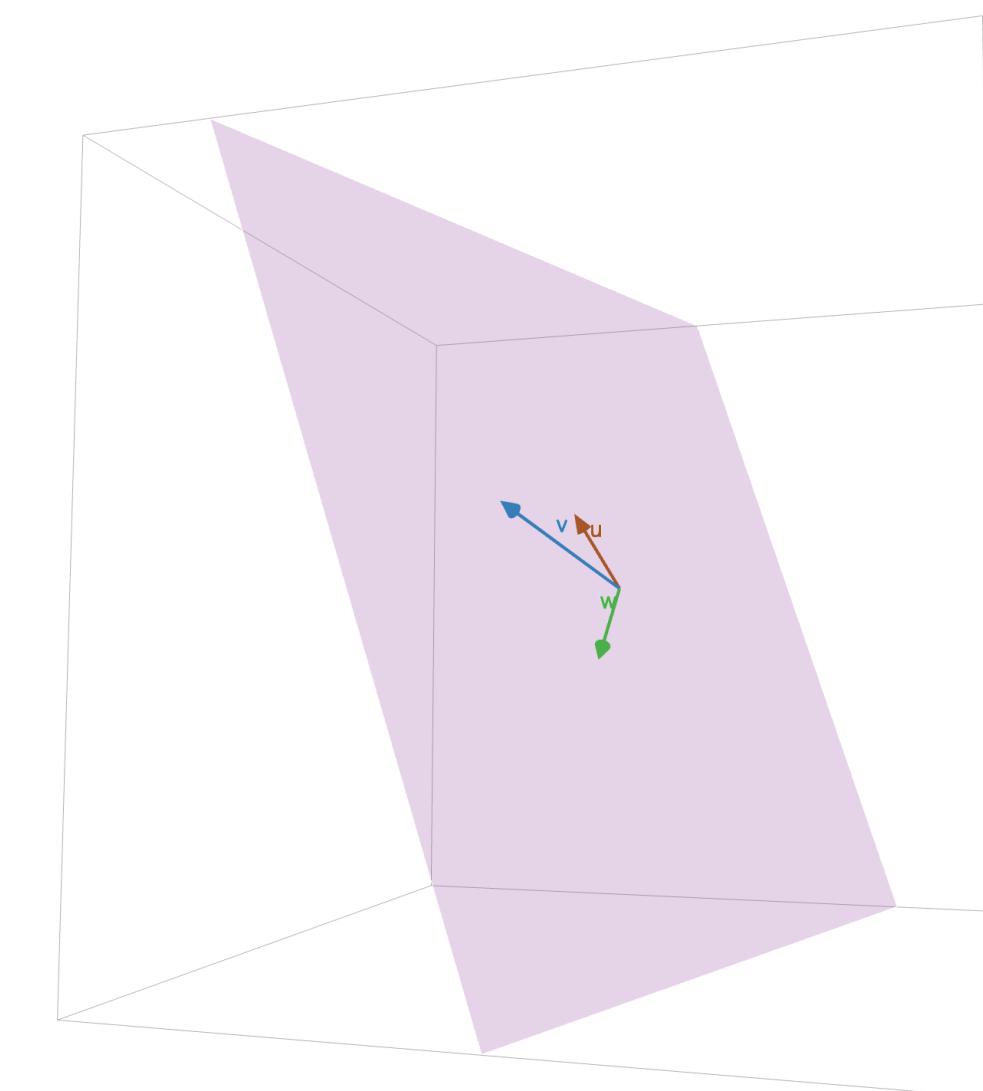


Two vectors are linearly dependent if and only if they are colinear

Three Vectors

Definition. A collection of vectors is **coplanar** if their span is a plane

Three vectors are linearly dependent if and only if they are colinear or coplanar



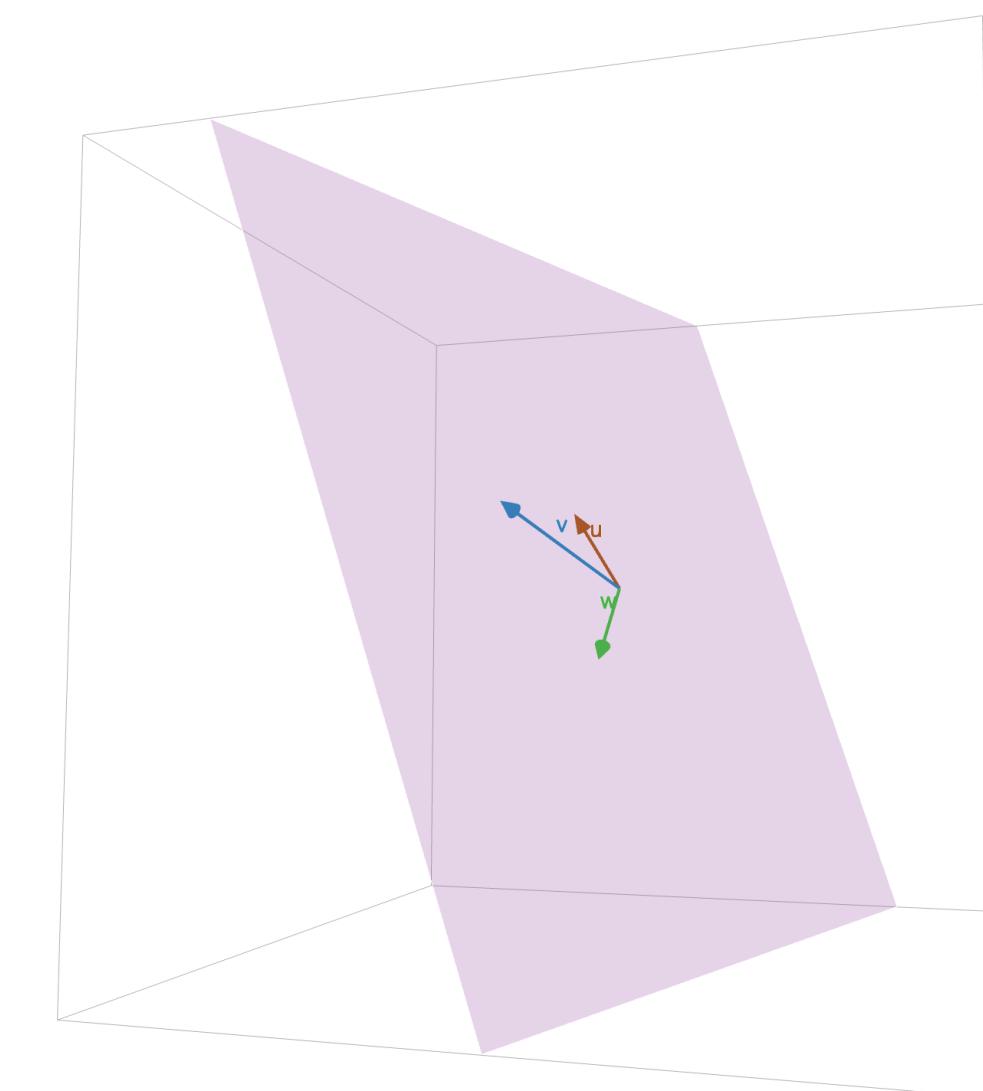
[image source](#)

Three Vectors

Definition. A collection of vectors is **coplanar** if their span is a plane

Three vectors are linearly dependent if and only if they are colinear or coplanar

This reasoning can be extended to more vectors, but we run out of terminology



[image source](#)

Yet Another Interpretation

Increasing Span Criterion

If v_1, v_2, \dots, v_n are linearly *independent* then we cannot write one of it's vectors as a linear combination of the others

But we get something stronger

Increasing Span Criterion

Theorem. v_1, v_2, \dots, v_n are linearly independent if and only if for all $i \leq n$,

$$v_i \notin \text{span}\{v_1, v_2, \dots, v_{i-1}\}$$

Increasing Span Criterion

Theorem. v_1, v_2, \dots, v_n are linearly independent if and only if for all $i \leq n$,

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As we add vectors, the span gets larger

Increasing Span Criterion

So in this case, our span keeps getting "bigger"

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So in this case, our span keeps getting "bigger"
 $\text{span}\{\}$ is a point $\{0\}$

$\text{span}\{v_1\}$ is a line

$\text{span}\{v_1, v_2\}$ is a plane

$\text{span}\{v_1, v_2, v_3\}$ is a 3d-hyperplane

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$\text{span}\{v_1, v_2, v_3\}$ is a 3d-hyperplane

$\text{span}\{v_1, v_2, v_3, v_4\}$ is a 4d-hyperlane

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 $\text{span}\{\}$ is a point $\{0\}$

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...

Characterization of Linear Dependence

Theorem. v_1, v_2, \dots, v_n are linearly dependent if and only there is an $i \leq n$,

$$v_i \in \text{span}\{v_1, v_2, \dots, v_{i-1}\}$$

Characterization of Linear Dependence

Theorem. v_1, v_2, \dots, v_n are linearly dependent if and only there is an $i \leq n$,

$$v_i \in \text{span}\{v_1, v_2, \dots, v_{i-1}\}$$

As we add vectors, we'll eventually find one in the span of the preceding ones

Characterization of Linear Dependence

$\text{span}\{\}$ is a point $\{0\}$

$\text{span}\{v_1\}$ is a line

$\text{span}\{v_1, v_2\}$ is a plane

$\text{span}\{v_1, v_2, v_3\}$ is still a plane

...

