

# Linear Independence

**Geometric Algorithms**  
**Lecture 6**

# Practice Problem

Do these three vectors span all of  $\mathbb{R}^3$ ?

$$\mathbf{v}_1 = \begin{bmatrix} -4 \\ 4 \\ 2 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} -3 \\ 6 \\ -3 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} -5 \\ 8 \\ -2 \end{bmatrix}$$

# Answer

$$\mathbf{v}_1 = \begin{bmatrix} -4 \\ 4 \\ 2 \end{bmatrix}$$

$$\mathbf{v}_2 = \begin{bmatrix} -3 \\ 6 \\ -3 \end{bmatrix}$$

$$\mathbf{v}_3 = \begin{bmatrix} -5 \\ 8 \\ -2 \end{bmatrix}$$

~~$$\vec{b} = \alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \alpha_3 \vec{v}_3$$~~

$$\begin{bmatrix} -4 & -3 & -5 \\ 4 & 6 & 8 \\ 2 & -3 & -2 \end{bmatrix} \sim$$

$$\begin{bmatrix} 2 & -3 & -2 \\ 2^{-2} & 3^{+3} & 4^{+2} \\ -4^{+4} & -3^{-6} & -5^{-4} \end{bmatrix} \sim$$

$$\begin{bmatrix} 2 & -3 & 2 \\ 0 & 6 & 6 \\ 0 & -9 & -9 \end{bmatrix}$$

2 pivots

3 rows

$$\sim \begin{bmatrix} 2 & -3 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

NO

# Outline

- » Motivate and define **linear independence**
- » See several perspectives on linear independence
- » If there's time: see an application of linear systems to **network flows**

# Keywords

linear independence

linear dependence

homogenous systems of linear equations

trivial and nontrivial solutions

# Homogeneous Linear Systems

# Recall: The Zero Vector

$$\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

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$$\mathbf{v} + \mathbf{0} = \mathbf{0} + \mathbf{v} = \mathbf{v}$$

$$c\mathbf{0} = \mathbf{0}$$

$$\mathbf{u} + -\mathbf{u} = \mathbf{0}$$

$$\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

# Recall: The Zero Vector

$$\mathbf{v} + \mathbf{0} = \mathbf{0} + \mathbf{v} = \mathbf{v}$$

$$c\mathbf{0} = \mathbf{0}$$

$$\mathbf{u} + -\mathbf{u} = \mathbf{0}$$

$$\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \Big| \textcolor{blue}{n}$$

the  
dimension is  
implicit in  
the notation

# Homogenous Linear Systems

**Definition.** A system of linear equations is called *homogeneous* if it can be expressed as

$$A\mathbf{x} = \mathbf{0}$$

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# Homogenous Linear Systems

**Definition.** A system of linear equations is called *homogeneous* if it can be expressed as

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = 0$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = 0$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = 0$$

# Trivial Solutions

**Definition.** For the matrix equation  $A\mathbf{x} = \mathbf{0}$  the solution  $\mathbf{x} = \mathbf{0}$  is called the ***trivial solution***

Any other solution is called ***nontrivial***

# Trivial Solutions

**Definition.** For the vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \dots + x_n\mathbf{a}_n = \mathbf{0}$$

the solution  $\mathbf{x} = \mathbf{0}$  is called the ***trivial solution***

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# Trivial Solutions

**Definition.** For the system of linear equations

$$\begin{aligned}a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= 0 \\a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= 0 \\&\vdots \\a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= 0\end{aligned}$$

the solution  $\mathbf{x} = \mathbf{0}$  is called the ***trivial solution***

Any other solution is called ***nontrivial***

# Questions about Homogeneous Systems

When does  $A\mathbf{x} = \mathbf{0}$  have only the **trivial solution**?

When does  $A\mathbf{x} = \mathbf{0}$  have **nontrivial solutions**?

What does it mean *geometrically* in each case?

# An Important Feature of Homogenous Systems

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 1 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & \blacksquare \\ 0 & -3 & -6 & \blacksquare \end{bmatrix}$$

# An Important Feature of Homogenous Systems

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*What do we know about the covered column?*

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*What do we know about the covered column?*

**It has to be all zeros**

# Linear Independence

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**Definition.** A set of vectors  $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$  is *linearly independent* if the vectors equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \dots + x_n\mathbf{v}_n = \mathbf{0}$$

has exactly one solution (the trivial solution)

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**The columns of  $A$  are linearly independent  
if  $A\mathbf{x} = \mathbf{0}$  has exactly one solution**

# Linear Dependence

**Definition.** A set of vectors  $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$  is *linearly dependent* if the vectors equation

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has a *nontrivial* solution

A set of vectors is linearly dependent if there is a nontrivial linear combination of the vectors which equals  $\mathbf{0}$

# Linear Dependence (Alternative)

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**Definition.** A set of vectors is **linearly dependent** if it is not linearly independent

$A\mathbf{x} = \mathbf{0}$  has a nontrivial solution

$\equiv$

$A\mathbf{x} = \mathbf{0}$  does not have only the trivial solution

# Examples

$$\left\{ \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Lin.  
Ind.

$$(-1)x_1 + x_2 = 0$$

|||

$$x_2 - x_1 = 0$$

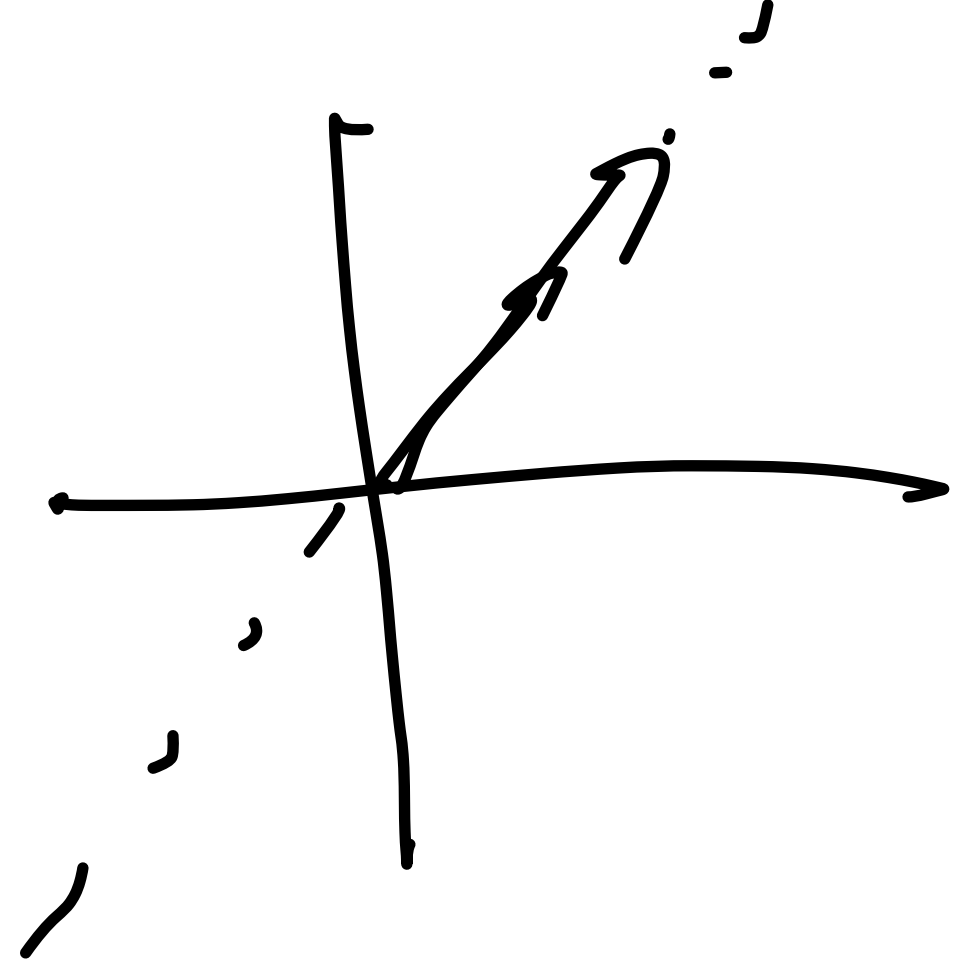
$$x_1 \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left. \begin{array}{l} 2x_1 = 0 \\ 3x_1 = 0 \end{array} \right\} \Rightarrow x_1 = 0$$

$$3x_1 = 0$$

$$\left. \begin{array}{l} x_2 - x_1 = 0 \\ 0 \end{array} \right\} \Rightarrow x_2 = 0$$

# Examples



$$\left\{ \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ 6 \\ -2 \end{bmatrix} \right\}$$

$$x_1 \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ 6 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 = -2$$

$$x_2 = \underline{1}$$

$$x_1 = -4$$

$$x_2 = 2$$

Lin.  
Dep.

# Examples

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Lin.  
Ind.

$$x_1 \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

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# Examples

$\{ \}$

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Lin.

Ind,

# **Another Interpretation of Linear Dependence**

demo  
(from ILA)

# Three Vectors in $\mathbb{R}^3$

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# Three Vectors in $\mathbb{R}^3$

It's possible for three vectors in  $\mathbb{R}^3$  to span all of  $\mathbb{R}^3$ , but it's not guaranteed

There may be vectors which lies in the plane spanned by two other vectors

Or even two vectors which lie in the span of one of the others

# Fundamental Concern

*How do we classify when a set of vectors does not span as much as it possibly can? When it is "smaller" than it could be?*

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*How do we classify when a set of vectors does not span as much as it possibly can? When it is "smaller" than it could be?*

**This is the role of linear dependence**

# Linear Dependence (Another Alternative)

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**Definition.** A set of vectors  $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$  is *linearly dependent* if it is nonempty and one of its vectors can be written as a linear combination of the others (not including itself)

# The Linear Combination Perspective

Suppose we have four vectors such that

$$\mathbf{v}_3 = 2\mathbf{v}_1 + 3\mathbf{v}_2 + \quad + 5\mathbf{v}_4$$

what do we know about the equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 + x_4\mathbf{v}_4 = \mathbf{0}$$

# The Linear Combination Perspective

$$\vec{0} \quad \cancel{\mathbf{v}_3} = \boxed{2}\mathbf{v}_1 + \boxed{3}\mathbf{v}_2 + \boxed{(-1)}\vec{\mathbf{v}_3} + \boxed{5}\mathbf{v}_4$$

$x_1 \qquad x_2 \qquad x_3 \qquad x_4$

Implies  $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 + x_4\mathbf{v}_4 = \mathbf{0}$  has a nontrivial solution:

$$(2, 3, -1, 5)$$

# The Vector Equation Perspective

Suppose  $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 + x_4\mathbf{v}_4 = \mathbf{0}$  has a nontrivial solution  $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$  where, say,  $\alpha_2 \neq 0$

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Suppose  $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 + x_4\mathbf{v}_4 = \mathbf{0}$  has a nontrivial solution  $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$  where, say,  $\alpha_2 \neq 0$

We can turn this into a linear combination

# The Vector Equation Perspective

$$\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \alpha_3 \mathbf{v}_3 + \alpha_4 \mathbf{v}_4 = \mathbf{0}$$

$$\frac{-\alpha_2 \vec{v}_2}{-\alpha_2} = \frac{\alpha_1 \vec{v}_1}{-\alpha_2} + \frac{\alpha_3 \vec{v}_3}{-\alpha_2} + \frac{\alpha_4 \vec{v}_4}{\alpha_2}$$

$$\vec{v}_2 = -\frac{\alpha_1}{\alpha_2} \vec{v}_1 - \frac{\alpha_3}{\alpha_2} \vec{v}_3 - \frac{\alpha_4}{\alpha_2} \vec{v}_4$$

Suppose  $x_1 \mathbf{v}_1 + x_2 \mathbf{v}_2 + x_3 \mathbf{v}_3 + x_4 \mathbf{v}_4 = \mathbf{0}$  has a nontrivial solution  $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$  where, say,  $\alpha_2 \neq 0$

# In All

**Theorem.** A set of vectors is linearly dependent if and only if it is nonempty and *at least* one of its vectors can be written as a linear combination of the others

P if and only if Q means  
P implies Q and Q implies P

# Linear Dependence Relation

**Definition.** If  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$  are linearly dependent, then a ***linear dependence relation*** is an equation of the form

$$\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \dots + \alpha_n \mathbf{v}_n = \mathbf{0}$$

A linear dependence relation  
*witnesses* the linear dependence

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**Question.** Write down a linear dependence relation for the vectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$

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**Solution.** Find a nontrivial solution to the equation

$$[\mathbf{v}_1 \quad \mathbf{v}_2 \quad \dots \quad \mathbf{v}_n] \mathbf{x} = \mathbf{0}$$

*(there will be a free variable you can choose to be nonzero)*

# Example

Write down the linear dependence relation for the following vectors

$$\mathbf{v}_1 = \begin{bmatrix} -4 \\ 4 \\ 2 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} -3 \\ 6 \\ -3 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} -5 \\ 8 \\ -2 \end{bmatrix}$$

# Answer

$$\begin{bmatrix} -4 & -3 & -5 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} -4 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$x_1 = -\frac{1}{2}x_3$$

$$x_2 = -x_3$$

$x_3$  is free

$$x_1 = -1$$

$$x_2 = -2$$

$$x_3 = 2$$

$$\sim \begin{bmatrix} -4 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -4 & -3 & -5 \\ 0 & 3 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$- \begin{bmatrix} 4 \\ 4 \\ 2 \end{bmatrix} - 2 \begin{bmatrix} -3 \\ 6 \\ -3 \end{bmatrix} + 2 \begin{bmatrix} -5 \\ 8 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$4 + 6 - 10 = 0$$

$$-4 - 12 + 16 = 0$$

$$-2 + 6 - 4 = 0$$

# Simple Cases

# The Empty Set

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$\{\}$  is linearly independent

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There is no nontrivial linear combination of the vectors equaling  $\mathbf{0}$

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There are none at all...

