

Matrix-Vector Equations

Geometric Algorithms
Lecture 5

Practice Problem

Is the vector $\begin{bmatrix} 9 \\ 3 \\ -14 \end{bmatrix}$ in $\text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ -4 \end{bmatrix} \right\}$?

Answer

$$\begin{bmatrix} 9 \\ 3 \\ -14 \end{bmatrix} \in \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ -4 \end{bmatrix} \right\} ?$$

$$x_1 \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 2 \\ -4 \end{bmatrix} = \begin{bmatrix} 9 \\ 3 \\ -14 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 3 & 9 \\ 0 & 1 & 2 & 3 \\ -2 & -1 & -4 & -14 \end{bmatrix}$$

$+2 \quad +2 \quad +6 \quad +18$

$$R_3 \leftarrow R_3 + 2R_1$$

$$\begin{bmatrix} 1 & 1 & 3 & 9 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 4 \end{bmatrix}$$

$-1 \quad -2 \quad -3$

$$R_3 \leftarrow R_3 - R_2$$

$$\begin{bmatrix} 1 & 1 & 3 & 9 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

NO SOLUTION

NOT IN SPAN

$$0 = 1$$

Outline

- » Motivate the study of matrix–vector equations
- » Formally define matrix–vector multiplication
- » Revisit spans
- » Take stock of our perspectives on systems of linear equations

Keywords

matrix-vector multiplication

the matrix equation

inner-product

row-column rule

Recap

Recall: Vector "Interface"

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`equality` what does it mean for two vectors
to be equal?

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addition what does $\mathbf{u} + \mathbf{v}$ (adding two vectors
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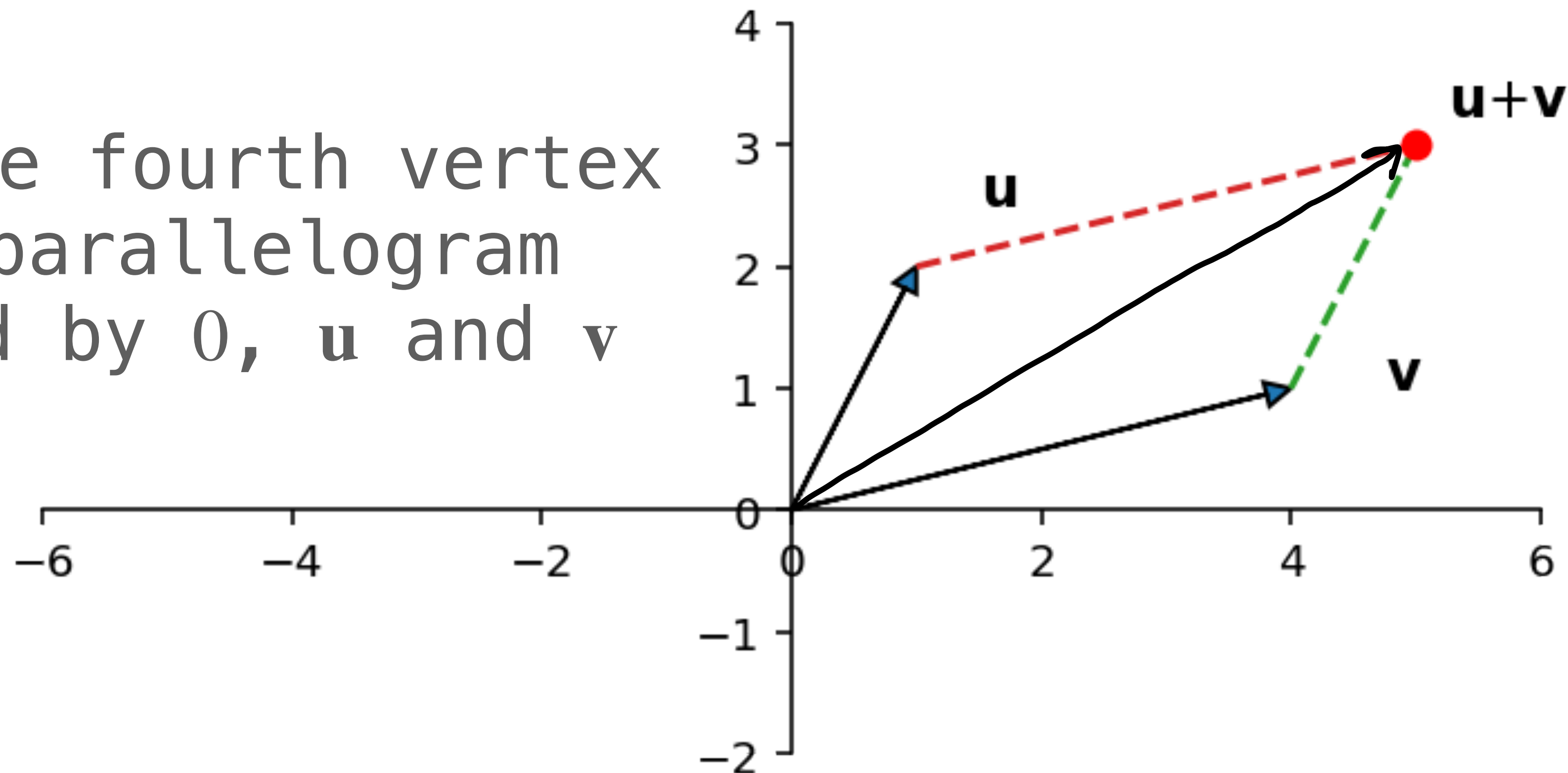
scaling what does $a\mathbf{v}$ (multiplying a vector by a real number) mean?

What properties do they need to satisfy?

Recall: Vector Addition (Geometrically)

in \mathbb{R}^2 it's called the *parallelogram rule*

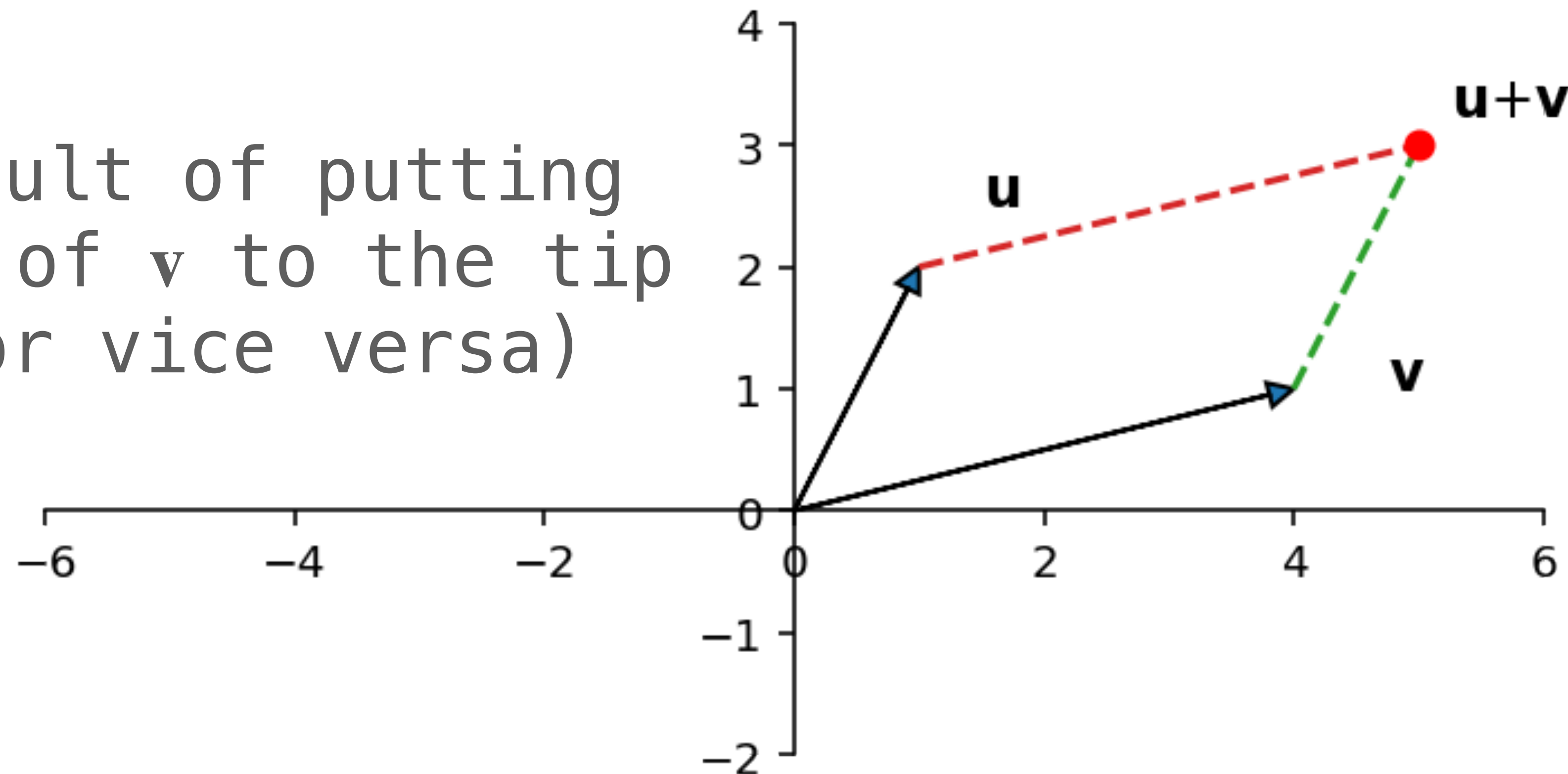
$\mathbf{u} + \mathbf{v}$ is the fourth vertex
of the parallelogram
generated by $\mathbf{0}$, \mathbf{u} and \mathbf{v}



