# Linear Transformations

## Geometric Algorithms
## Lecture 7

CAS CS 132

# Practice Problem

Find three vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ in $\mathbb{R}^3$ such that

» every pair of vectors (i.e., $\{\mathbf{v}_1, \mathbf{v}_2\}$, $\{\mathbf{v}_1, \mathbf{v}_3\}$, $\{\mathbf{v}_2, \mathbf{v}_3\}$) is linearly independent

» $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent

# Objectives

» Introduce matrix transformations

» Define linear transformations

» Start looking at the geometry of linear transformations

# Keywords

Transformations

Domain, Codomain

Image, Range

Matrix Transformations

Linear Transformations

Additivity, Homogeneity

Dilation, Contraction, Shearing, Rotation

# Recap

# Recap: Homogenous Linear Systems

**Definition.** A linear system is ***homogeneous*** if it can be expressed as

$$A\mathbf{x} = \mathbf{0}$$

# Recap: Linear Independence

**Definition.** A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is ***linearly independent*** if the vectors equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \ldots + x_n\mathbf{v}_n = \mathbf{0}$$

has exactly one solution (the trivial solution)

# Recap: Linear Dependence

**Definition.** A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is ***linearly dependent*** if the vectors equation

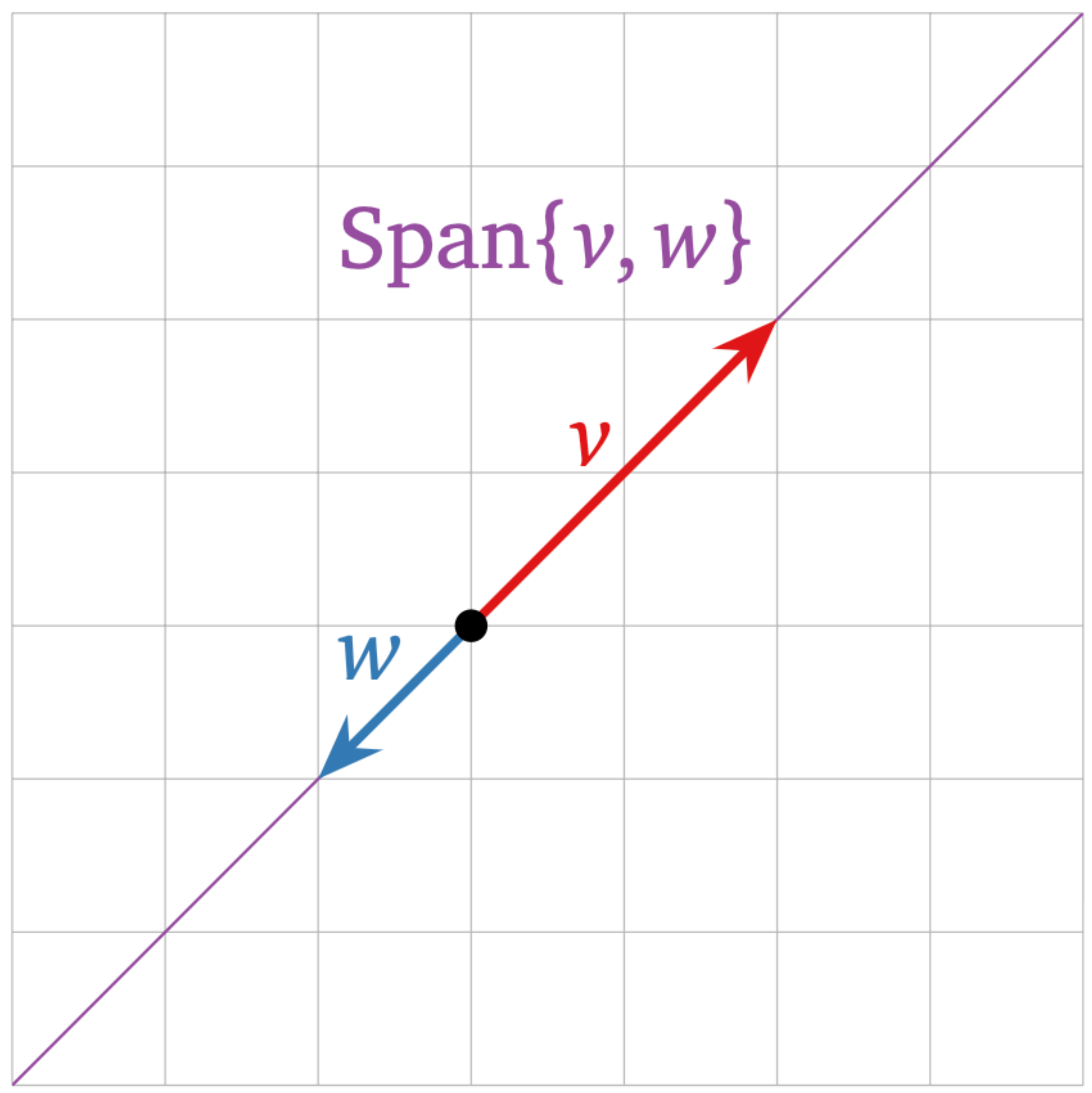
$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \ldots + x_n\mathbf{v}_n = \mathbf{0}$$

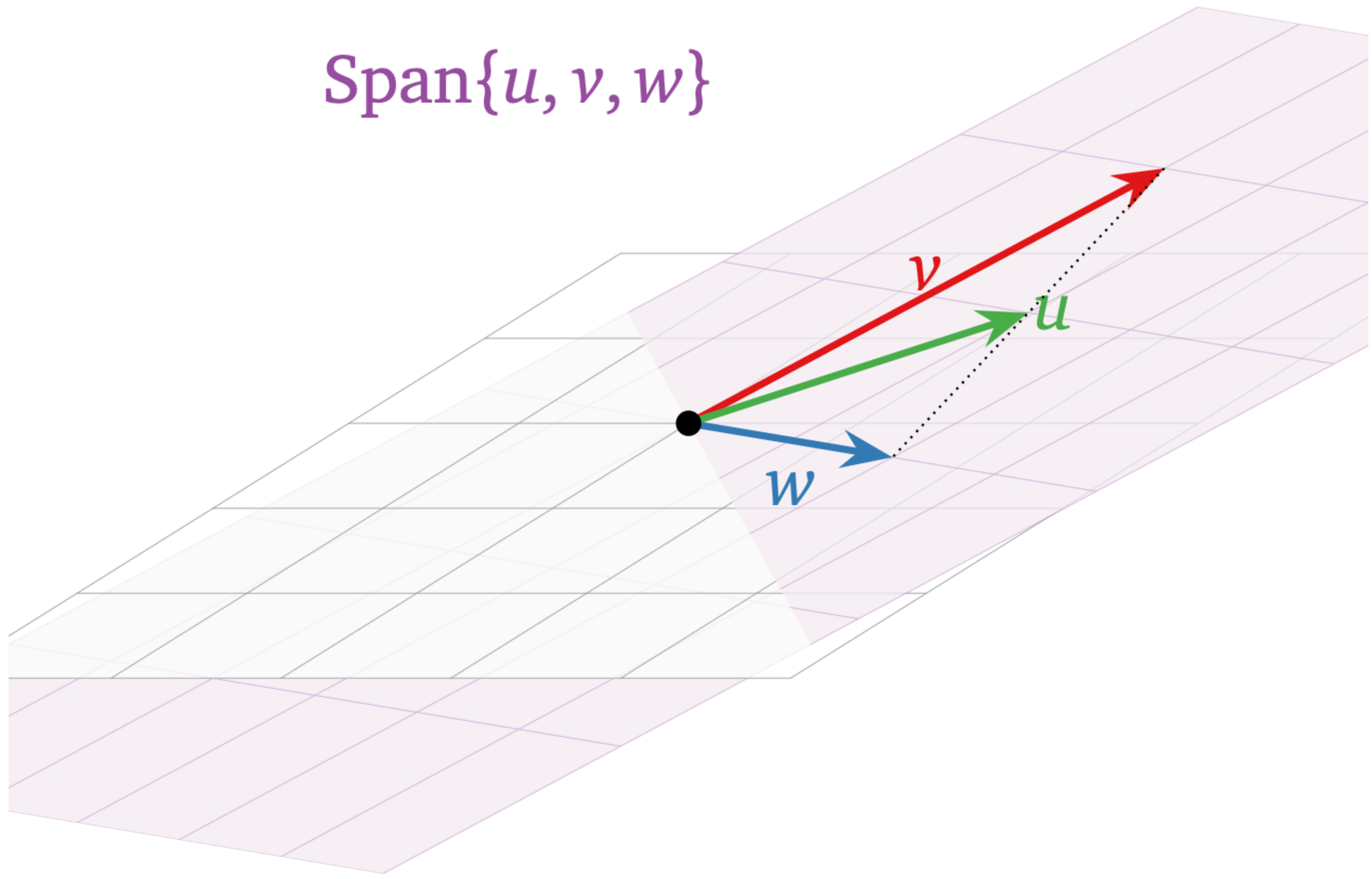
has a nontrivial solution

# Recap: Linear Dependence

**Definition.** A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is ***linearly dependent*** if one of its vectors can be written as a linear combination of the others (not including itself)

# Linear Dependence (Pictorally)



# Recall: Linear Dependence Relation

**Definition.** If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly dependent, then a ***linear dependence relation*** is an equation of the form

$$\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \ldots + \alpha_n \mathbf{v}_n = \mathbf{0}$$

A linear dependence relation *witnesses* the linear dependence

# Recap: Increasing Span

**Theorem.** $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly dependent if and only if there is an $i \leq n$,

$$\mathbf{v}_i \in \mathsf{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{i-1}\}$$

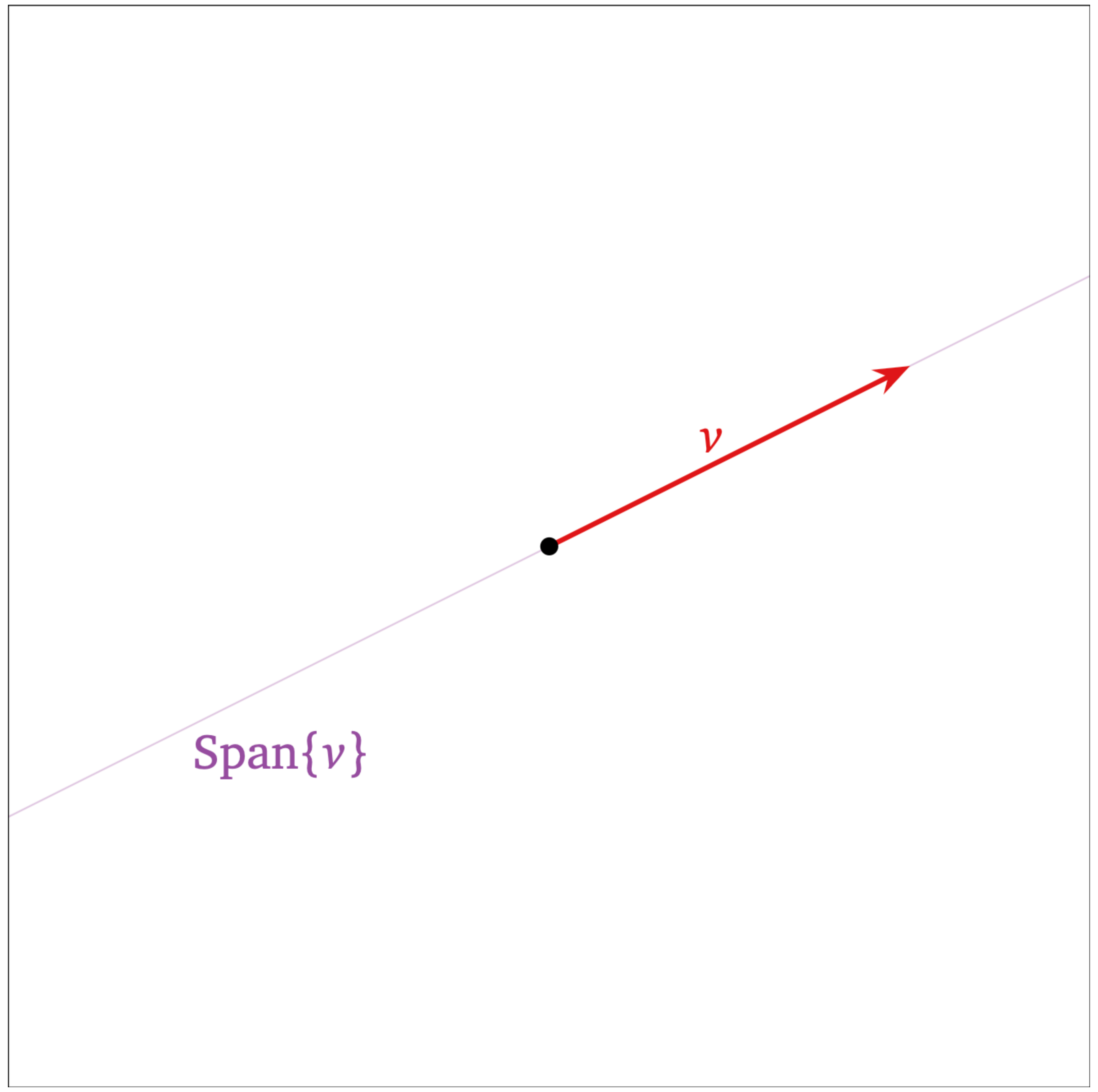
# Recap: Increasing Span

**Theorem.** $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly dependent if and only there is an $i \leq n$,

$$\mathbf{v}_i \in \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{i-1}\}$$

