

# **Vector Equations (Workshop)**

**Geometric Algorithms  
Lecture 4**

# Keywords

vector

vector addition

vector scaling/multiplication

the zero vector

vector equations

linear combinations

span

# Vectors

# Column Vectors

**Definition.** a *column vector* is a matrix with a single column, e.g.,

# A Note on Matrix Size

an  $(m \times n)$  matrix is a matrix with  $m$  rows and  $n$  columns

$$\begin{array}{c} m \end{array} \left[ \begin{array}{ccccc} * & * & \dots & * & * \\ * & * & \dots & * & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ * & * & \dots & * & * \\ * & * & \dots & * & * \end{array} \right]$$

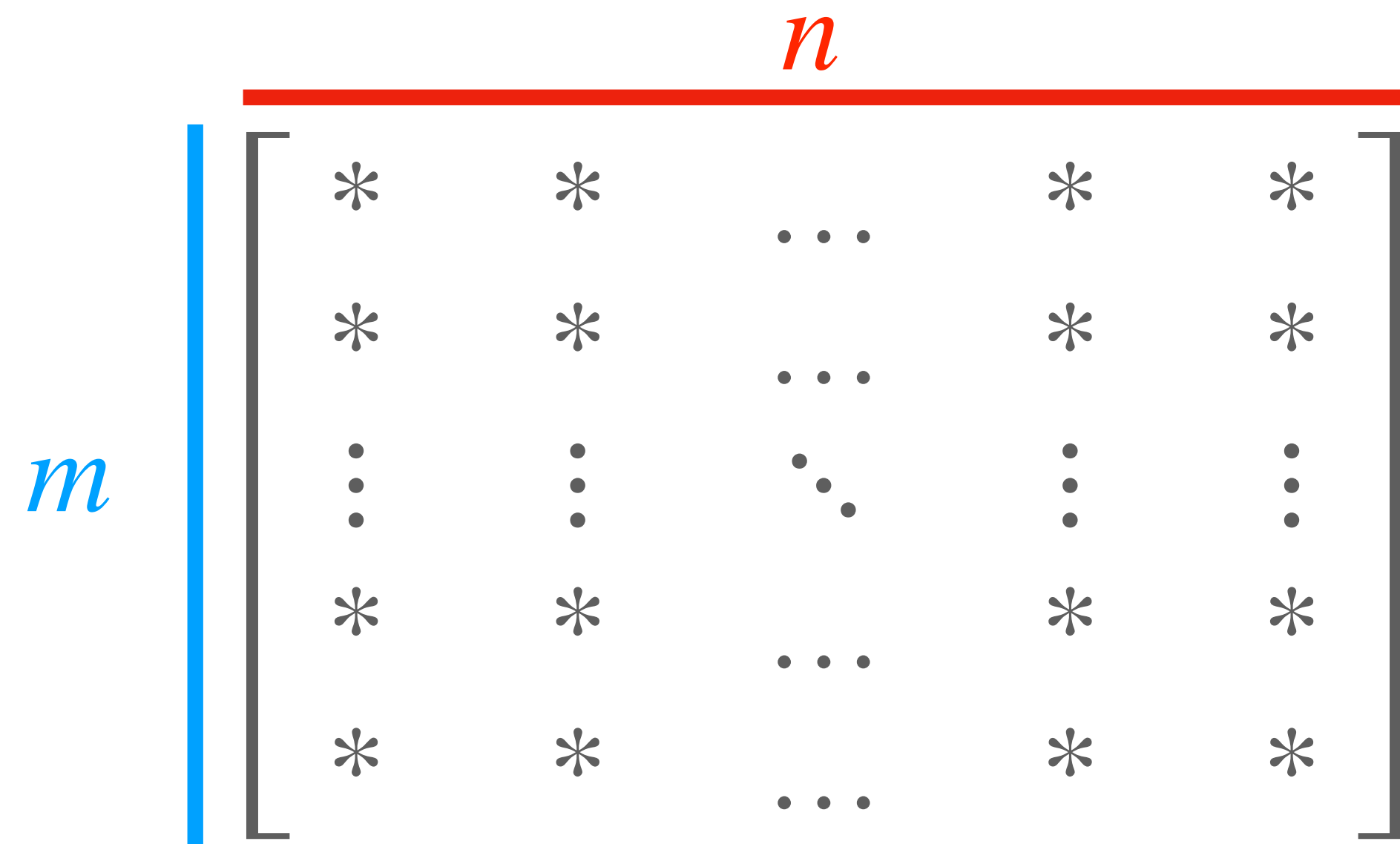
A diagram of a general  $m \times n$  matrix. A blue vertical line to the left of the matrix is labeled with the variable  $m$  in blue. A red horizontal line above the matrix is labeled with the variable  $n$  in red. The matrix is enclosed in large square brackets and contains five rows and five columns of elements. The elements are represented by asterisks (\*), with ellipses (...) indicating continuation in both dimensions. The diagonal element at row 3, column 3 is marked with a small dot.