Vector Addition (Geometrically)

or the *tip-to-tail rule*

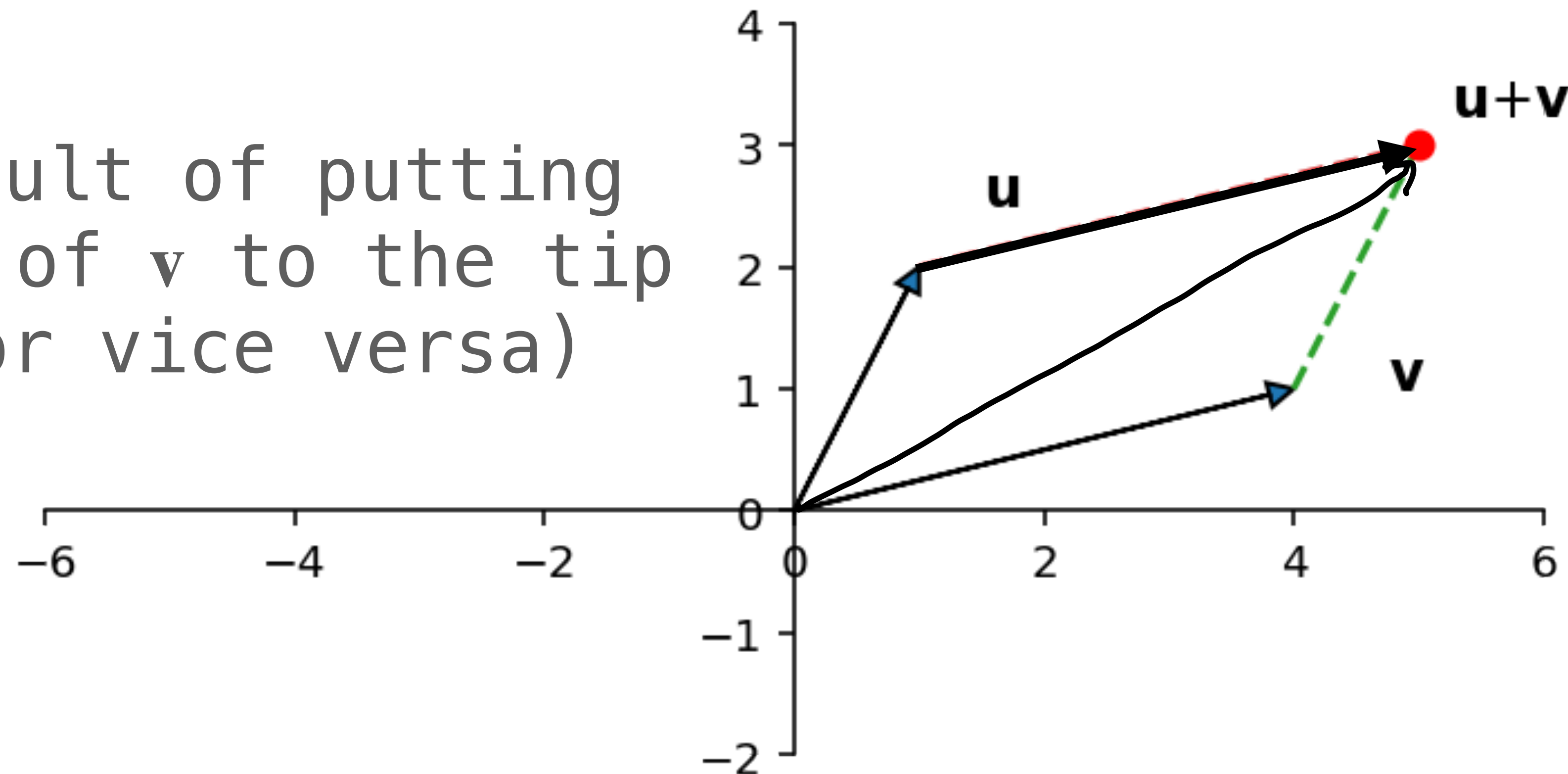
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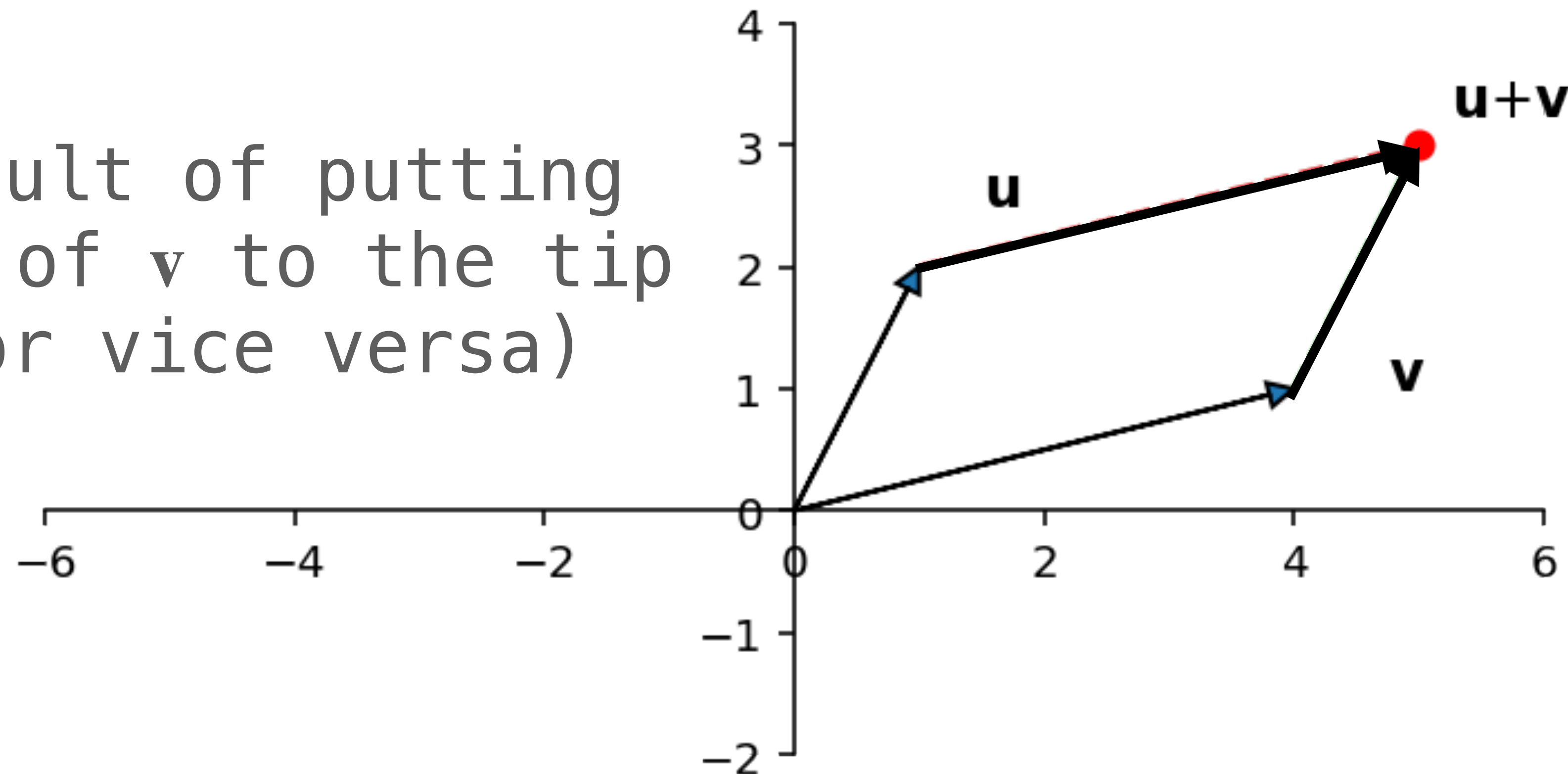
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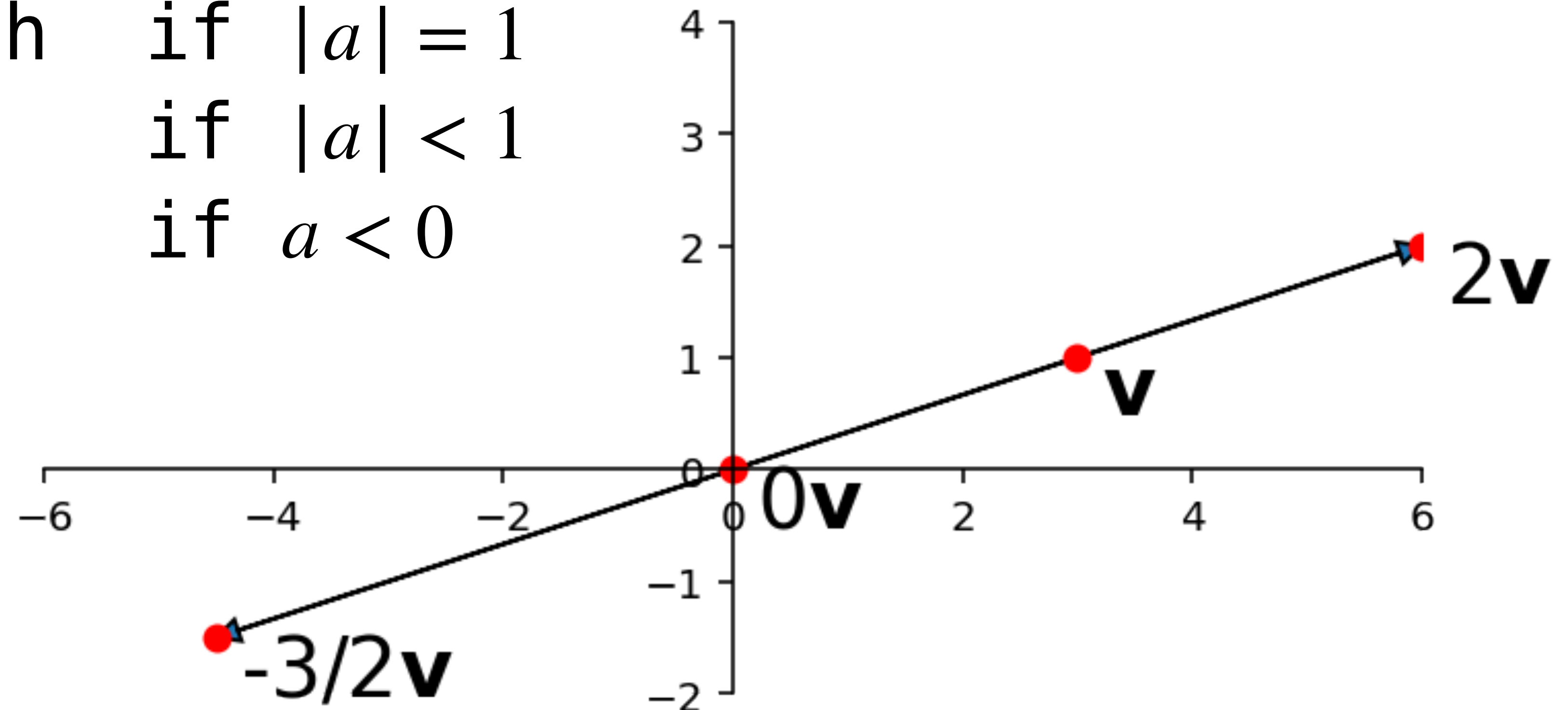
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Recall Vector Scaling (Geometrically)

longer	if $ a > 1$
the same length	if $ a = 1$
shorter	if $ a < 1$
reversed	if $a < 0$