Characterization of Linear Dependence

$\text{span}\{\}$ is a point $\{0\}$

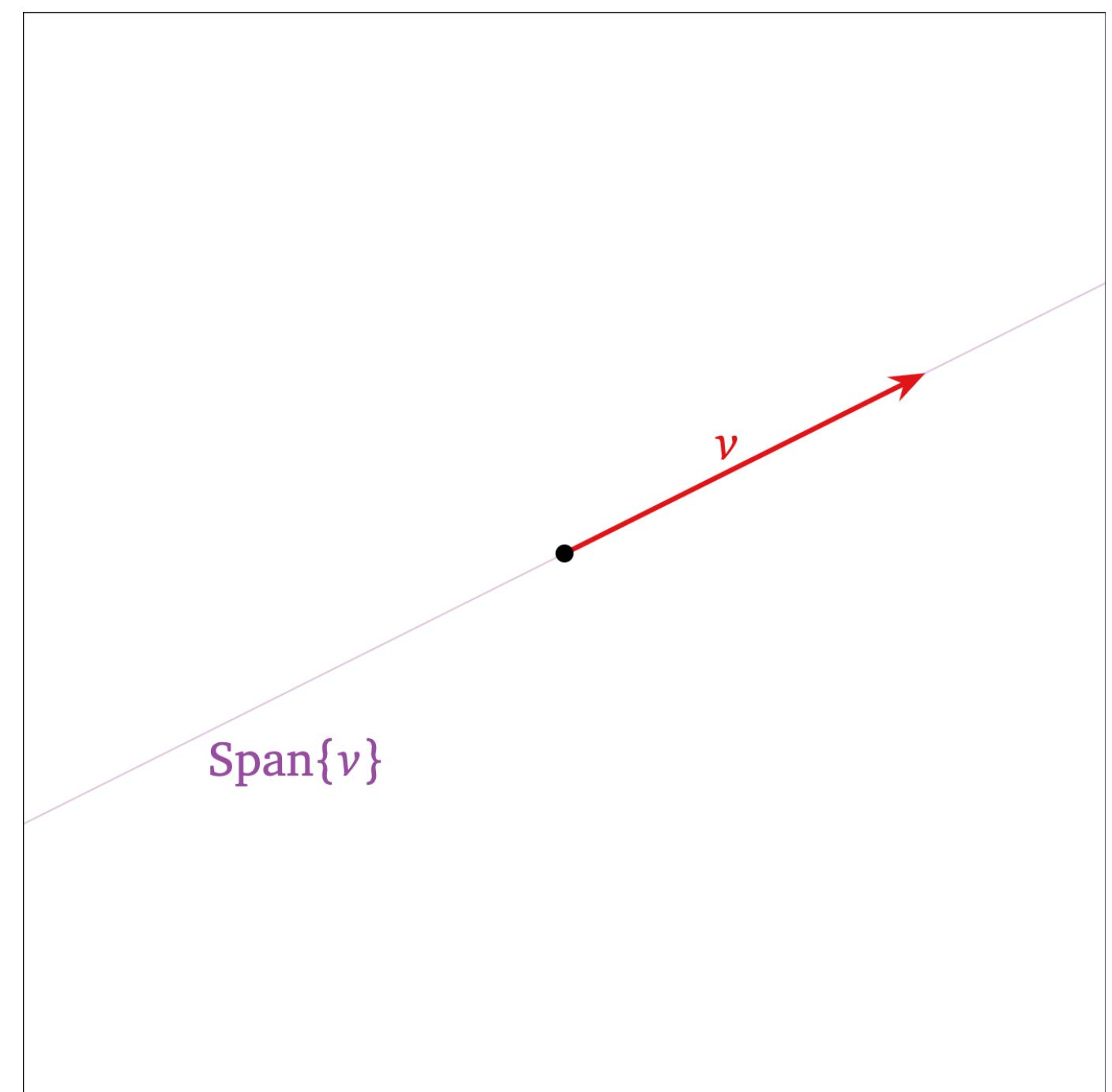
$\text{span}\{v_1\}$ is a line

$\text{span}\{v_1, v_2\}$ is a plane

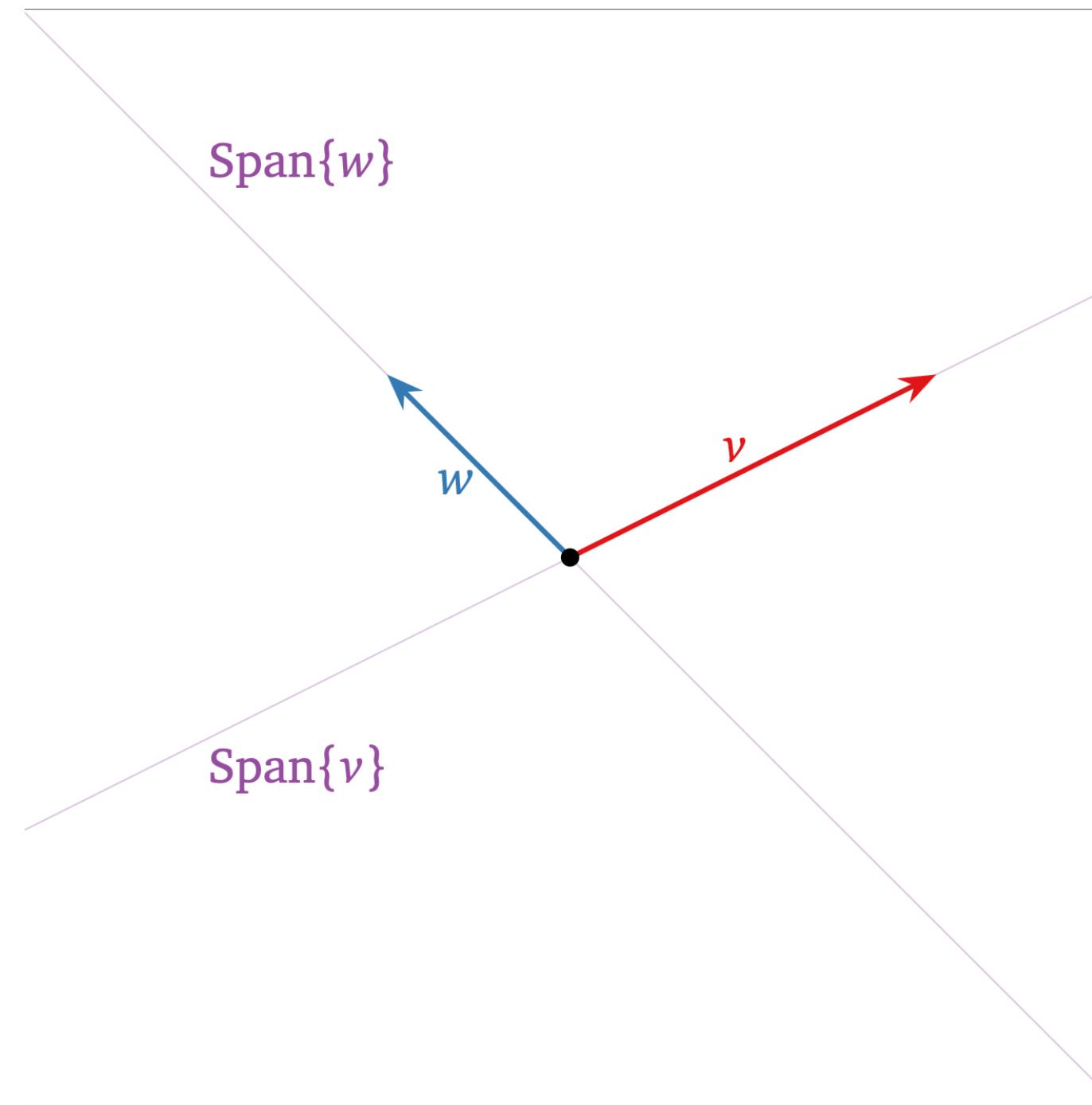
$\text{span}\{v_1, v_2, v_3\}$ is still a plane

...
(this is an example, it may take a lot more vectors before
we find one in the span of the preceding vectors)

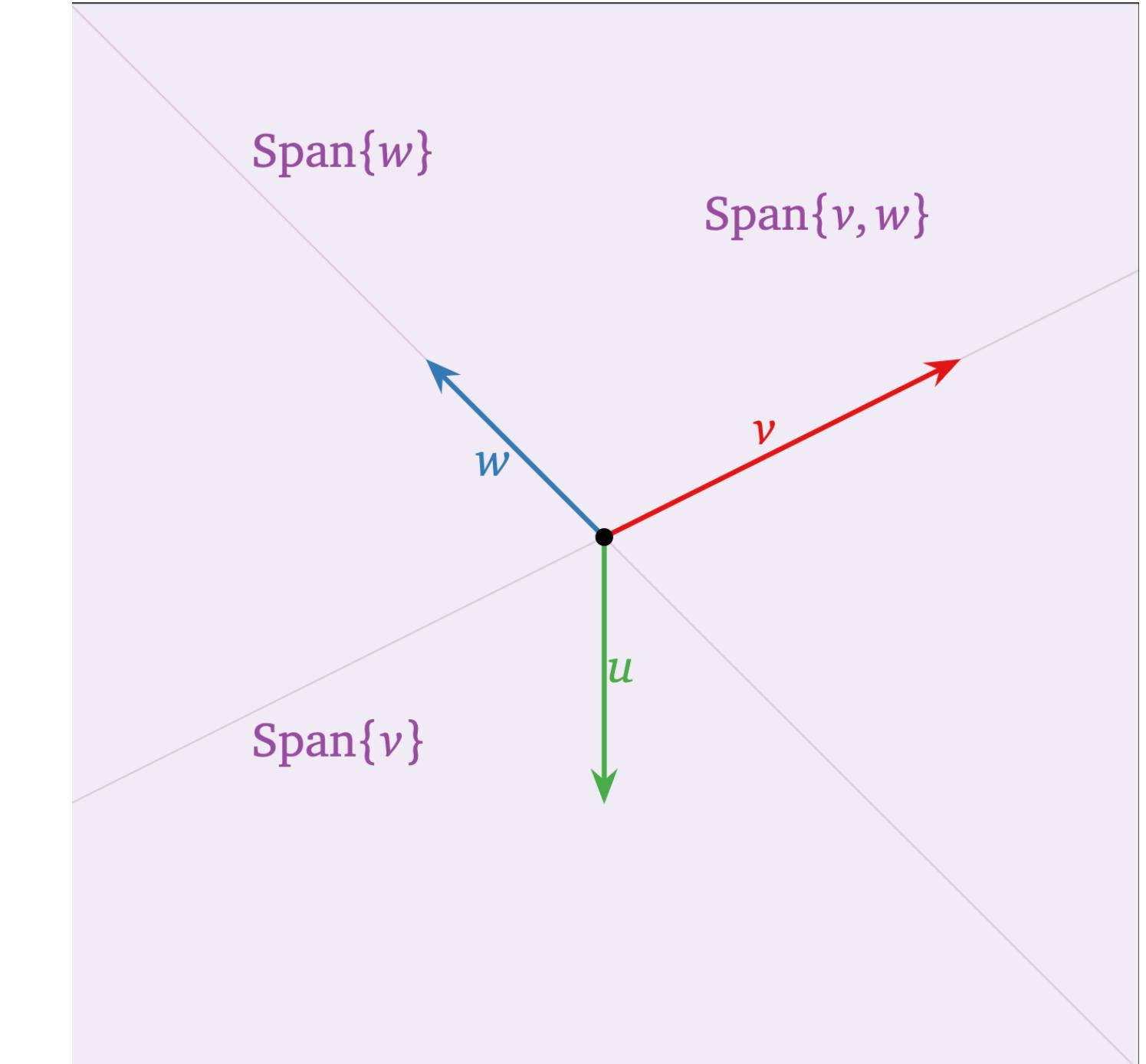
As a Picture



span of 1 vector
a line



span of 2 vector
a plane



span of 3 vector
still a plane

Characterization of Linear Dependence

Corollary. If v_1, v_2, \dots, v_k are linearly dependent, then for any vector v_{k+1} , the vectors $v_1, v_2, \dots, v_k, v_{k+1}$ are linearly dependent

If we add a vector to a linearly dependent set, it remains linearly dependent

Question

Are the following vectors linearly independent?

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\mathbf{v}_2 = \begin{bmatrix} 2023 \\ 0 \end{bmatrix}$$

$$\mathbf{v}_3 = \begin{bmatrix} 0.1 \\ 7 \end{bmatrix}$$

Answer: No

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\mathbf{v}_2 = \begin{bmatrix} 2023 \\ 0 \end{bmatrix}$$

$$\mathbf{v}_3 = \begin{bmatrix} 0.1 \\ 7 \end{bmatrix}$$

Any three vectors can at most span a plane

The first two are not colinear, so they span a plane (\mathbb{R}^2)

Linear Independence and Free Variables

Linear Dependence Relations (Again)

When finding a linear dependence relation, we came across a system which a free variable

$$\begin{bmatrix} -4 & -3 & -5 & 0 \\ 0 & 3 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

we can take x_3 to be free

Pivots and Linear Dependence

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Theorem. The columns of a matrix A are linearly independent if and only if A has a pivot in every column

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Remember that we choose our free variables to be the ones whose columns don't have pivots

Pivots and Linear Dependence

Theorem. The columns of a matrix A are linearly independent if and only if A has a pivot in every column

Remember that we choose our free variables to be the ones whose columns don't have pivots

Free variables allow for infinitely many (nontrivial) solution

Recall: General Form Solutions

$$x_1 = -(0.5)x_3$$

$$x_2 = -x_3$$

x_3 is free

Recall: General Form Solutions

$$x_1 = -0.5$$

$$x_2 = -1$$

$$x_3 = 1$$

Recall: General Form Solutions

$$x_1 = 0.5$$

$$x_2 = 1$$

$$x_3 = -1$$

Recall: General Form Solutions

$$x_1 = 1$$

$$x_2 = 2$$

$$x_3 = -2$$

Recall: General Form Solutions

$$x_1 = 1$$

$$x_2 = 2$$

$$x_3 = -2$$

The point: the solution is not unique

How To: Linear Independence

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Question. Is the set of vectors $\{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n\}$ linearly independent?

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Solution. Check if $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \dots + x_n\mathbf{a}_n = \mathbf{0}$ has a unique solution

How To: Linear Independence

Question. Is the set of vectors $\{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n\}$ linearly independent?

Solution. Check if $[\mathbf{a}_1 \ \mathbf{a}_2 \ \dots \ \mathbf{a}_n] \mathbf{x} = \mathbf{0}$ has a unique solution

How To: Linear Independence

Question. Is the set of vectors $\{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n\}$ linearly independent?

Solution. Check if the general form solution of
 $[\mathbf{a}_1 \ \mathbf{a}_2 \ \dots \ \mathbf{a}_n \ 0]$ has any free variables

How To: Linear Independence

Question. Is the set of vectors $\{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n\}$ linearly independent?

Solution. Reduce $[\mathbf{a}_1 \ \mathbf{a}_2 \ \dots \ \mathbf{a}_n]$ to echelon form and check if has a pivot position in every column

Example: Recap Problem Again

$$\mathbf{v}_1 = \begin{bmatrix} -4 \\ 4 \\ 2 \end{bmatrix}$$

$$\mathbf{v}_2 = \begin{bmatrix} -3 \\ 6 \\ -3 \end{bmatrix}$$

$$\mathbf{v}_3 = \begin{bmatrix} -5 \\ 8 \\ -2 \end{bmatrix}$$

The reduced echelon form of $[\mathbf{v}_1 \ \mathbf{v}_2 \ \mathbf{v}_3]$ is

$$\left[\begin{array}{ccc|c} 1 & 0 & 0.5 & \\ 0 & 1 & 1 & \\ 0 & 0 & 0 & \end{array} \right]$$

column
without a
pivot

Linear Independence and Full Span

The columns of a $(m \times n)$ matrix span all of \mathbb{R}^n if there is a pivot in every row

The columns of a matrix are linearly independent if there is a pivot in every column

Tall Matrices

If $m > n$ then the columns cannot span \mathbb{R}^m

$$\begin{bmatrix} * & \cdots & * \\ * & \cdots & * \\ * & \cdots & * \\ \vdots & \vdots & \vdots \\ * & \cdots & * \\ * & \cdots & * \\ * & \cdots & * \end{bmatrix}$$

Tall Matrices

If $m > n$ then the columns cannot span \mathbb{R}^m

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix}$$

This matrix has at most 3 pivots, but 4 rows

Wide Matrices

If $m < n$ then the columns cannot be linearly independent

$$\begin{bmatrix} * & * & * & \cdots & * & * & * \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ * & * & * & \cdots & * & * & * \end{bmatrix}$$

Wide Matrices

If $m < n$ then the columns cannot be linearly independent

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix}$$

This matrix has at most 3 pivots, but 4 columns

A Warning

The columns of a $(m \times n)$ matrix span all of \mathbb{R}^n if there is a pivot in every row