As we add vectors, we'll eventually find one in the span of the preceding ones

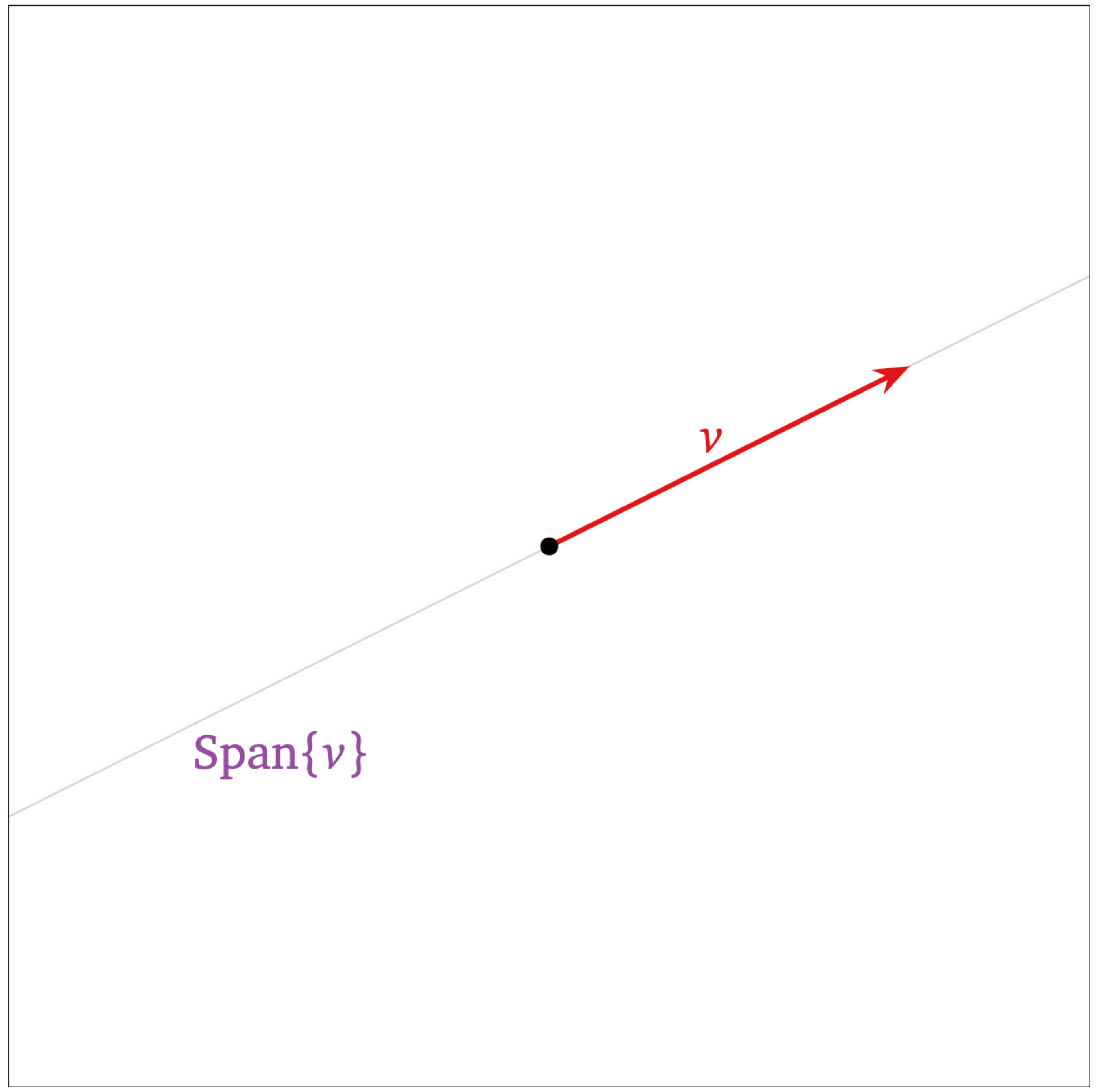
# Recap: Increasing Span



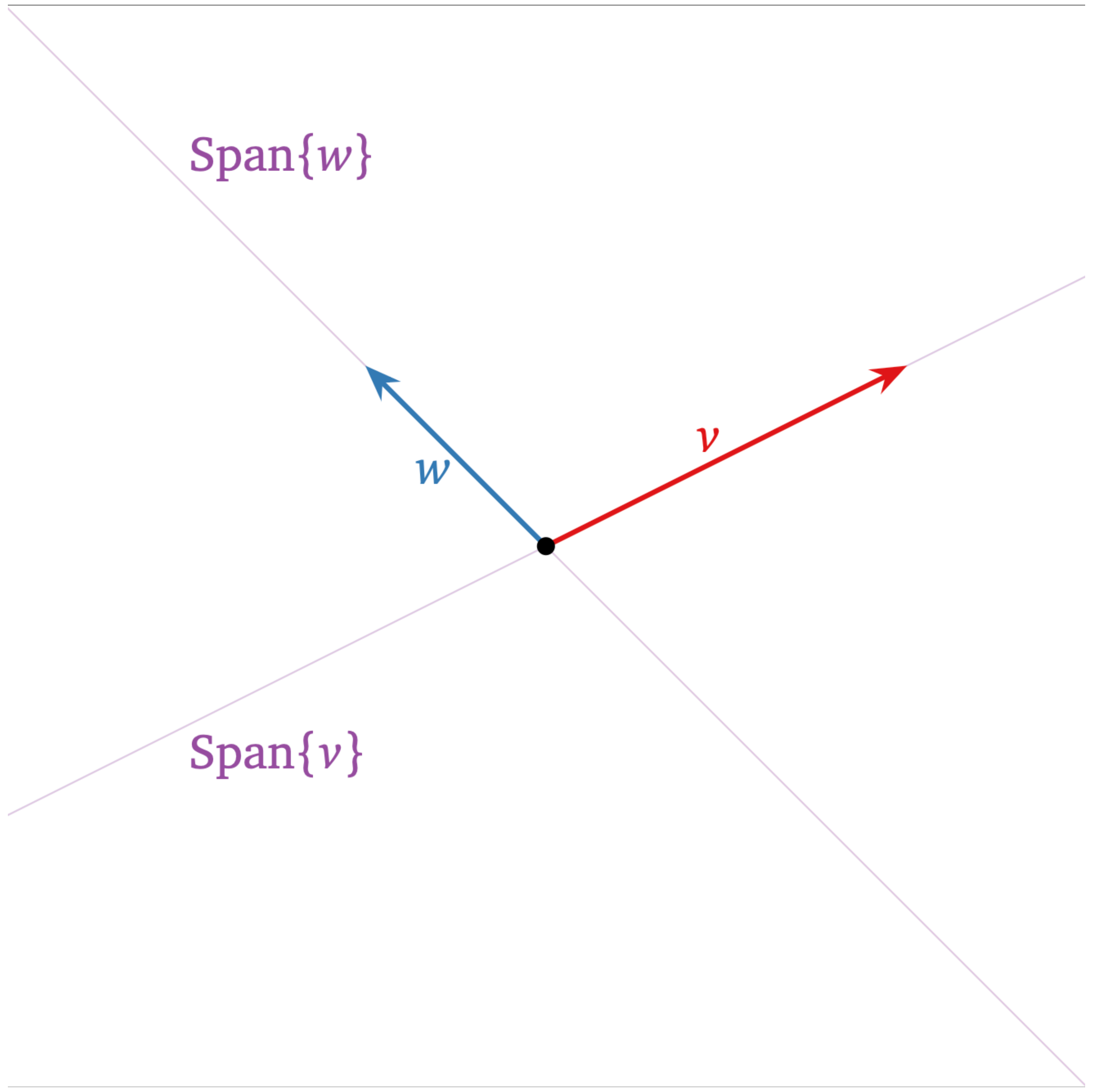

span of 1 vector
a line

image source

# Recap: Increasing Span



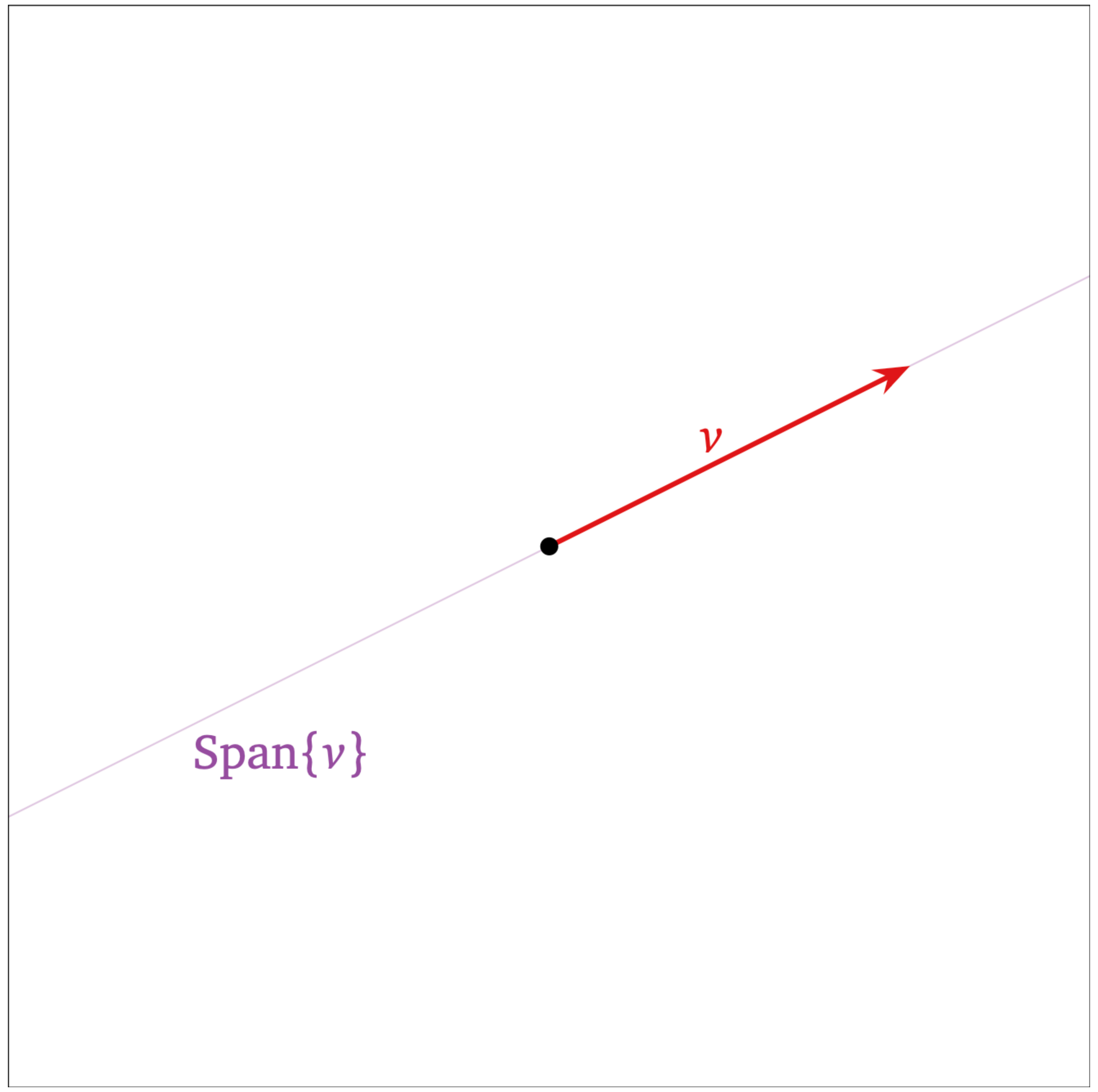

span of 1 vector
a line



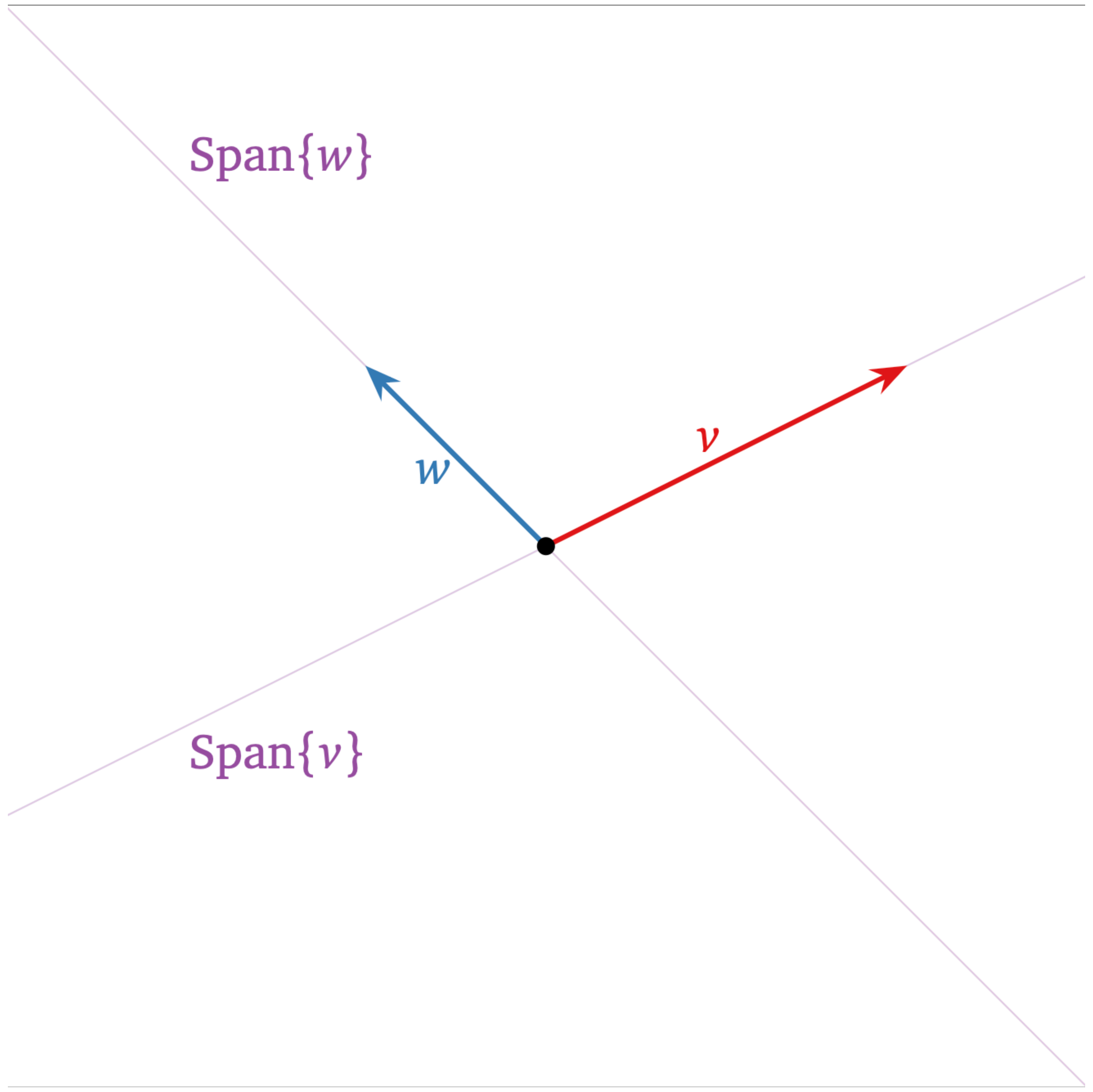

span of 2 vector
a plane

image source

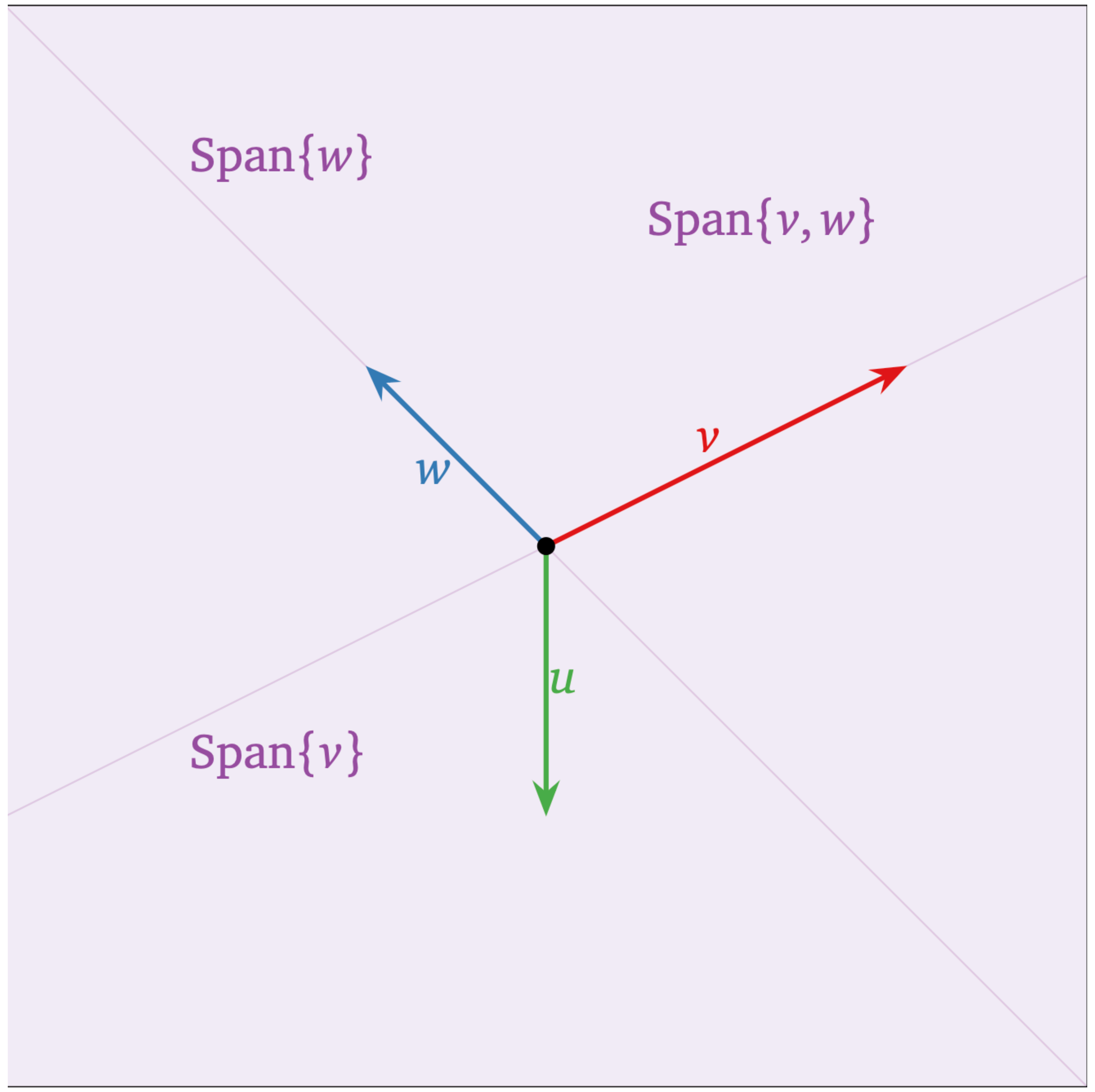
# Recap: Increasing Span




span of 1 vector
a line




span of 2 vector
a plane




span of 3 vector
still a plane

image source

# Recap: Pivots and Linear Dependence

**Theorem.** The columns of a matrix $A$ are linearly independent if and only if $A$ has a pivot in every <u>column</u>

Free variables allow for infinitely many (nontrivial) solution

# Recap: Example

$$\mathbf{v}_1 = \begin{bmatrix} -4 \\ 4 \\ 2 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} -3 \\ 6 \\ -3 \end{bmatrix} \qquad \mathbf{v}_3 = \begin{bmatrix} -5 \\ 8 \\ -2 \end{bmatrix}$$

The reduced echelon form of $[\mathbf{v}_1 \;\; \mathbf{v}_2 \;\; \mathbf{v}_3]$ is

$$\begin{bmatrix} 1 & 0 & 0.5 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

column without a pivot

# Recap: Linear Independence and Full Span

The columns of a $(m \times n)$ matrix span all of $\mathbb{R}^n$ if there is a pivot in every row

The columns of a matrix are linearly independent if there is a pivot in every column

Don't confuse these!

# Matrix Transformations

# Recall: Spans (with Matrices)

**Definition.** The ***span*** of a set of vectors is the set of all possible linear combinations

$$\text{span}\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\} = \{[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \cdots \quad \mathbf{a}_n]\,\mathbf{v} : \mathbf{v} \in \mathbb{R}^n\}$$