Recall Vector Scaling (Geometrically)

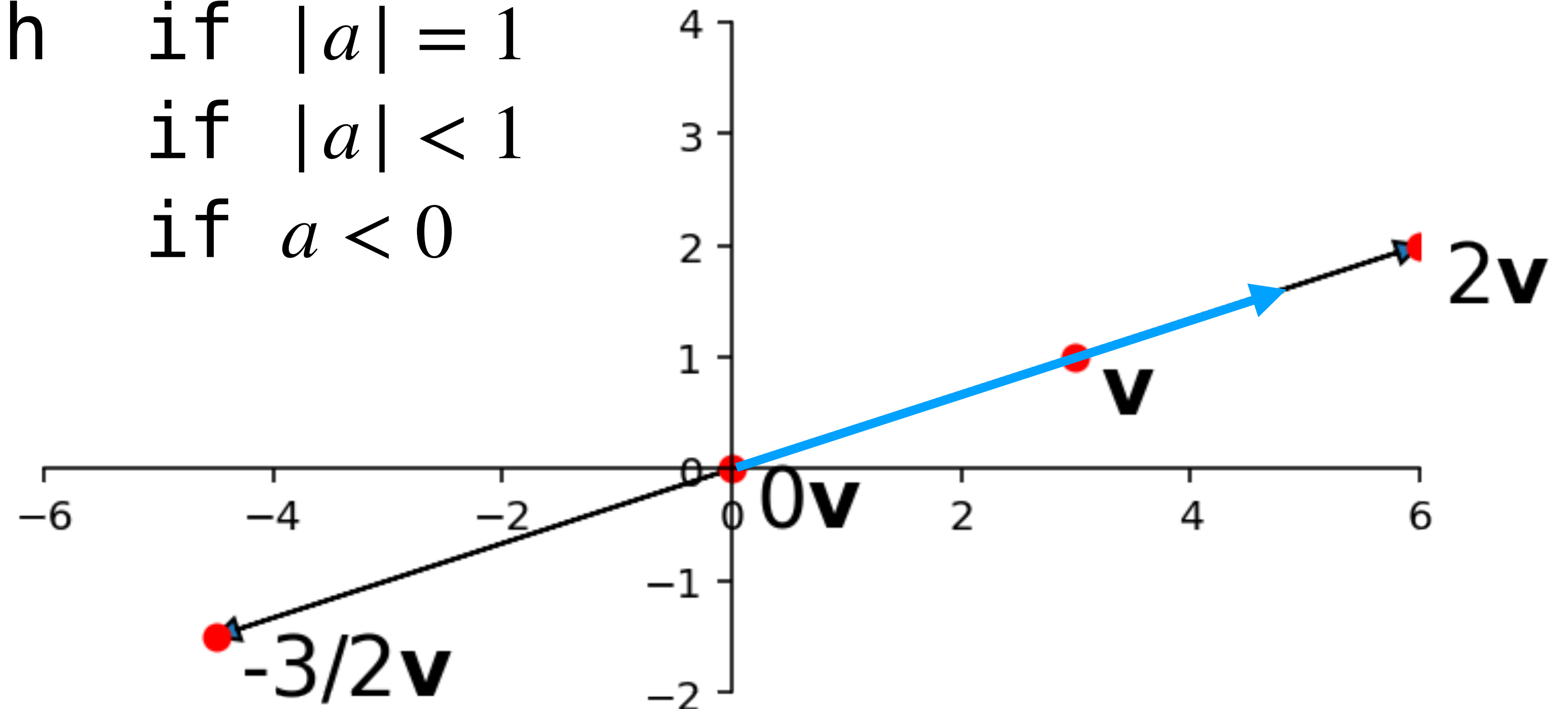
longer
the same length
shorter
reversed

if $|a| > 1$

if $|a| = 1$

if $|a| < 1$

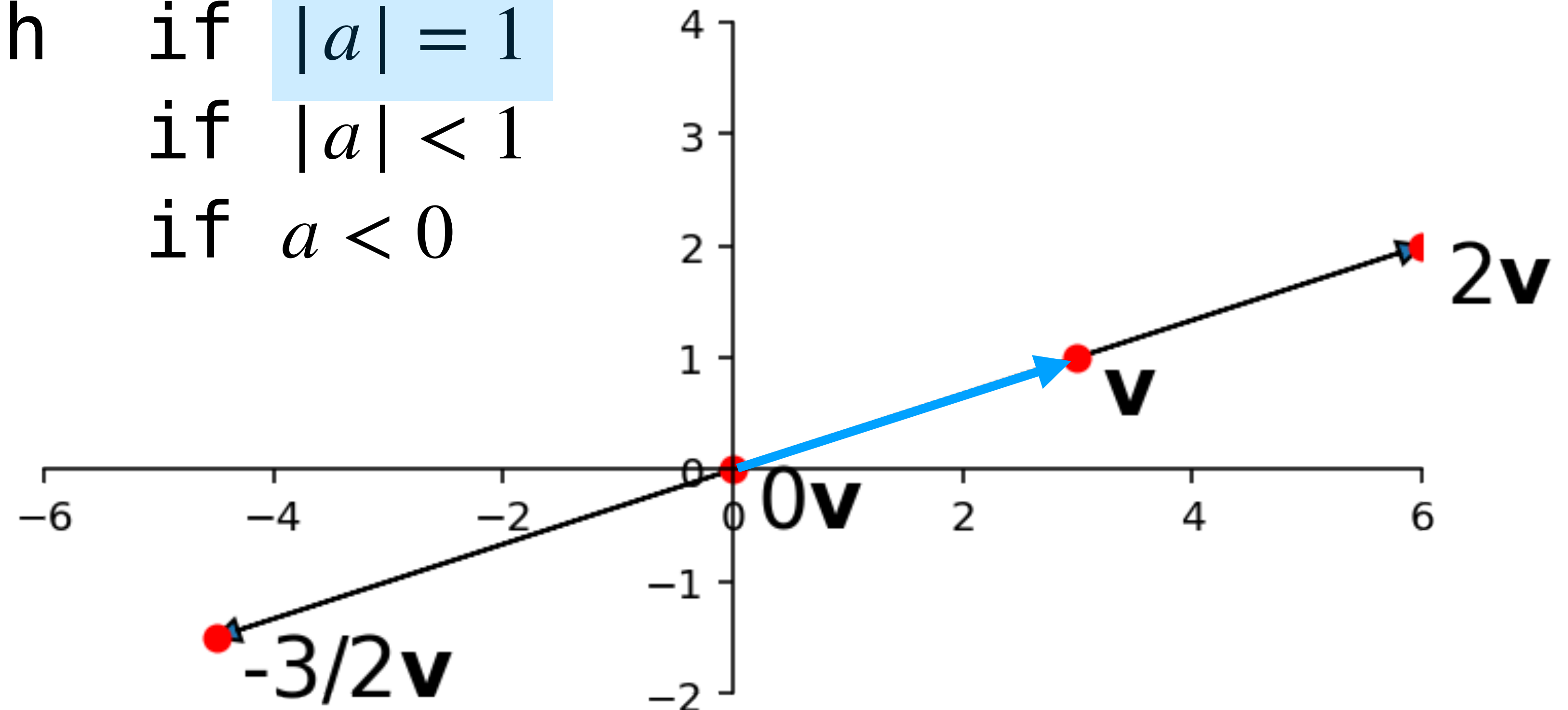
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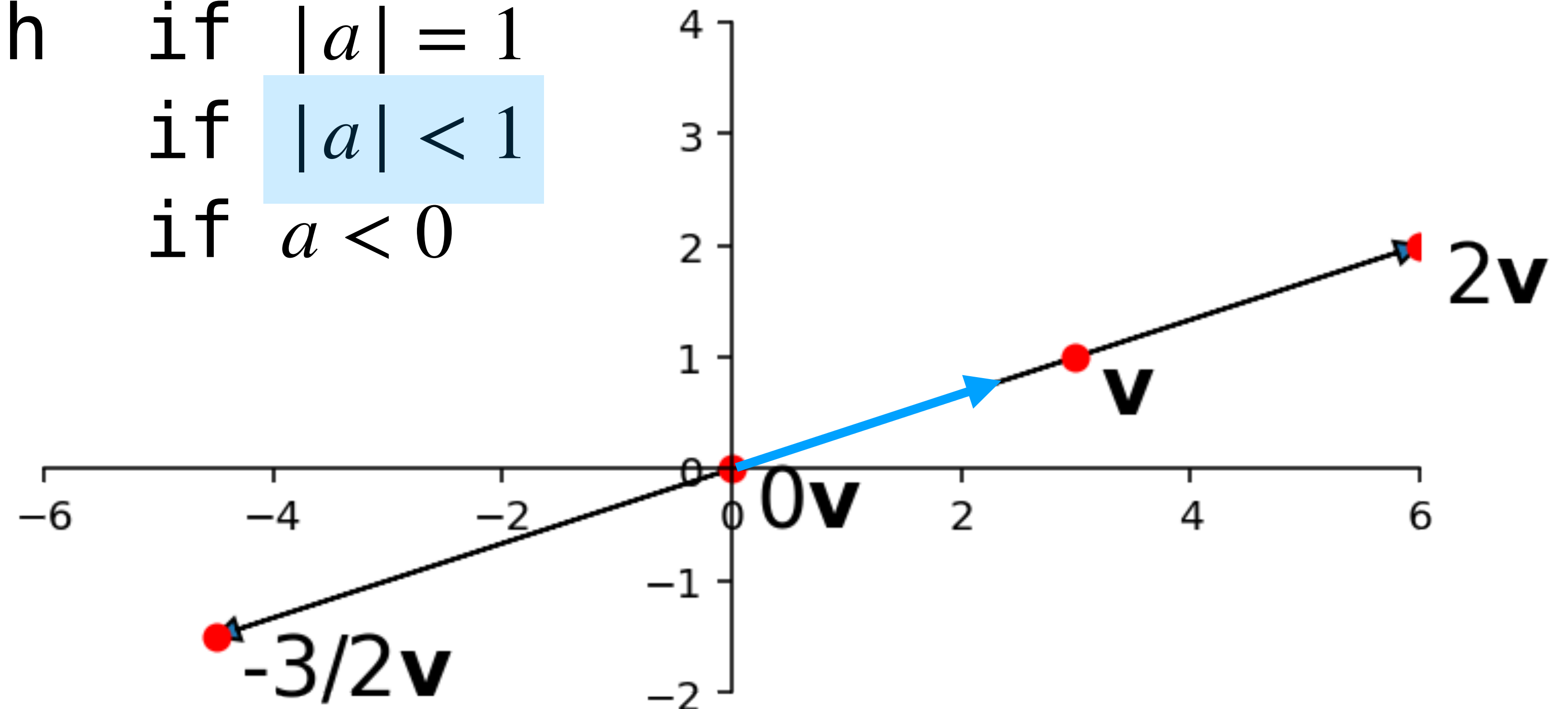
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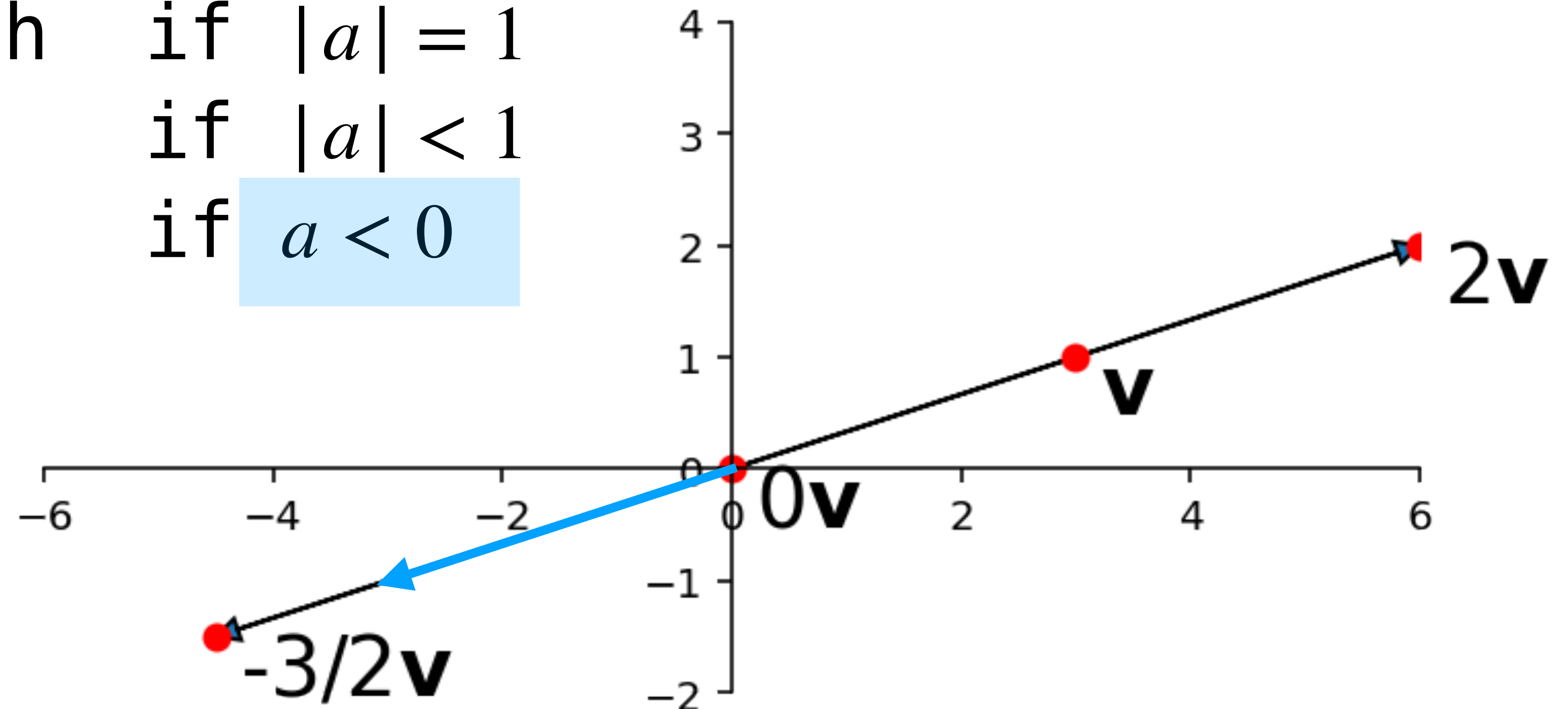
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Recall Vector Scaling (Geometrically)