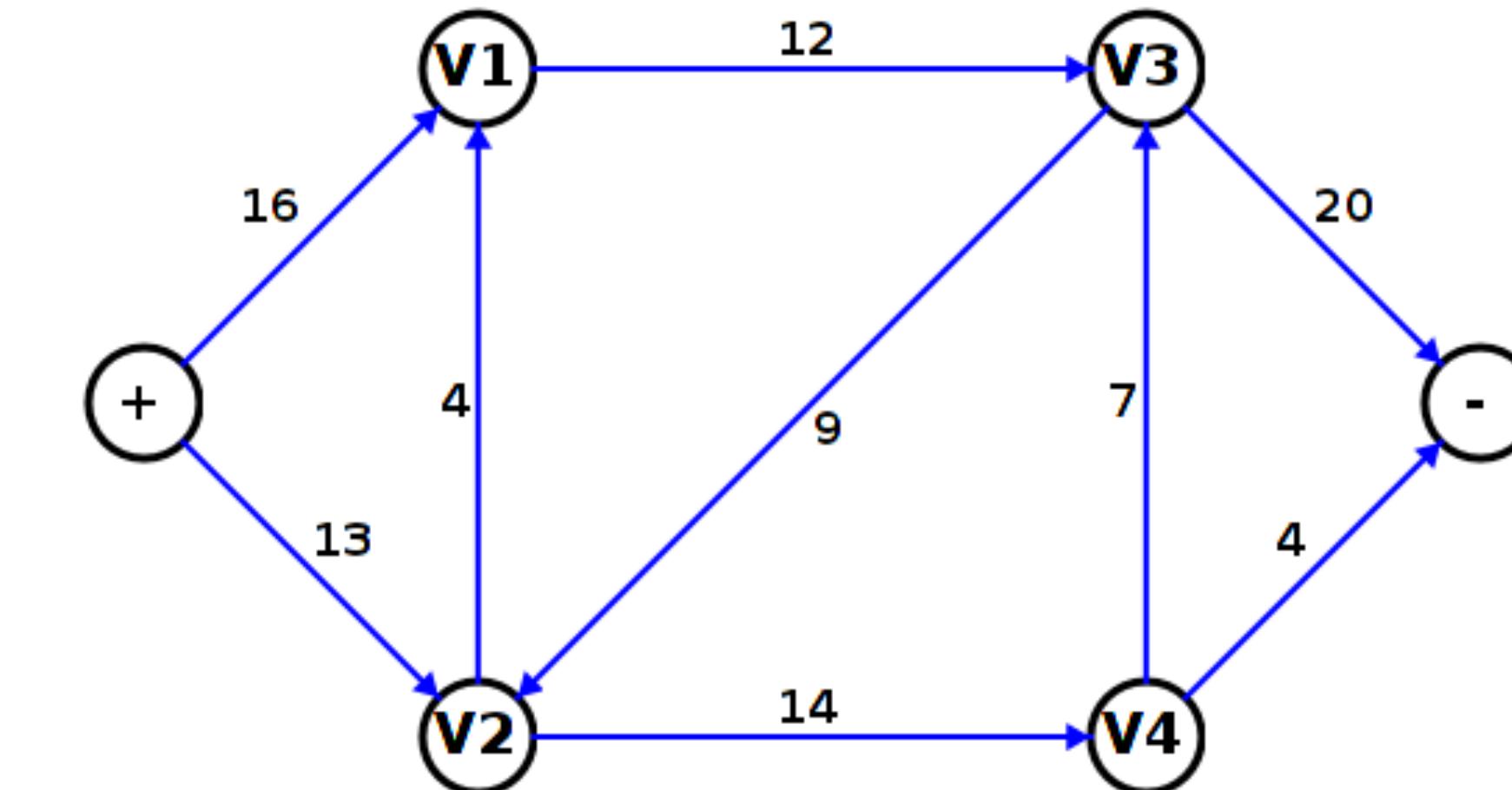
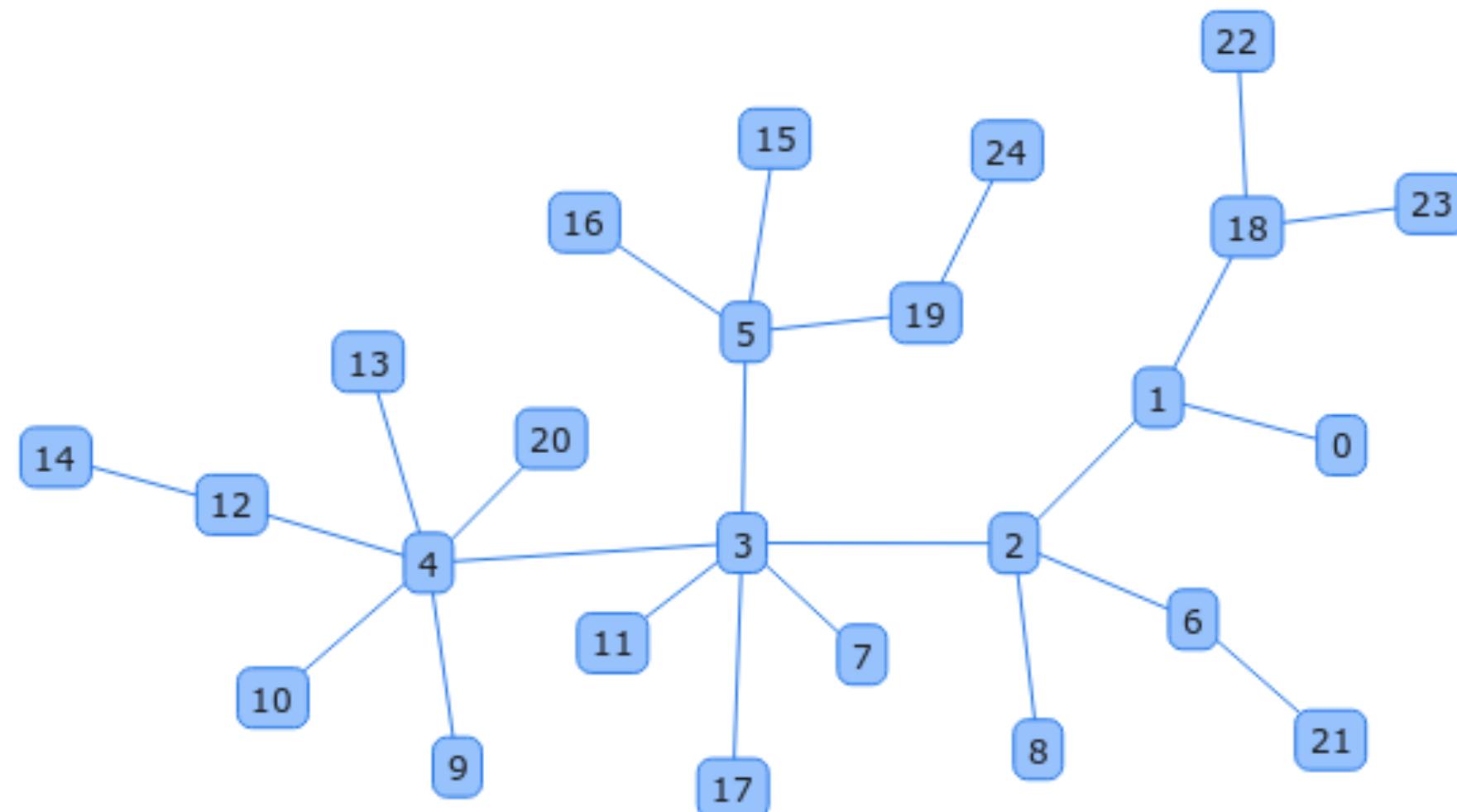
The columns of a matrix are linearly independent if there is a pivot in every column

Don't confuse these!

Application: Networks and Flow

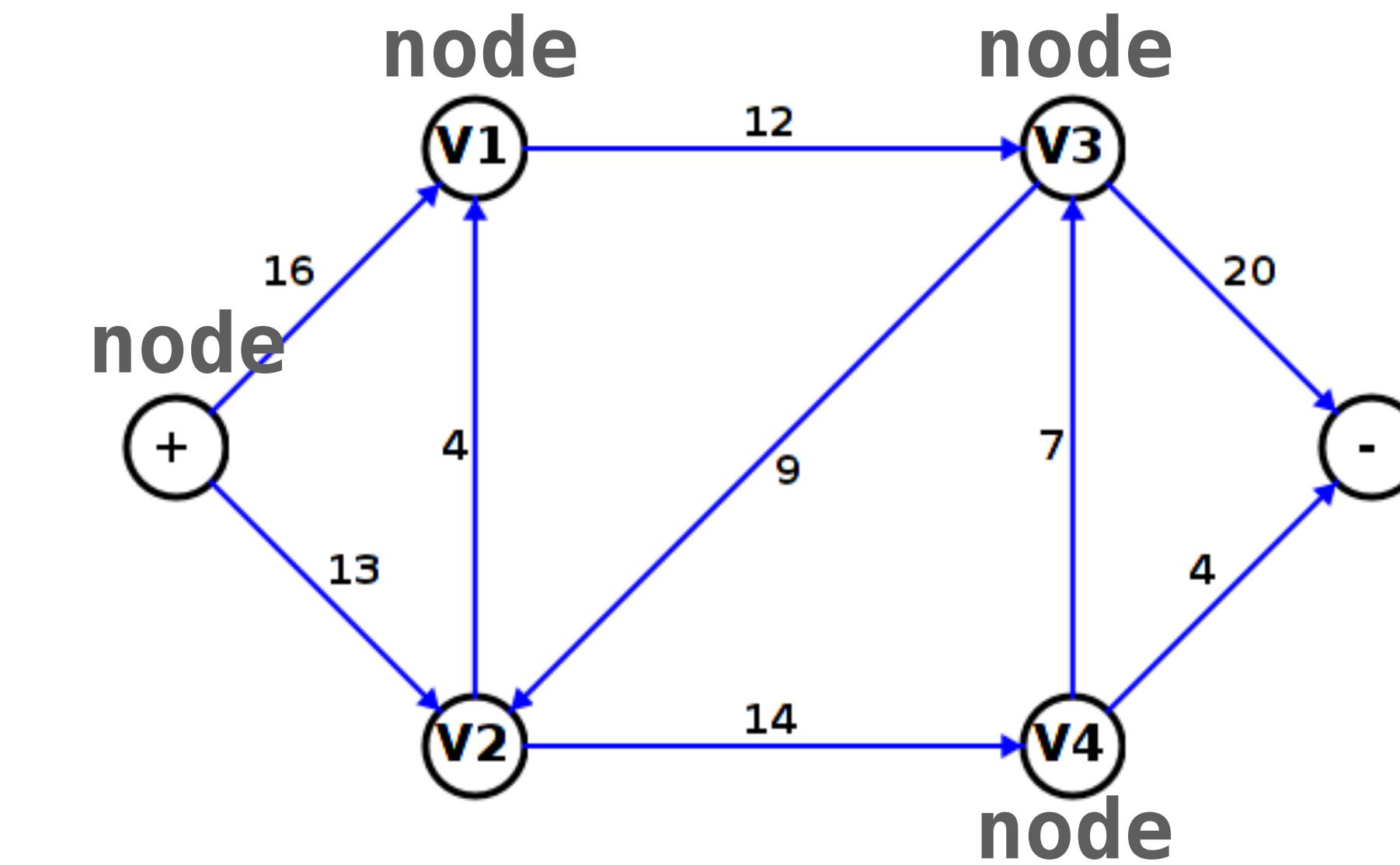
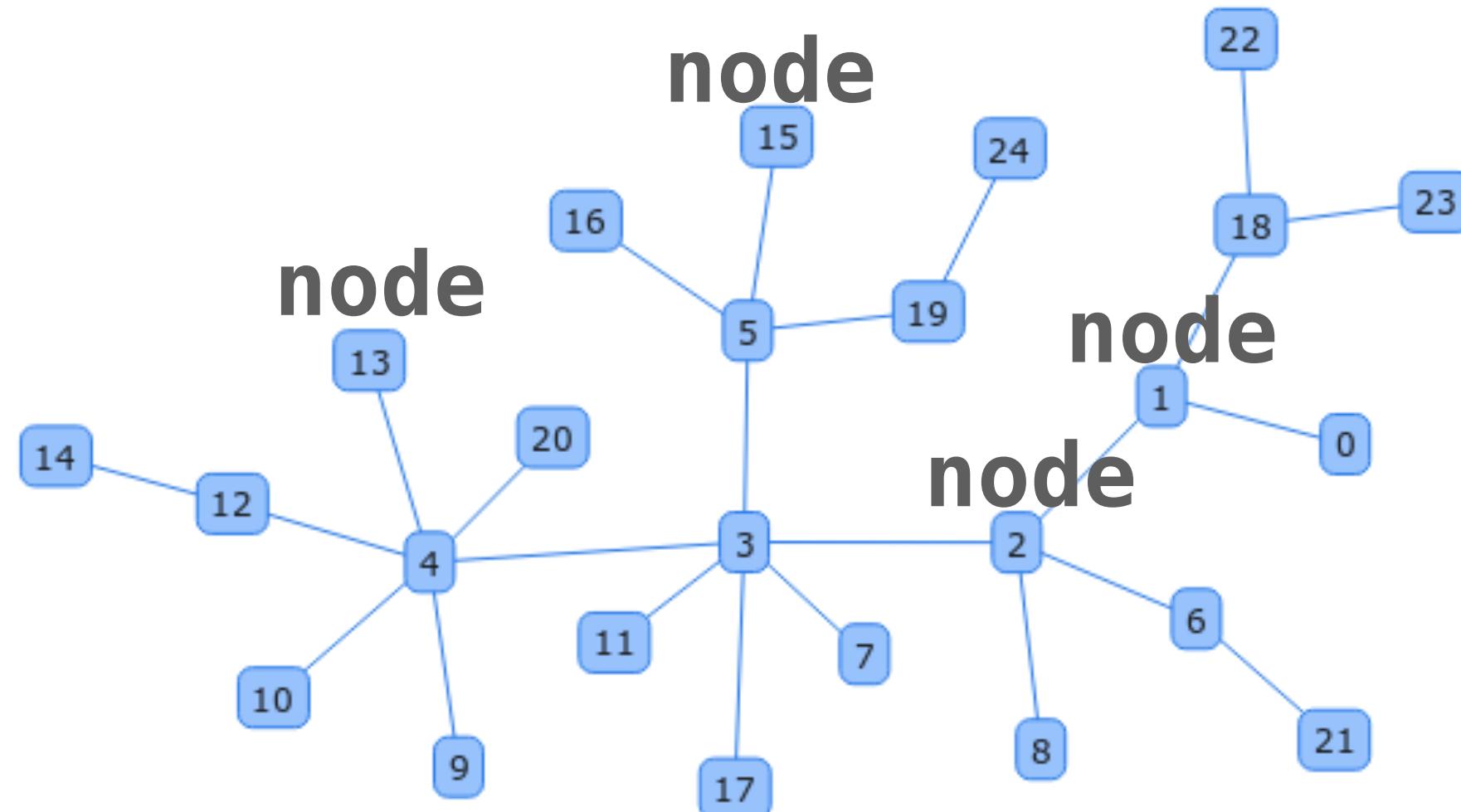
Graphs/Networks