# Matrices as Transformations

Matrices allow us to *transform* vectors into vectors the span of its columns

$$\mathbf{x} \mapsto A\mathbf{x}$$

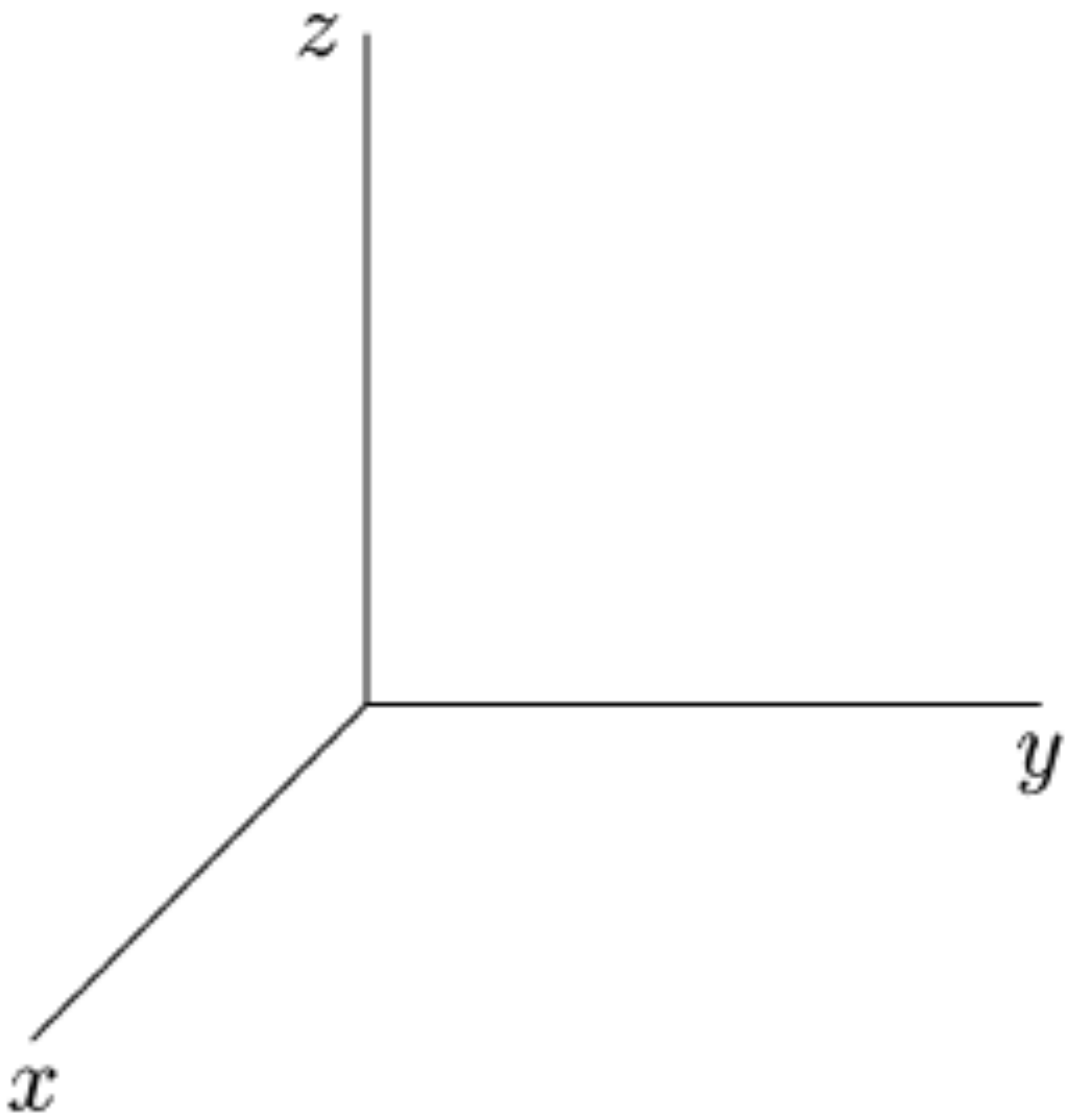
map a vector $\mathbf{v}$ to the vector $A\mathbf{v}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} =$$

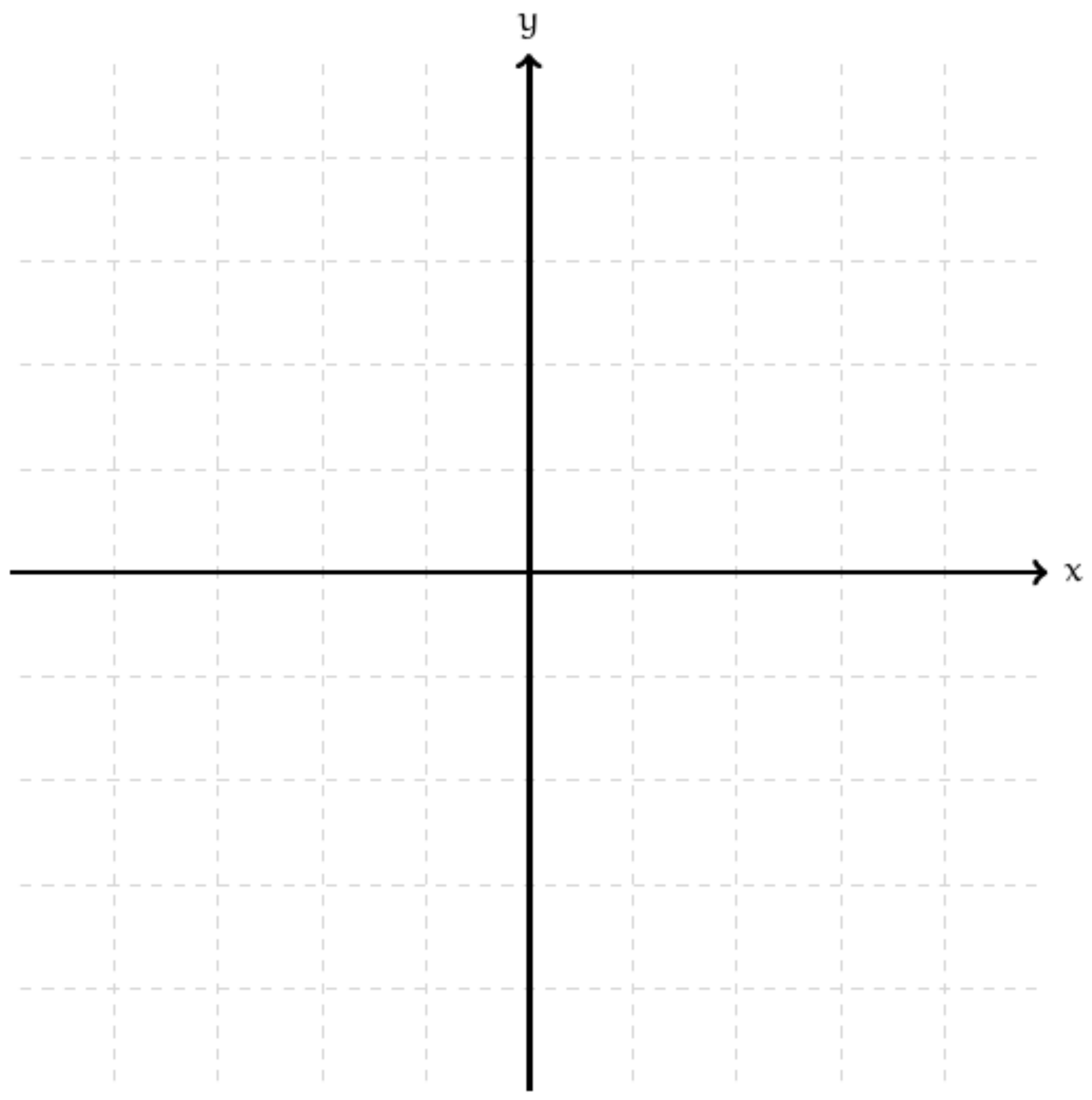
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} =$$

$$\begin{bmatrix}1 & 0 & 0\\ 0 & 2 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} = \begin{bmatrix}x_1\\ 2x_2\end{bmatrix}$$




$$\begin{bmatrix}1 & 0 & 0\\ 0 & 2 & 0\end{bmatrix}\mathbf{x}$$



!!Important!!

The vector may be a different size after translation

# Recall: Matrix-Vector Multiplication and Dimension

matrix-vector multiplication only works if the number of *columns* of the matrix matches the dimension of the vector

$$\begin{array}{cc} & n \\ m & \begin{bmatrix} * & \cdots & * \\ * & \cdots & * \\ \vdots & \ddots & \vdots \\ * & \cdots & * \\ * & \cdots & * \end{bmatrix} \end{array} \; \begin{array}{cc} n & \begin{bmatrix} * \\ \vdots \\ * \end{bmatrix} \end{array} = \begin{array}{cc} m & \begin{bmatrix} * \\ * \\ \vdots \\ * \\ * \end{bmatrix} \end{array}$$

$$(m \times n) \qquad \mathbb{R}^n \qquad \mathbb{R}^m$$

# Motivating Questions

What kind of functions can we define this way?

How do we interpret what the transformation does to a set of vectors?

How does this relate to matrix equations?

# Motivating Questions

What kind of functions can we define this way?

How do we interpret what the transformation does to a set of vectors?

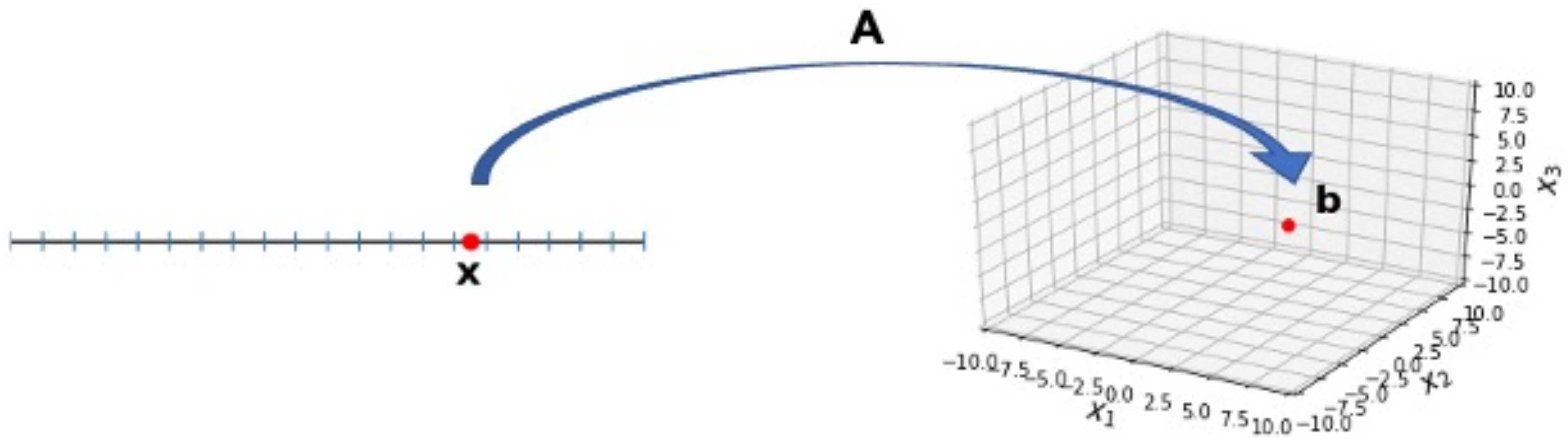
How does this relate to matrix equations?