# The Empty Set

$\{\}$  is linearly independent

There is no nontrivial linear combination of the vectors equaling  $\mathbf{0}$

There are none at all...  $\text{span } \{\} = \{ \vec{0} \}$

$\mathbf{0}$  is in every span, even the span of the empty set

# One Vector

A single vector  $\mathbf{v}$  is linearly independent if and only if  $\mathbf{v} \neq \mathbf{0}$

*(Note that  $x_1 \mathbf{0} = \mathbf{0}$  has many nontrivial solutions)*

$$x_1 = 2 \quad 2 \vec{0} = \vec{0}$$

# The Zero Vector and Linear Dependence

If a set of vectors  $V$  contains the  $\mathbf{0}$ , then it is linearly dependent

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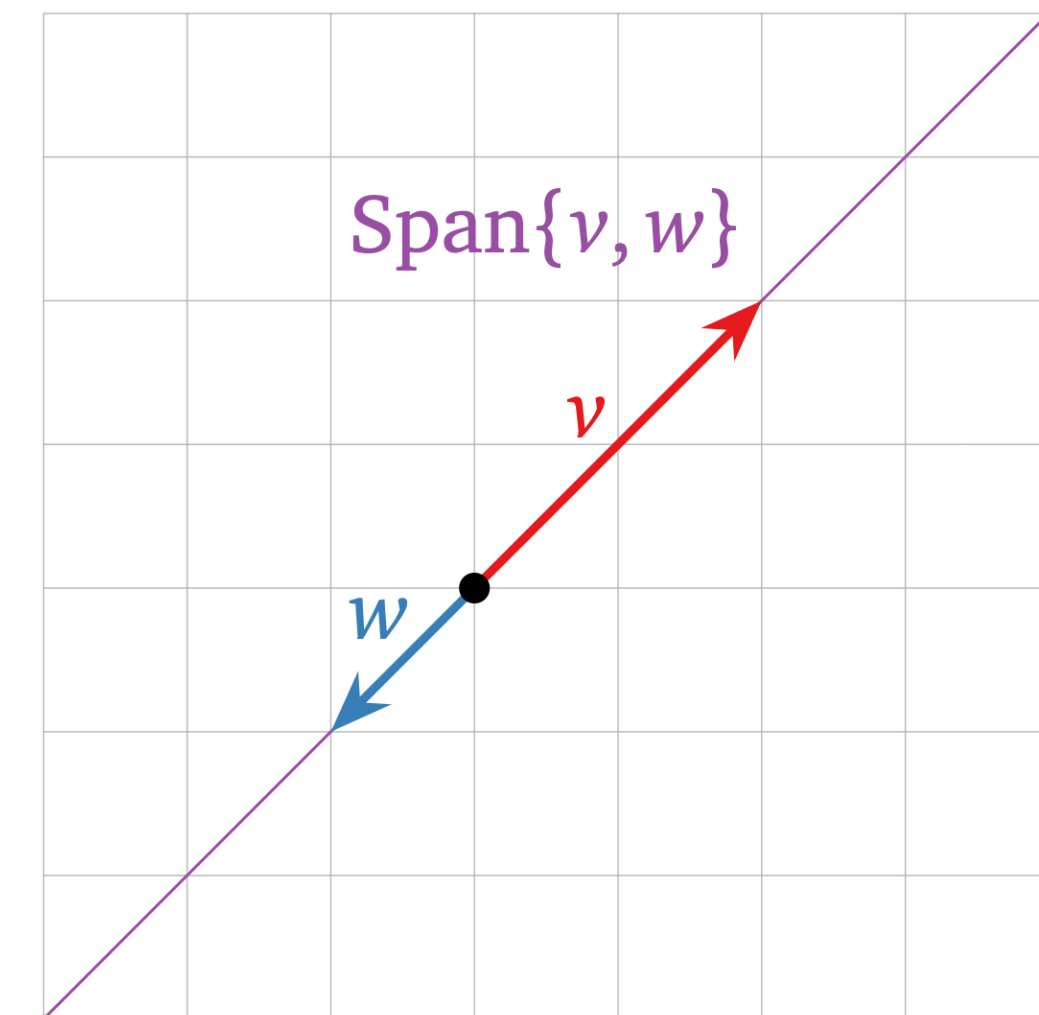
$$(1)\mathbf{0} + 0\mathbf{v}_2 + 0\mathbf{v}_2 + \dots + 0\mathbf{v}_n = \mathbf{0}$$

There is a very simple nontrivial solution

# Two Vectors

**Definition.** Two vectors are *colinear* if they are scalar multiples of each other

e.g.,  $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$  and  $\begin{bmatrix} 1.5 \\ 1.5 \\ 3 \end{bmatrix}$  or  $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$  and  $\begin{bmatrix} -1 \\ -1 \end{bmatrix}$

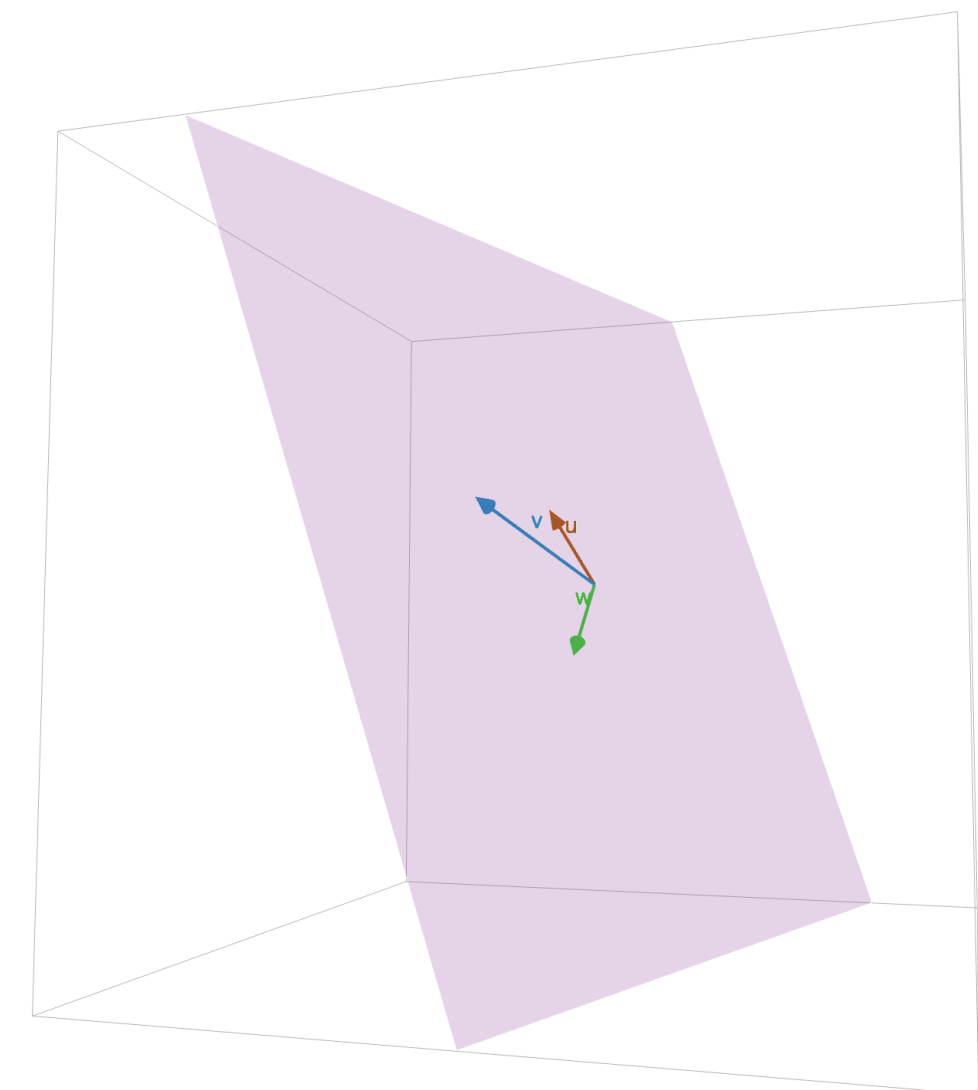


Two vectors are linearly dependent if and only if they are colinear

# Three Vectors

**Definition.** A collection of vectors is **coplanar** if their span is a plane

Three vectors are linearly dependent if and only if they are colinear or coplanar

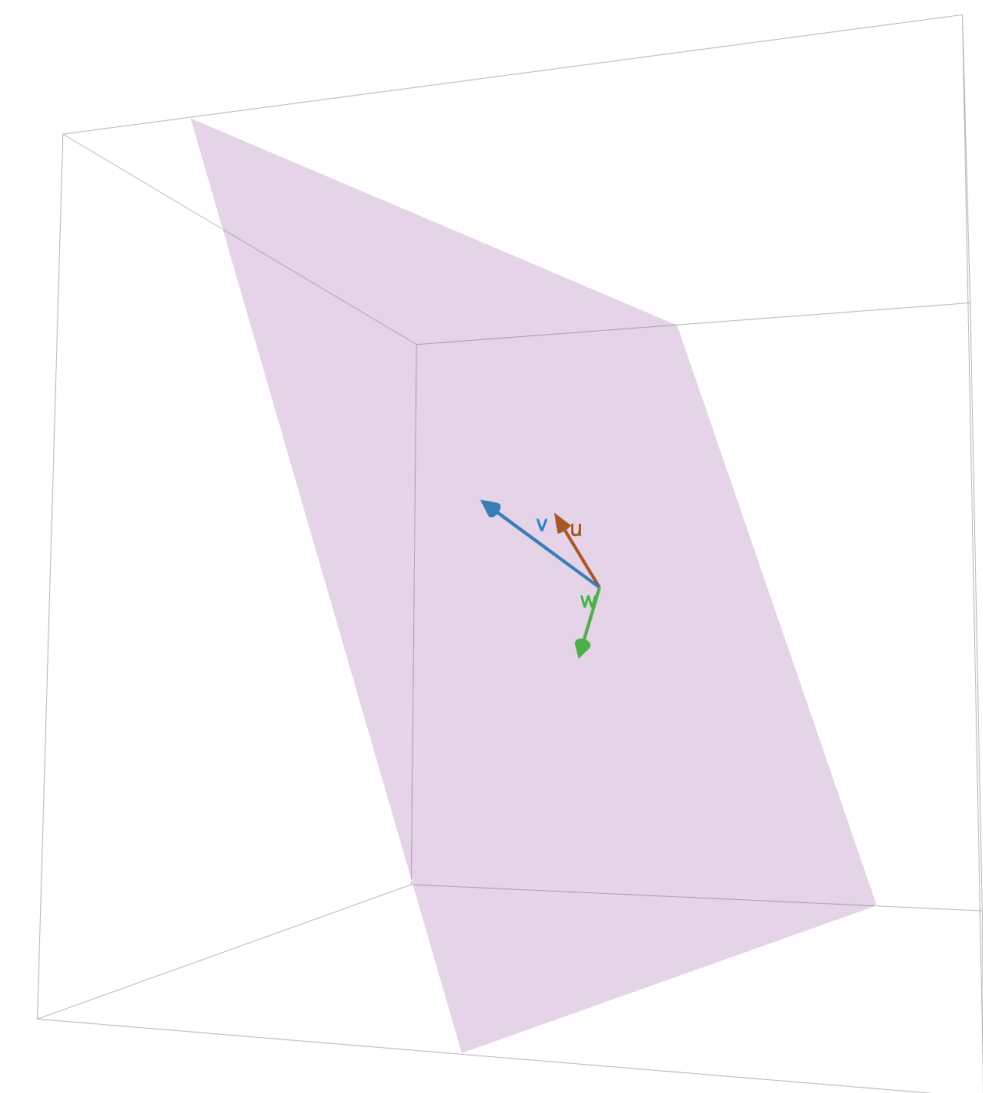


# Three Vectors

**Definition.** A collection of vectors is **coplanar** if their span is a plane

Three vectors are linearly dependent if and only if they are colinear or coplanar

This reasoning can be extended to more vectors, but we run out of terminology



**Yet Another Interpretation**

# Increasing Span Criterion

If  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$  are linearly *independent* then we cannot write one of it's vectors as a linear combination of the others

**But we get something stronger**

# Increasing Span Criterion

**Theorem.**  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$  are linearly independent if and only if for all  $i \leq n$ ,

$$\mathbf{v}_i \notin \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{i-1}\}$$

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As we add vectors, the span gets larger

# Increasing Span Criterion

So in this case, our span keeps getting "bigger"

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$\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$  is a 4d-hyperplane

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$\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$  is a 4d-hyperplane

...

# Characterization of Linear Dependence

**Theorem.**  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$  are linearly  
dependent if and only there is an  
 $i \leq n$ ,

$$\mathbf{v}_i \in \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{i-1}\}$$

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As we add vectors, we'll eventually find  
one in the span of the preceding ones

# Characterization of Linear Dependence

$\text{span}\{\}$  is a point  $\{\mathbf{0}\}$

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$\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$  is still a plane

...

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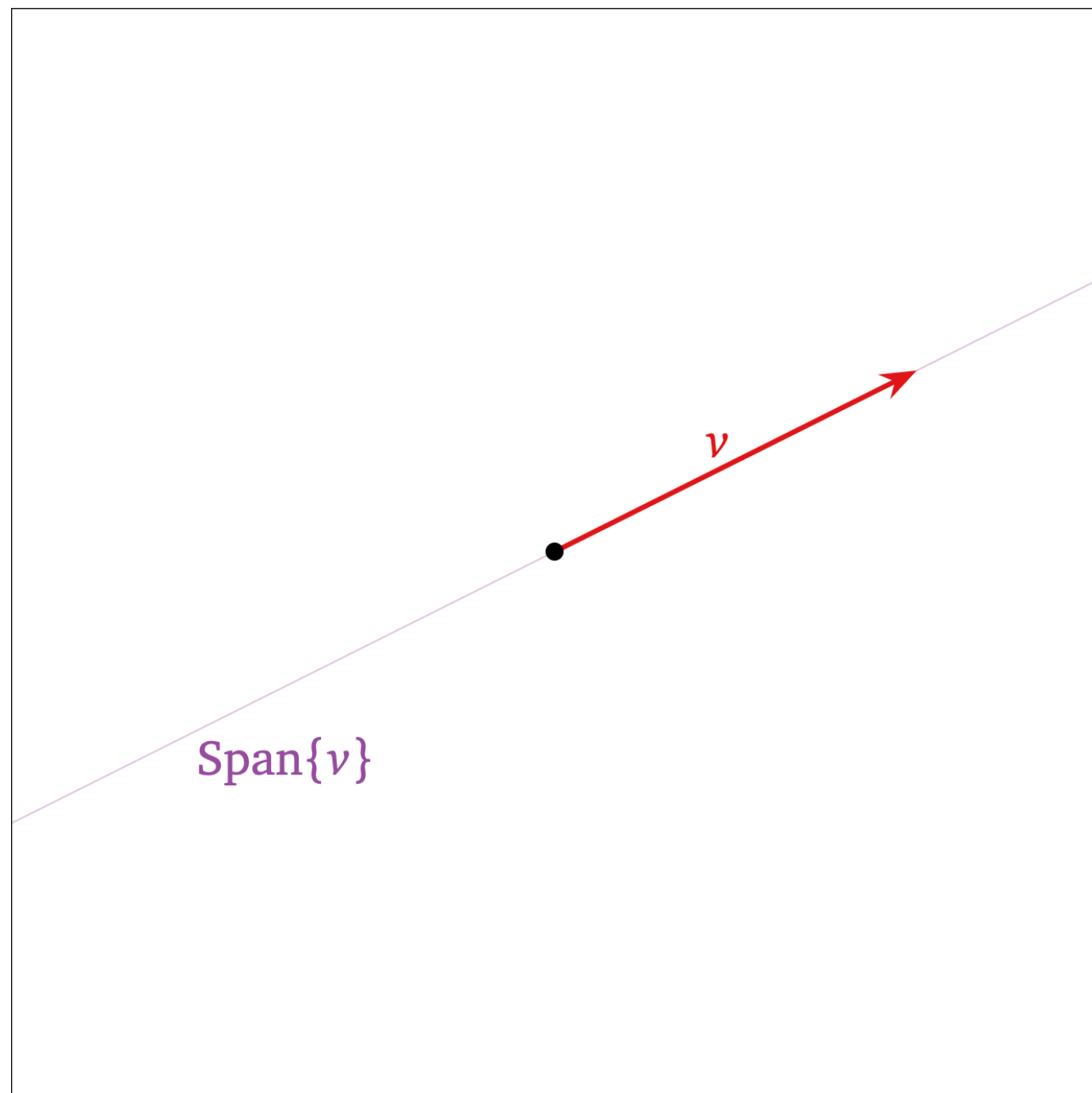
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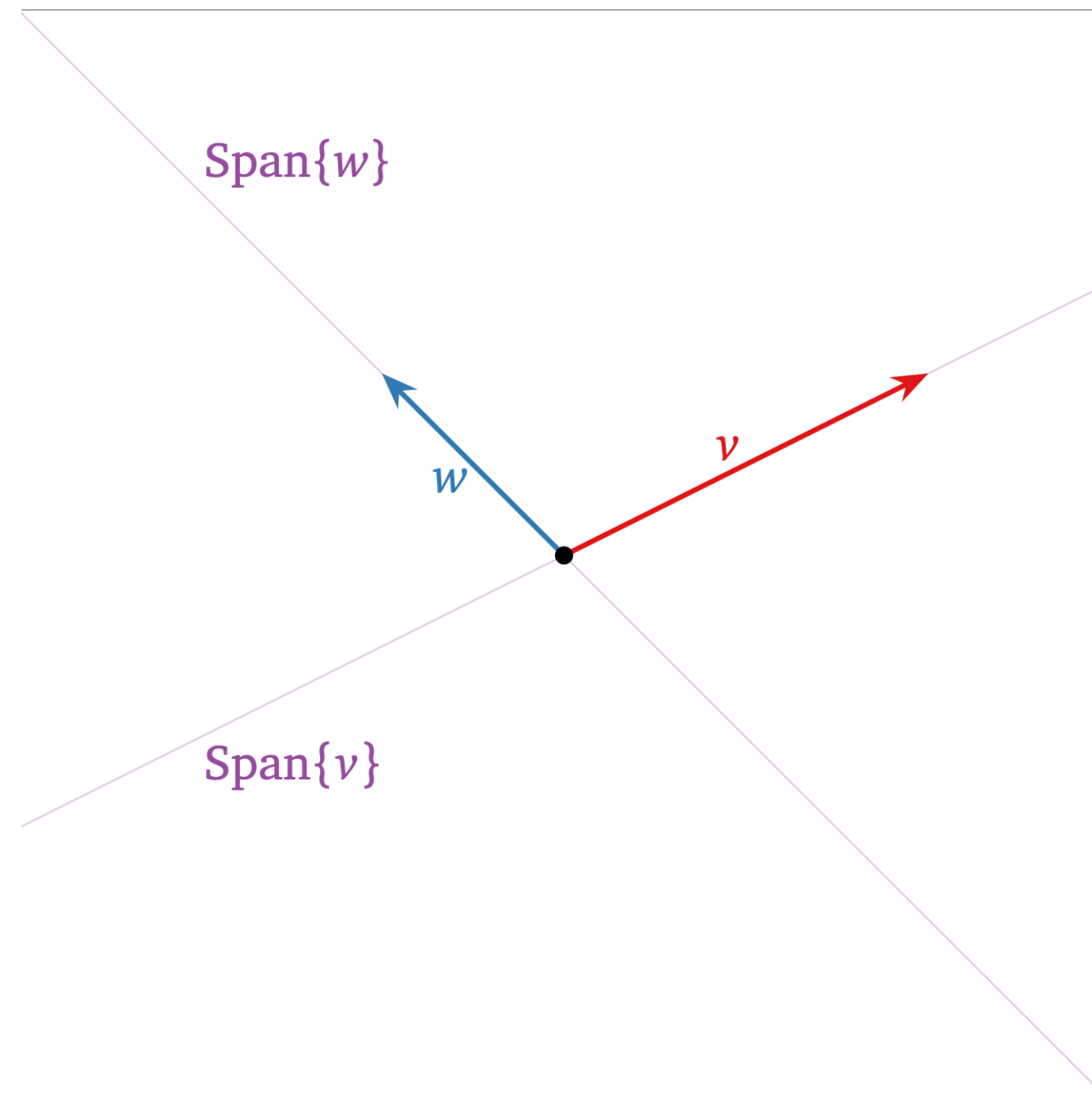
...