A *graph/network* is a mathematical object representing collection of *nodes* and *edges* connecting them



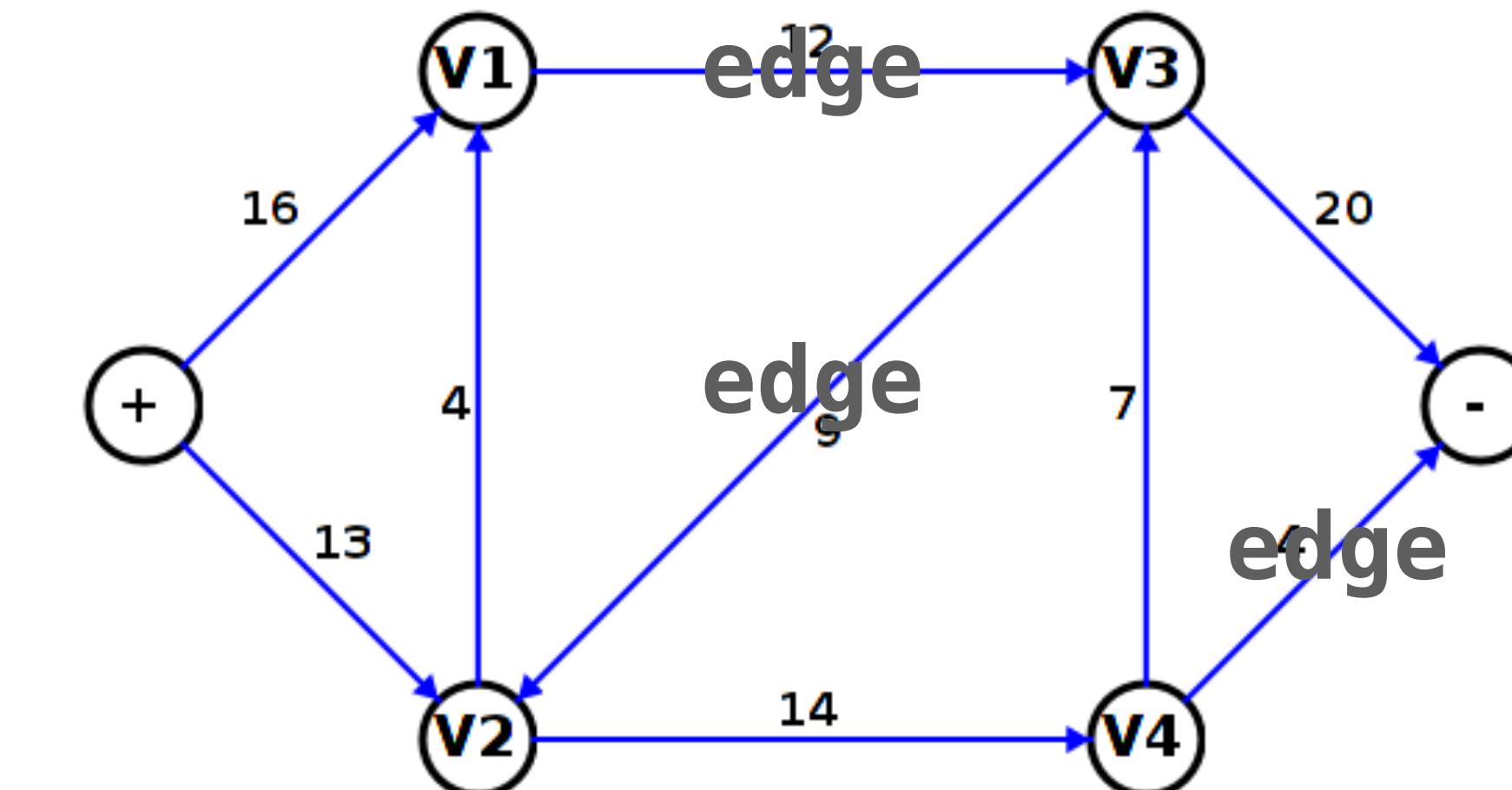
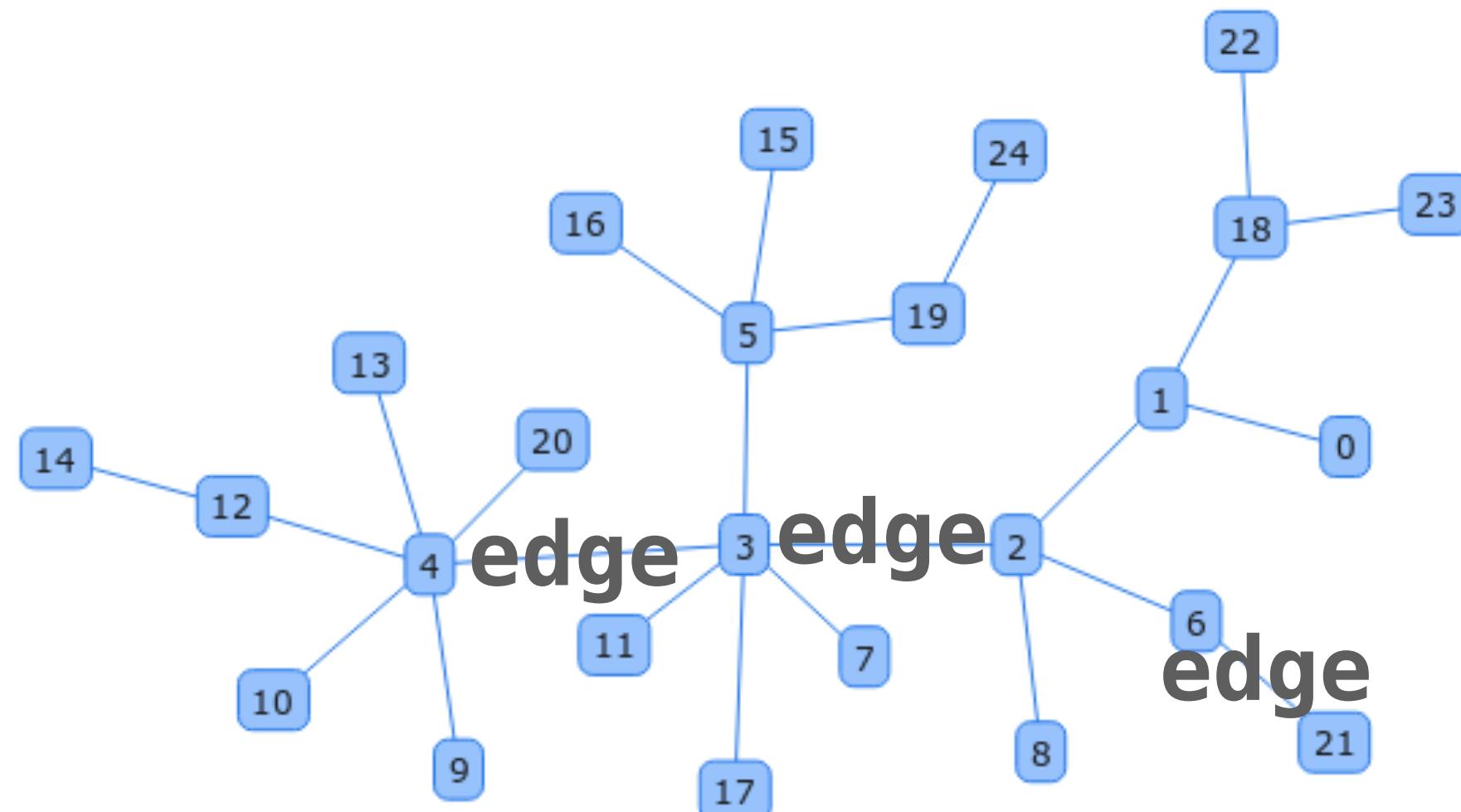
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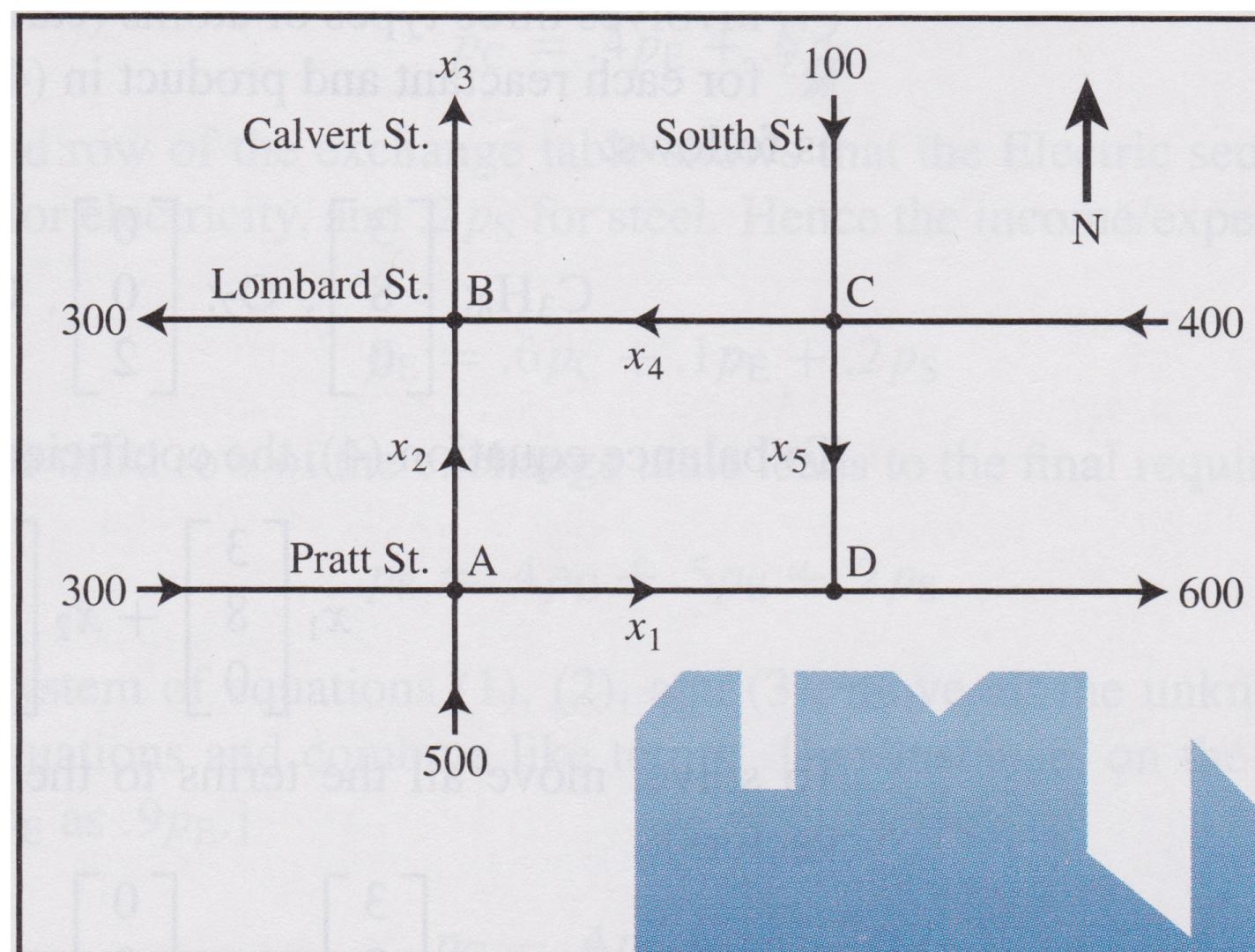
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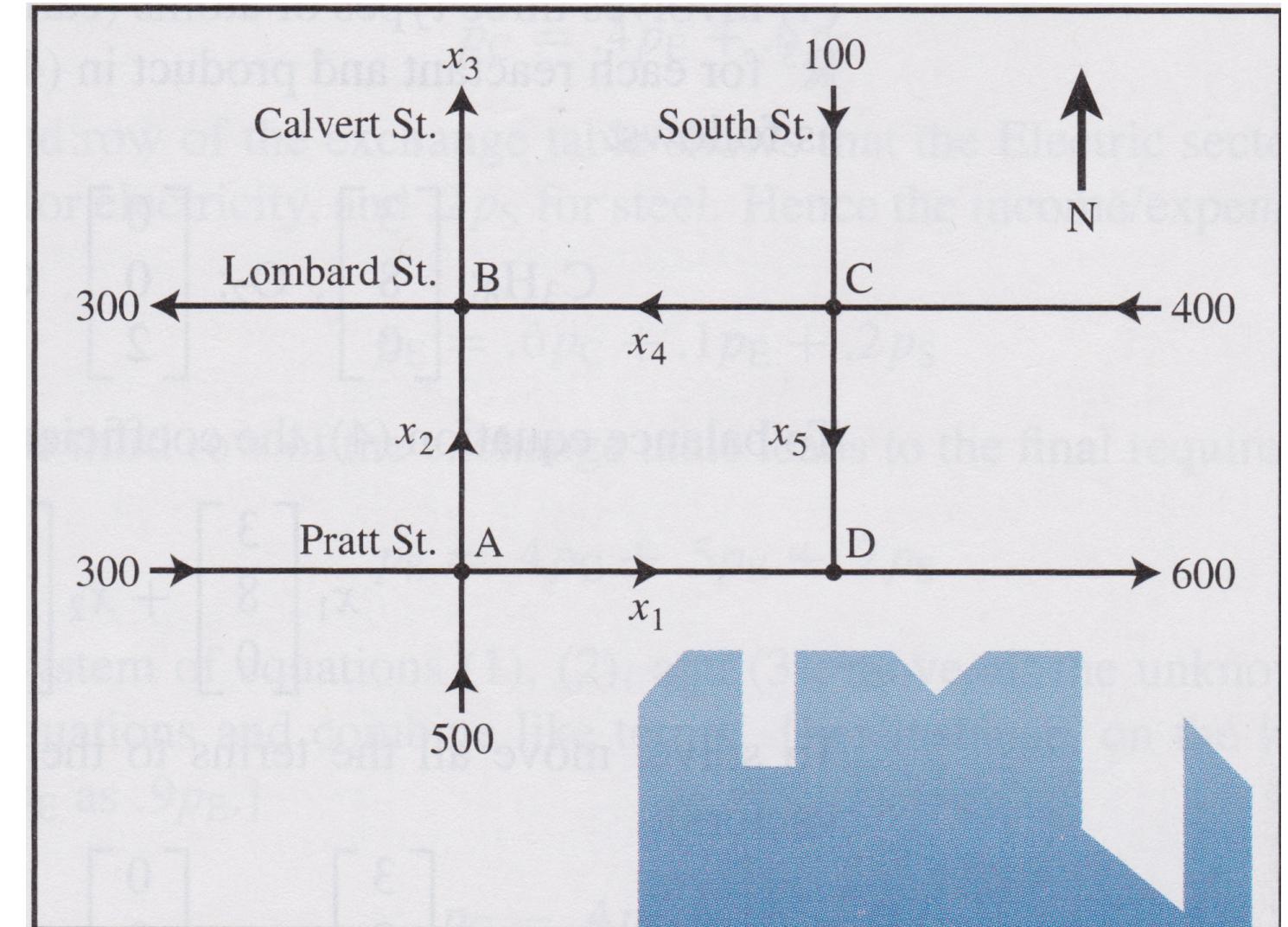
Directed Graphs