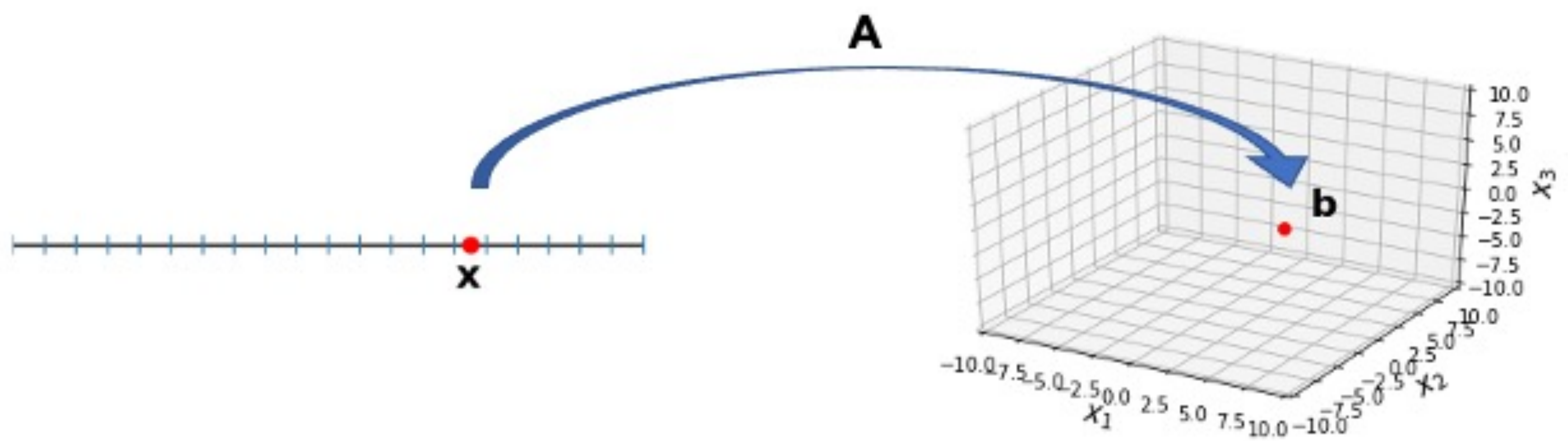
# A New Interpretation of the Matrix Equation

| | | |
|---|---|---|
| $A\mathbf{x} = \mathbf{b}$? | $\equiv$ | is there a vector which $A$ transforms into $\mathbf{b}$? |
| Solve $A\mathbf{x} = \mathbf{b}$ | $\equiv$ | find a vector which $A$ transforms into $\mathbf{b}$ |

# Question

Suppose a matrix transforms a vector according to the following picture. What is the size of the matrix?





A
x
b
$x_1$
$x_2$
$x_3$

$$\mathbb{R}^n \rightarrow \mathbb{R}^n$$

Mapping between the same space can be viewed as a way of moving around points

# Transformations

# Transformations in General

**Definition.** A ***transformation*** $T$ from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a function which maps every vector $\mathbf{v}$ in $\mathbb{R}^n$ to a vector $T(\mathbf{v})$ in $\mathbb{R}^m$

# Transformations in General

**Definition.** A ***transformation*** $T$ from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a function which maps every vector $\mathbf{v}$ in $\mathbb{R}^n$ to a vector $T(\mathbf{v})$ in $\mathbb{R}^m$

$$T : \mathbb{R}^n \to \mathbb{R}^m$$

# Transformations in General

**Definition.** A ***transformation*** $T$ from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a function which maps every vector $\mathbf{v}$ in $\mathbb{R}^n$ to a vector $T(\mathbf{v})$ in $\mathbb{R}^m$

$$T : \underbrace{\boxed{\mathbb{R}^n}}_{\text{domain}} \to \underbrace{\boxed{\mathbb{R}^m}}_{\text{codomain}}$$

# Image and Range

# Image and Range

**Definition.** For a vector $\mathbf{v}$, the ***image*** of $\mathbf{v}$ under the transformation $T$ is the vector $T(\mathbf{v})$

# Image and Range

**Definition.** For a vector $\mathbf{v}$, the ***image*** of $\mathbf{v}$ under the transformation $T$ is the vector $T(\mathbf{v})$

**Definition.** The ***range*** of a transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is the set of all possible images under $T$

# Image and Range

**Definition.** For a vector $\mathbf{v}$, the ***image*** of $\mathbf{v}$ under the transformation $T$ is the vector $T(\mathbf{v})$

**Definition.** The ***range*** of a transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is the set of all possible images under $T$

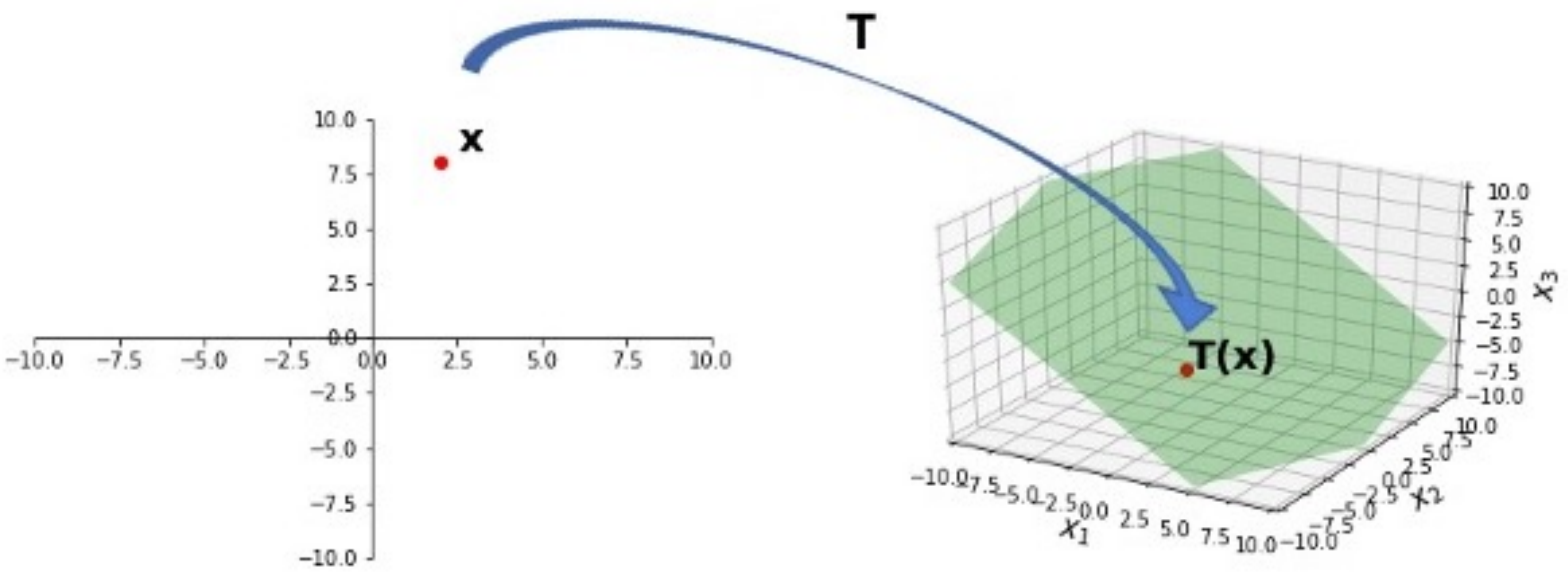
$$\operatorname{ran}(T) = \{T(\mathbf{v}) : v \in \mathbb{R}^n\}$$

$$\text{image of } \mathbf{v} \text{ under } T \equiv \text{output of } T \text{ applied to } \mathbf{v}$$

$$\text{range of } T \equiv \text{all possible output of } T$$

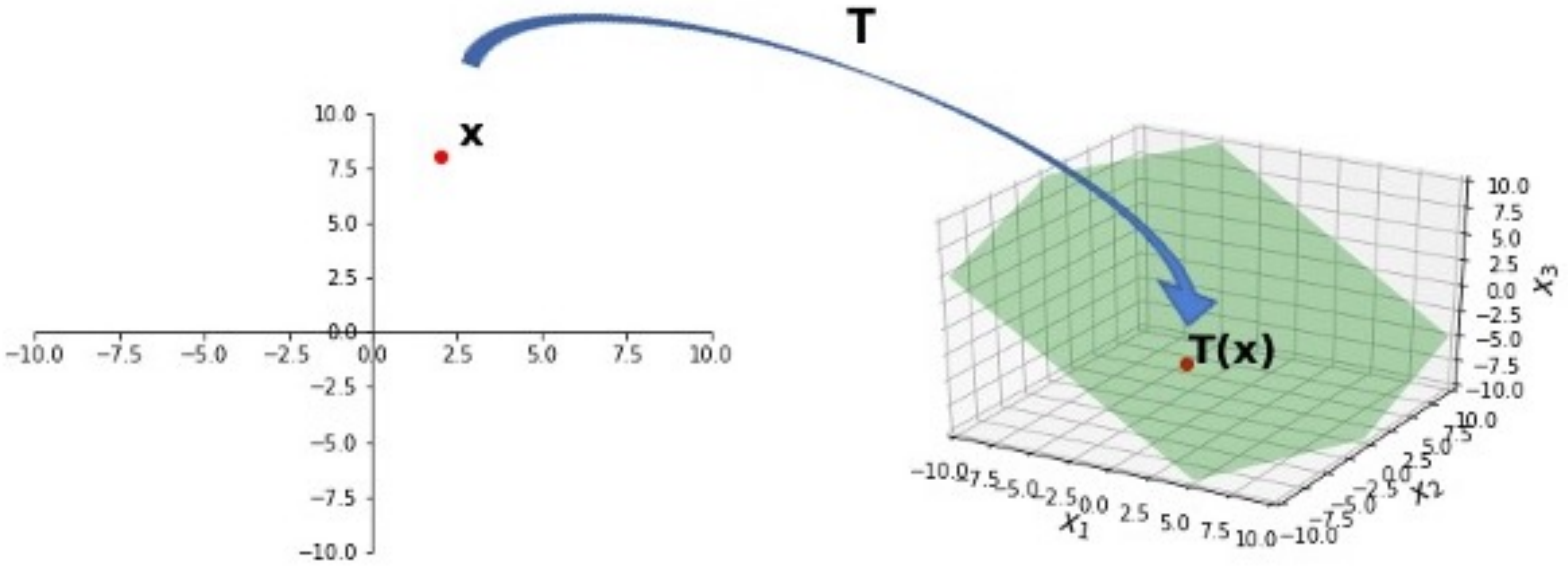
# Codomain and Range

The codomain and range of a transformation may or may not be the same



# Codomain and Range

The codomain and range of a transformation may or may not be the same



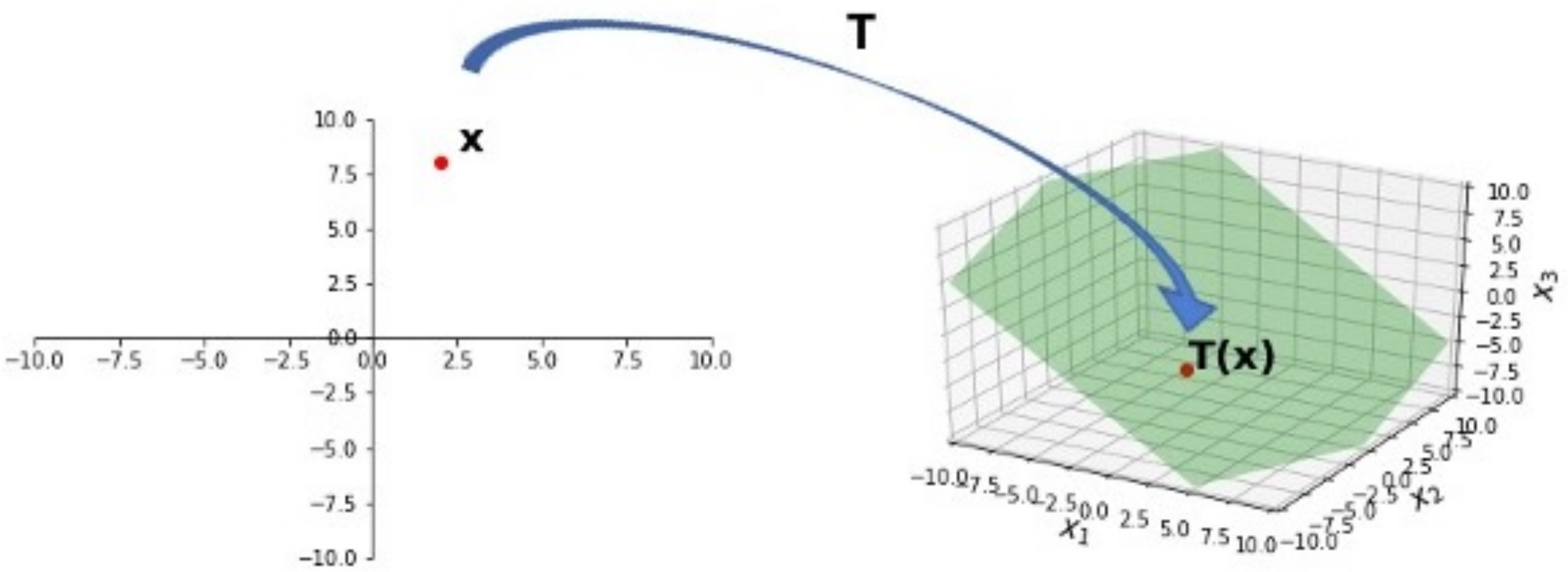

range: just the green plane

domain: $\mathbb{R}^2$

codomain: $\mathbb{R}^3$

# Codomain and Range

The codomain and range of a transformation may or may not be the same




range: just the green plane

domain: $\mathbb{R}^2$

codomain: $\mathbb{R}^3$

The range is always contained in the codomain

# Example

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mapsto \begin{bmatrix} x_1^2 \\ x_2 \\ 0 \end{bmatrix}$$

# Matrix Transformations

# Transformation of a Matrix

# Transformation of a Matrix

The ***transformation of a*** $(m \times n)$ ***matrix*** $A$ is the function $T : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$T(\mathbf{v}) = A\mathbf{v}$$

# Transformation of a Matrix

The ***transformation of a*** $(m \times n)$ ***matrix*** $A$ is the function $T : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$T(\mathbf{v}) = A\mathbf{v}$$

given $\mathbf{v}$, return $A$ multiplied by $\mathbf{v}$

# Transformation of a Matrix

The ***transformation of a*** $(m \times n)$ ***matrix*** $A$ is the function $T : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$T(\mathbf{v}) = A\mathbf{v}$$

given $\mathbf{v}$, return $A$ multiplied by $\mathbf{v}$

e.g. $T(\mathbf{v}) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \mathbf{v}$

# Range and Span

# Range and Span

The span of the columns of a matrix $A$ is the set of all possible *images* under $A$

# Range and Span

The span of the columns of a matrix $A$ is the set of all possible *images* under $A$

$$\text{span}\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\} = \text{ran}([\mathbf{a}_1 \quad \mathbf{a}_2 \quad \ldots \quad \mathbf{a}_n])$$

# Range and Span

The span of the columns of a matrix $A$ is the set of all possible *images* under $A$

$$\mathrm{span}\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\} = \mathrm{ran}([\mathbf{a}_1 \quad \mathbf{a}_2 \quad \cdots \quad \mathbf{a}_n])$$

The transformation of a vector $\mathbf{v}$ under the matrix $A$ always lies in the span of its columns

# Motivating Questions

What kind of functions can we define this way?