(this is an example, it may take a lot more vectors before we find one in the span of the preceding vectors)

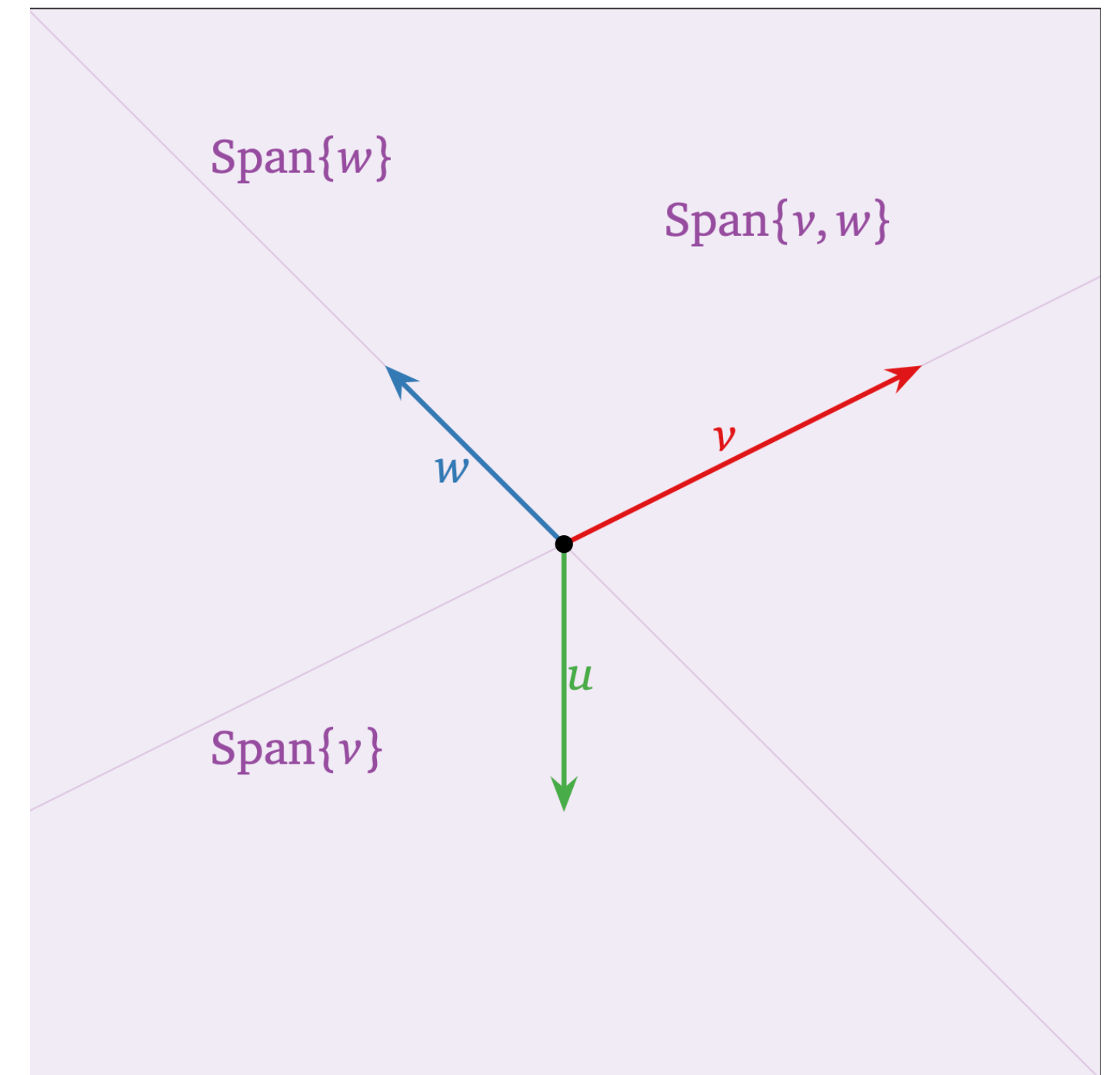
# As a Picture



span of 1 vector  
a line



span of 2 vector  
a plane



span of 3 vector  
still a plane

# Characterization of Linear Dependence

**Corollary.** If  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$  are linearly dependent, then for any vector  $\mathbf{v}_{k+1}$ , the vectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k, \mathbf{v}_{k+1}$  are linearly dependent

If we add a vector to a linearly dependent set, it remains linearly dependent

# Question

Are the following vectors linearly independent?

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 2023 \\ 0 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} 0.1 \\ 7 \end{bmatrix}$$

**Answer: No**

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 2023 \\ 0 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} 0.1 \\ 7 \end{bmatrix}$$

Any three vectors can at most span a plane

The first two are not colinear, so they span a plane ( $\mathbb{R}^2$ )

# **Linear Independence and Free Variables**

# Linear Dependence Relations (Again)

When finding a linear dependence relation, we came across a system which has a free variable

$$\begin{bmatrix} -4 & -3 & -5 & 0 \\ 0 & 3 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

we can take  $x_3$  to be free

# Pivots and Linear Dependence

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Remember that we choose our free variables to be the ones whose columns don't have pivots

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**Theorem.** The columns of a matrix  $A$  are linearly independent if and only if  $A$  has a pivot in every column

Remember that we choose our free variables to be the ones whose columns don't have pivots

Free variables allow for infinitely many  
(nontrivial) solutions

# Recall: General Form Solutions

$$x_1 = - (0.5)x_3$$

$$x_2 = - x_3$$

$x_3$  is free

# Recall: General Form Solutions

$$x_1 = -0.5$$

$$x_2 = -1$$

$$x_3 = 1$$

# Recall: General Form Solutions

$$x_1 = 0.5$$

$$x_2 = 1$$

$$x_3 = -1$$

# Recall: General Form Solutions

$$x_1 = 1$$

$$x_2 = 2$$

$$x_3 = -2$$

# Recall: General Form Solutions

$$x_1 = 1$$

$$x_2 = 2$$

$$x_3 = -2$$

The point: the solution is not unique

# How To: Linear Independence

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**Question.** Is the set of vectors  $\{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n\}$  linearly independent?

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**Solution.** Check if  $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \dots + x_n\mathbf{a}_n = \mathbf{0}$  has a unique solution

# How To: Linear Independence

**Question.** Is the set of vectors  $\{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n\}$  linearly independent?

**Solution.** Check if  $[\mathbf{a}_1 \ \mathbf{a}_2 \ \dots \ \mathbf{a}_n] \mathbf{x} = \mathbf{0}$  has a unique solution

# How To: Linear Independence

**Question.** Is the set of vectors  $\{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n\}$  linearly independent?

**Solution.** Check if the general form solution of  $[\mathbf{a}_1 \ \mathbf{a}_2 \ \dots \ \mathbf{a}_n \ \mathbf{0}]$  has any free variables

# How To: Linear Independence

**Question.** Is the set of vectors  $\{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n\}$  linearly independent?

**Solution.** Reduce  $[\mathbf{a}_1 \ \mathbf{a}_2 \ \dots \ \mathbf{a}_n]$  to echelon form and check if has a pivot position in every column

# Example: Recap Problem Again

$$\mathbf{v}_1 = \begin{bmatrix} -4 \\ 4 \\ 2 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} -3 \\ 6 \\ -3 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} -5 \\ 8 \\ -2 \end{bmatrix}$$

The reduced echelon form of  $[\mathbf{v}_1 \ \mathbf{v}_2 \ \mathbf{v}_3]$  is

$$\begin{bmatrix} 1 & 0 & 0.5 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

column  
without a  
pivot

# Linear Independence and Full Span

The columns of a  $(m \times n)$  matrix span all of  $\mathbb{R}^n$  if there is a pivot in every row

The columns of a matrix are linearly independent if there is a pivot in every column

# Tall Matrices

If  $m > n$  then the columns cannot span  $\mathbb{R}^m$

A diagram of a tall matrix, represented by a large square bracket on the left and right. The matrix has 8 rows and 3 columns. The elements are arranged as follows:

- Row 1:  $*$ ,  $\dots$ ,  $*$
- Row 2:  $*$ ,  $\cdot$ ,  $*$
- Row 3:  $*$ ,  $\dots$ ,  $*$
- Row 4:  $\vdots$ ,  $\vdots$ ,  $\vdots$
- Row 5:  $*$ ,  $\dots$ ,  $*$
- Row 6:  $*$ ,  $\dots$ ,  $*$
- Row 7:  $*$ ,  $\dots$ ,  $*$

Red circles highlight the following elements:

- The first element in the first row ( $*$ ).
- The second element in the second row ( $\cdot$ ).
- The third element in the third row ( $\cdot$ ).
- The third element in the fourth row ( $\vdots$ ).

# Tall Matrices

If  $m > n$  then the columns cannot span  $\mathbb{R}^m$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix}$$

This matrix has at most 3 pivots, but 4 rows

# Wide Matrices

If  $m < n$  then the columns cannot be linearly independent

$$\begin{bmatrix} * & * & * & \dots & * & * & * \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ * & * & * & \dots & * & * & * \end{bmatrix}$$

# Wide Matrices

If  $m < n$  then the columns cannot be linearly independent

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix}$$

This matrix has at most 3 pivots, but 4 columns

# A Warning

The columns of a  $(m \times n)$  matrix span all of  $\mathbb{R}^n$  if there is a pivot in every row