Today we focus on *directed* graphs, in which edges have a specified direction



Think of
these as
one-way
streets

Flow



We are often interested in how much "stuff" we can push through the edges

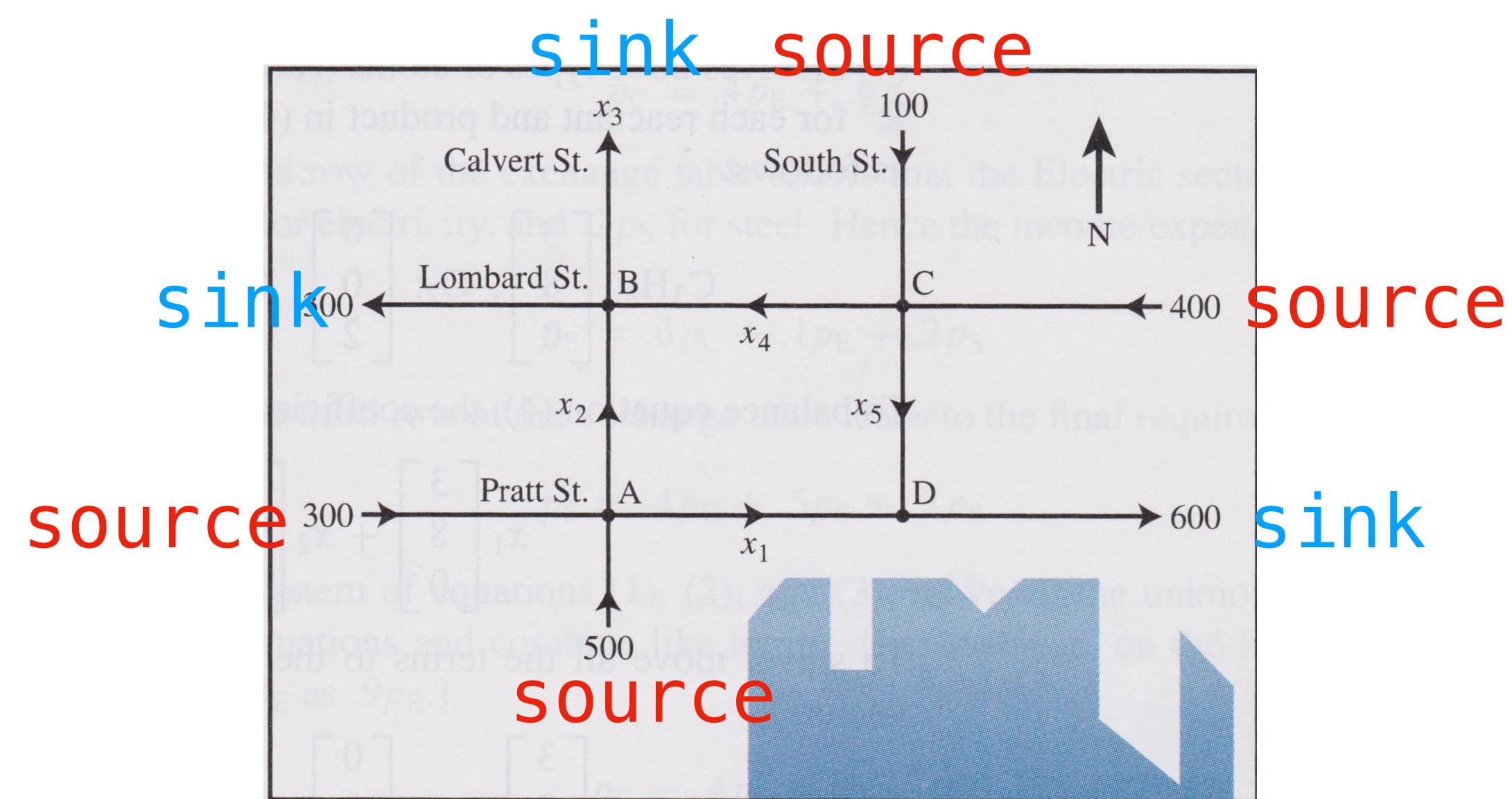
In the above example, the "stuff" is cars/hr

I like to imagine water moving through a pipe, and splitting at joints in the pipe

Flow Network

A **flow network** is a directed graph with specified **source** and **sink** nodes

Flow comes out of and goes into sources and sinks. They are assigned a flow value (or variable)



Flow

Flow

Definition. The *flow* of a graph is an assignment of nonnegative values to the edges so that the following holds

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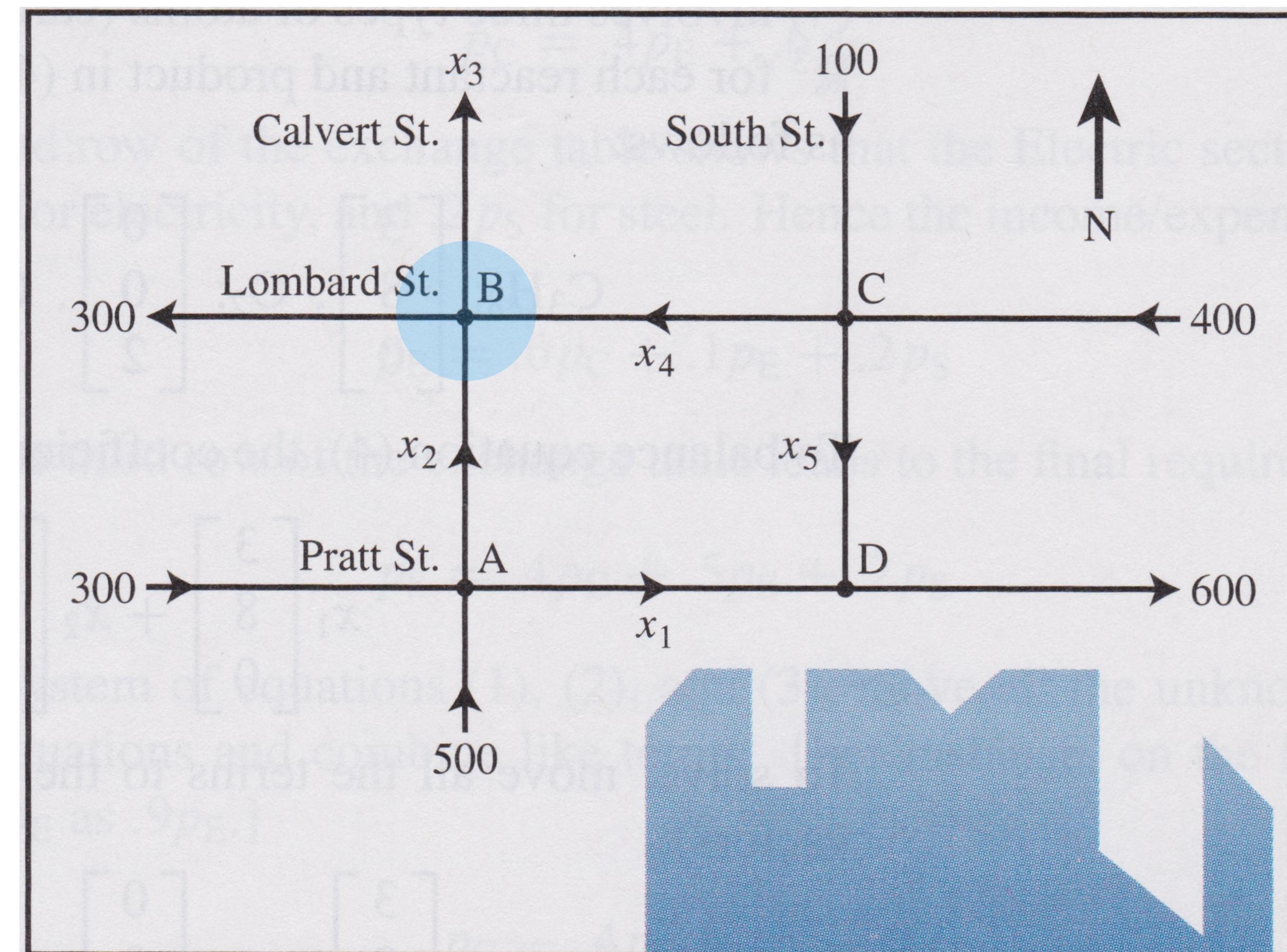
source/sink constraint: flow into a sink/out of a source is nonnegative