How do we interpret what the transformation does to a set of vectors?

How does this relate to matrix equations?

# Linear Transformations

# Recall: Algebraic Properties

Matrix–vector multiplication satisfies the following two properties:

1. $A(\mathbf{u}+\mathbf{v}) = A\mathbf{u} + A\mathbf{v}$ (additivity)

2. $A(c\mathbf{v}) = c(A\mathbf{v})$ (homogeneity)

# Example

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \left( \begin{bmatrix} 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) =$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} =$$

# Example

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \left( 2 \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right) =$$

# Linear Transformations

**Definition.** A transformation $T:\mathbb{R}^m \to \mathbb{R}^n$ is ***linear*** if it satisfies the following two properties

1. $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ (additivity)

2. $T(c\mathbf{v}) = cT(\mathbf{v})$ (homogeneity)

# Linear Transformations

**Definition.** A transformation $T : \mathbb{R}^m \to \mathbb{R}^n$ is ***linear*** if it satisfies the following two properties

1. $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ (additivity)

2. $T(c\mathbf{v}) = cT(\mathbf{v})$ (homogeneity)

Matrix transformations are linear transformations

# Example: Identity

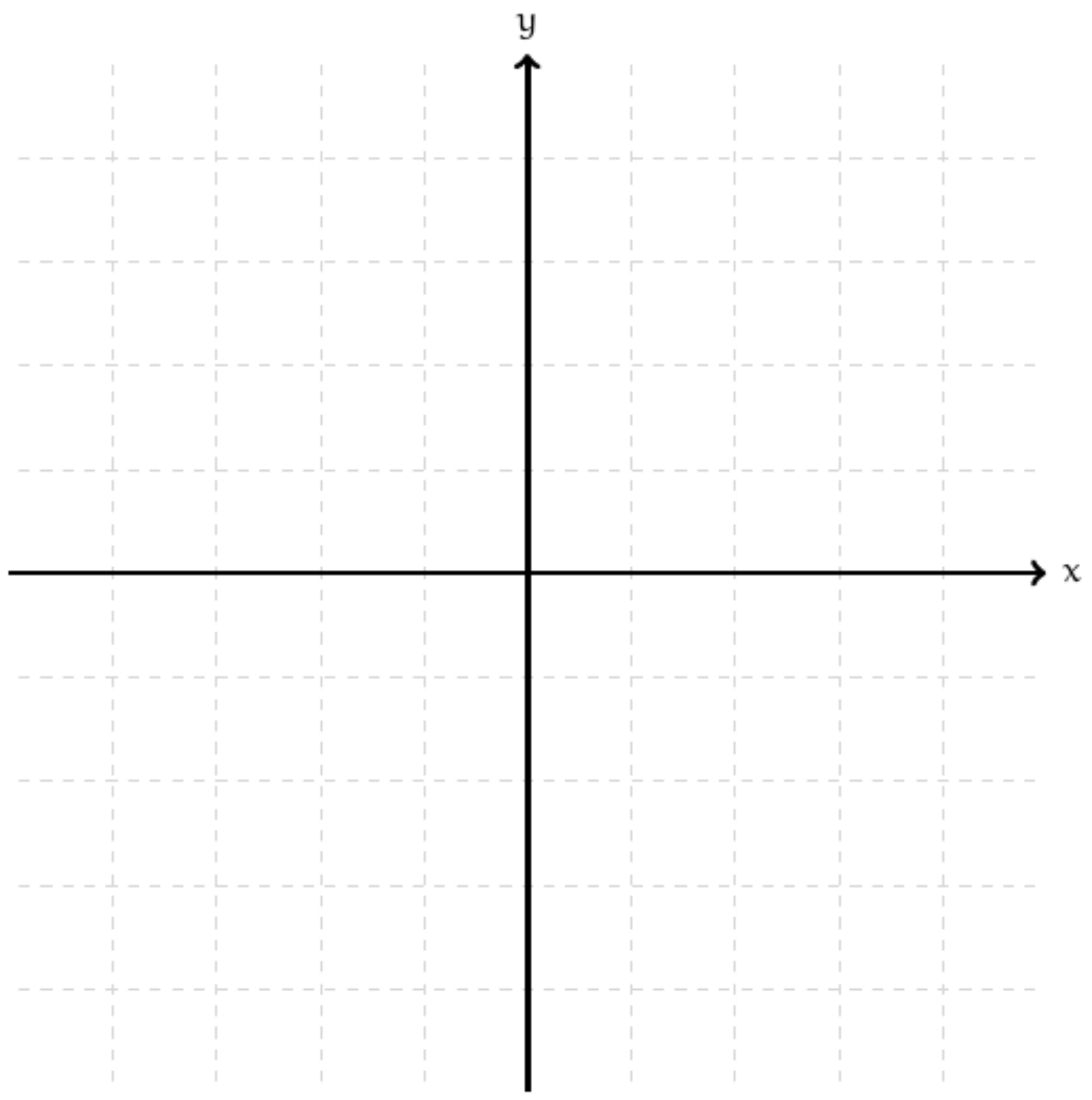
$$T(\mathbf{v}) = \mathbf{v}$$

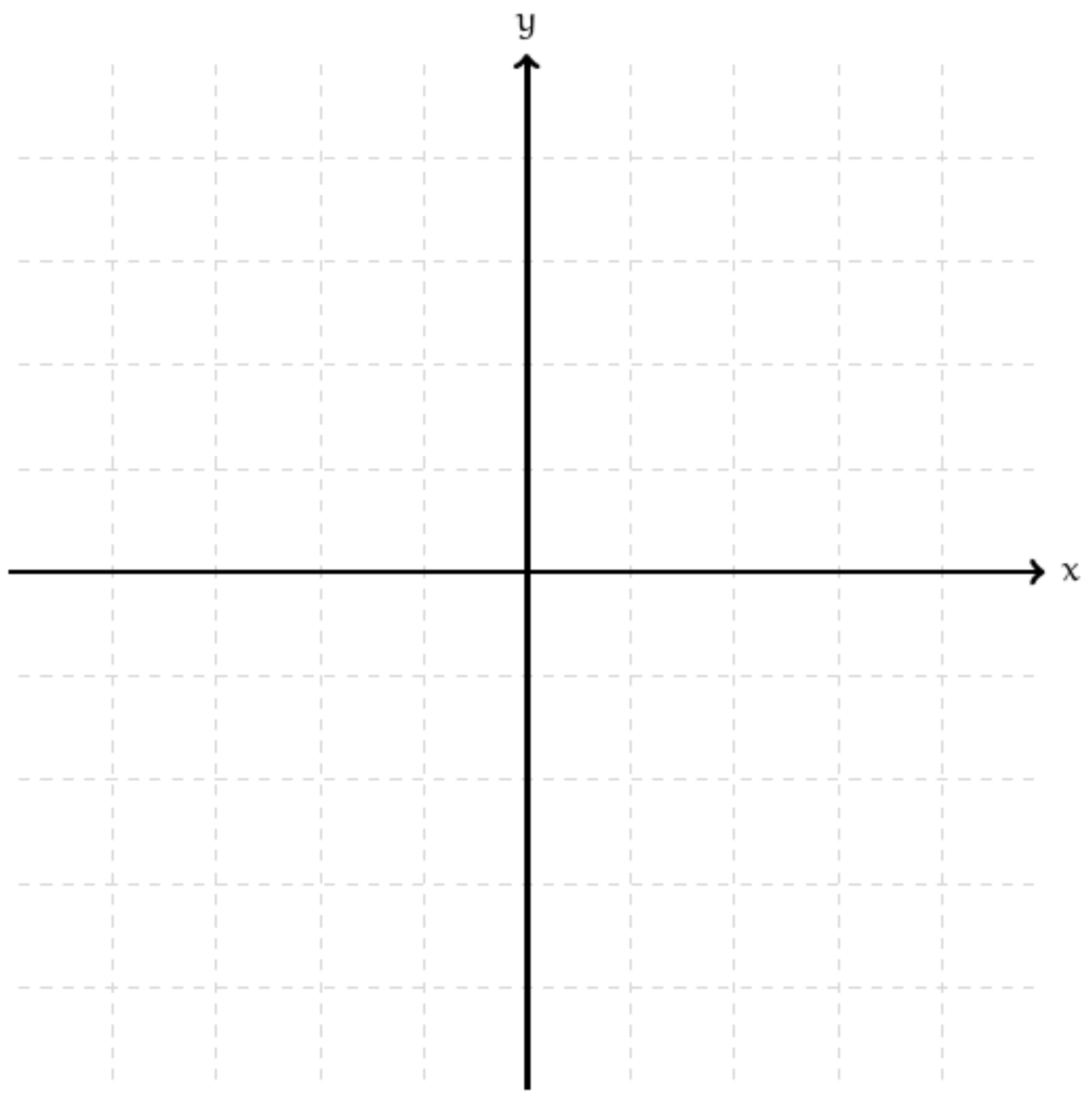
# Example: Zero

$$T(\mathbf{v}) = \mathbf{0}$$

# Example: Rotation

We'll see this on Thursday, but we can reason about it geometrically for now.



# Example: Indefinite Integrals

$$T(f) = \int f(x)dx$$

Disclaimer:
Advanced
Material

the same goes for derivatives
(how are functions vectors???)

# Example: Expectation

$$T(X) = \mathbb{E}[X]$$

Disclaimer: Advanced Material

This is exactly <u>linearity</u> of expectation.

(how are random variables vectors???)