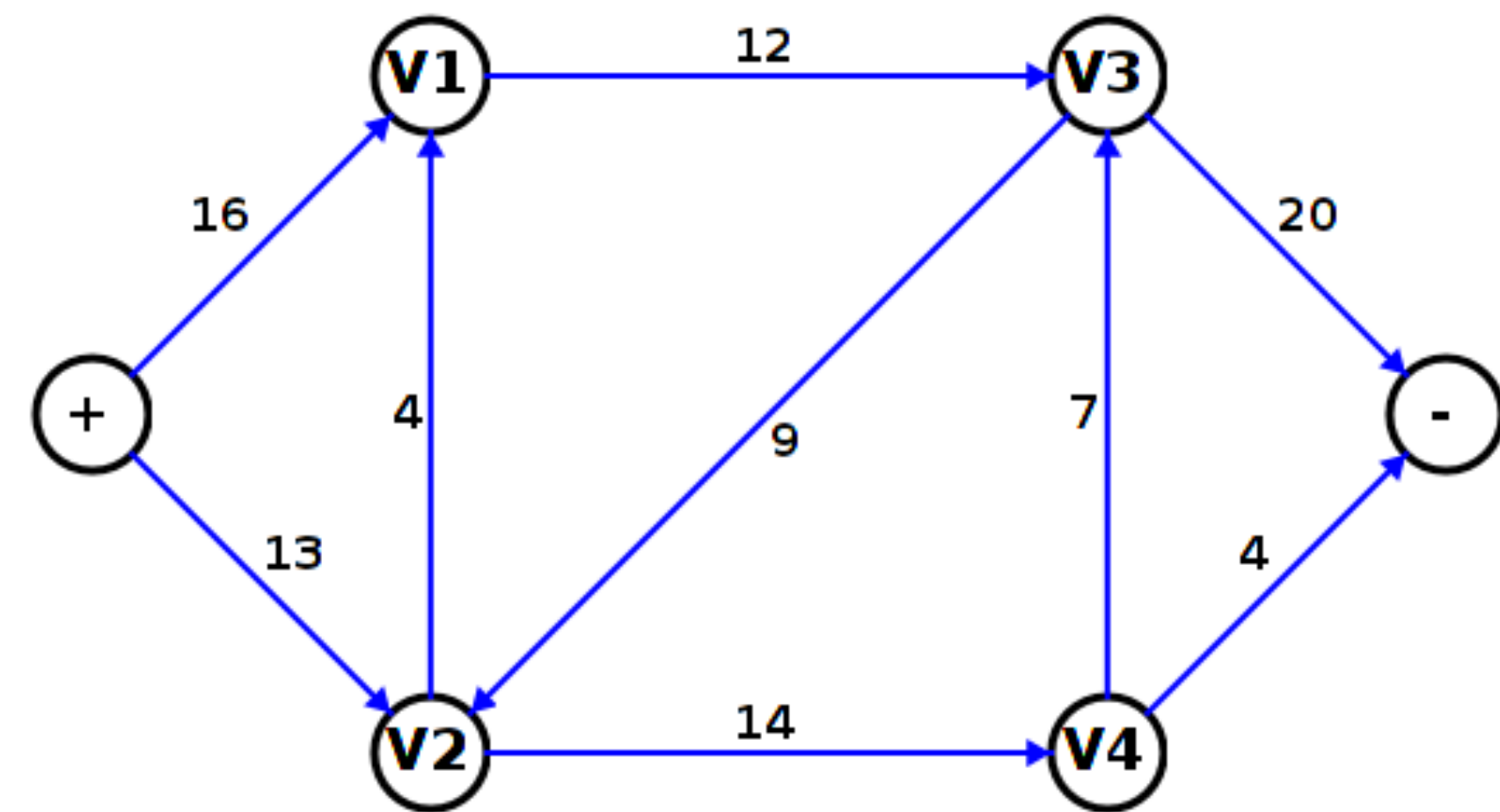
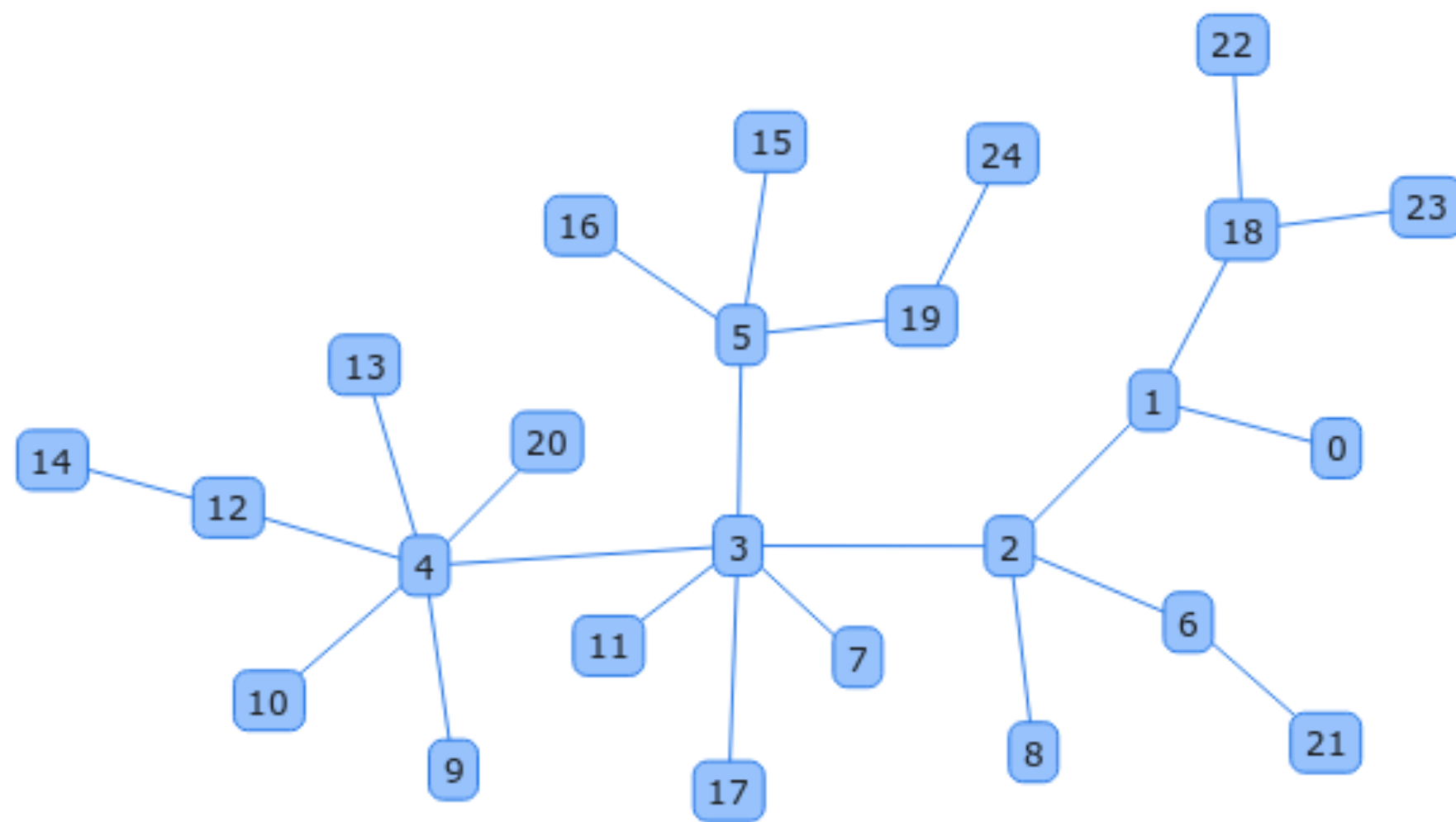
The columns of a matrix are linearly independent if there is a pivot in every column

Don't confuse these!

# **Application: Networks and Flow**

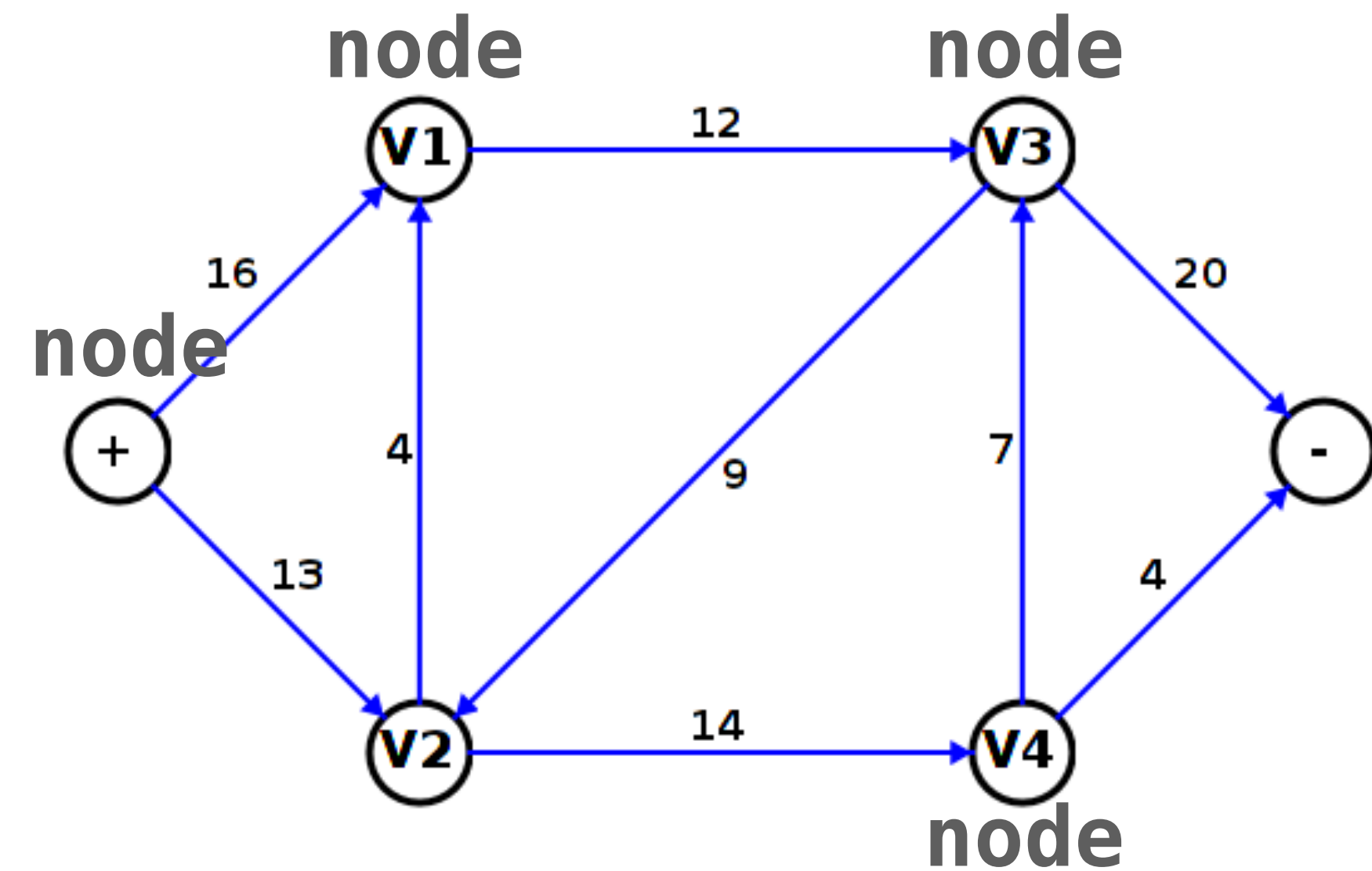
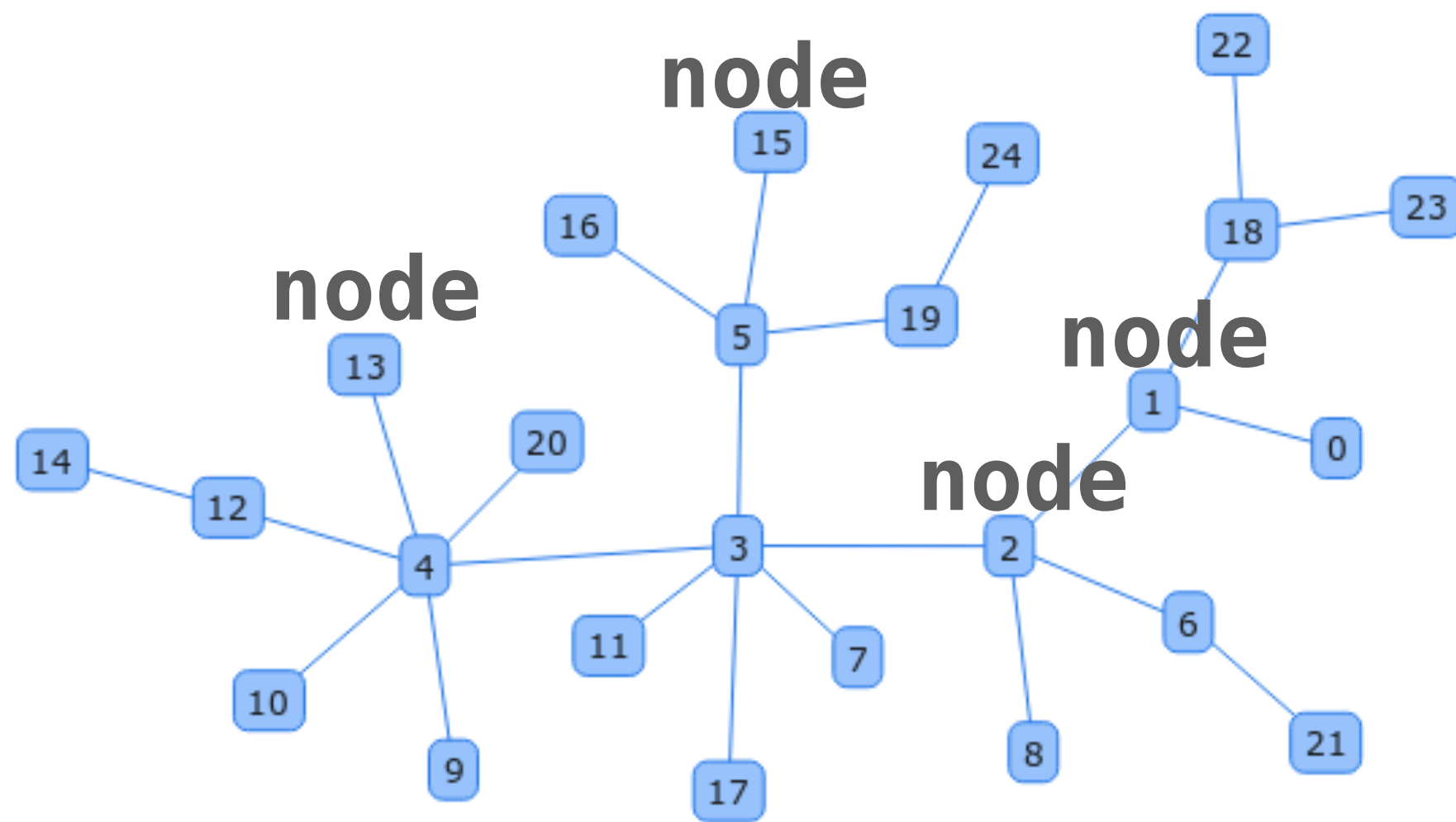
# Graphs/Networks

A *graph/network* is a mathematical object representing collection of *nodes* and *edges* connecting them