Flow Conservation

Flow in = Flow out

$$x_2 + x_4 = 300 + x_3$$

$$100 + 400 = x_4 + x_5$$

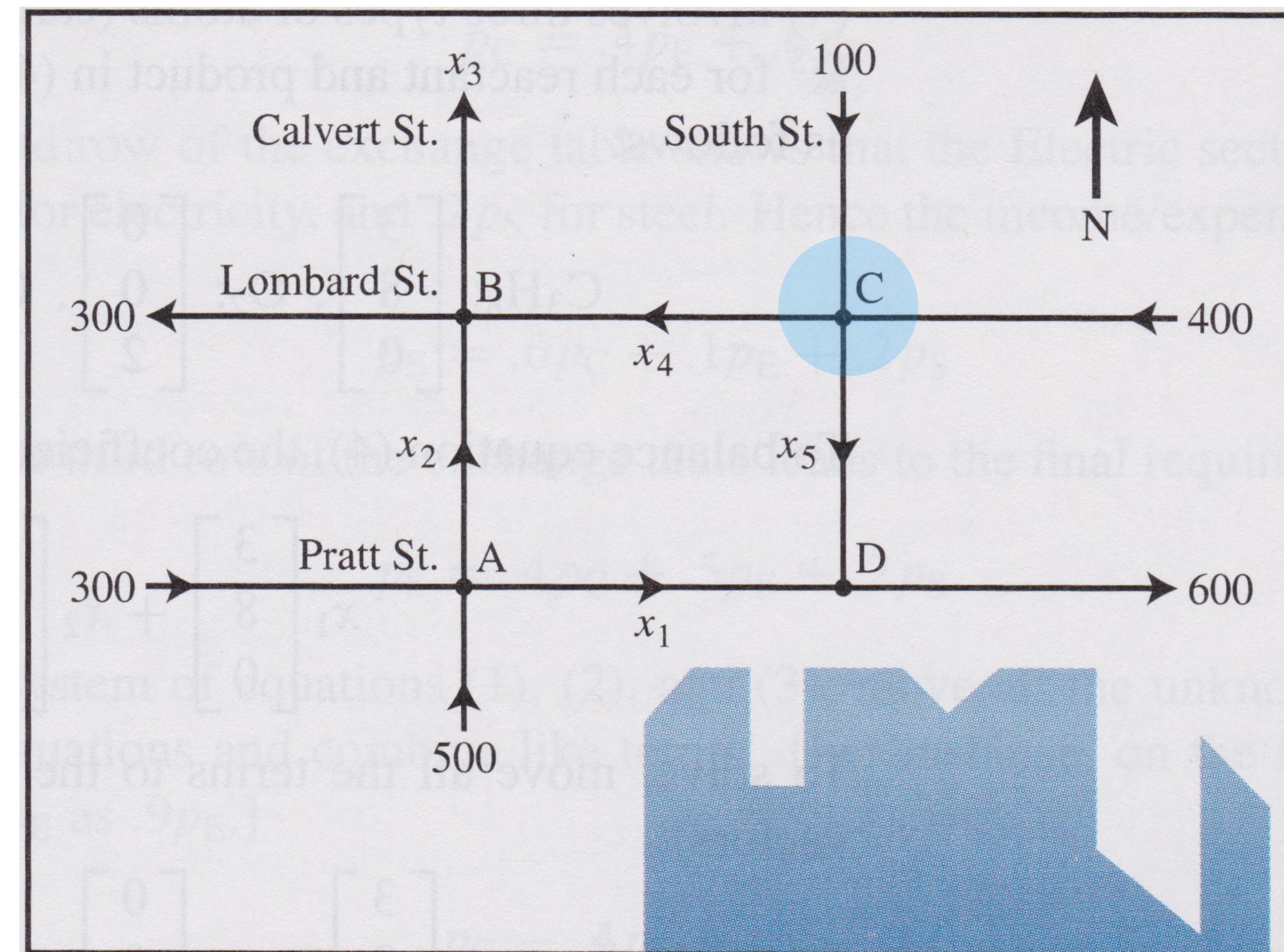


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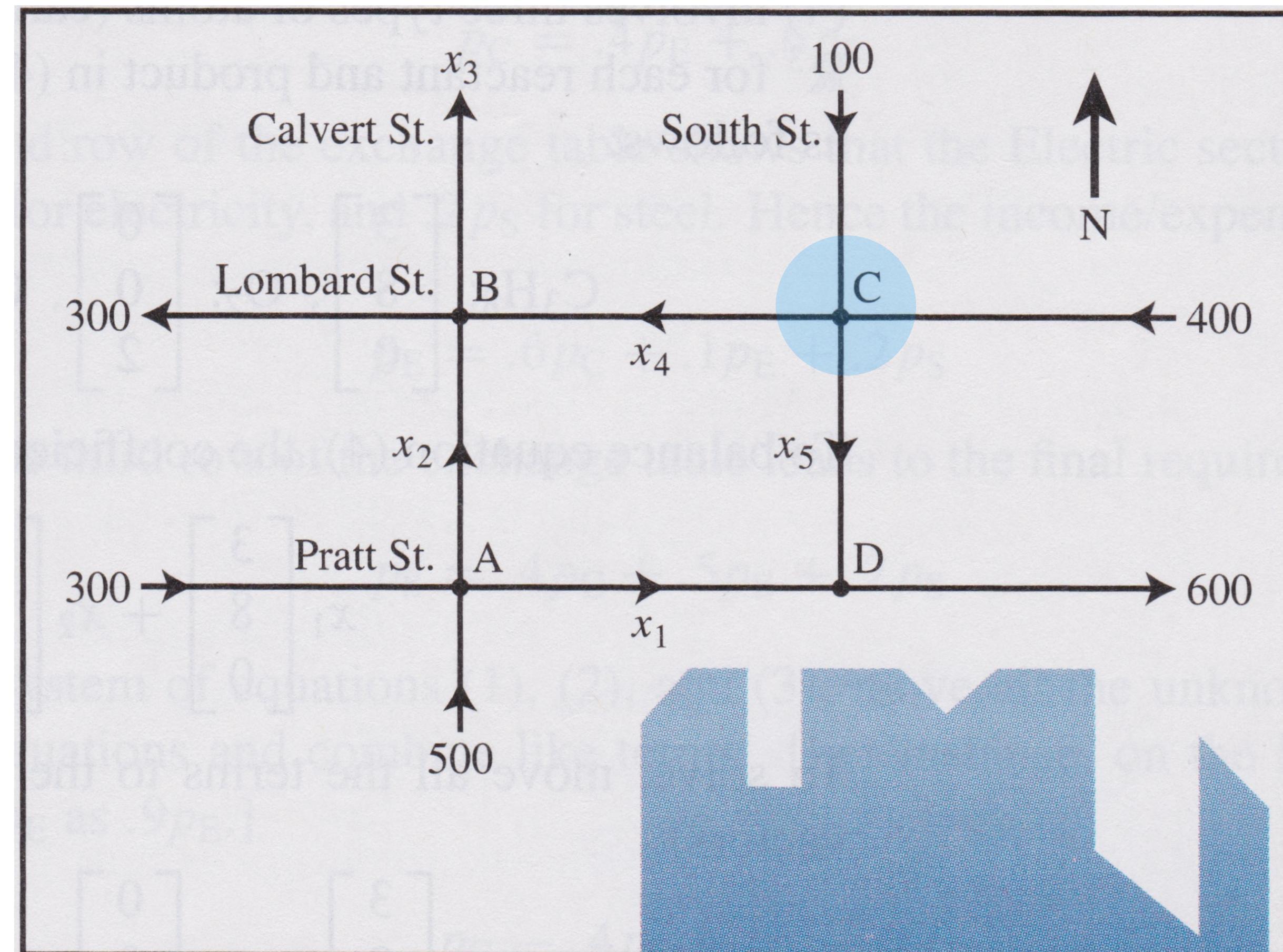
Flow Conservation

Flow in = Flow out

$$x_2 + x_4 = 300 + x_3$$

$$100 + 400 = x_4 + x_5$$

Every node
determines a linear
equation



How To: Network Flow

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Question. Find a general solution for the flow of a given graph

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Question. Find a general solution for the flow of a given graph

Solution. Write down the linear equations determined by flow conservation at non-source and non-sink nodes, and then solve