longer if $|a| > 1$
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Recall: Linear Combinations

Definition. a *linear combination* of vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ is a vector of the form

$$\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \dots + \alpha_n \mathbf{v}_n$$

where $\alpha_1, \alpha_2, \dots, \alpha_n$ are in \mathbb{R}

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where $\alpha_1, \alpha_2, \dots, \alpha_n$ are in \mathbb{R}
weights

Recall: The Fundamental Concern

Can \mathbf{u} be written as a linear combination of

$\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$?

That is, are there weights $\alpha_1, \alpha_2, \dots, \alpha_n$ such that

$$\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \dots \alpha_n \mathbf{v}_n = \mathbf{u}?$$

Recall: The Fundamental Connection

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}$$

augmented matrix

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

system of linear equations

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

vector equation

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Motivation

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augmented matrix

Why not view
these as a
vector too?

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vector equation

Solutions as Vectors

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A solution is an ordered list of numbers

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So it can be represented as a vector

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A solution is an ordered list of numbers

So it can be represented as a vector

Can we view a linear system as a single equation with matrices and vectors?

How do matrices and vectors "interface"?

Matrix-Vector Multiplication

Matrix-Vector "Interface"

multiplication

what does $A\mathbf{v}$ mean when A is a matrix and \mathbf{v} is a vector?

Matrix-Vector Multiplication (Pictorially)

AS

Matrix-Vector Multiplication (Pictorially)

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix}$$

Matrix-Vector Multiplication (Pictorially)

The diagram illustrates matrix-vector multiplication pictorially. On the left, a matrix is represented by a large left square bracket followed by four vertical columns. The first column contains a_{11} , a_{21} , a vertical ellipsis, and a_{m1} . The second column contains a_{12} , a_{22} , a vertical ellipsis, and a_{m2} . The third column contains an ellipsis, a diagonal ellipsis, and an ellipsis. The fourth column contains a_{1n} , a_{2n} , a vertical ellipsis, and a_{mn} . A large right square bracket follows the fourth column. To the right of the matrix is a vector, represented by a large left square bracket followed by four light blue rectangular boxes. The first box contains s_1 , the second contains s_2 , the third contains a vertical ellipsis, and the fourth contains s_n . A large right square bracket follows the fourth box.

$$\left[\begin{array}{c} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{array} \begin{array}{c} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{array} \begin{array}{c} \dots \\ \dots \\ \ddots \\ \dots \end{array} \begin{array}{c} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{array} \right] \left[\begin{array}{c} s_1 \\ s_2 \\ \vdots \\ s_n \end{array} \right]$$

Matrix-Vector Multiplication (Pictorially)

$$s_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + s_2 \begin{bmatrix} a_{21} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \dots + s_n \begin{bmatrix} a_{m1} \\ a_{m2} \\ \vdots \\ a_{mn} \end{bmatrix}$$

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a linear combination of the columns where
s defines the weights

Why keeping track of matrix size is important

this only works if the number of *columns* of the matrix matches the number of *rows* of the vector

$$\begin{bmatrix} * & \dots & * \\ * & \dots & * \\ \vdots & \ddots & \vdots \\ * & \dots & * \\ * & \dots & * \end{bmatrix} \begin{bmatrix} * \\ \vdots \\ * \end{bmatrix} = \begin{bmatrix} * \\ * \\ \vdots \\ * \\ * \end{bmatrix}$$