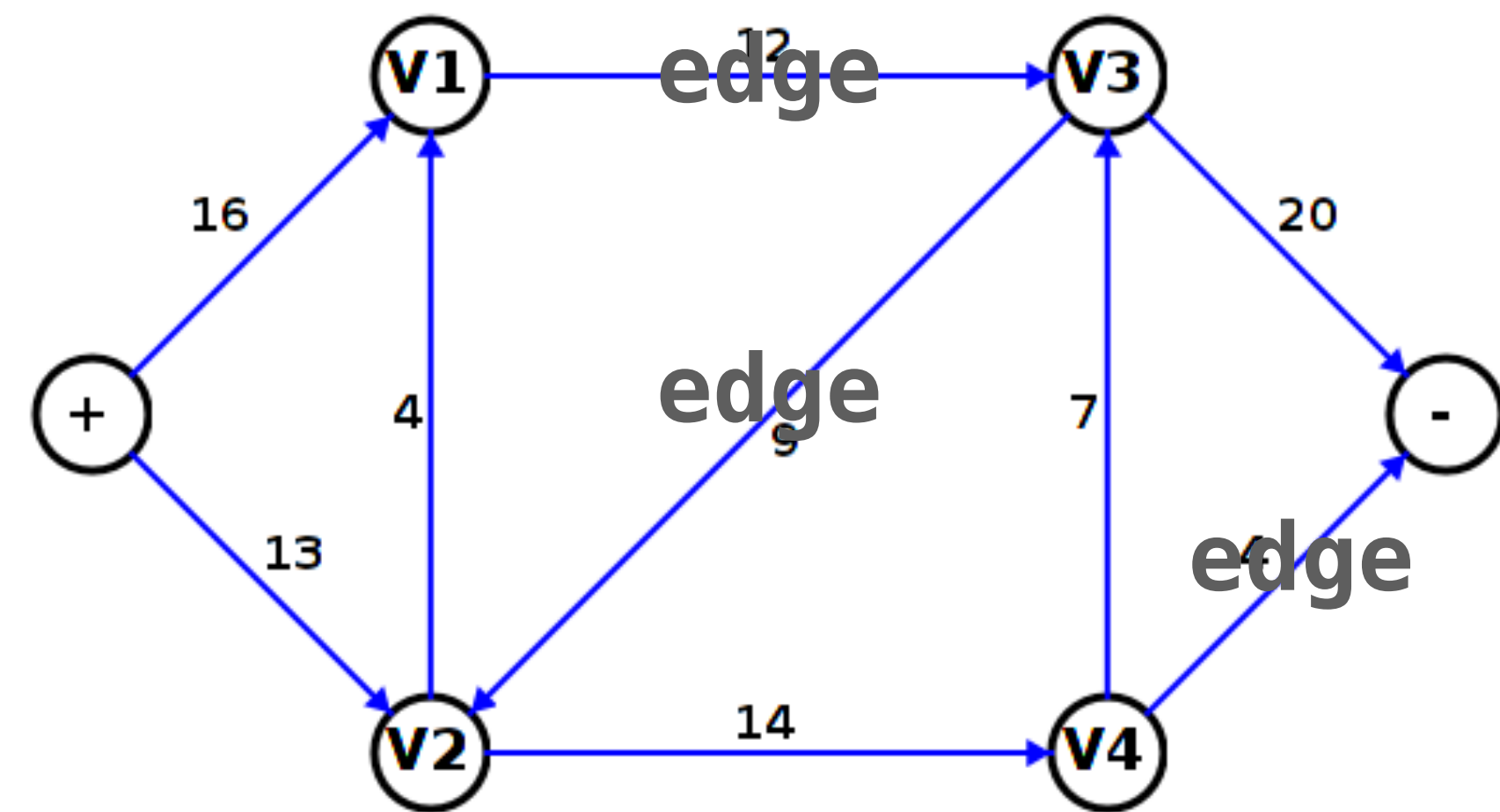
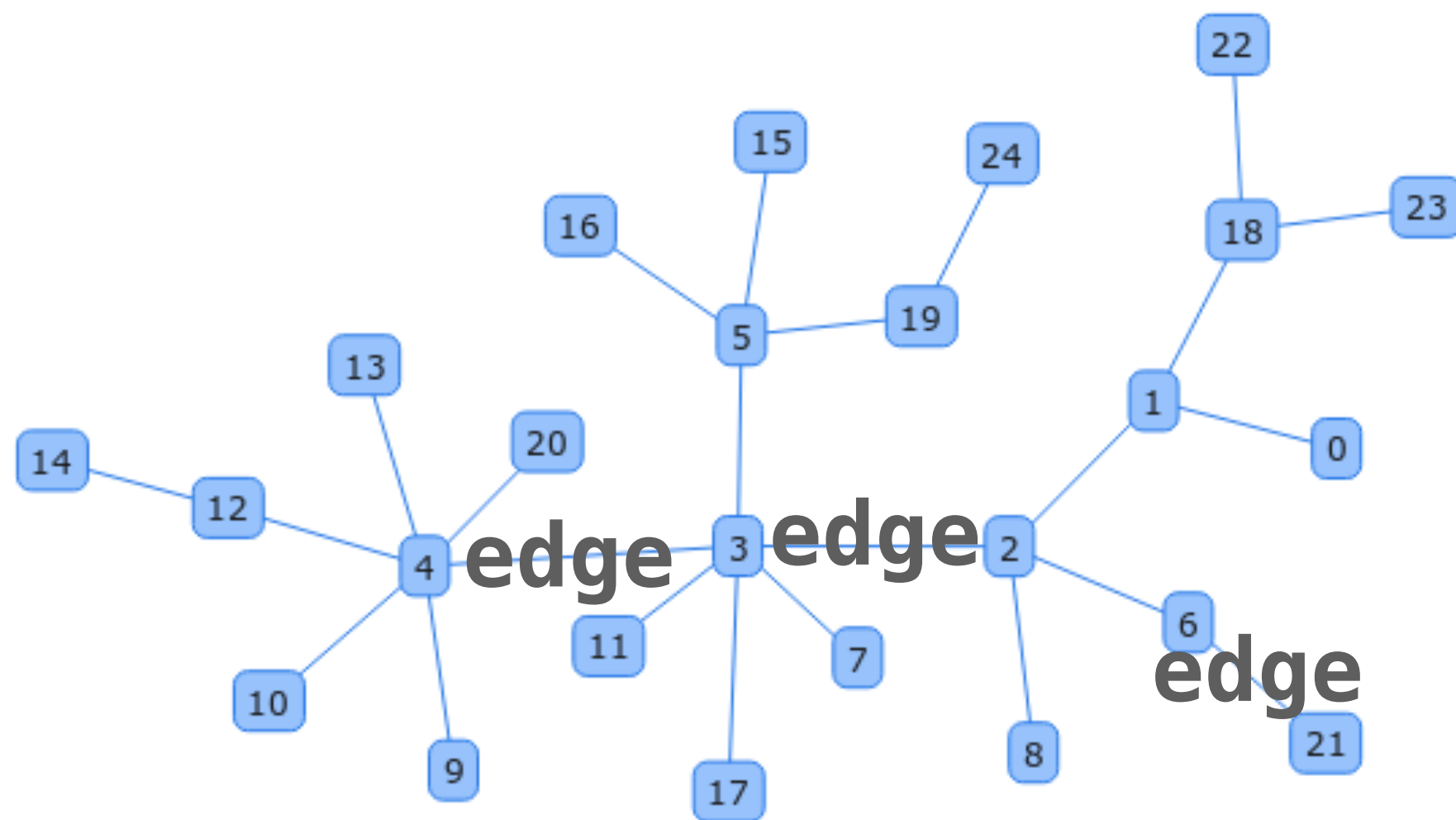
# Graphs/Networks

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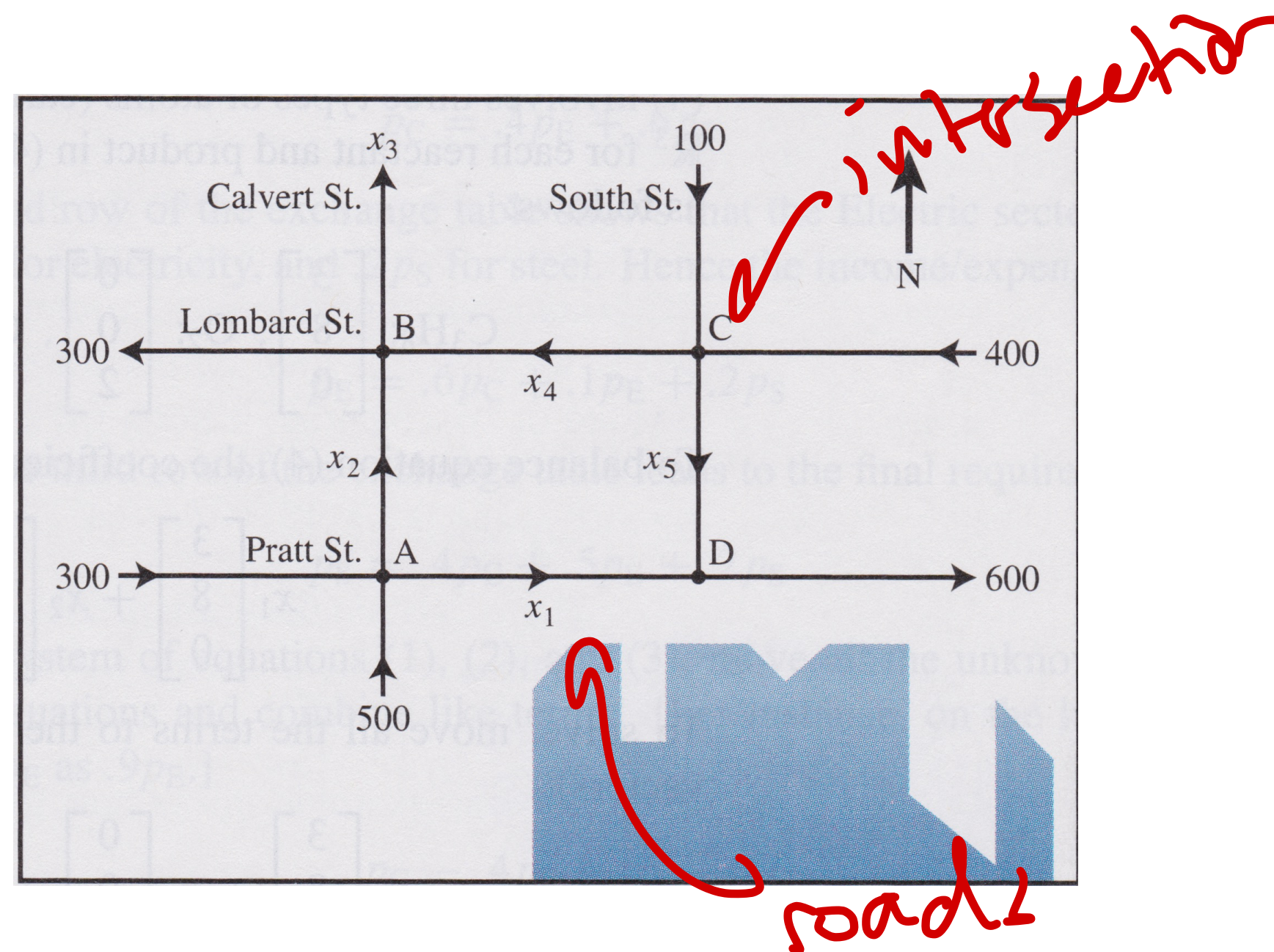
# Graphs/Networks

A *graph/network* is a mathematical object representing collection of *nodes* and *edges* connecting them



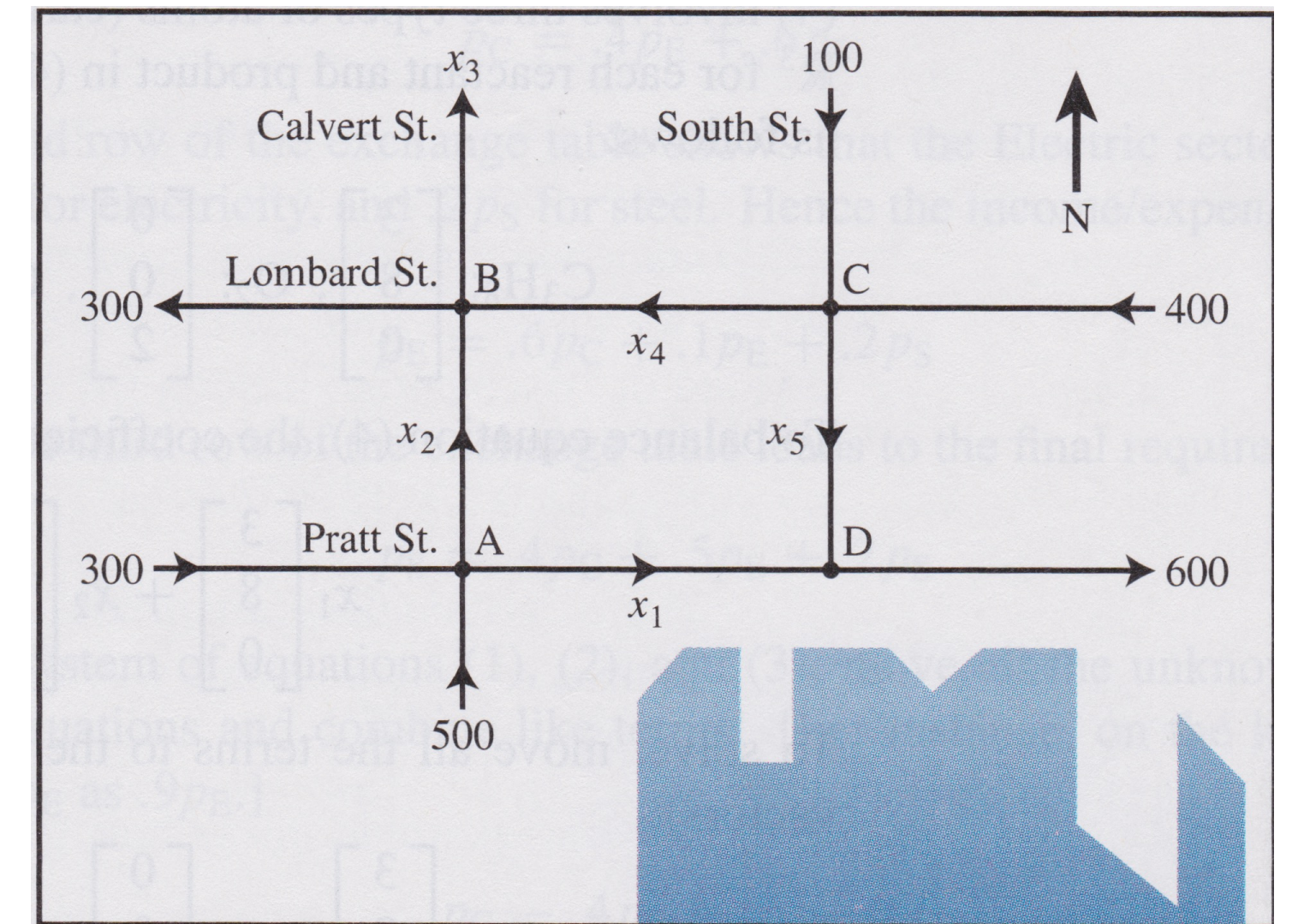
# Directed Graphs

Today we focus on *directed* graphs, in which edges have a specified direction



Think of  
these as  
one-way  
streets

# Flow



We are often interested in how much "stuff" we can push through the edges

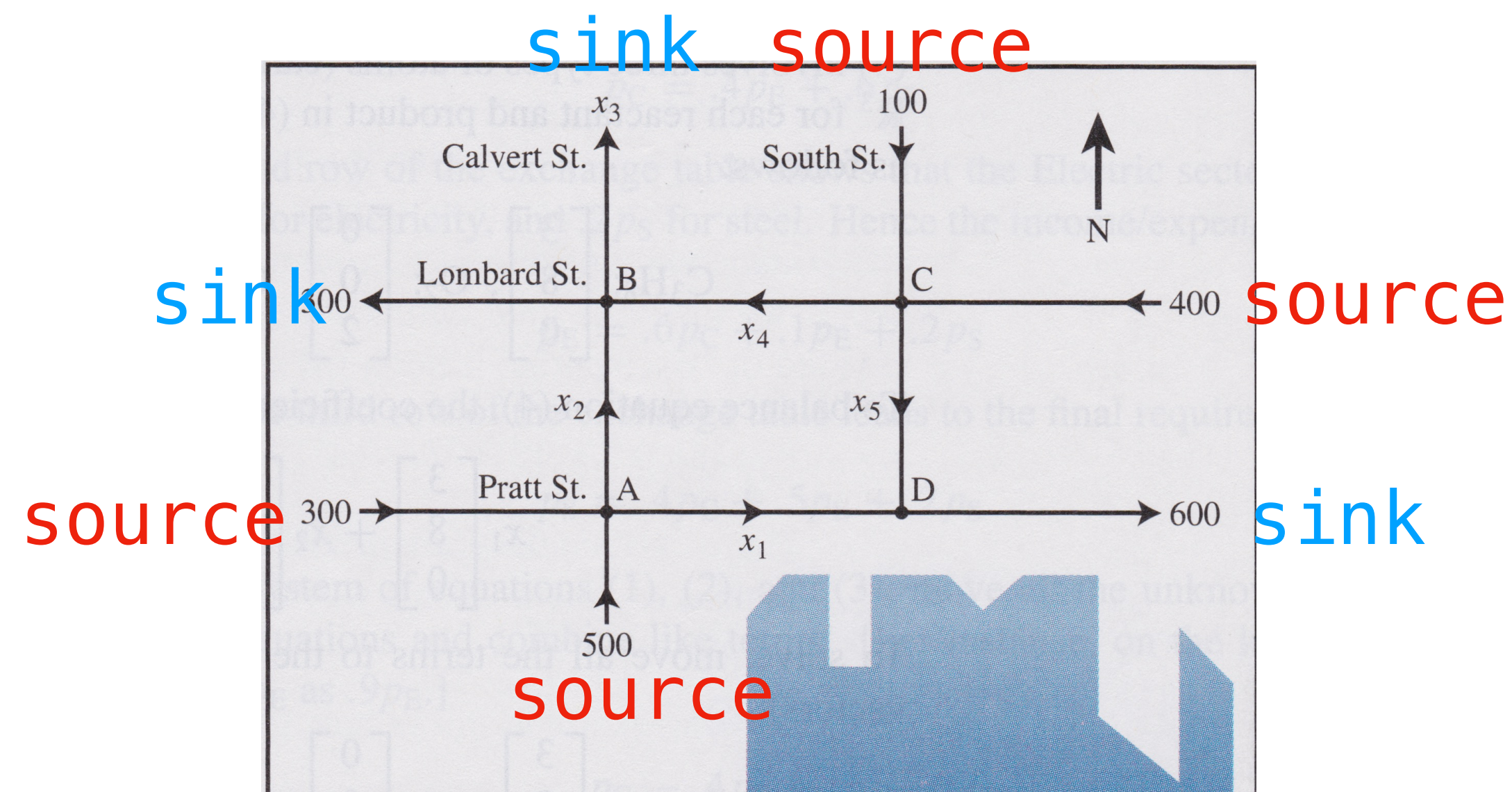
In the above example, the "stuff" is cars/hr