$$4 \left[ \begin{array}{c} 2 \\ 3 \\ 0.1 \\ -2 \end{array} \right]$$

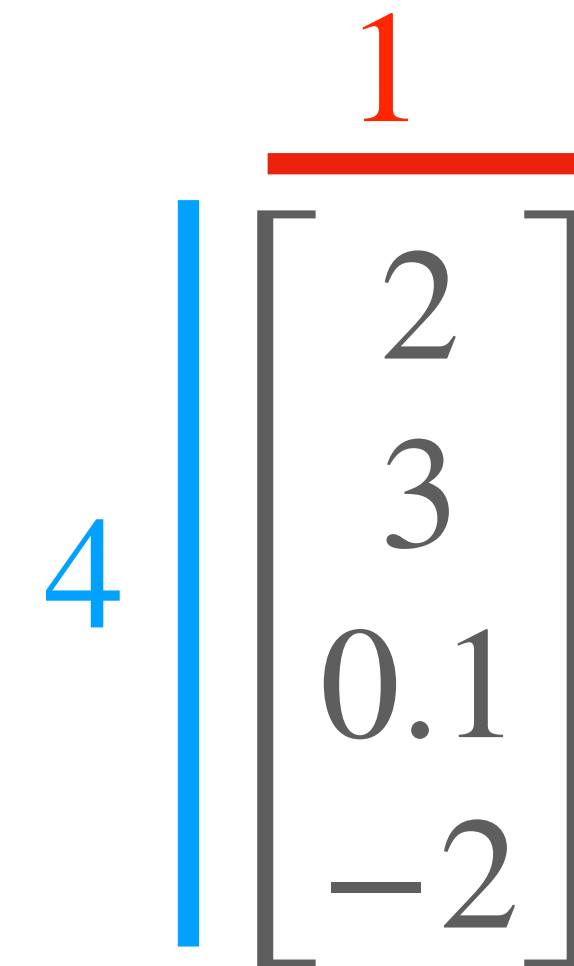
A diagram of a specific  $4 \times 1$  matrix. A blue vertical line to the left of the matrix is labeled with the number 4 in blue. A red horizontal line above the matrix is labeled with the number 1 in red. The matrix is enclosed in large square brackets and contains four rows and one column of numerical elements: 2, 3, 0.1, and -2.

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an  $(m \times n)$  matrix is a matrix with  $m$  rows and  $n$  columns



A diagram of a general matrix with  $m$  rows and  $n$  columns. A blue vertical line to the left of the matrix is labeled  $m$ . A red horizontal line above the matrix is labeled  $n$ . The matrix is enclosed in large square brackets and contains asterisks (\*) and ellipses (...) to represent its entries.