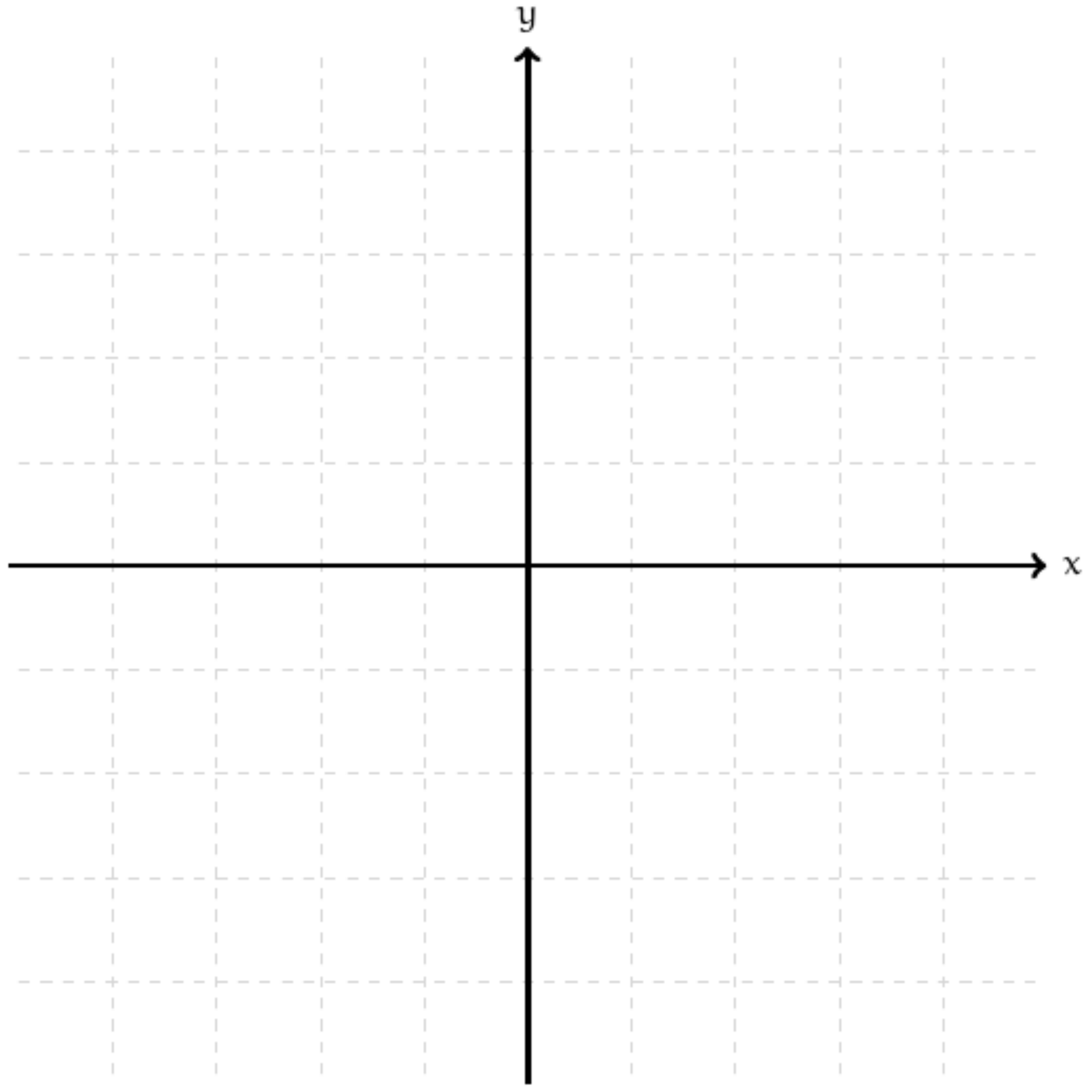
# Non-Example: Squares

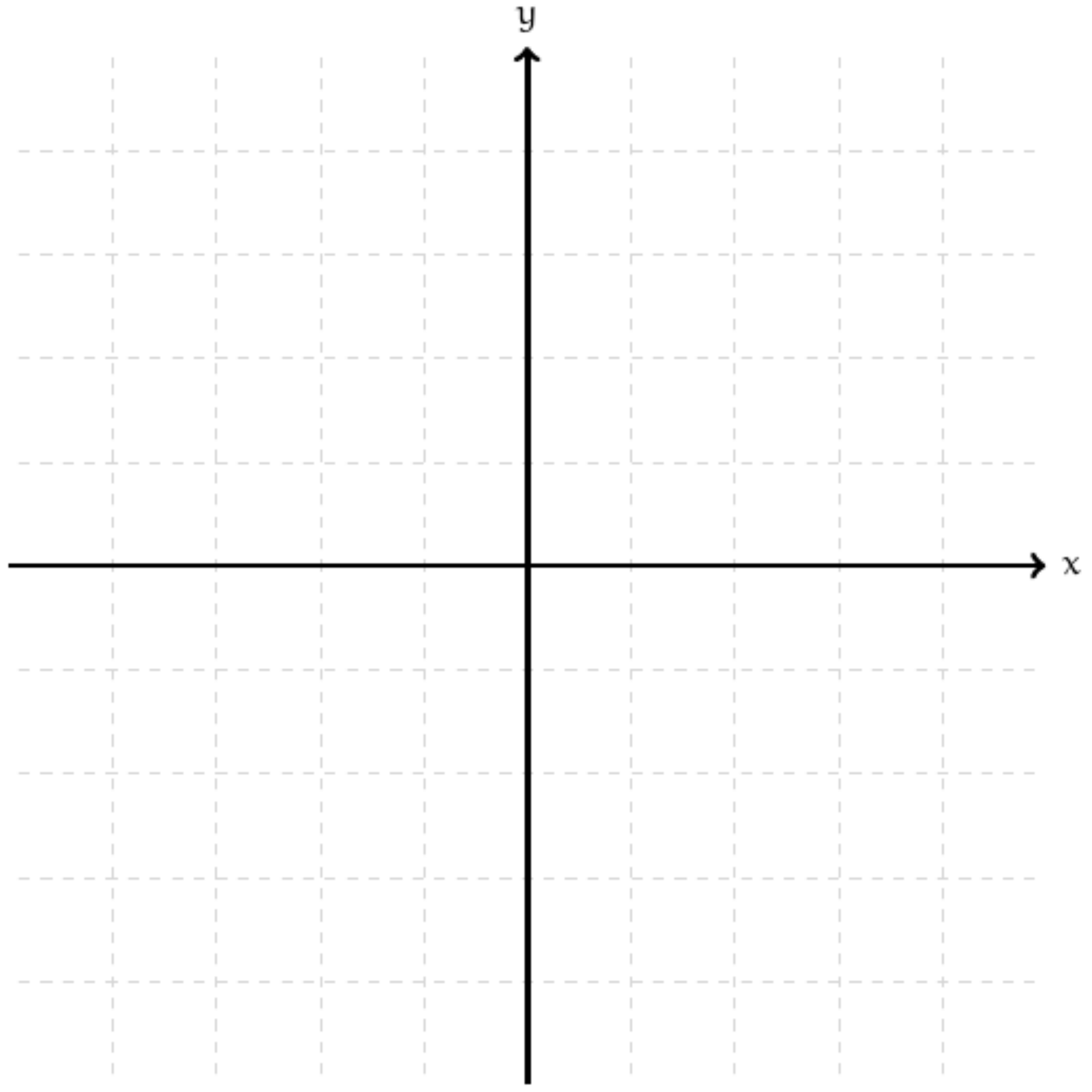
$$T(x) = x^2$$

Note that $T : \mathbb{R}^1 \to \mathbb{R}^1$

# Non-Example: Translation

$$\mathbf{x} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$



# Properties of Linear Transformations

# The Zero Vector

$$T(\mathbf{0}) = \;???$$

# The Zero Vector

$$T(\mathbf{0}) = \mathbf{0}$$

# The Zero Vector

$$T(\mathbf{0}) = \mathbf{0}$$

The zero vector is *fixed* by linear transformations

# The Zero Vector

$$T(\mathbf{0}) = \mathbf{0}$$

Note: These may be different dimensions!

The zero vector is *fixed* by linear transformations

# A Single Condition

$$T(a\mathbf{v} + b\mathbf{u}) = aT(\mathbf{v}) + bT(\mathbf{u})$$

# A Single Condition

$$T(a\mathbf{v} + b\mathbf{u}) = aT(\mathbf{v}) + bT(\mathbf{u})$$

We can combine our linearity conditions

# A Single Condition

**Theorem.** A transformation $T : \mathbb{R}^m \to \mathbb{R}^n$ is linear if and only if for any vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^m$ and any real numbers $a$ and $b$,

$$T(a\mathbf{u} + b\mathbf{v}) = aT(\mathbf{u}) + bT(\mathbf{v})$$

# A Single Condition

**Theorem.** A transformation $T : \mathbb{R}^m \to \mathbb{R}^n$ is linear if and only if for any vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^m$ and any real numbers $a$ and $b$,

$$T(a\mathbf{u} + b\mathbf{v}) = aT(\mathbf{u}) + bT(\mathbf{v})$$

It's often easiest to show this single condition

# Linear Combinations

$$T(a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \ldots + a_n\mathbf{v}_n)$$

And we can generalize to any linear combination

# Geometry of Matrix Transformations

# Motivating Questions

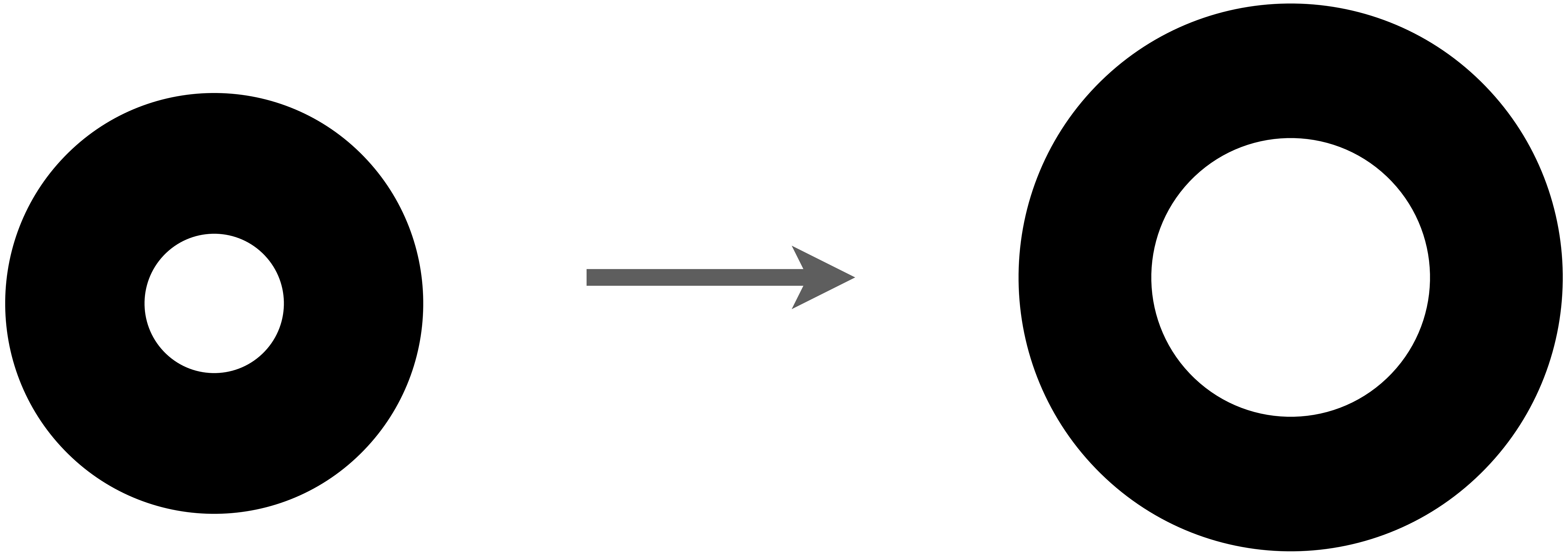
What kind of functions can we define this way?

How do we interpret what the transformation does to a set of vectors?

How does this relate to matrix equations?

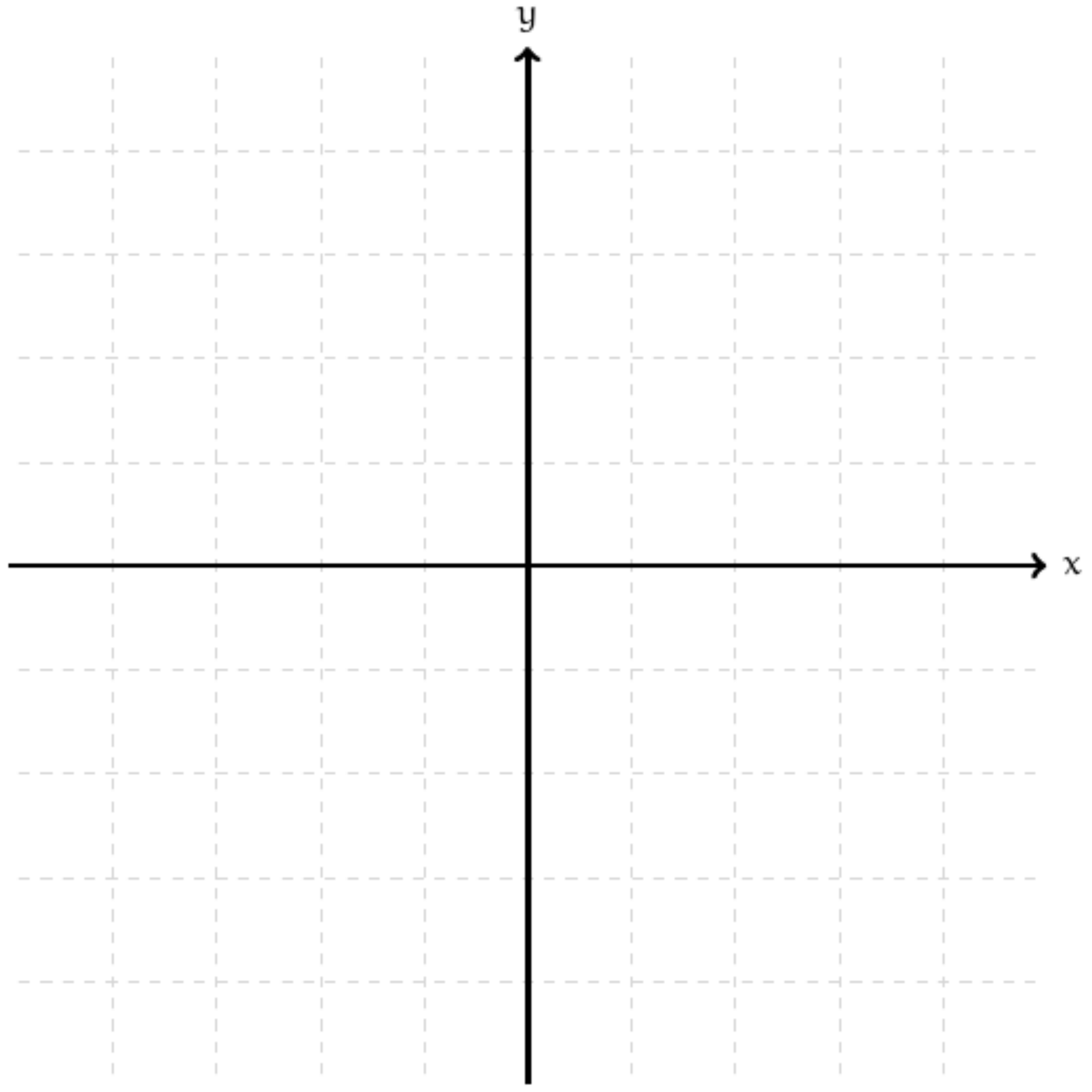
Matrix transformations change the "shape" of a set of set of vectors (points)

# Example: Dilation

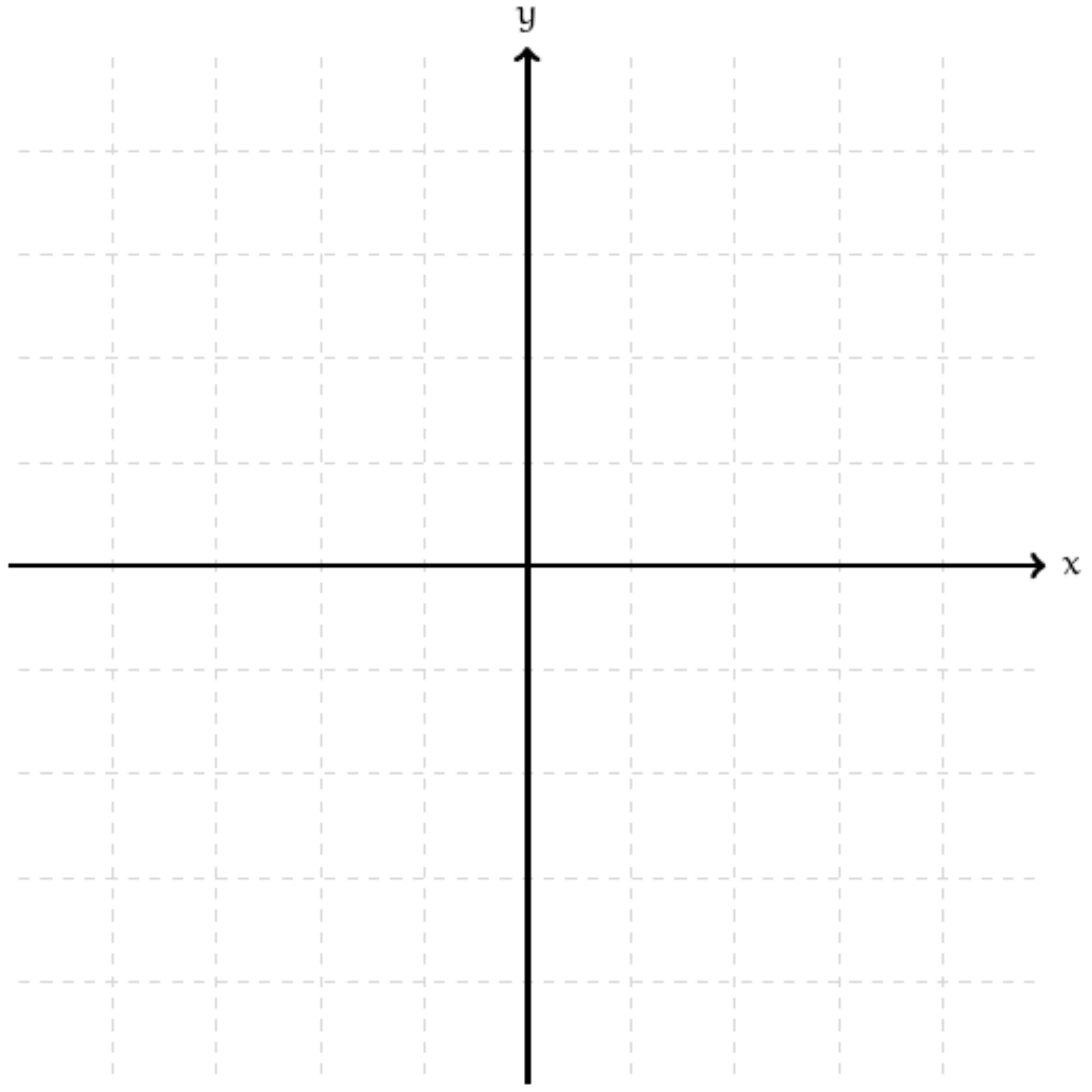


# Example: Dilation

$$\begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} rx_1 \\ rx_2 \end{bmatrix}$$