I like to imagine water moving through a pipe, and splitting at joints in the pipe

# Flow Network

A ***flow network*** is a directed graph with specified **source** and **sink** nodes

Flow comes out of and goes into sources and sinks. They are assigned a flow value (or variable)



**Flow**

# Flow

**Definition.** The *flow* of a graph is an assignment of nonnegative values to the edges so that the following holds

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**conservation:** flow into a node = flow out of a node

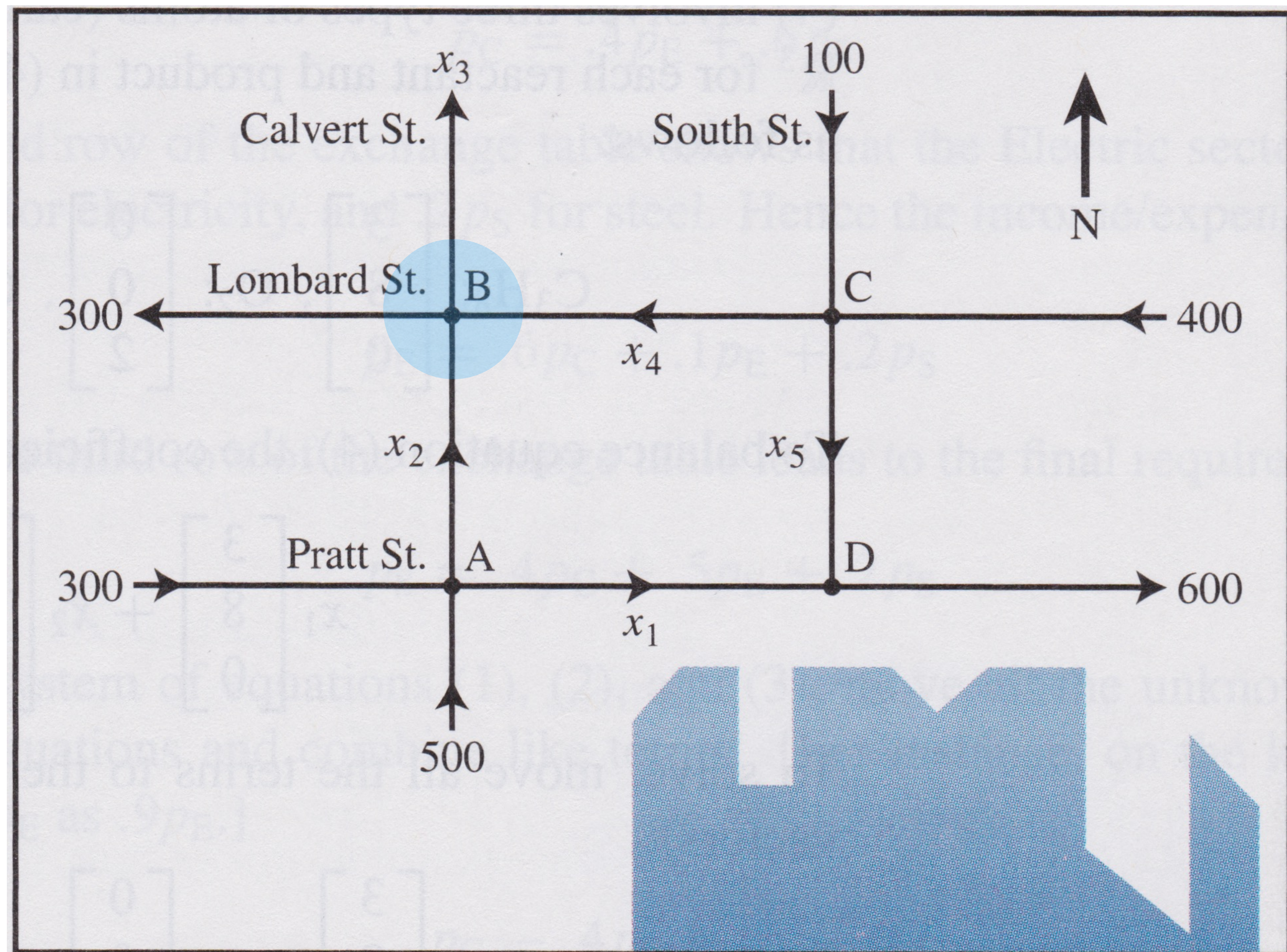
**source/sink constraint:** flow into a sink/out of a source is nonnegative

# Flow Conservation

Flow in = Flow out

$$x_2 + x_4 = 300 + x_3$$

$$100 + 400 = x_4 + x_5$$

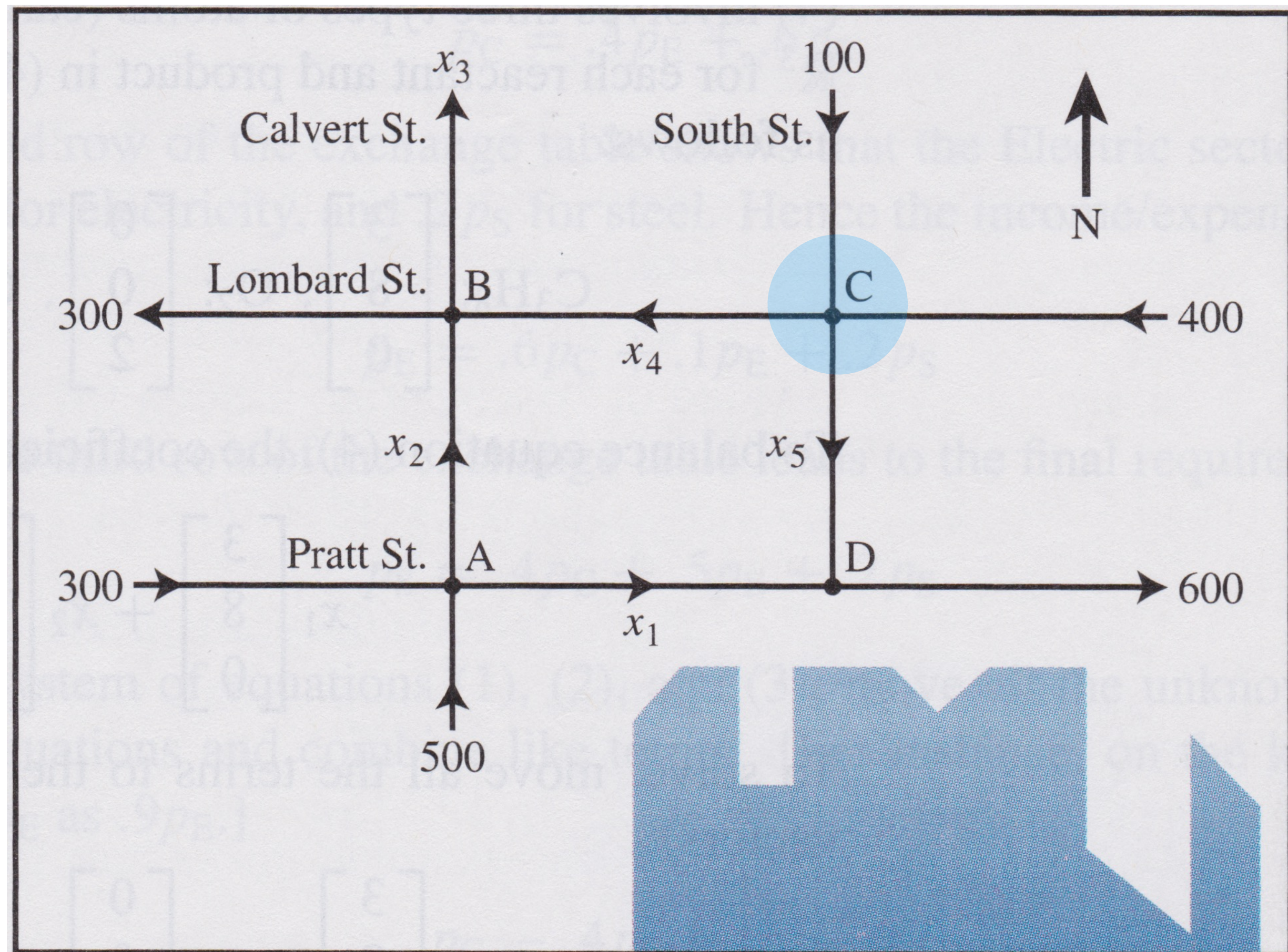


# Flow Conservation

Flow in = Flow out

$$x_2 + x_4 = 300 + x_3$$

$$100 + 400 = x_4 + x_5$$



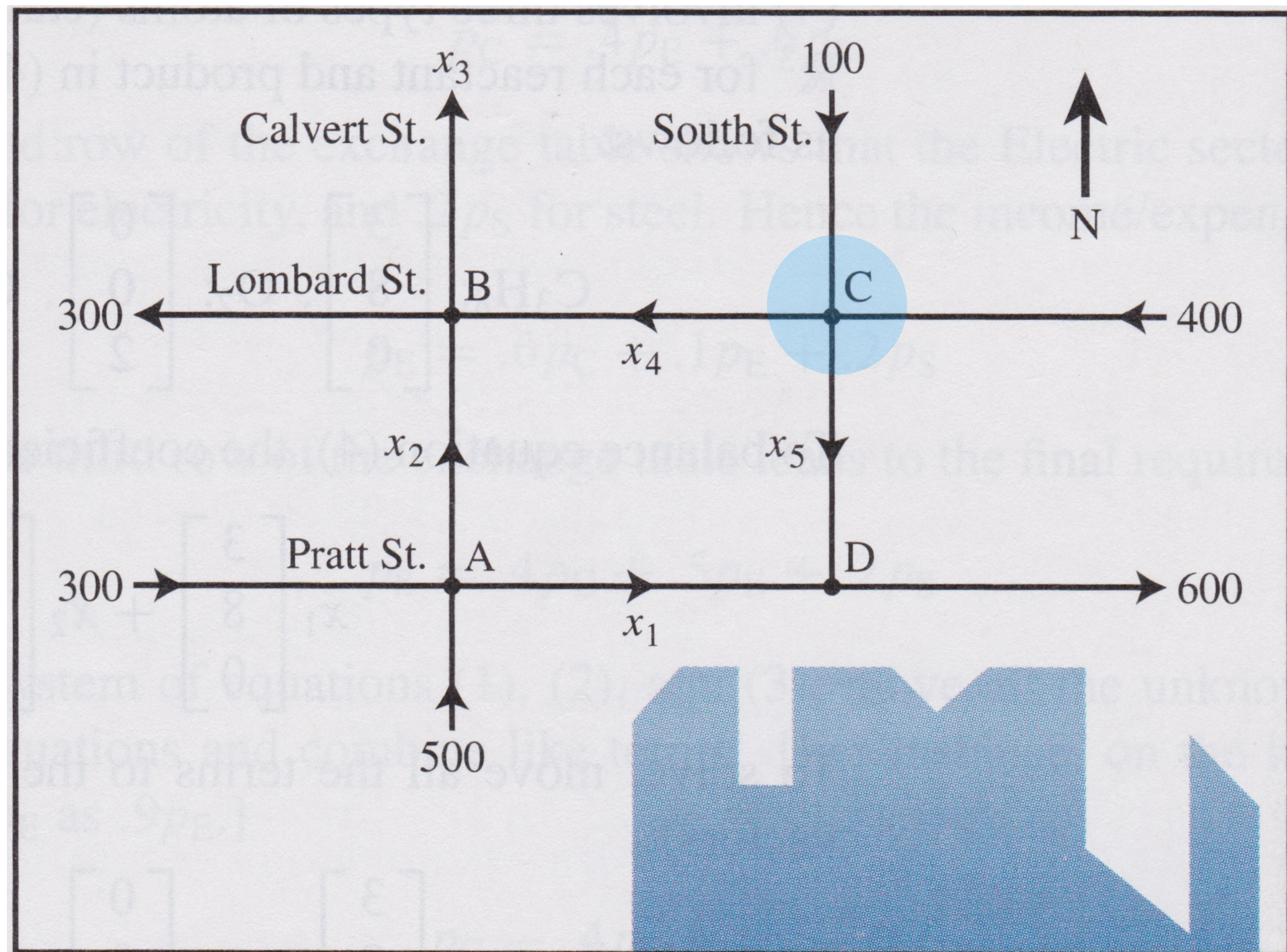
# Flow Conservation

Flow in = Flow out

$$x_2 + x_4 = 300 + x_3$$

$$100 + 400 = x_4 + x_5$$

Every node  
determines a linear  
equation



# How To: Network Flow

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**Question.** Find a general solution for the flow of a given graph

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**Question.** Find a general solution for the flow of a given graph

**Solution.** Write down the linear equations determined by flow conservation at non-source and non-sink nodes, and then solve