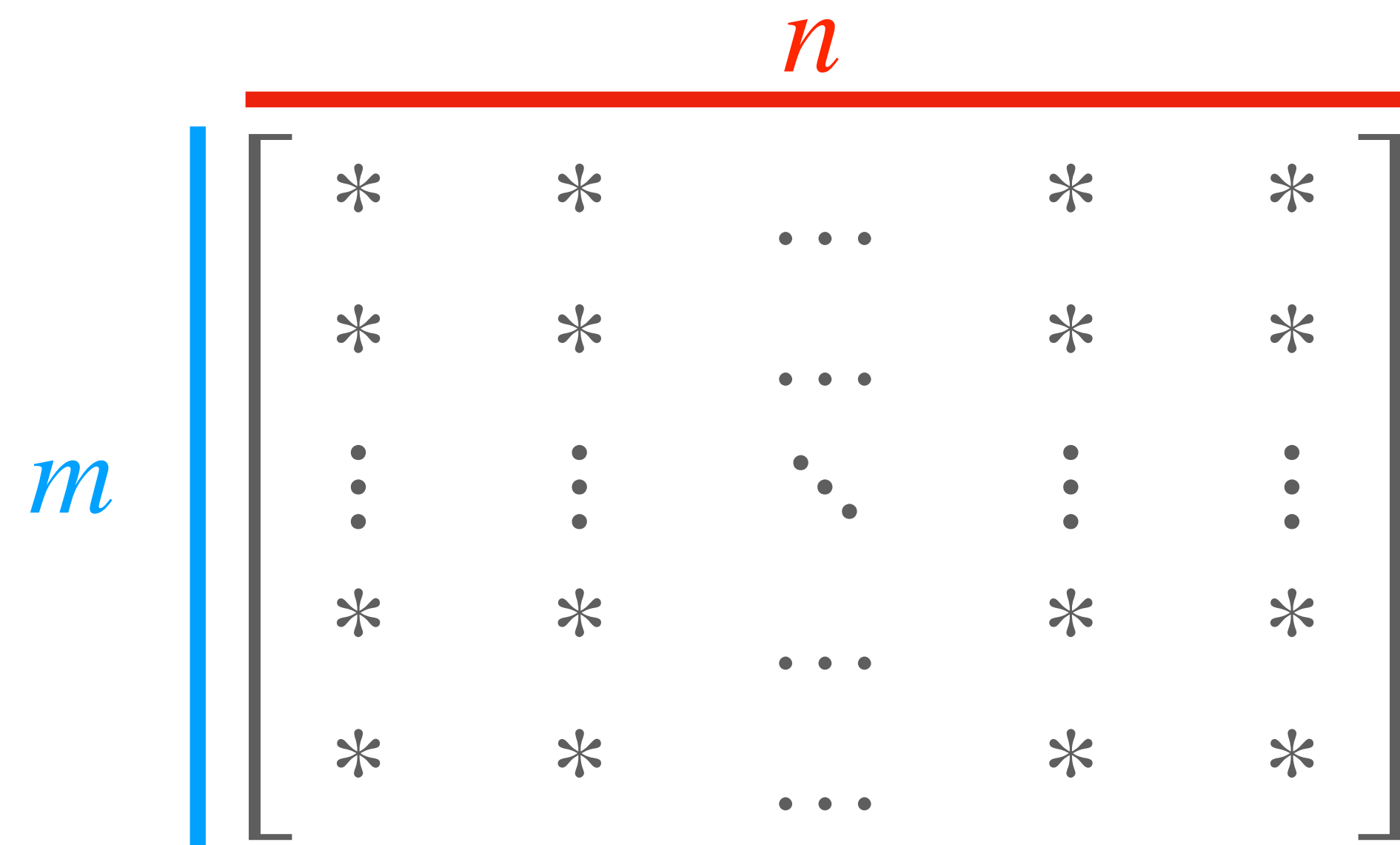


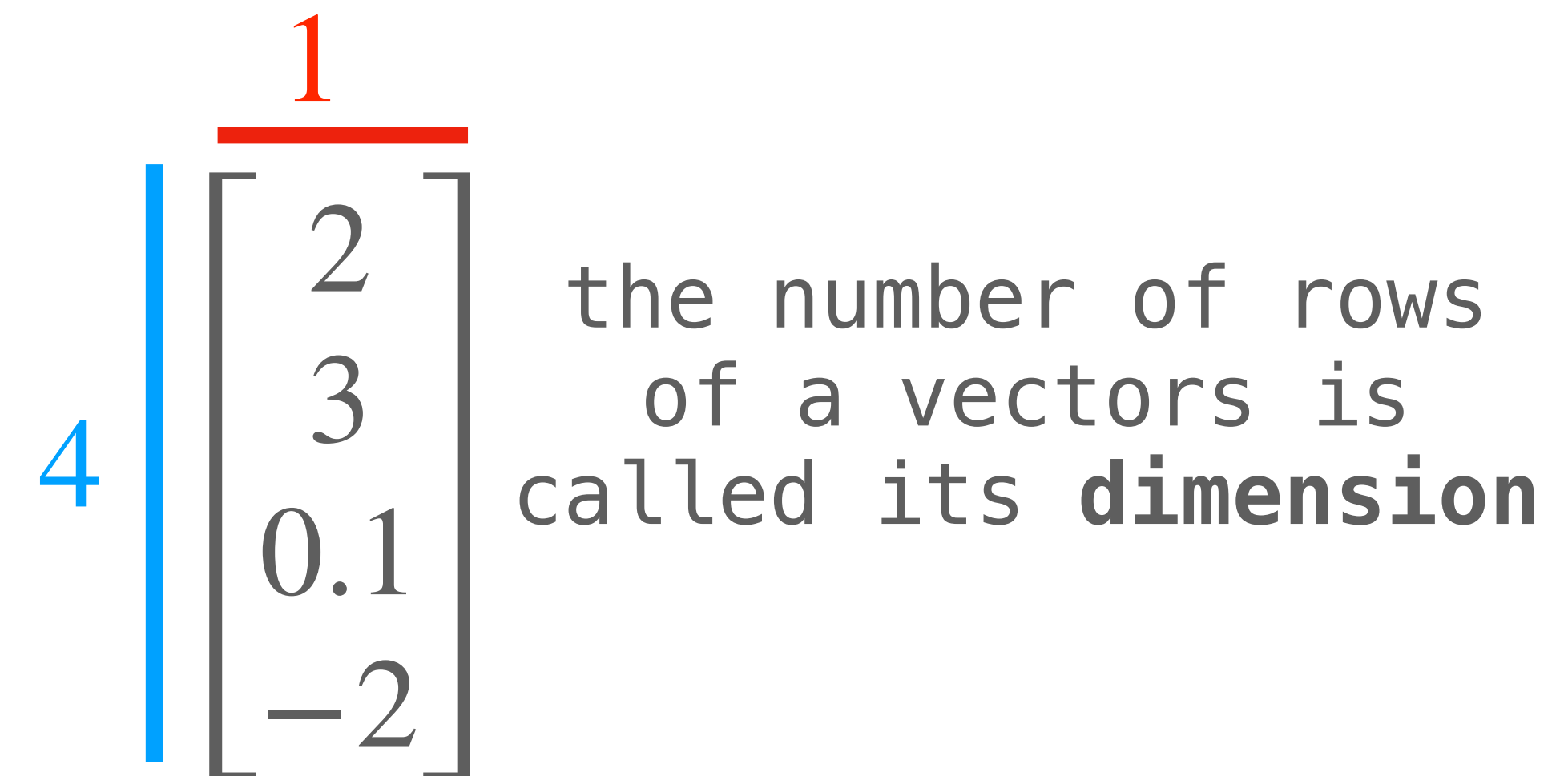
A diagram of a specific column vector with 4 rows and 1 column. A blue vertical line to the left of the vector is labeled 4. A red horizontal line above the vector is labeled 1. The vector is enclosed in large square brackets and contains the numerical values 2, 3, 0.1, and -2.