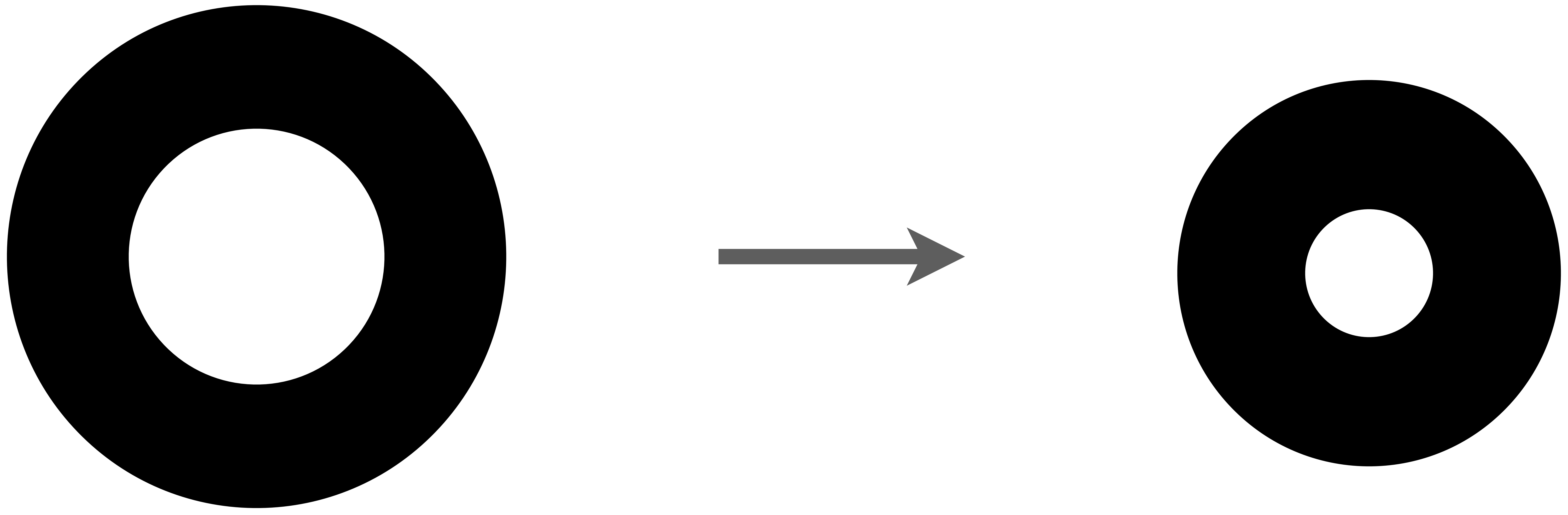



$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \mathbf{x}$$



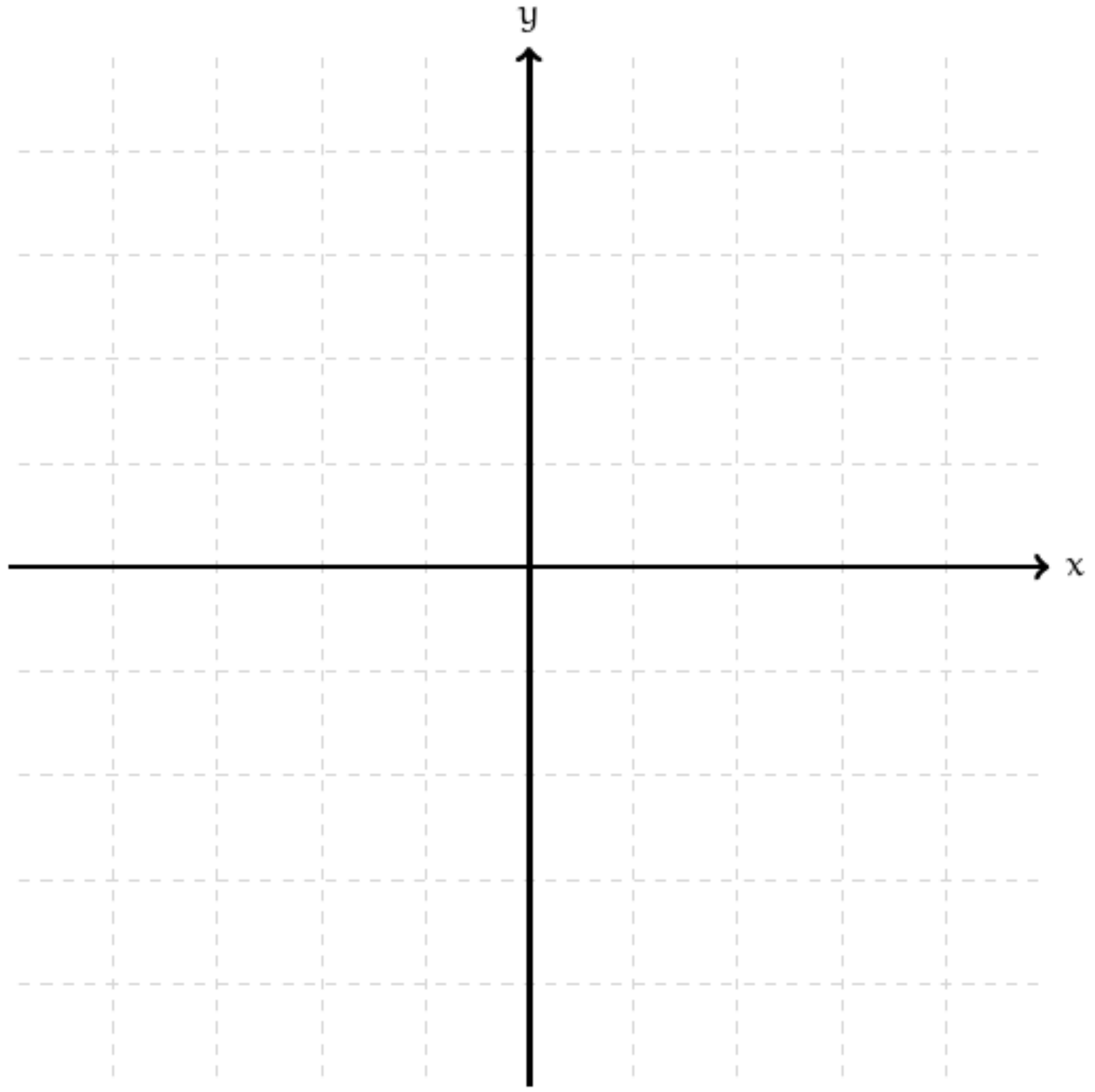

if $r > 1$, then the transformation pushes points away from the origin.

# Example: Contraction

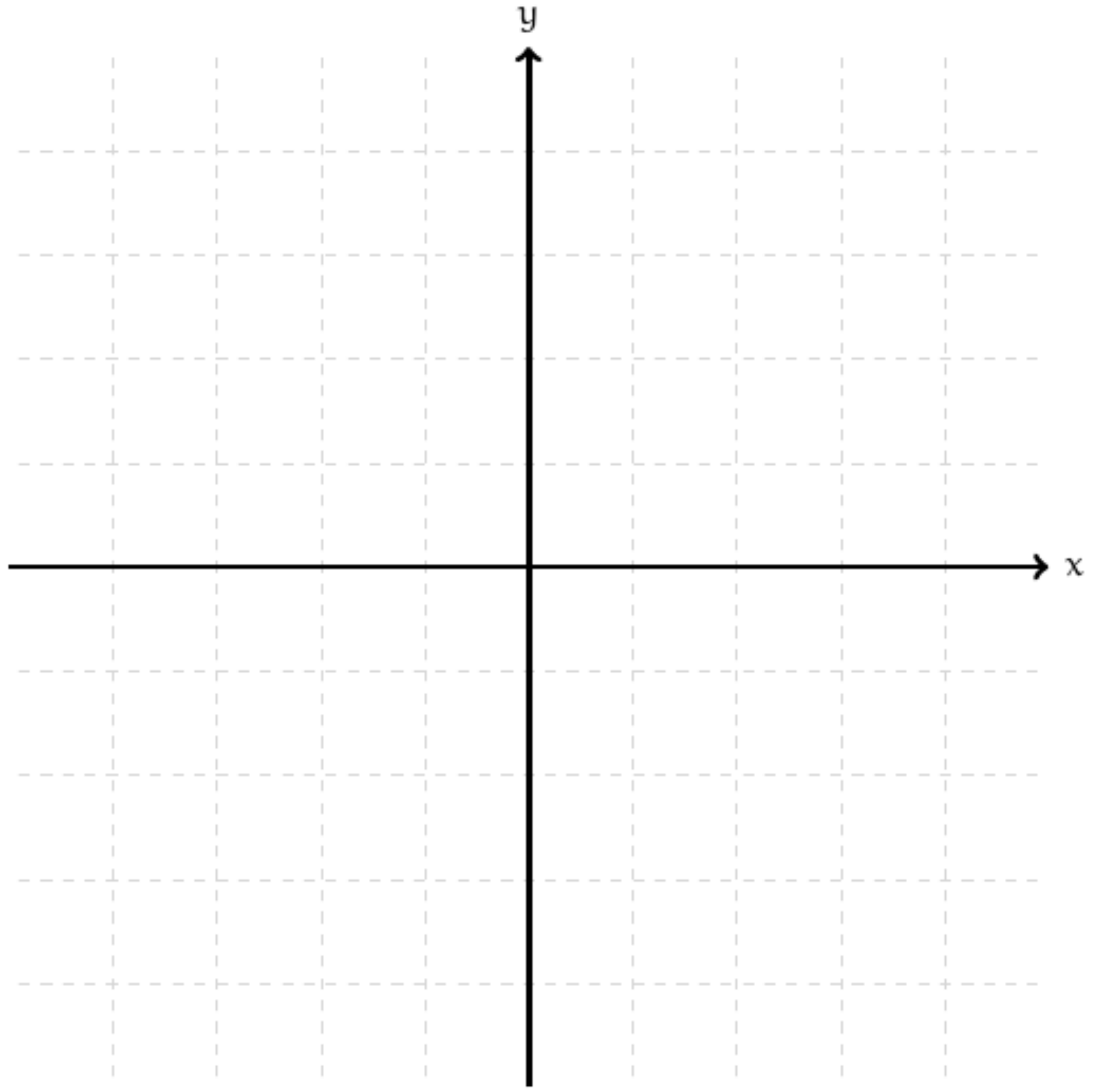


# Example: Contraction

$$\begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} rx_1 \\ rx_2 \end{bmatrix}$$




$$\begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix} \mathbf{x}$$

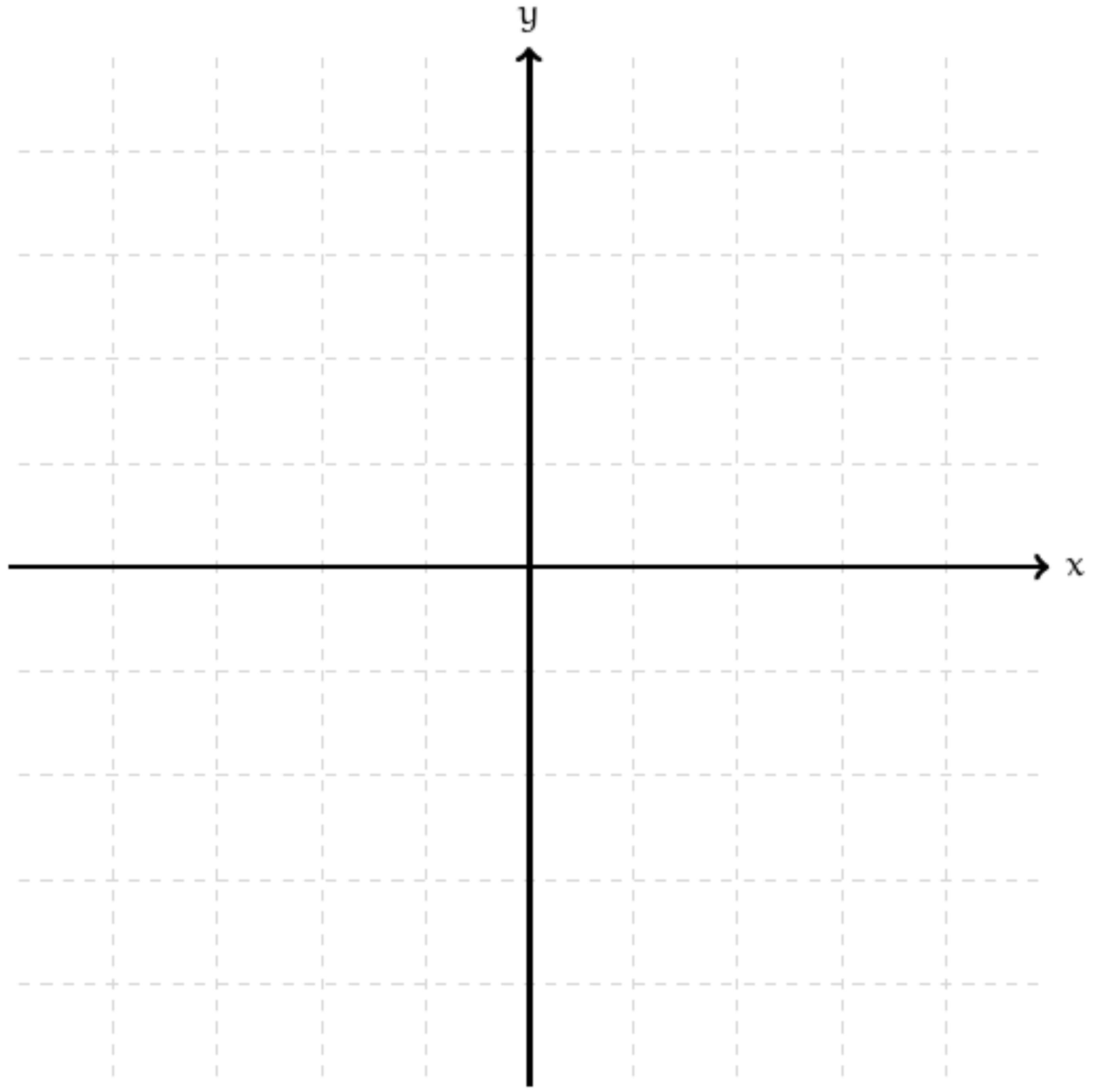



if $0 \le r \le 1$, then the transformation pulls points towards the origin.