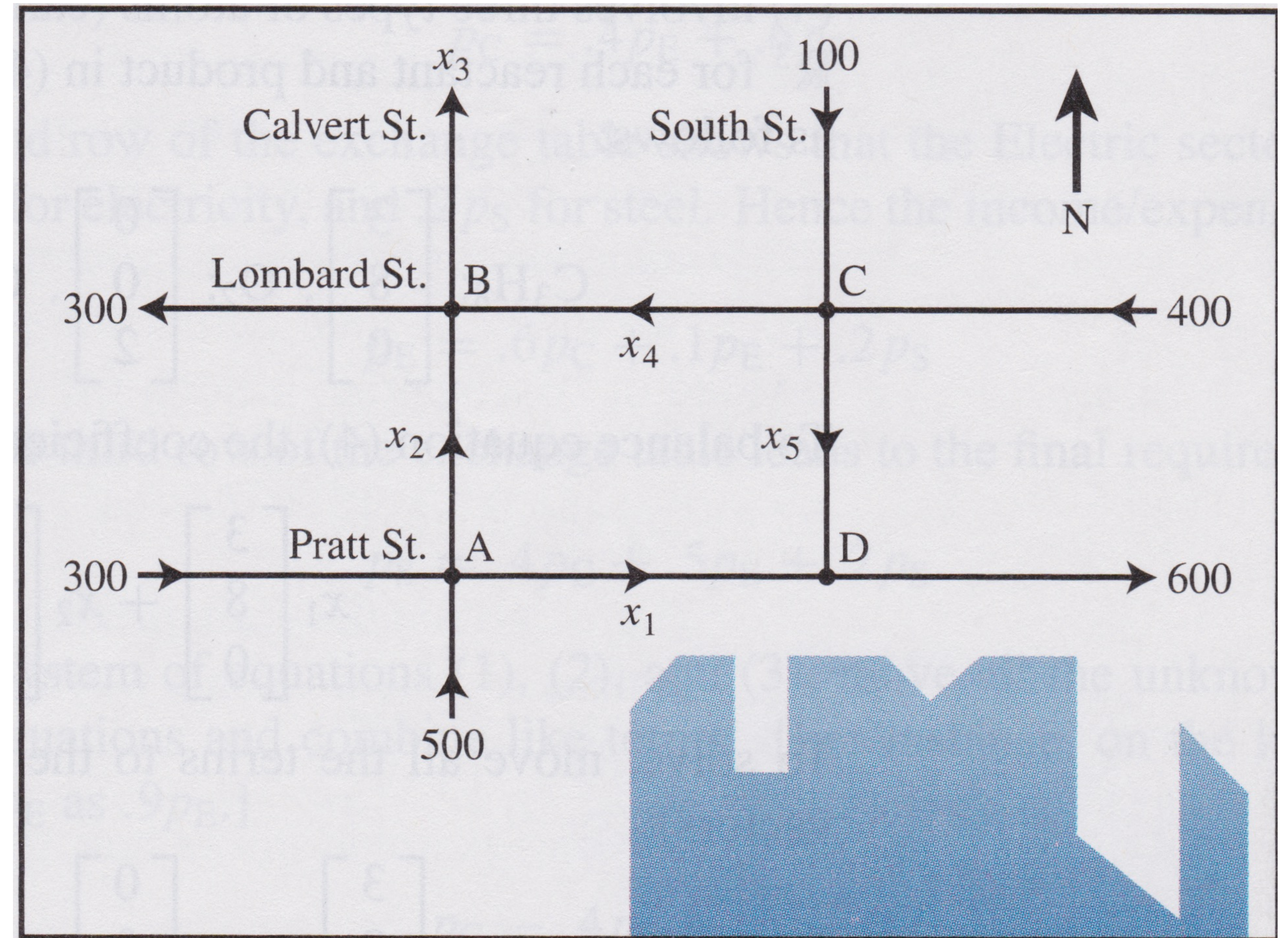
# Example

(A)  $500 + 300 = x_1 + x_2$

(B)  $x_2 + x_4 = 300 + x_3$

(C)  $100 + 400 = x_4 + x_5$

(D)  $x_1 + x_5 = 600$



# Example

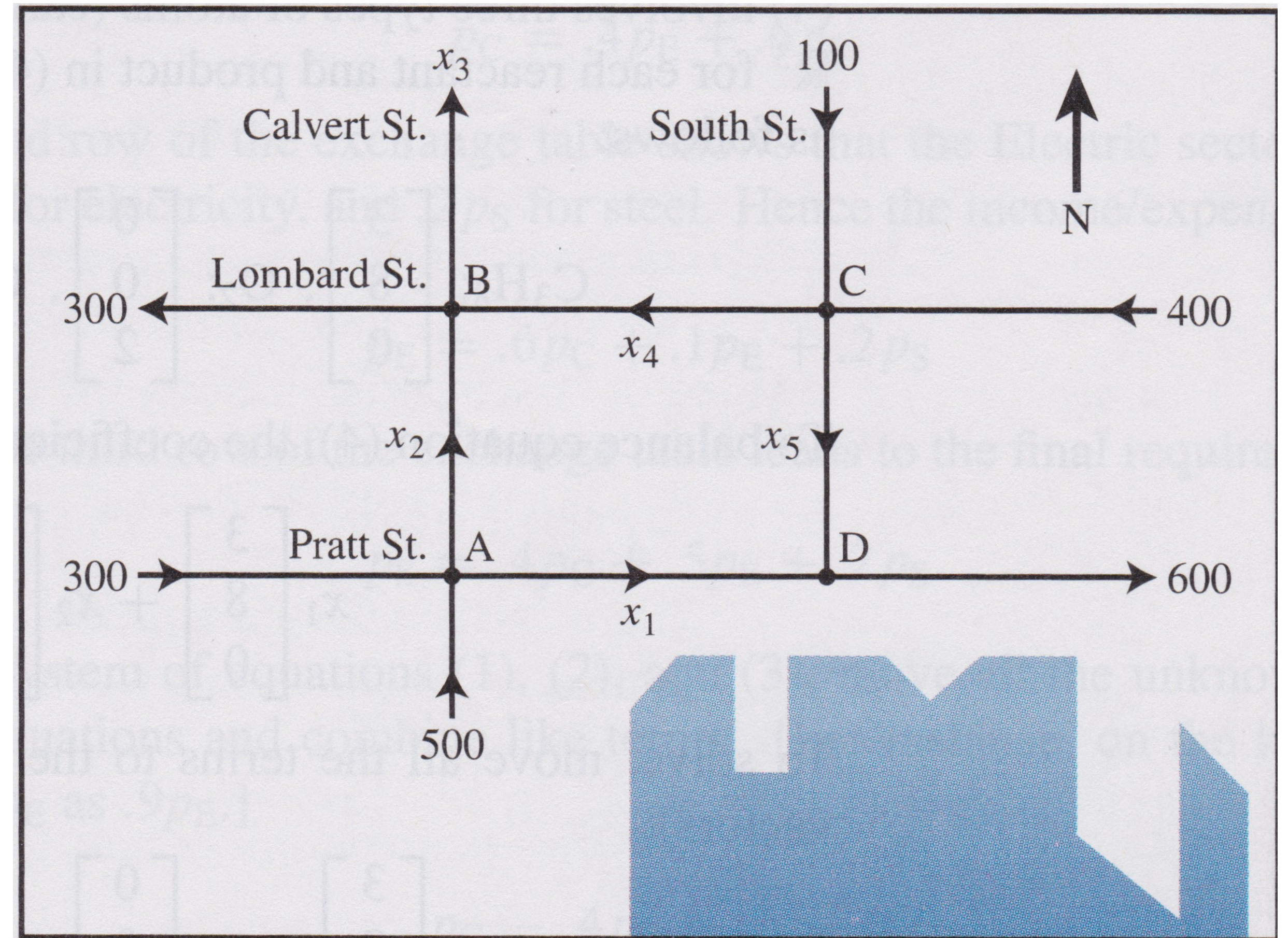
(A)  $500 + 300 = x_1 + x_2$

(B)  $x_2 + x_4 = 300 + x_3$

(C)  $100 + 400 = x_4 + x_5$

(D)  $x_1 + x_5 = 600$

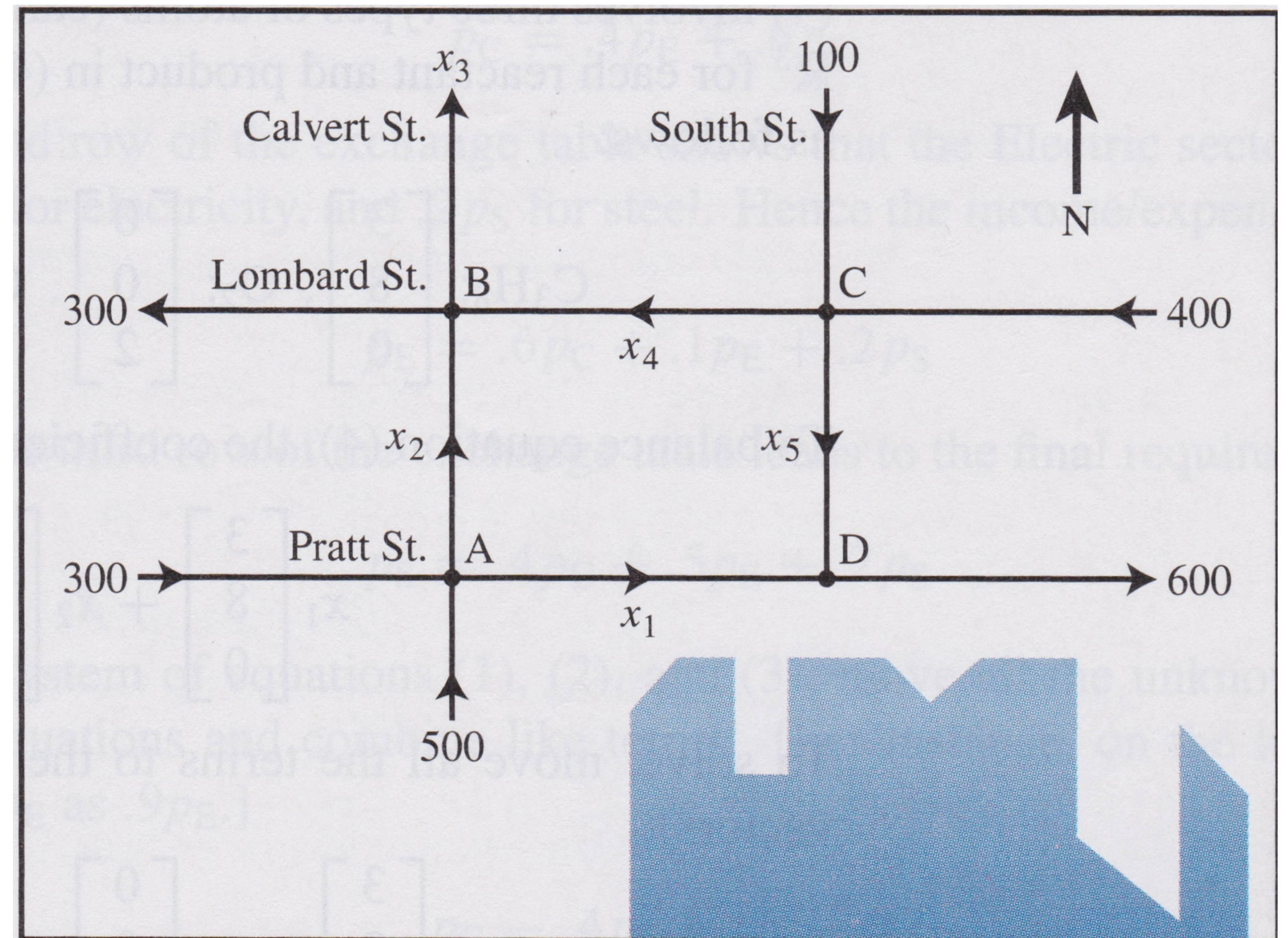
System of Linear Equations



# Example

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 800 \\ 0 & 1 & -1 & 1 & 0 & 300 \\ 0 & 0 & 0 & 1 & 1 & 500 \\ 1 & 0 & 0 & 0 & 1 & 600 \end{bmatrix}$$

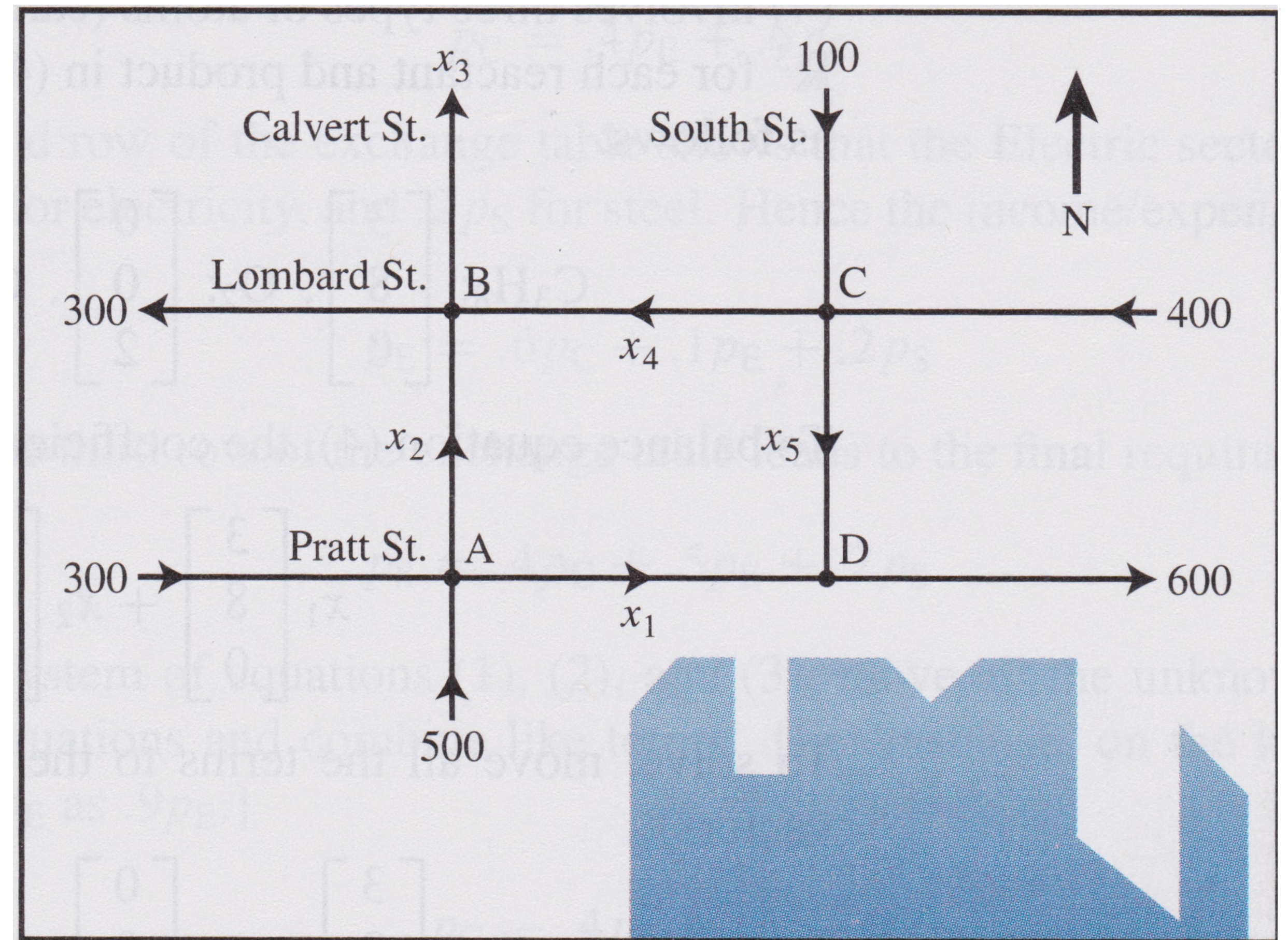
Augmented Matrix



# Example

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 600 \\ 0 & 1 & 0 & 0 & -1 & 200 \\ 0 & 0 & 1 & 0 & 0 & 400 \\ 0 & 0 & 0 & 1 & 1 & 500 \end{bmatrix}$$

Reduced Echelon Form



Note that global flow is conserved.