$(m \times n) \qquad (n \times 1) \qquad (m \times 1)$

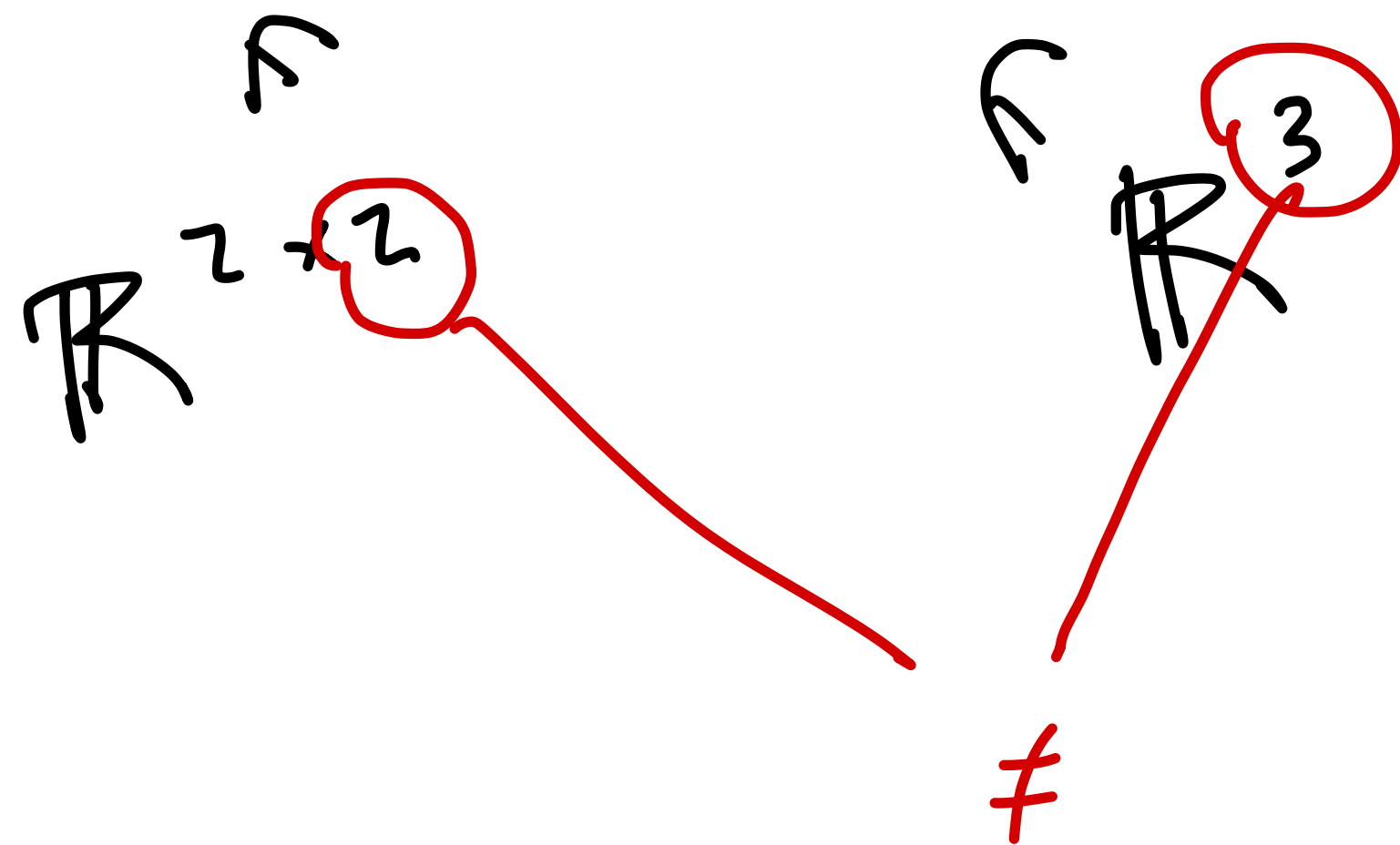
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$$\begin{array}{ccc} \begin{array}{c} \textcolor{blue}{m} \end{array} \begin{array}{c} \textcolor{red}{n} \\ \left[\begin{array}{ccc} * & \dots & * \\ * & \dots & * \\ \vdots & \ddots & \vdots \\ * & \dots & * \\ * & \dots & * \end{array} \right] \end{array} & \begin{array}{c} \textcolor{red}{n} \begin{array}{c} \textcolor{purple}{1} \\ \left[\begin{array}{c} * \\ \vdots \\ * \end{array} \right] \end{array} & = & \begin{array}{c} \textcolor{blue}{m} \begin{array}{c} \textcolor{purple}{1} \\ \left[\begin{array}{c} * \\ * \\ \vdots \\ * \\ * \end{array} \right] \end{array} \end{array} \\ \begin{array}{c} (\textcolor{blue}{m} \times \textcolor{red}{n}) \end{array} & \begin{array}{c} (\textcolor{red}{n} \times \textcolor{purple}{1}) \end{array} & & \begin{array}{c} (\textcolor{blue}{m} \times \textcolor{purple}{1}) \end{array} \end{array}$$

Non-Example

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + 2 \begin{bmatrix} 2 \\ 4 \end{bmatrix} + 3 ???$$



Non-Example

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THESE DON'T MATCH

(2×2)

(3×1)

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THESE MATCH
 (2×2) (2×1)

Matrix-Vector Multiplication

Definition. Given a $(m \times n)$ matrix A with columns $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$, and a vector \mathbf{v} in \mathbb{R}^n , we define

$$A\mathbf{v} = \begin{bmatrix} | & | & & | \\ \mathbf{a}_1 & \mathbf{a}_2 & \dots & \mathbf{a}_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = v_1\mathbf{a}_1 + v_2\mathbf{a}_2 + \dots v_n\mathbf{a}_n$$

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$A\mathbf{v}$ is a linear combination of the columns of A with weights given by \mathbf{v}

Algebraic Properties

The algebraic properties of matrix–vector multiplication are **very important**

1. $A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v}$ (additivity)
2. $A(c\mathbf{v}) = c(A\mathbf{v})$ (homogeneity)

Derivation of (1) for A in $\mathbb{R}^{n \times 3}$

$$[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3] \left(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \right)$$

Derivation of (1) for A in $\mathbb{R}^{n \times 3}$

$$[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3] \begin{pmatrix} \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 \end{bmatrix} \end{pmatrix}$$

by vector addition

Derivation of (1) for A in $\mathbb{R}^{n \times 3}$

$$(u_1 + v_1)\mathbf{a}_1 + (u_2 + v_2)\mathbf{a}_2 + (u_3 + v_3)\mathbf{a}_3$$

by matrix vector multiplication

Derivation of (1) for A in $\mathbb{R}^{n \times 3}$

$$u_1 \mathbf{a}_1 + v_1 \mathbf{a}_1 + u_2 \mathbf{a}_2 + v_2 \mathbf{a}_2 + u_3 \mathbf{a}_3 + v_3 \mathbf{a}_3$$

by vector scaling (distribution)

Derivation of (1) for A in $\mathbb{R}^{n \times 3}$

$$(u_1 \mathbf{a}_1 + u_2 \mathbf{a}_2 + u_3 \mathbf{a}_3) + (v_1 \mathbf{a}_1 + v_2 \mathbf{a}_2 + v_3 \mathbf{a}_3)$$

by rearranging

Derivation of (1) for A in $\mathbb{R}^{n \times 3}$

$$[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3] \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3] \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

by matrix vector multiplication

Derivation of (1) for A in $\mathbb{R}^{n \times 3}$

$$[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3] \left(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \right)$$

equals

$$[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3] \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3] \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

fin

A Common Error

$$A\mathbf{v} \neq \mathbf{v}A$$

Important. We multiply matrices and vectors
with the matrix *on the left*

Looking forward a bit

$$\begin{bmatrix} * & \dots & * \\ * & \dots & * \\ \vdots & \ddots & \vdots \\ * & \dots & * \\ * & \dots & * \end{bmatrix} \begin{bmatrix} * \\ \vdots \\ * \end{bmatrix} = \begin{bmatrix} * \\ * \\ \vdots \\ * \\ * \end{bmatrix}$$

Remember. column vectors are matrices with 1 column

Eventually we'll be able to view all of these as
matrix operations

Example

$$\begin{matrix} 2 \times 3 & 3 \\ \begin{bmatrix} 2 & -3 & 4 \\ -1 & 1 & 0 \end{bmatrix} & \begin{bmatrix} 5 \\ 5 \\ 4 \end{bmatrix} \end{matrix}$$

Compute the above matrix-vector multiplication

$$= 5 \begin{bmatrix} 2 \\ -1 \end{bmatrix} + 5 \begin{bmatrix} -3 \\ 1 \end{bmatrix} + 4 \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 10 \\ -5 \end{bmatrix} + \begin{bmatrix} -15 \\ 5 \end{bmatrix} + \begin{bmatrix} 16 \\ 0 \end{bmatrix} = \begin{bmatrix} 11 \\ 0 \end{bmatrix}$$

A Tip

$$\begin{bmatrix} 2 & -3 & 4 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ 5 \\ 4 \end{bmatrix} = \begin{bmatrix} ? \\ ? \end{bmatrix}$$

A Tip

$$\begin{bmatrix} 2 & -3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 5 \\ 4 \end{bmatrix} = \begin{bmatrix} ? \\ ? \end{bmatrix}$$

$$5(2) + 5(-3) + 4(4) = 11$$

A Tip

$$\sim \begin{bmatrix} 2 & -3 & 4 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ 5 \\ 4 \end{bmatrix} = \begin{bmatrix} 11 \\ ? \end{bmatrix} \sim$$

$$5(-1) + 5(1) + 4(0) = 0$$

A Tip

$$\begin{bmatrix} 2 & -3 & 4 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ 5 \\ 4 \end{bmatrix} = \begin{bmatrix} 11 \\ 0 \end{bmatrix}$$

Calculating $A\mathbf{v}$

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix} = \begin{bmatrix} ? \\ ? \\ \vdots \\ ? \end{bmatrix}$$

$$v_1 = a_{11}s_1 + a_{12}s_2 + \dots + a_{1n}s_n = \sum_{i=1}^n a_{1i}s_i$$

Calculating $A\mathbf{v}$

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix} = \begin{bmatrix} v_1 \\ ? \\ \vdots \\ ? \end{bmatrix}$$

$$v_2 = a_{21}s_1 + a_{22}s_2 + \dots + a_{2n}s_n = \sum_{i=1}^n a_{2i}s_i$$

Calculating $A\mathbf{v}$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ ? \end{bmatrix}$$

$$v_m = a_{m1}s_1 + a_{m2}s_2 + \cdots + a_{mn}s_n = \sum_{i=1}^n a_{mi}s_i$$

Calculating $A\mathbf{v}$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix}$$

Row-Column Rule

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^n a_{1i}s_i \\ \sum_{i=1}^n a_{2i}s_i \\ \vdots \\ \sum_{i=1}^n a_{mi}s_i \end{bmatrix}$$

Inner product: $[a_1 \ a_2 \ \cdots \ a_n] \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix} = \sum_{i=1}^n a_i s_i$

Inner Product

Definition. The **inner product** of vectors **u** and **v** in \mathbb{R}^n is defined the

$$\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u} \cdot \mathbf{v} = \sum_{i=1}^n u_i v_i$$

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = 1(4) + 2(5) + 3(6) \\ = 4 + 10 + 18 = 32$$

Row-Column Rule

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^n a_{1i}s_i \\ \sum_{i=1}^n a_{2i}s_i \\ \vdots \\ \sum_{i=1}^n a_{mi}s_i \end{bmatrix}$$

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The i th entry of the As is the inner product of the i th row of A and s

The Matrix Equation

Recall: Vector Equations

$$x_1 \mathbf{a}_1 + x_2 \mathbf{a}_2 + \dots + x_n \mathbf{a}_n = \mathbf{b}$$

Recall: Vector Equations

$$x_1 \mathbf{a}_1 + x_2 \mathbf{a}_2 + \dots + x_n \mathbf{a}_n = \mathbf{b}$$

Question. Can \mathbf{b} be written as a linear combination of $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$?