$\mathbb{R}^{m \times n}$  is set of matrices with  $\mathbb{R}$  entries

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an  $(m \times n)$  matrix is a matrix with  $m$  rows and  $n$  columns


$$\begin{matrix} m \\ \left[ \begin{array}{ccccc} * & * & \dots & * & * \\ * & * & \dots & * & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ * & * & \dots & * & * \\ * & * & \dots & * & * \end{array} \right] \end{matrix}$$


$$\begin{matrix} 4 \\ \left[ \begin{array}{c} 2 \\ 3 \\ 0.1 \\ -2 \end{array} \right] \end{matrix}$$

the number of rows  
of a vectors is  
called its **dimension**

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# Vector Operations



# Vector "Interface"

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equality      what does it mean for two vectors to be equal?

addition      what does  $\mathbf{u} + \mathbf{v}$  (adding two vectors mean?

scaling      what does  $a\mathbf{v}$  (multiplying a vector by a real number) mean?

# Vector Equality

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

is the same as

$$\begin{aligned} a_1 &= b_1 \\ a_2 &= b_2 \\ &\vdots \\ a_n &= b_n \end{aligned}$$

# Vector Addition

adding two vectors means adding their entries

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \vdots \\ a_n + b_n \end{bmatrix}$$

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**!! IMPORTANT !!**

**WE CAN ONLY ADD VECTORS OF THE SAME SIZE**

# Vector Scaling

scaling/multiplying a vector by a number means multiplying each of it's elements

$$a \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} ab_1 \\ ab_2 \\ \vdots \\ ab_n \end{bmatrix}$$



# Algebraic Properties

For any vectors  $\mathbf{u}, \mathbf{v}, \mathbf{w}$  and any real numbers  $c, d$ :

$$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$$

$$c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$$

$$(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$$

$$(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$$

$$\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$$

$$c(d\mathbf{u}) = (cd)\mathbf{u}$$

$$\mathbf{u} + (-\mathbf{u}) = -\mathbf{u} + \mathbf{u} = \mathbf{0}$$

$$1\mathbf{u} = \mathbf{u}$$

demo  
(from ILA)

# Linear Combinations

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**Definition.** a *linear combination* of vectors

$$\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$$

is a vector of the form

$$\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \dots + \alpha_n \mathbf{v}_n$$

where  $\alpha_1, \alpha_2, \dots, \alpha_n$  are in  $\mathbb{R}$

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weights

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Looks suspiciously like  
a linear equation...

where  $\alpha_1, \alpha_2, \dots, \alpha_n$  are in  $\mathbb{R}$   
weights

demo  
(from ILA)

# **Vector Equations and Linear Systems**



# The Fundamental Connection

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$$x_1 \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$$

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$$\begin{bmatrix} x_1 + 2x_2 \\ (-2)x_1 + 5x_2 \\ -5x_1 + 6x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$$



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we get a system  
of linear  
equations we  
know how to  
solve

# General Vector Equations

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{1m} \end{bmatrix} + x_2 \begin{bmatrix} a_{21} \\ a_{21} \\ \vdots \\ a_{2m} \end{bmatrix} + \dots + x_n \begin{bmatrix} a_{n1} \\ a_{n2} \\ \vdots \\ a_{nm} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

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by vector scaling

# General Vector Equations

$$\begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

by vector addition

# General Vector Equations

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m$$

by vector equality

# The Fundamental Connection

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}$$

augmented matrix

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system of linear equations

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vector equation

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this is notation for  
building a matrix  
out of column  
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read:  $\mathbf{u}$  is an element of  $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$

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demo  
(from ILA)

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you know how to do this now

demo  
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workshop