# Example

$$x_1 = 600 - x_5$$

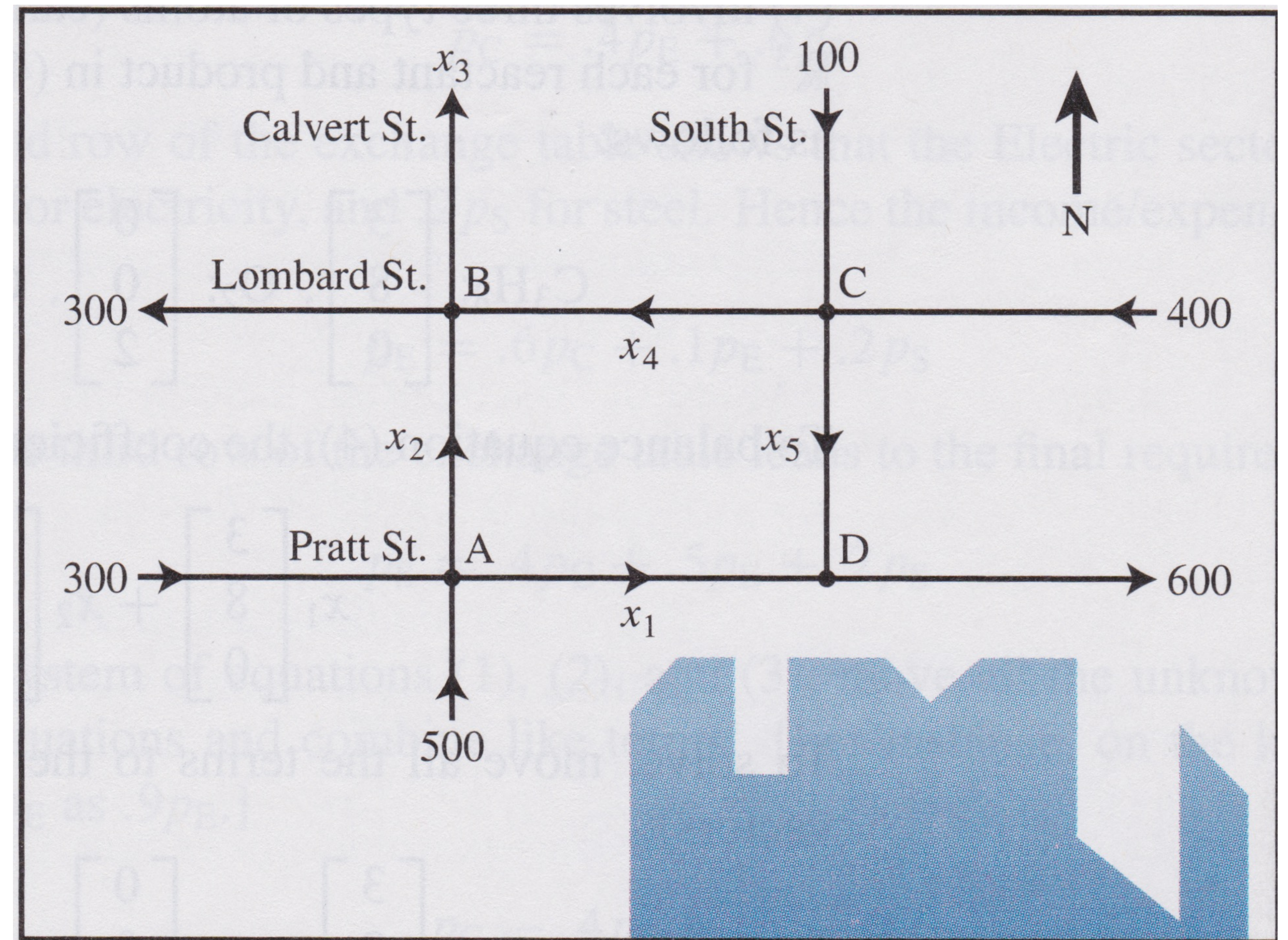
$$x_2 = 200 + x_5$$

$$x_3 = 400$$

$$x_4 = 500 - x_5$$

$x_5$  is free

General Solution



# How To: Max Flow Value for an Edge

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**Question.** Find the maximum value of a flow variable in a given flow network

# How To: Max Flow Value for an Edge

**Question.** Find the maximum value of a flow variable in a given flow network

**Solution.** Remember that flow values must be positive. Look at the general form solution and see what makes this hold

# Example

$$x_1 = 600 - x_5$$

$$x_2 = 200 + x_5$$

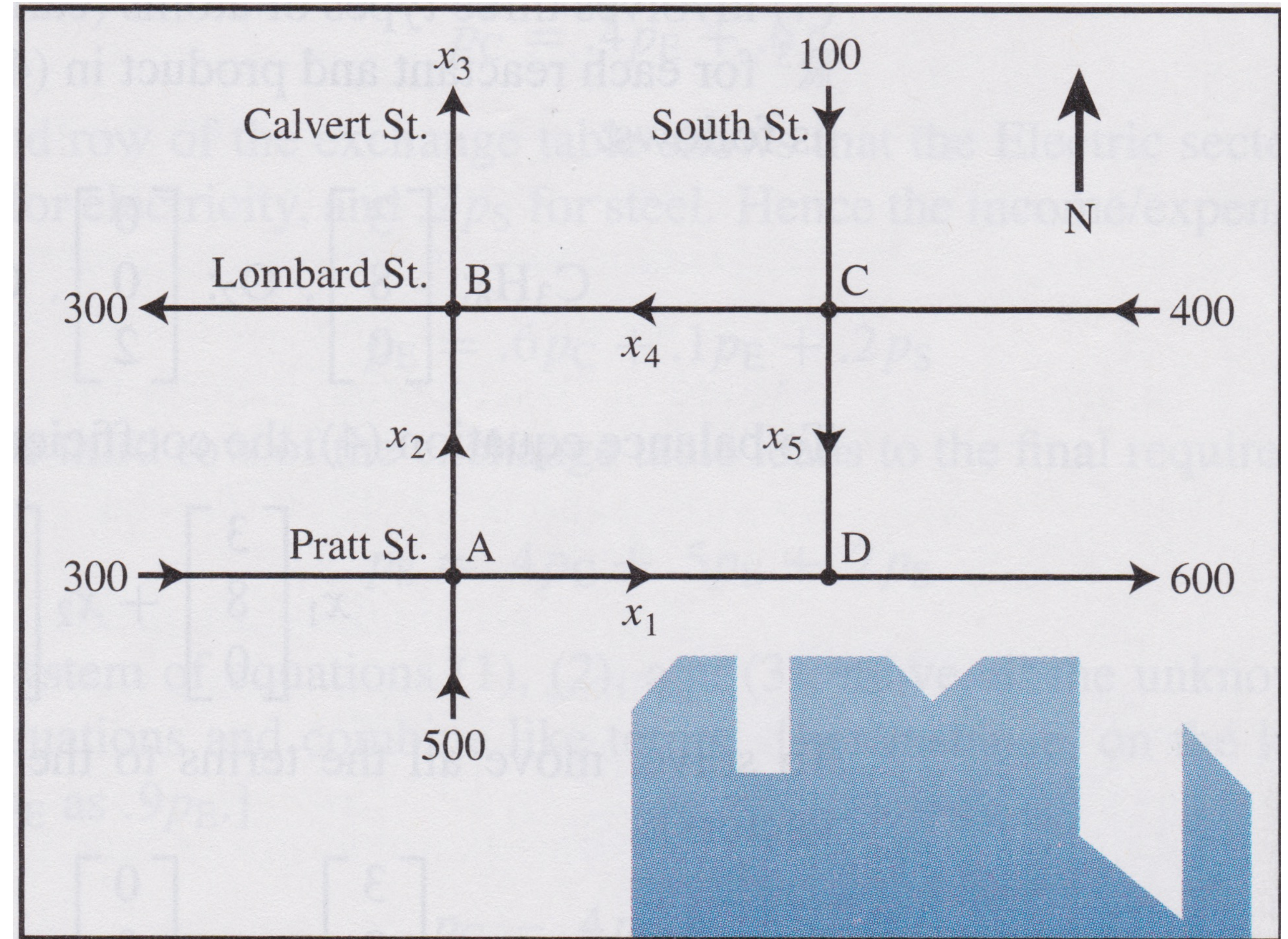
$$x_3 = 400$$

$$x_4 = 500 - x_5$$

$x_5$  is free

$$x_4 \geq 0 \text{ implies } x_5 \leq 500$$

$$x_1 \geq 0 \text{ implies } x_5 \leq 600$$



# Example

$$x_1 = 600 - x_5$$

$$x_2 = 200 + x_5$$

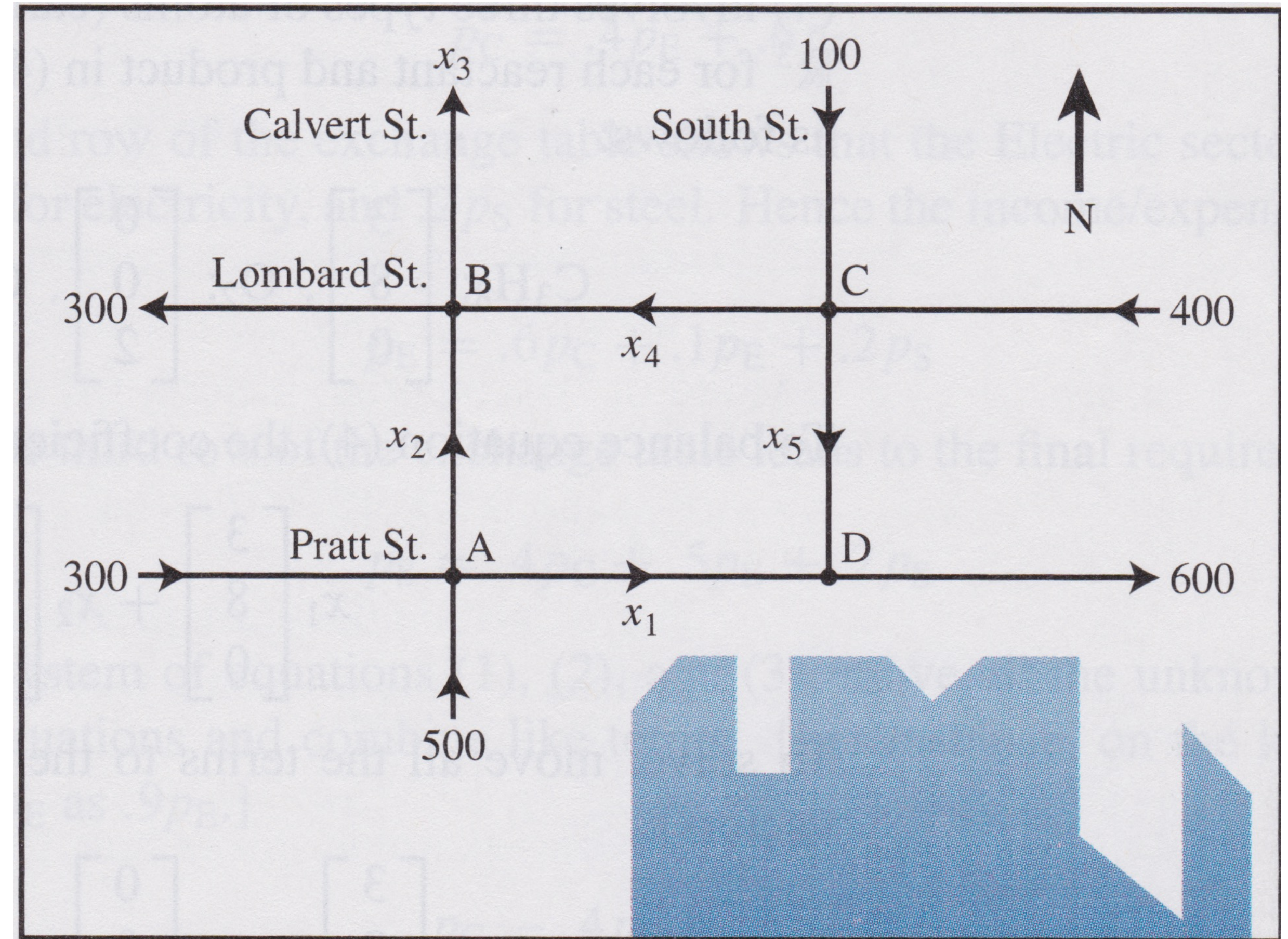
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$$x_4 \geq 0 \text{ implies } x_5 \leq 500$$

$$x_1 \geq 0 \text{ implies } x_5 \leq 600$$



# Example

$$x_1 = 600 - x_5$$

$$x_2 = 200 + x_5$$

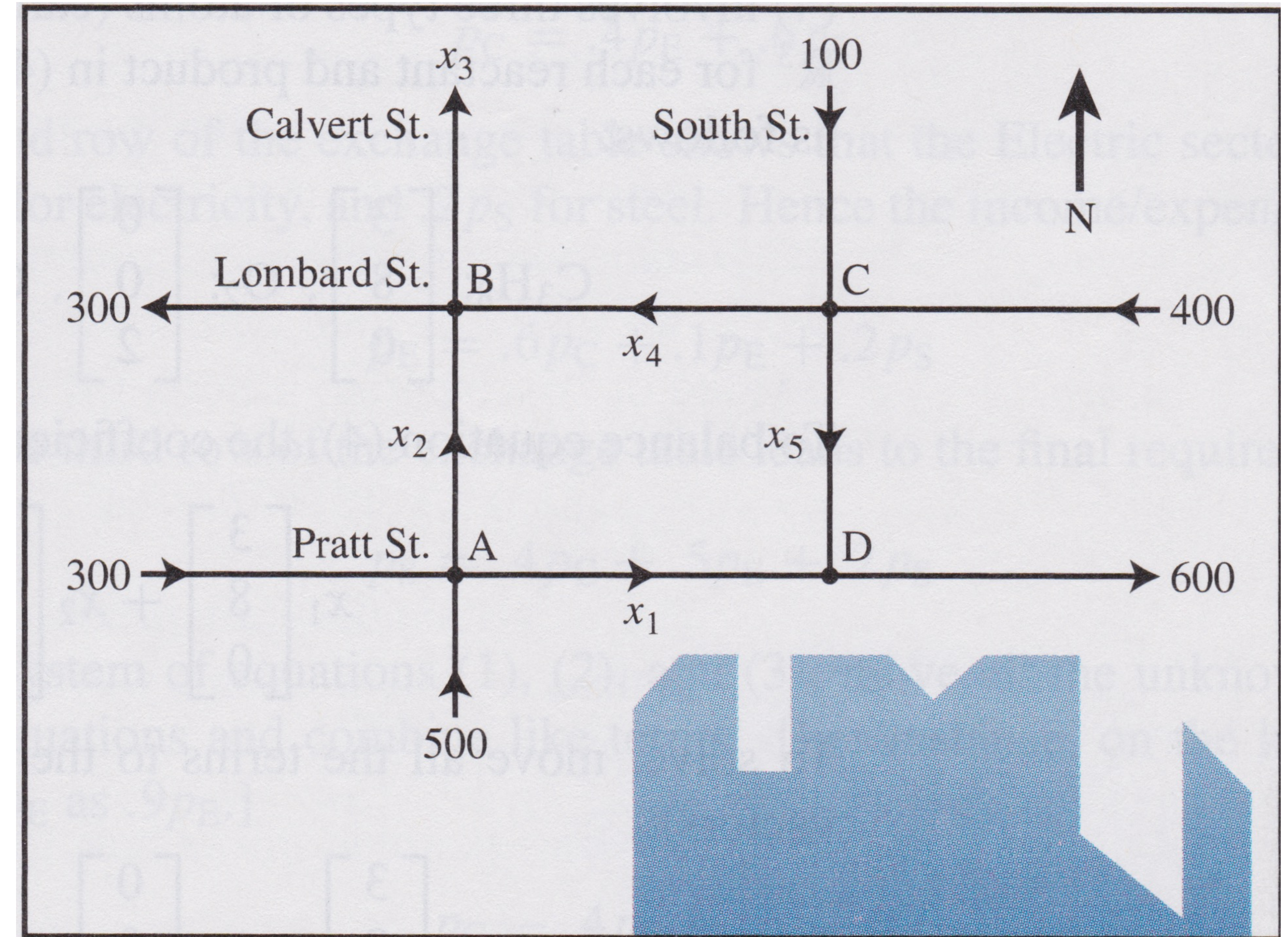
$$x_3 = 400$$

$$x_4 = 500 - x_5$$

$x_5$  is free

$$x_4 \geq 0 \text{ implies } x_5 \leq 500$$

$$x_1 \geq 0 \text{ implies } x_5 \leq 600$$



The maximum value of  $x_5$  is 500

# Summary

**Linear independence** helps us understand when a span is "as large as it can be"

We can reduce this seeing if a single homogeneous equation has a **unique solution**

**Network flows** define linear systems we can solve