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we can use the same idea for matrix–vector
multiplication

The Matrix Equation

$$A\mathbf{x} = [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \dots \quad \mathbf{a}_n] \mathbf{x} = \mathbf{b}$$

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Can \mathbf{b} be written as a linear combination of **the columns of A** ?

The Matrix Equation

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Can \mathbf{b} be written as a linear combination of **the columns of A** ?

The Idea. write the "vector part" of our matrix-vector multiplication as an *unknown*

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

How To: The Matrix Equation

Question. Does $A\mathbf{x} = \mathbf{b}$ have a solution?

Question. Is $A\mathbf{x} = \mathbf{b}$ consistent?

Question. Write down a solution to the equation $A\mathbf{x} = \mathbf{b}$

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(augmented matrix)

$$[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \dots \quad \mathbf{a}_n \quad \mathbf{b}]$$

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(augmented matrix)

$$[\mathbf{a}_1 \ \mathbf{a}_2 \ \dots \ \mathbf{a}_n \ \mathbf{b}]$$

!!they all have the same solution set!!

How To: The Matrix Equation

Question. Write down a solution to the equation $A\mathbf{x} = \mathbf{b}$

Solution.

Use Gaussian elimination (or other means) to convert $[\mathbf{a}_1 \ \mathbf{a}_2 \ \dots \ \mathbf{a}_n \ \mathbf{b}]$ to reduced echelon form

Then read off a solution from the reduced echelon form

Example

$$\begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 2 \\ -2 & -1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 3 \\ -14 \end{bmatrix}$$

Determine if the above matrix-vector equation has a solution

$$x_1 \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 2 \\ -4 \end{bmatrix} = \begin{bmatrix} 9 \\ 3 \\ -14 \end{bmatrix}$$

Full Span

Recall: Span

Recall: Span

Definition. the *span* of a set of vectors is the set of all possible linear combinations of them

$$\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\} = \{\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \dots\alpha_n\mathbf{v}_n : \alpha_1, \alpha_2, \dots, \alpha_n \text{ are in } \mathbb{R}\}$$

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$\mathbf{u} \in \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ exactly when \mathbf{u} can be expressed as a linear combination of those vectors

Spans (with Matrices)

Definition. The *span* of the vectors $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$ is:

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the span of the columns of a matrix A
is the set of vectors resulting
from multiplying A by any vector

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the span of the columns of a matrix A
is the set of vectors resulting
from multiplying A by any vector

(we will soon start thinking of A as a way of *transforming* vectors)

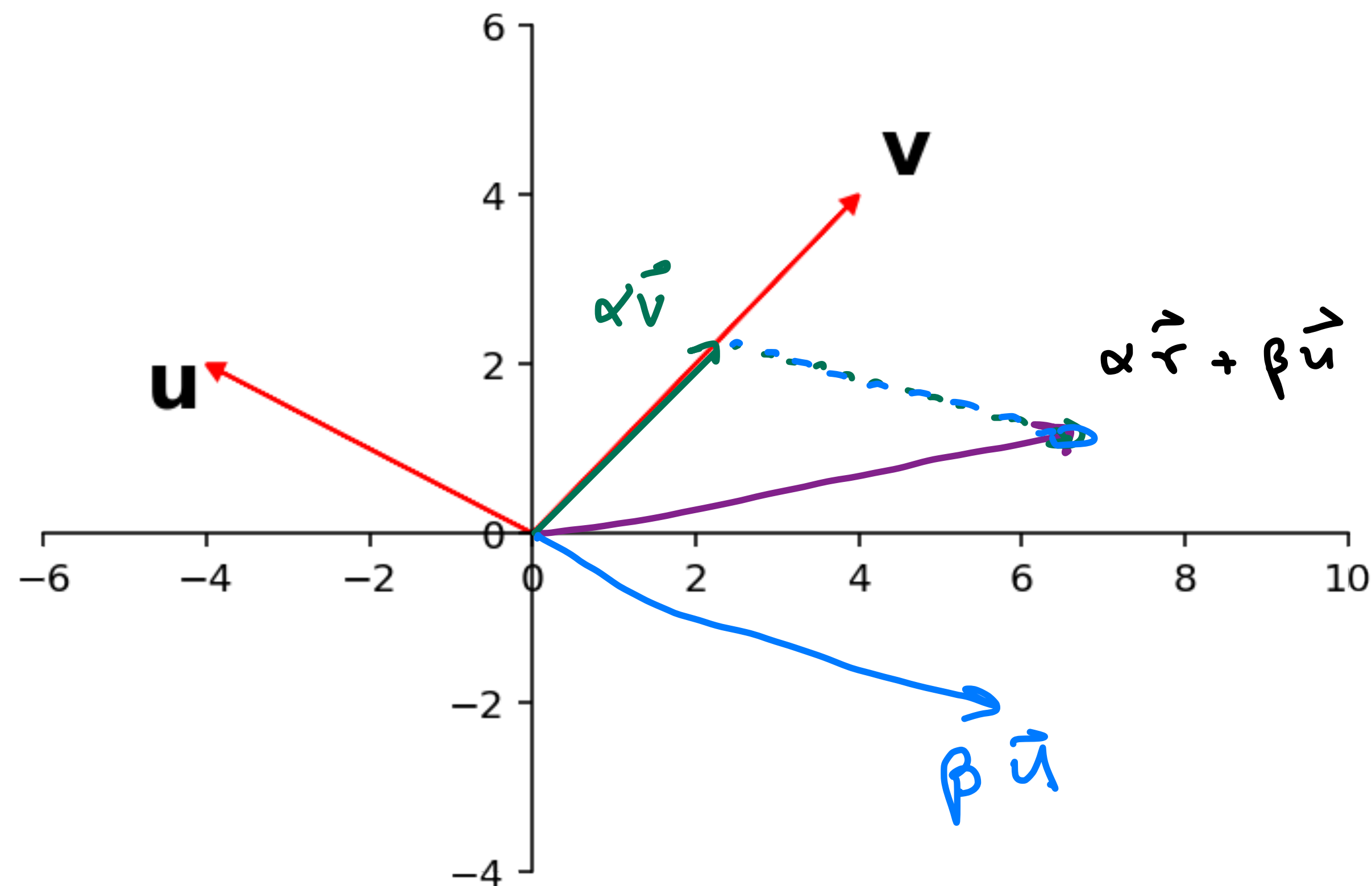
Spanning all of \mathbb{R}^2

Spanning all of \mathbb{R}^2

if two (or more) vectors in \mathbb{R}^2 span a plane,
they must span all of \mathbb{R}^2 . They "fill up" \mathbb{R}^2