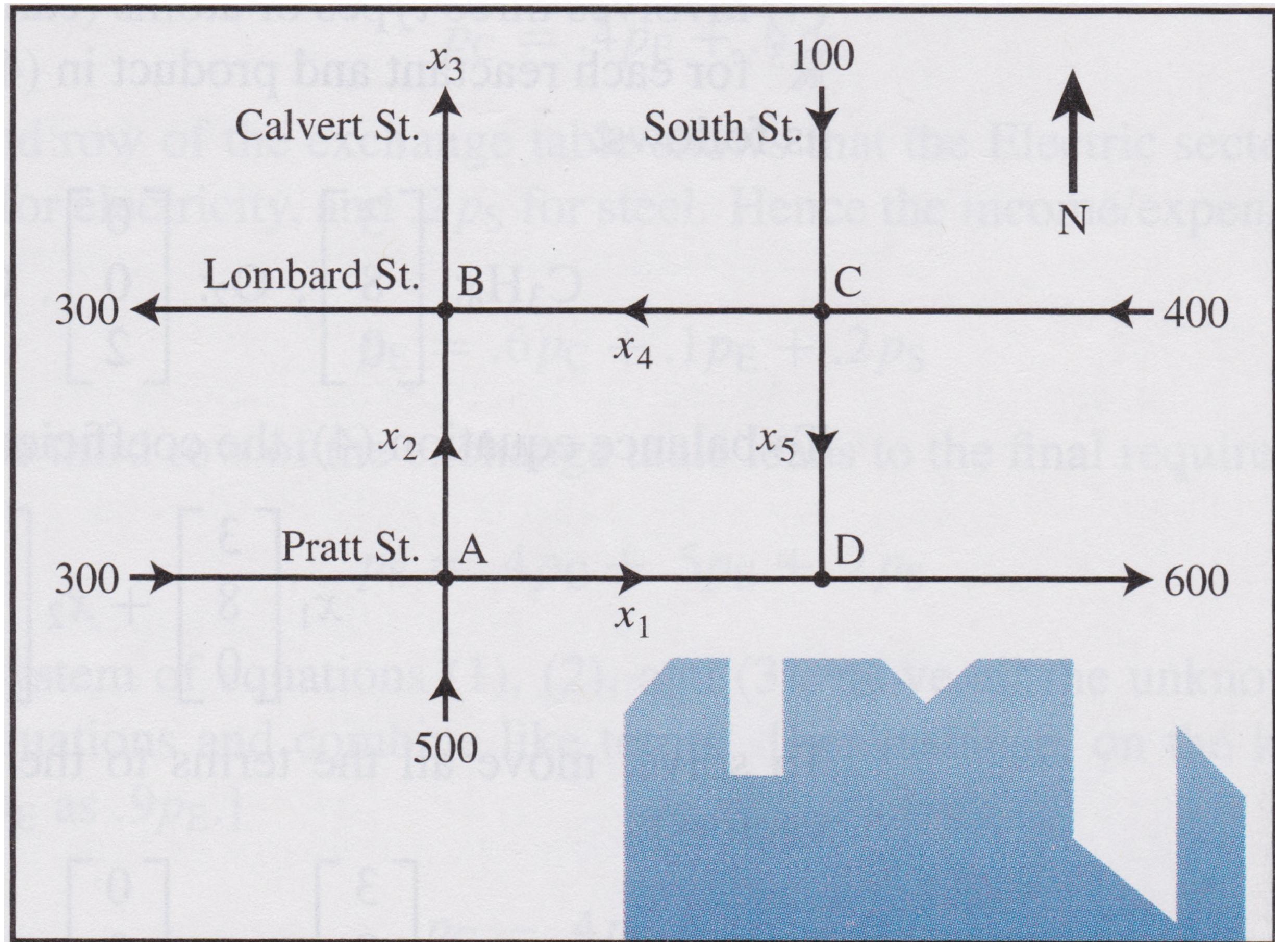
Example

(A) $500 + 300 = x_1 + x_2$

(B) $x_2 + x_4 = 300 + x_3$

(C) $100 + 400 = x_4 + x_5$

(D) $x_1 + x_5 = 600$



Example

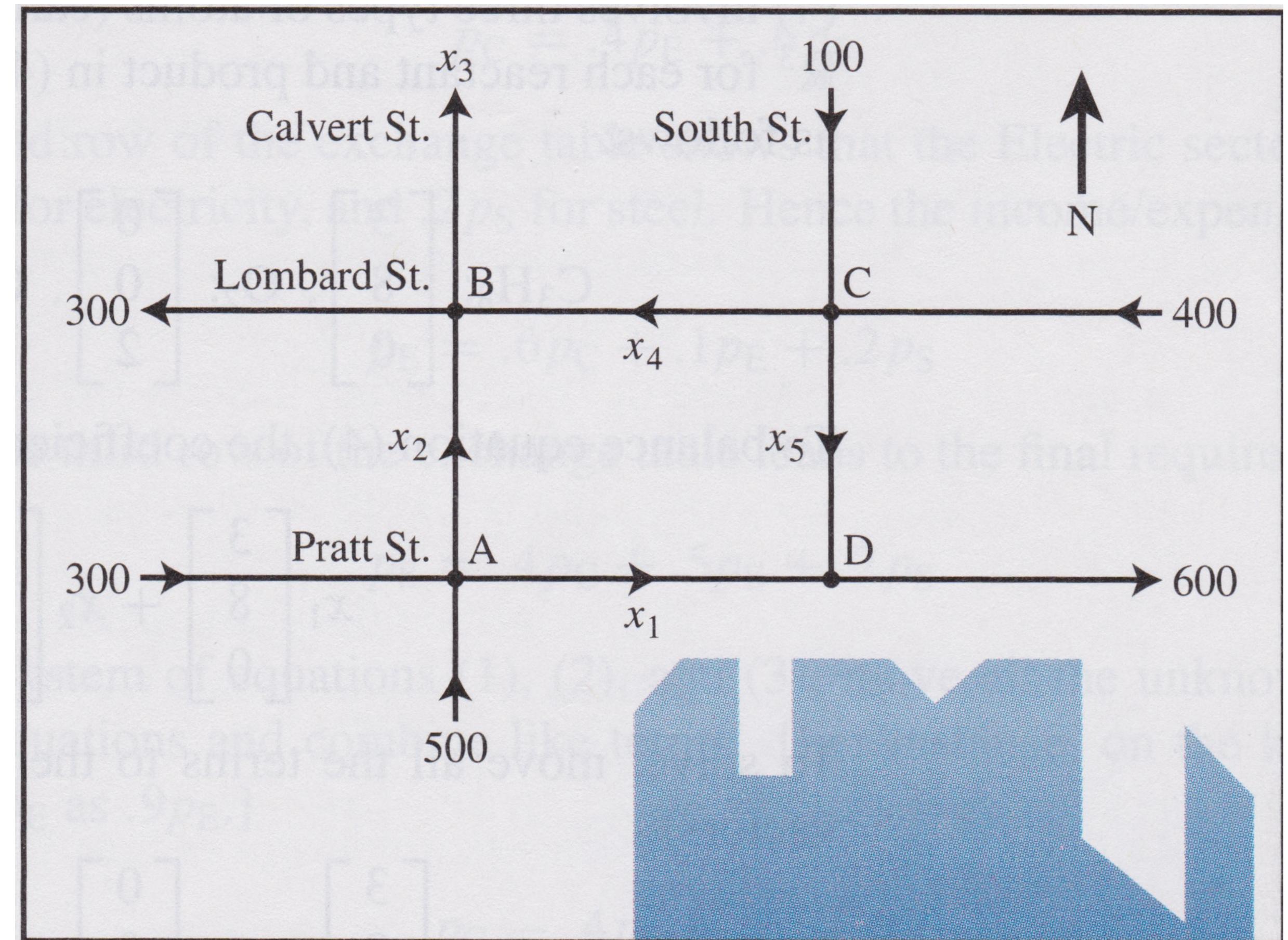
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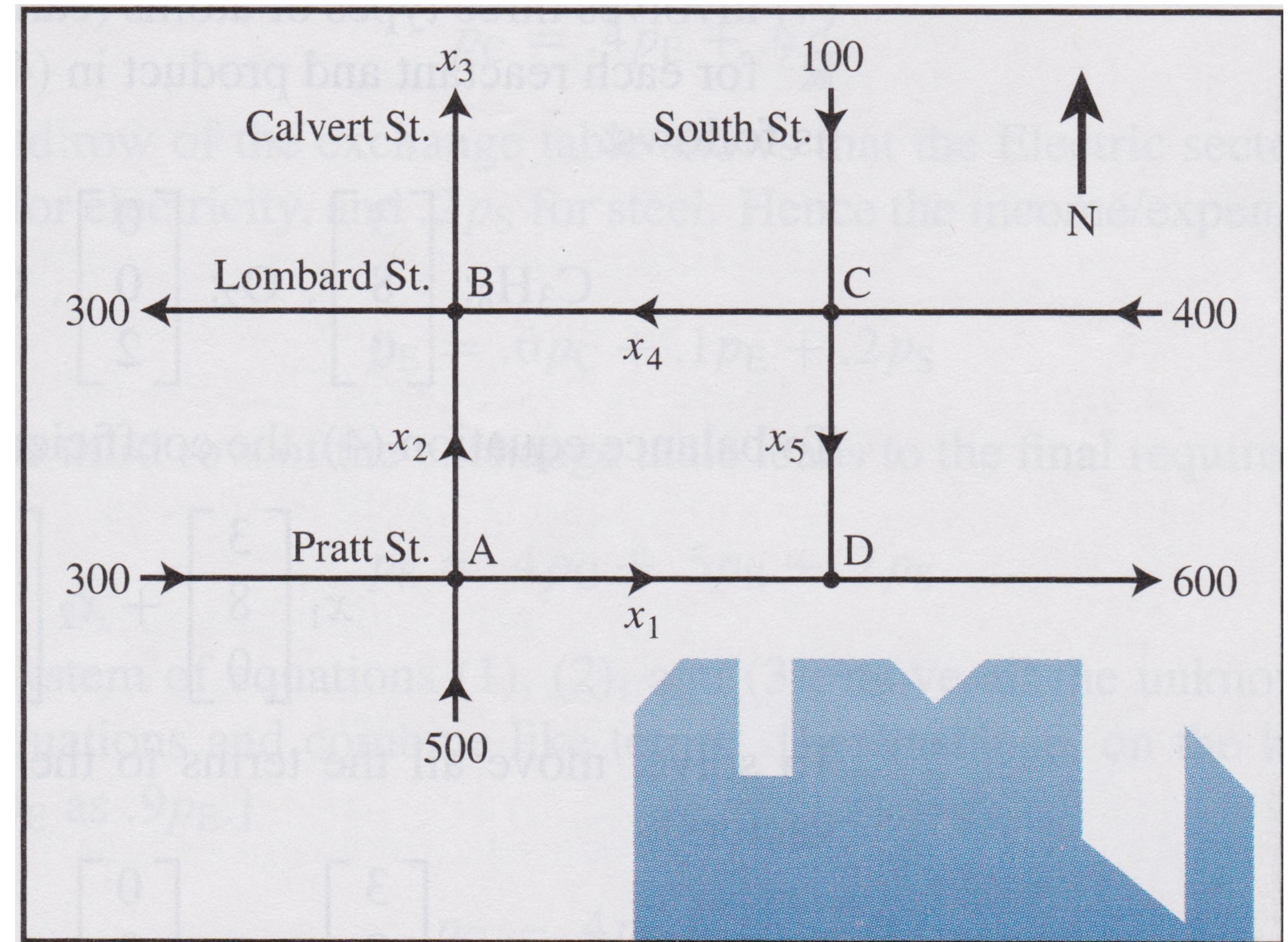
System of Linear Equations



Example

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 800 \\ 0 & 1 & -1 & 1 & 0 & 300 \\ 0 & 0 & 0 & 1 & 1 & 500 \\ 1 & 0 & 0 & 0 & 1 & 600 \end{bmatrix}$$

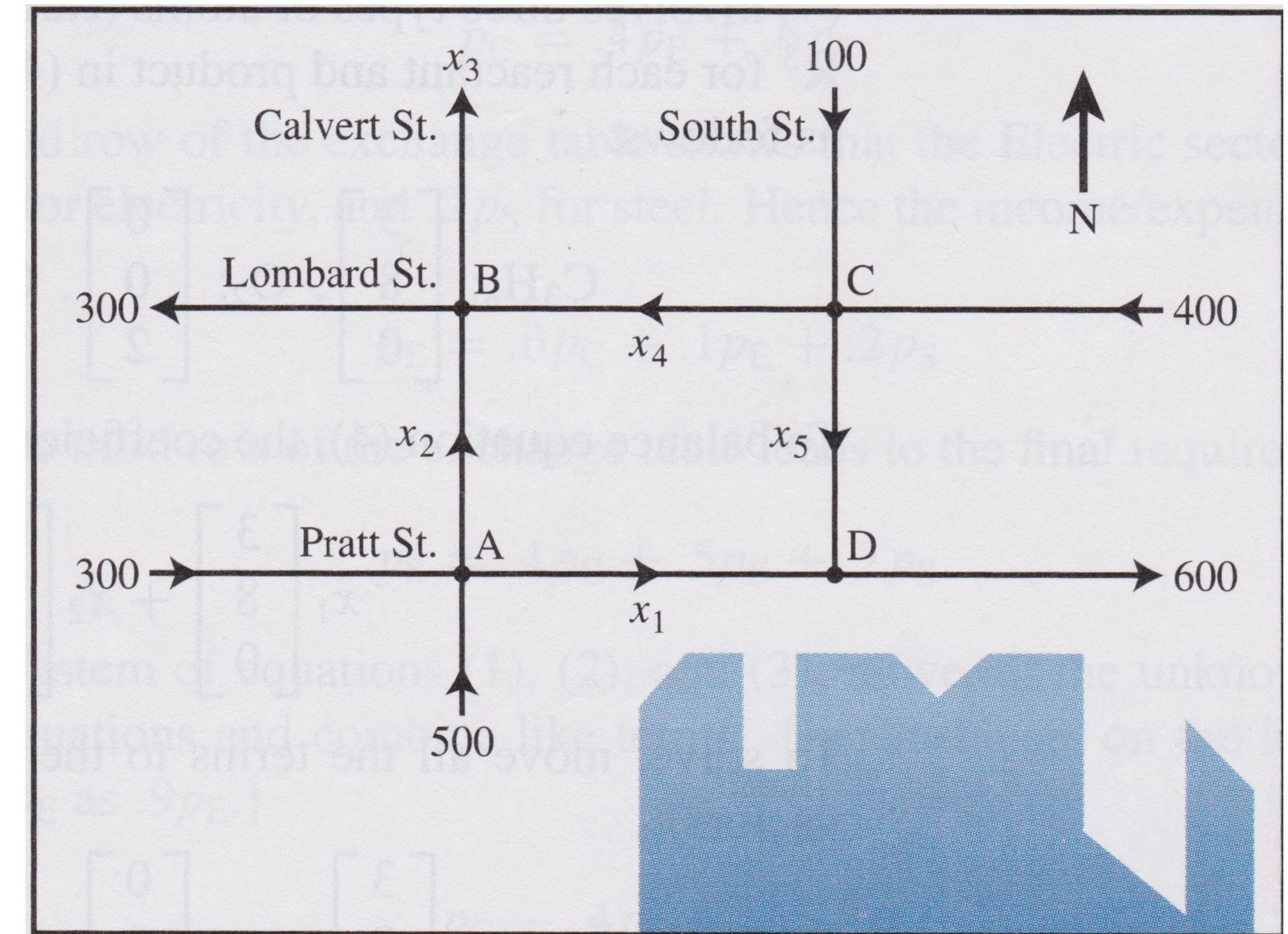
Augmented Matrix



Example

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 600 \\ 0 & 1 & 0 & 0 & -1 & 200 \\ 0 & 0 & 1 & 0 & 0 & 400 \\ 0 & 0 & 0 & 1 & 1 & 500 \end{bmatrix}$$

Reduced Echelon Form



Note that global flow is conserved.

Example

$$x_1 = 600 - x_5$$

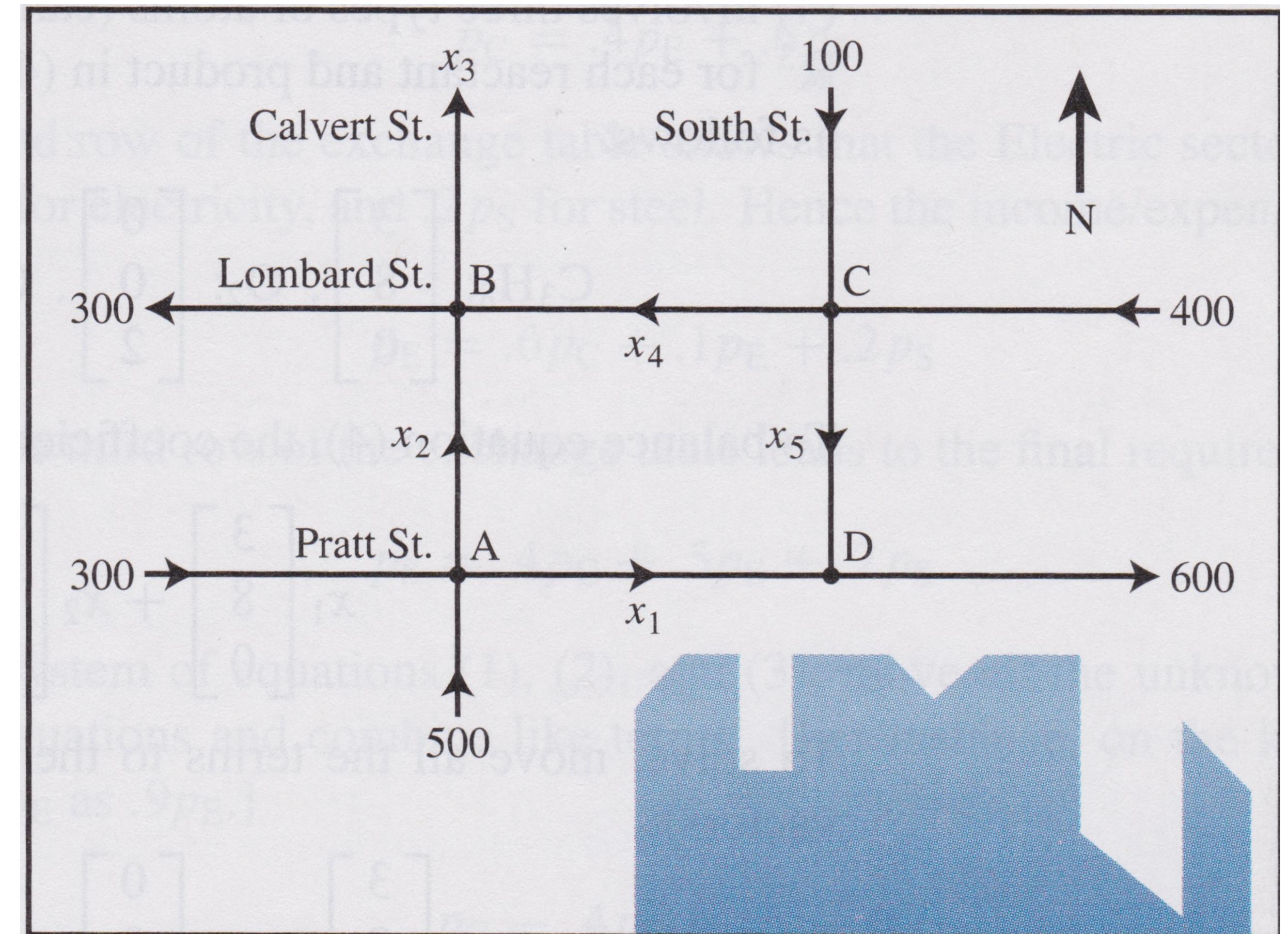
$$x_2 = 200 + x_5$$

$$x_3 = 400$$

$$x_4 = 500 - x_5$$

x_5 is free

General Solution



How To: Max Flow Value for an Edge

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Question. Find the maximum value of a flow variable in a given flow network