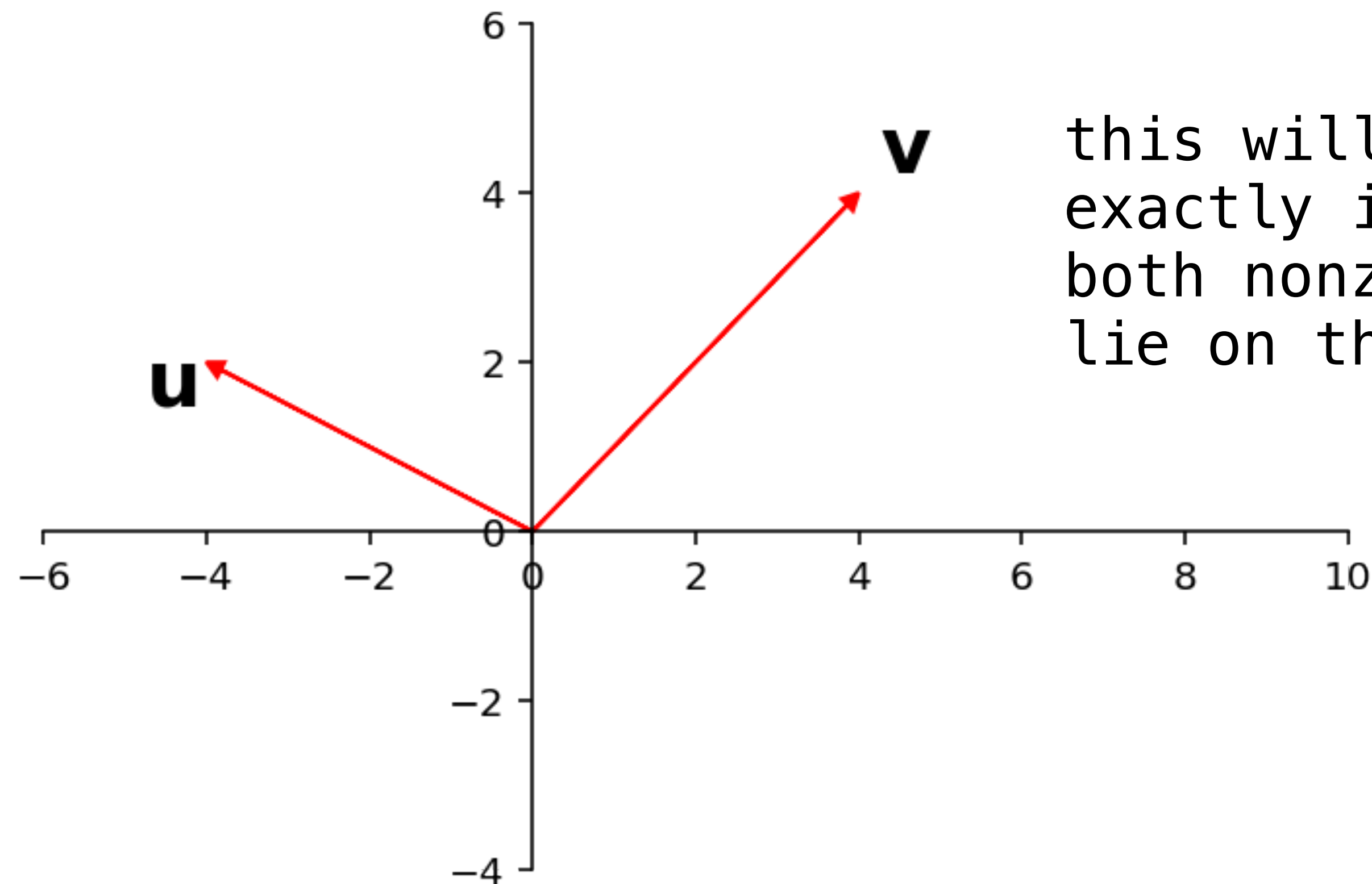
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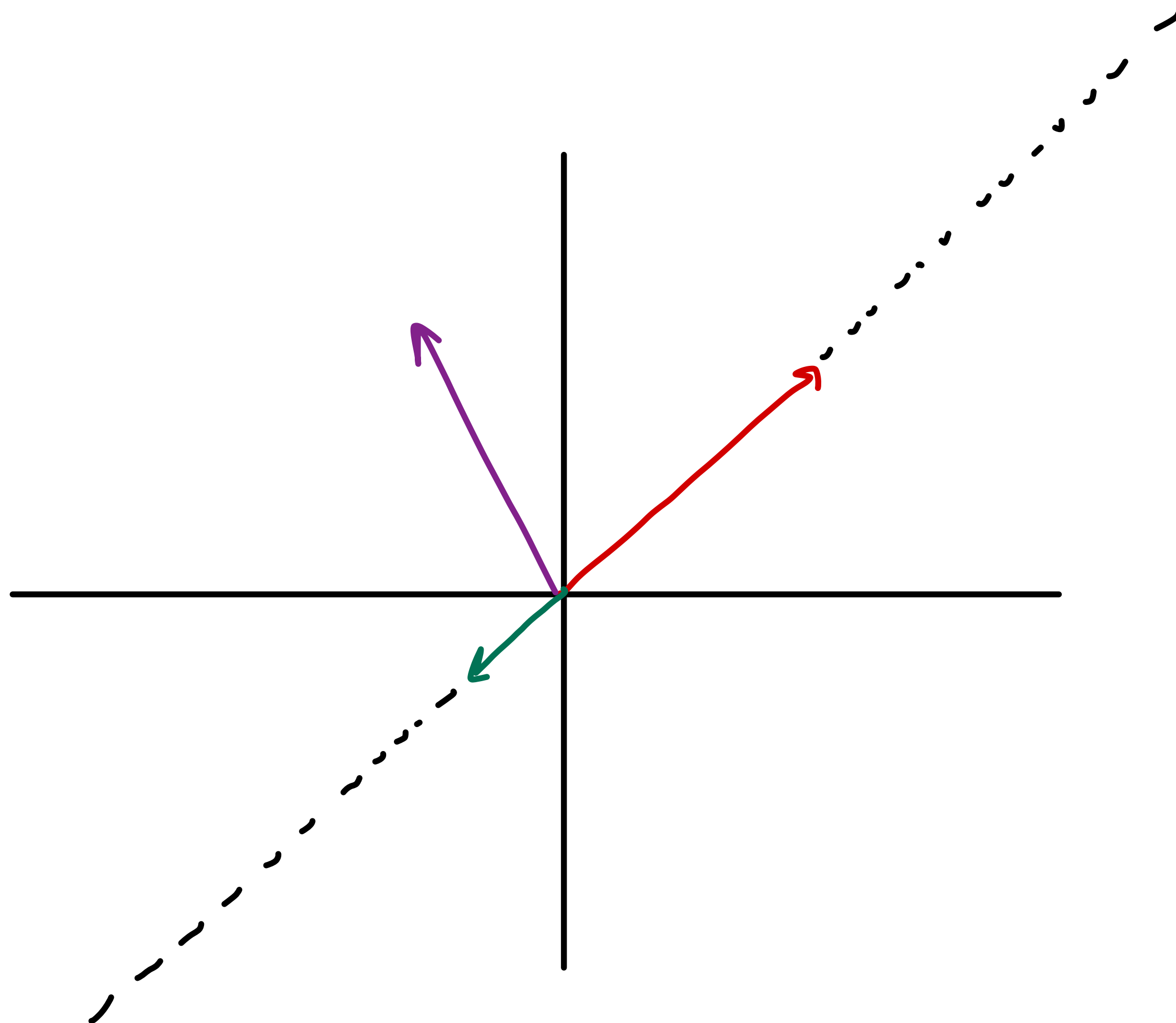


Spanning all of \mathbb{R}^2

if two (or more) vectors in \mathbb{R}^2 span a plane,
they must span all of \mathbb{R}^2 . They "fill up" \mathbb{R}^2



this will happen
exactly if they are
both nonzero and don't
lie on the same line



What about \mathbb{R}^n ?

When do a set of vectors span all of \mathbb{R}^n ?

When do a set of vectors "fill up" \mathbb{R}^n ?

A Few Questions

Can two vectors in \mathbb{R}^3 span all of \mathbb{R}^3 ?

Do five vectors in \mathbb{R}^3 necessarily span all of \mathbb{R}^3 ?

A Thought Experiment

Suppose I give you the augmented matrix of a linear system but I cover up the last column

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & \blacksquare \\ 2 & 1 & 0 & \blacksquare \end{array} \right]$$

A Thought Experiment

Then we reduce it to echelon form

$$\begin{bmatrix} 1 & 2 & 3 & \blacksquare \\ 2 & 1 & 0 & \blacksquare \end{bmatrix}$$

A Thought Experiment

Then we reduce it to echelon form

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & \blacksquare \\ 2 & 1 & 0 & \blacksquare \end{array} \right]$$

$$R_2 \leftarrow R_2 - 2R_1$$

A Thought Experiment

then we reduce it to echelon form

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \end{bmatrix} \begin{array}{c} \blacksquare \\ \blacksquare \end{array}$$

A Thought Experiment

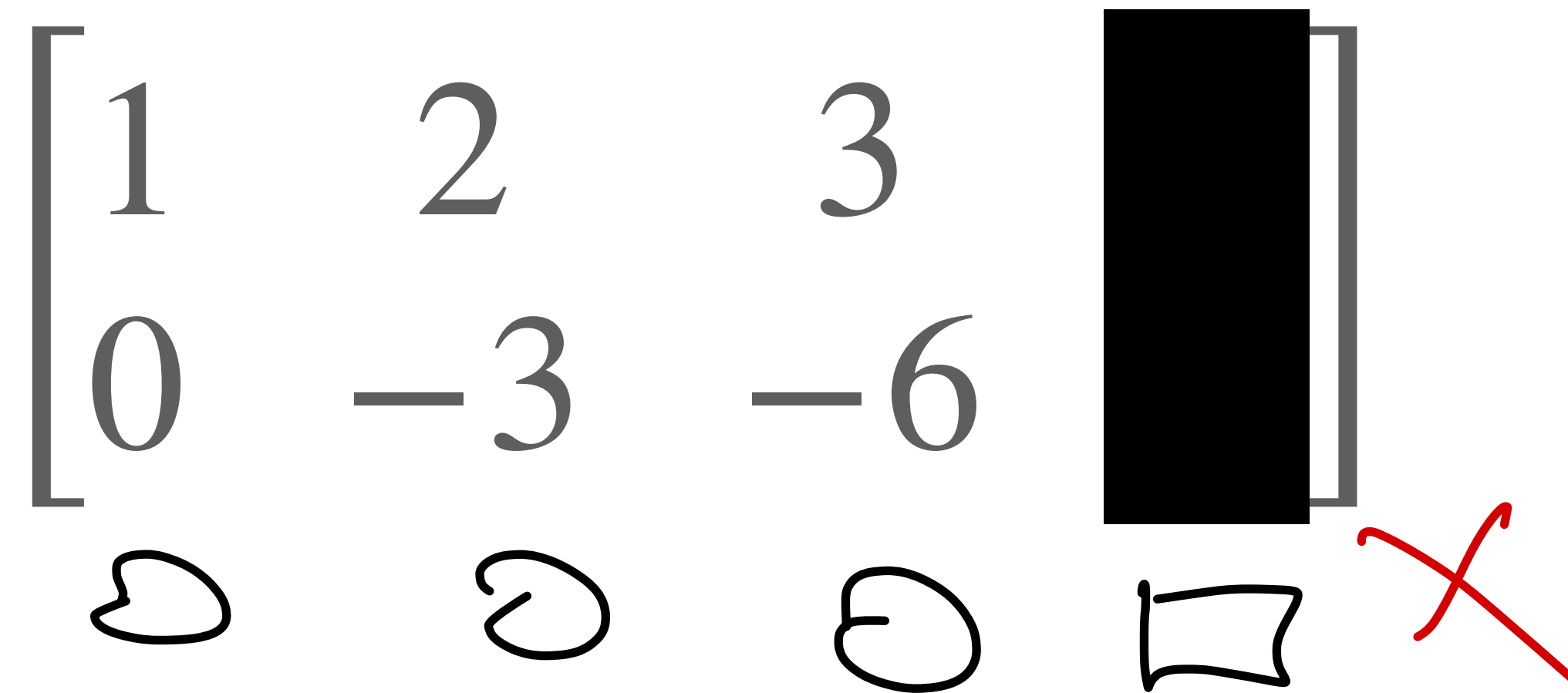
then we reduce it to echelon form

$$\begin{bmatrix} 1 & 2 & 3 & \blacksquare \\ 0 & -3 & -6 & \blacksquare \end{bmatrix}$$

Does it have a solution?

A Thought Experiment

then we reduce it to echelon form

$$\begin{bmatrix} 1 & 2 & 3 & \blacksquare \\ 0 & -3 & -6 & \blacksquare \\ 0 & 0 & 0 & \square \end{bmatrix}$$


Yes. It doesn't have an inconsistent row

A Thought Experiment

what about this system?

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 2 & 4 \end{bmatrix} \begin{bmatrix} \\ \end{bmatrix}$$

A Thought Experiment

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what about this system?

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \\ \end{bmatrix}$$

A Thought Experiment

$$\begin{bmatrix} 1 & 1 & 2 & 4 \\ 2 & 1 & 4 & 8 \end{bmatrix}$$

what about this system?

vs.

$$\begin{bmatrix} 1 & 1 & 2 & 4 \\ 2 & 1 & 4 & 7 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 2 & \blacksquare \\ 0 & 0 & 0 & \blacksquare \end{bmatrix}$$

it depends...