How To: Max Flow Value for an Edge

Question. Find the maximum value of a flow variable in a given flow network

Solution. Remember that flow values must be positive. Look at the general form solution and see what makes this hold

Example

$$x_1 = 600 - x_5$$

$$x_2 = 200 + x_5$$

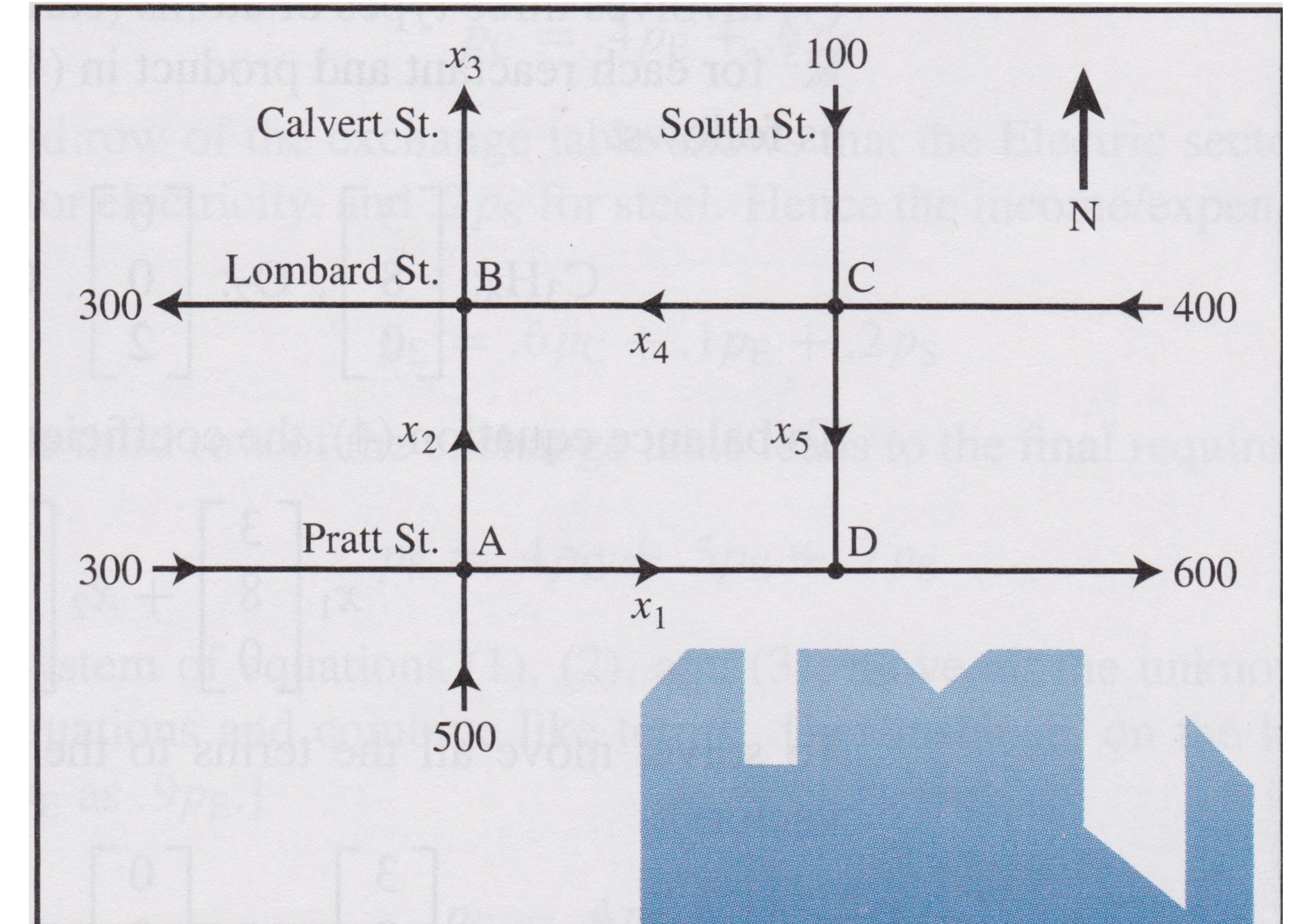
$$x_3 = 400$$

$$x_4 = 500 - x_5$$

x_5 is free

$x_4 \geq 0$ implies $x_5 \leq 500$

$x_1 \geq 0$ implies $x_5 \leq 600$



Example

$$x_1 = 600 - x_5$$

$$x_2 = 200 + x_5$$

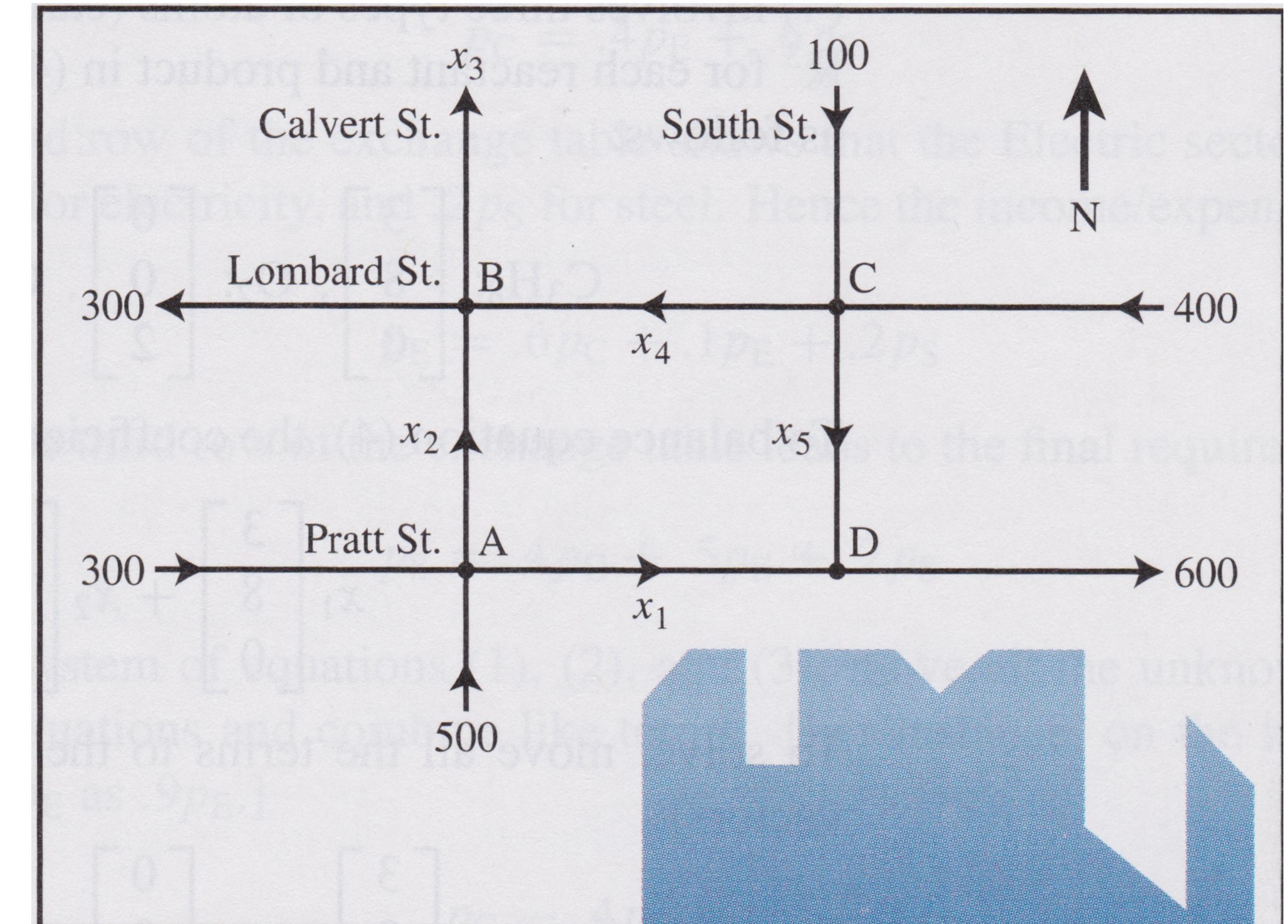
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Example

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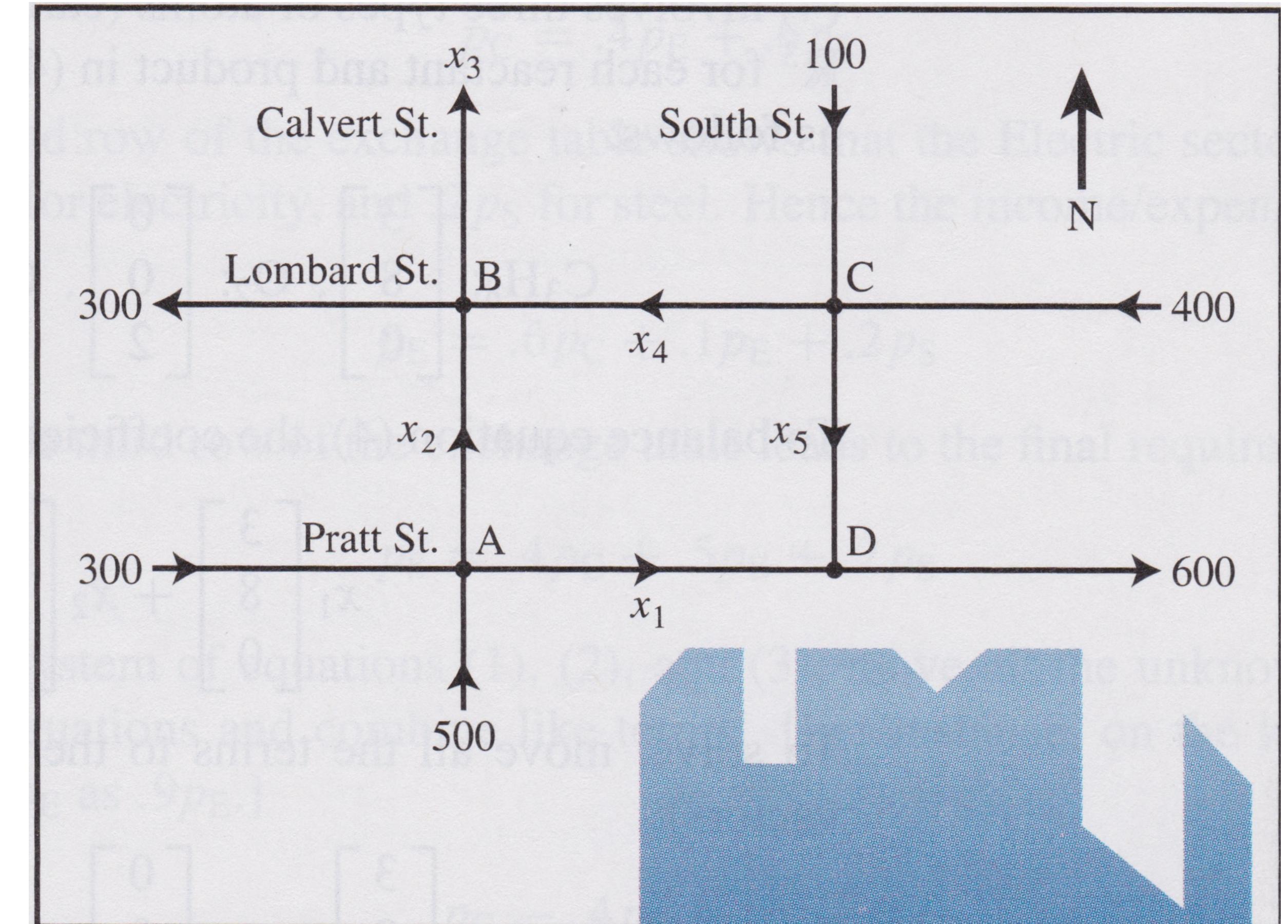
$$x_3 = 400$$

$$x_4 = 500 - x_5$$

x_5 is free

$x_4 \geq 0$ implies $x_5 \leq 500$

$x_1 \geq 0$ implies $x_5 \leq 600$



The maximum value of x_5 is 500

Summary

Linear independence helps us understand when a span is "as large as it can be"

We can reduce this seeing if a single homogeneous equation has a **unique solution**

Network flows define linear systems we can solve