Pivots and Spanning \mathbb{R}^m

$$\begin{bmatrix} 1 & 2 & 3 & \blacksquare \\ 2 & 1 & 0 & \blacksquare \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & \blacksquare \\ 0 & -3 & -6 & \blacksquare \end{bmatrix}$$

Pivots and Spanning \mathbb{R}^m

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If it doesn't matter what the last column is,
then **every choice must be possible**

$$x_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

Pivots and Spanning \mathbb{R}^m

$$\begin{bmatrix} 1 & 2 & 3 & \blacksquare \\ 2 & 1 & 0 & \blacksquare \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & \blacksquare \\ 0 & -3 & -6 & \blacksquare \end{bmatrix}$$

If it doesn't matter what the last column is,
then **every choice must be possible**

**Every vector in \mathbb{R}^2 can be written as a linear
combination of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} 3 \\ 0 \end{bmatrix}$**

Spanning \mathbb{R}^m

Theorem. For any $m \times n$ matrix, the following are logically equivalent

1. For every \mathbf{b} in \mathbb{R}^m , $A\mathbf{x} = \mathbf{b}$ has a solution
2. The columns of A span \mathbb{R}^m
3. A has a pivot position in every row

Spanning \mathbb{R}^m

Theorem. For any $m \times n$ matrix, the following are logically equivalent

1. For every \mathbf{b} in \mathbb{R}^m , $A\mathbf{x} = \mathbf{b}$ has a solution
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HOW TO: Spanning \mathbb{R}^m

Question. Does the set of vectors $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$ from \mathbb{R}^m span all of \mathbb{R}^m ?

Solution. Reduce $[\mathbf{a}_1 \ \mathbf{a}_2 \ \dots \ \mathbf{a}_n]$ to echelon form and check if every row has a pivot

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Solution. Reduce $[\mathbf{a}_1 \ \mathbf{a}_2 \ \dots \ \mathbf{a}_n]$ to echelon form and check if every row has a pivot

!! We only need the echelon form !!

Question

Do $\begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 2023 \end{bmatrix}$ span all of \mathbb{R}^3 ?

Answer: No

the matrix

$$\begin{bmatrix} 2 & 0 \\ 2 & 1 \\ 3 & 2023 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

cannot have more than 2 pivot positions

Not spanning \mathbb{R}^m

$$\begin{bmatrix} 1 & 1 & 2 & \blacksquare \\ 2 & 2 & 4 & \blacksquare \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 & \blacksquare \\ 0 & 0 & 0 & \blacksquare \end{bmatrix}$$

Not spanning \mathbb{R}^m

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In this case the choice matters

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We can't make the last column $[0 \ 0 \ 0 \ \blacksquare]$ for nonzero \blacksquare

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In this case the choice matters

We can't make the last column $[0 \ 0 \ 0 \ \blacksquare]$ for nonzero \blacksquare

But we can make the last column parameters to find equations that must hold

Not spanning \mathbb{R}^m

$$x_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 2 & b_1 \\ 2 & 2 & 4 & b_2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 & b_1 \\ 0 & 0 & 0 & b_2 - 2b_1 \end{bmatrix}$$

$$\xrightarrow{R_2 \leftarrow R_2 - 2R_1}$$

Not spanning \mathbb{R}^m

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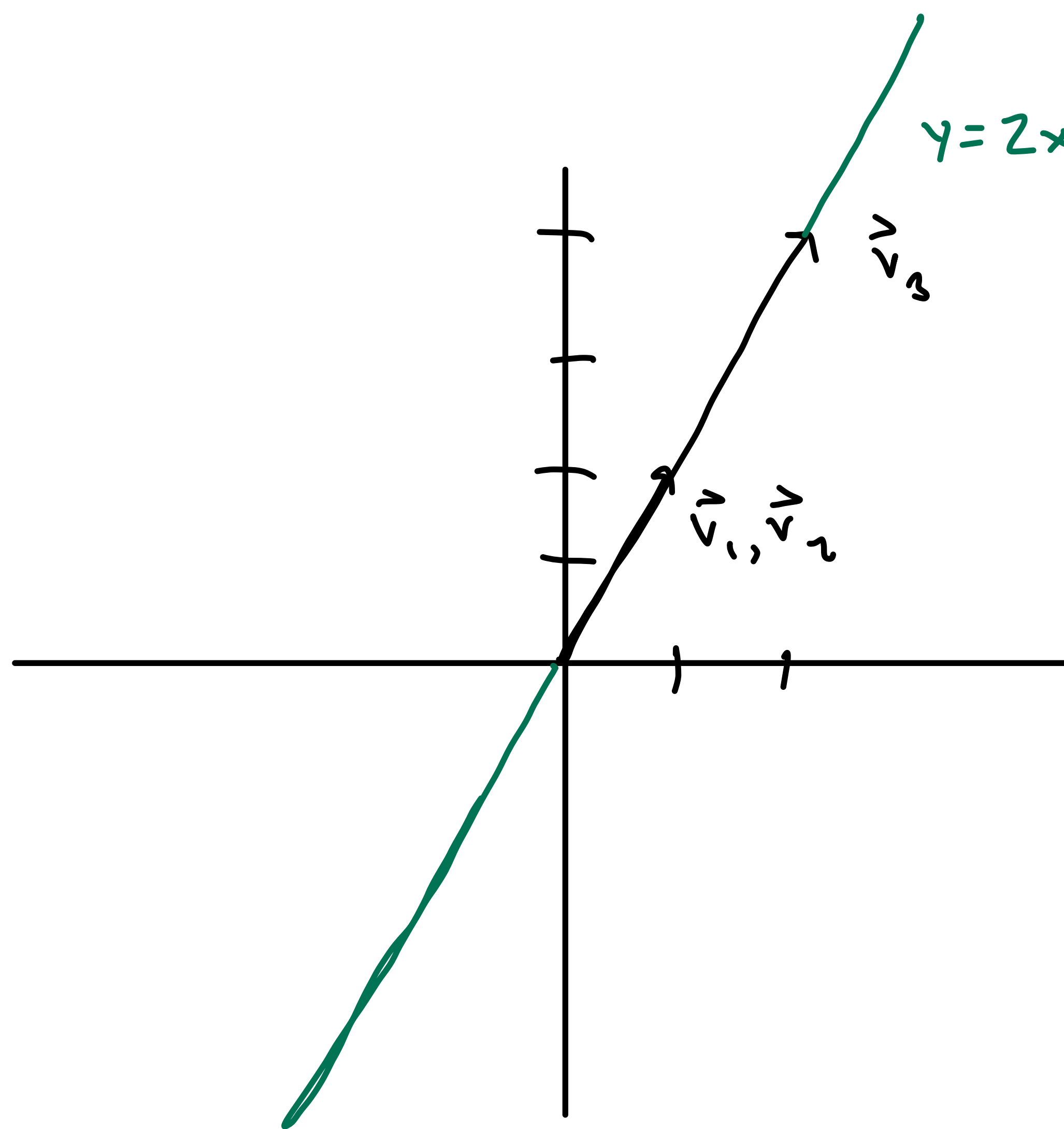
As long as $(-2)b_1 + b_2 = 0$, the system is consistent

Not spanning \mathbb{R}^m

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As long as $(-2)b_1 + b_2 = 0$, the system is consistent

**This gives use a linear equation which describes
the span of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$**



$$b_2 - 2b_1 = 0$$

$$y - 2x = 0$$

$$y = 2x$$

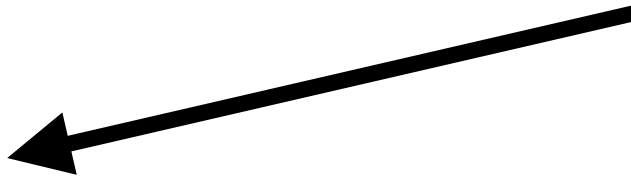
Question (Understanding Check)

True or False, the echelon form of any matrix has at most one row of the form $[0 \ 0 \ \dots \ 0 \ \blacksquare]$ where \blacksquare is nonzero.

Answer: True

$$\begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

leading
entry not
to the
right



this is not in echelon form

Question (More Challenging)

Give a linear equation for the span of the
vectors $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}$.

Answer

$$\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \quad \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & b_1 \\ 2 & -1 & b_2 \\ 0 & -1 & b_3 \end{bmatrix} \sim$$

$$\begin{bmatrix} * & * & \emptyset \\ \emptyset & + & \emptyset \\ 0 & 0 & \boxed{+} \end{bmatrix}$$

relationship
between b_1, b_2, b_3

Answer

$$\begin{bmatrix} 1 & -1 & b_1 \\ 2 & -1 & b_2 \\ 0 & -1 & b_3 \end{bmatrix}$$

Answer

$$\begin{bmatrix} 1 & -1 & b_1 \\ 0 & 2 & b_2 - 2b_1 \\ 0 & -1 & b_3 \end{bmatrix}$$

$$R_2 \leftarrow R_2 - 2R_1$$

Answer

$$\begin{bmatrix} 1 & -1 & b_1 \\ 0 & 2 & b_2 - 2b_1 \\ 0 & 0 & b_3 + (1/2)(b_2 - 2b_1) \end{bmatrix}$$

$$R_3 \leftarrow R_3 - (1/2)R_2$$

Answer

$$\begin{bmatrix} 1 & -1 & b_1 \\ 0 & 2 & b_2 - 2b_1 \\ 0 & 0 & b_3 + (1/2)(b_2 - 2b_1) \end{bmatrix}$$

$$R_3 \leftarrow R_3 - (1/2)R_2$$

Answer

$$0 = b_3 + (1/2)(b_2 - 2b_1)$$

Answer

$$b_1 - (1/2)b_2 - b_3 = 0$$

Answer

$$x_1 - (1/2)x_2 - x_3 = 0$$

Taking Stock

Four Representations

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}$$

augmented matrix

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

matrix equation

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

system of linear equations

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

vector equation

Four Representations

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}$$

augmented matrix

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

matrix equation

they all have the same solution sets

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

system of linear equations

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

vector equation

Summary

Matrix and vectors can be multiplied together to get new vectors

The matrix equation is another representation of systems of linear equations

Looking forward: Matrices *transform* vectors