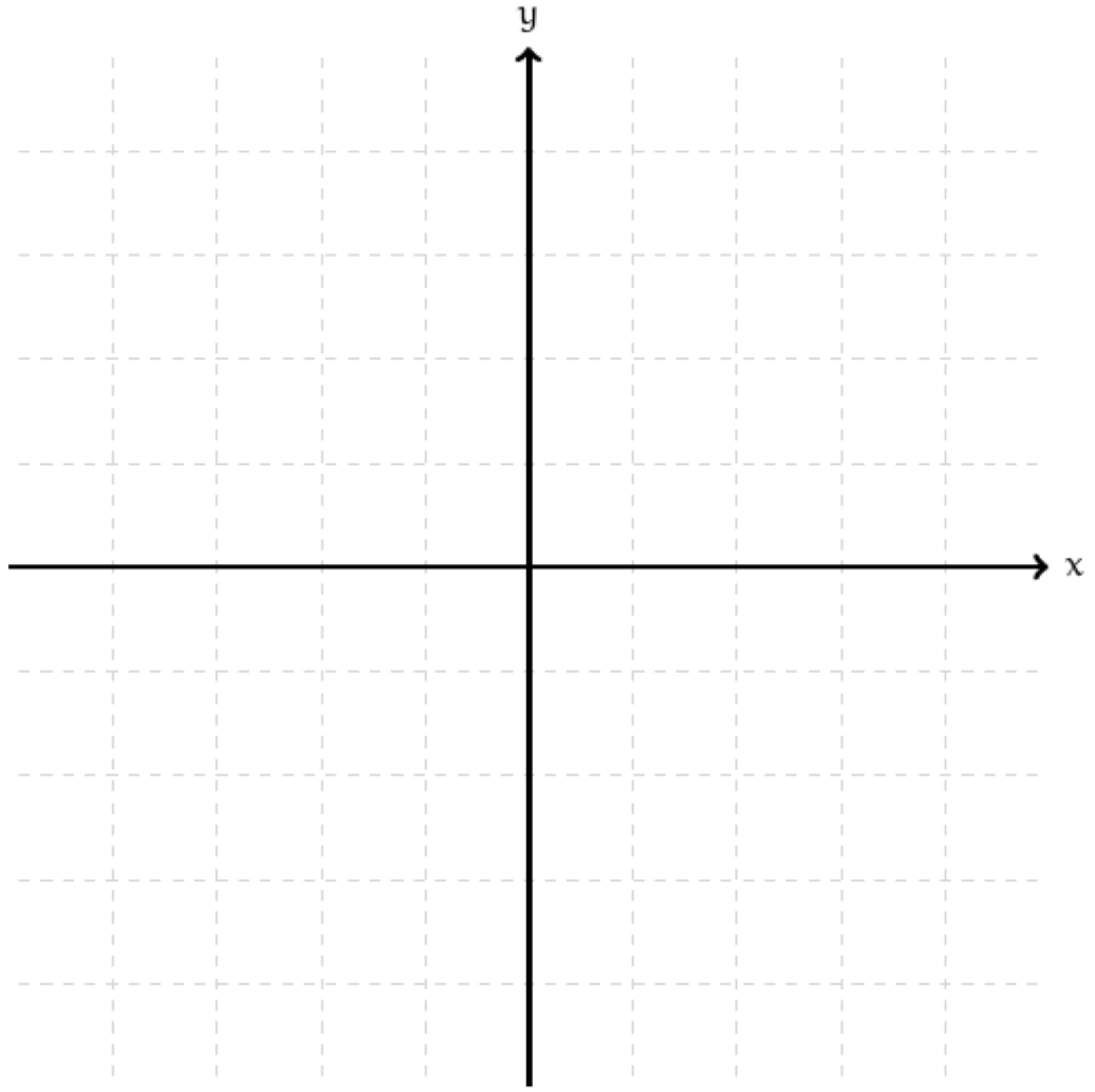
# Example: Shearing

# Example: Shearing

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ x_2 \end{bmatrix}$$

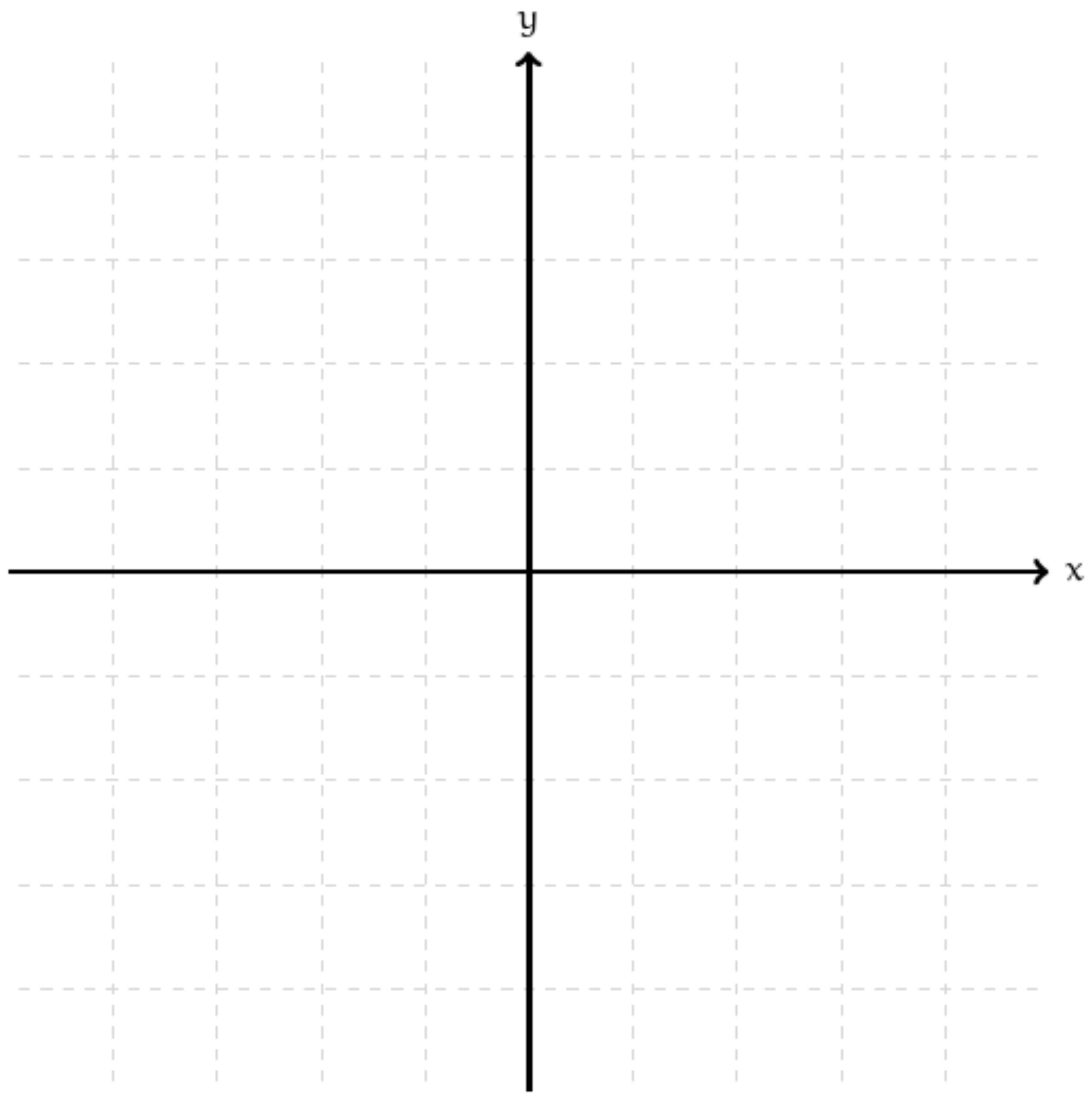



$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \mathbf{x} \longrightarrow$$

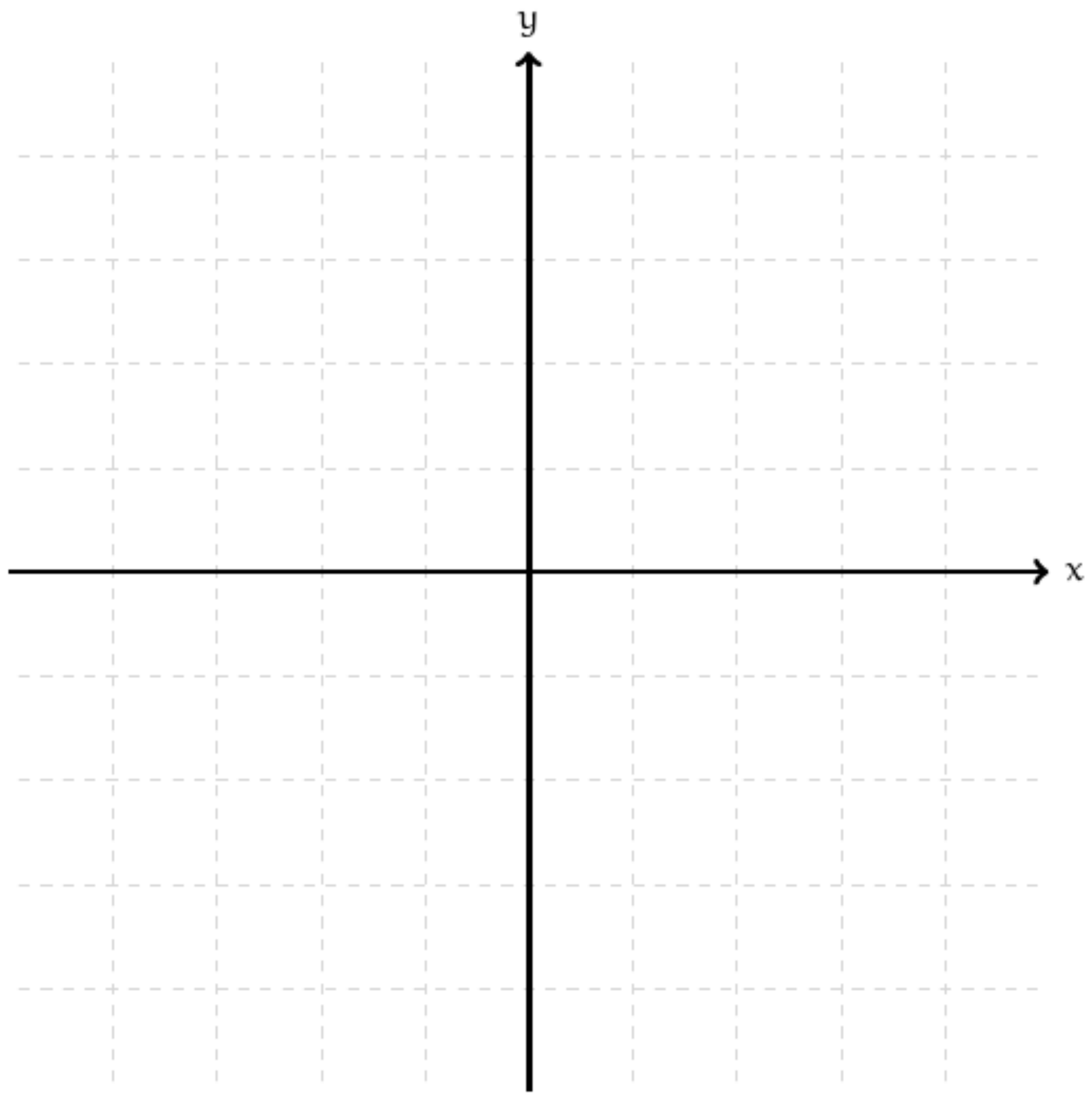



Imagine shearing like with rocks or metal.

# Question

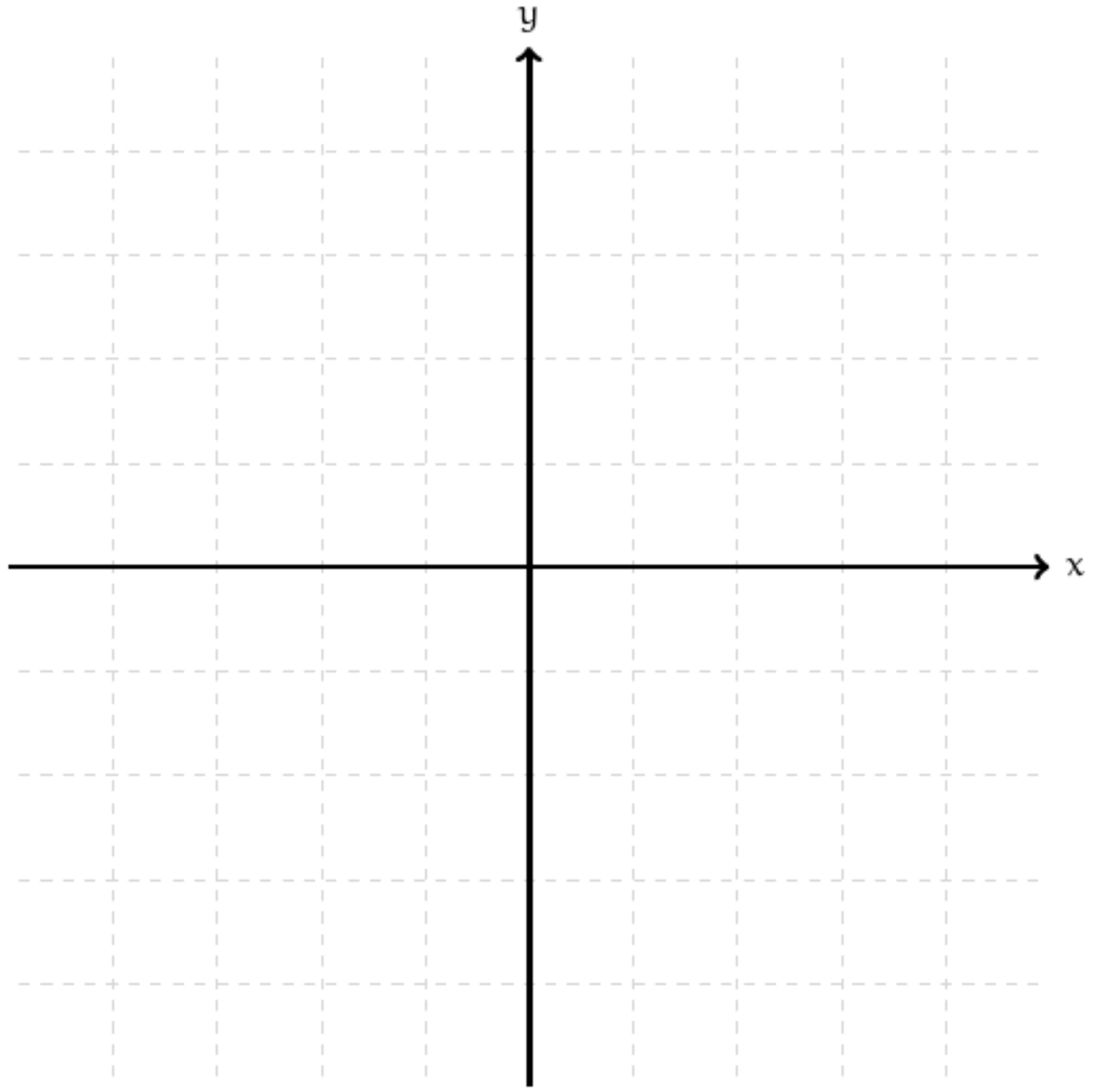



$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \mathbf{x}$$

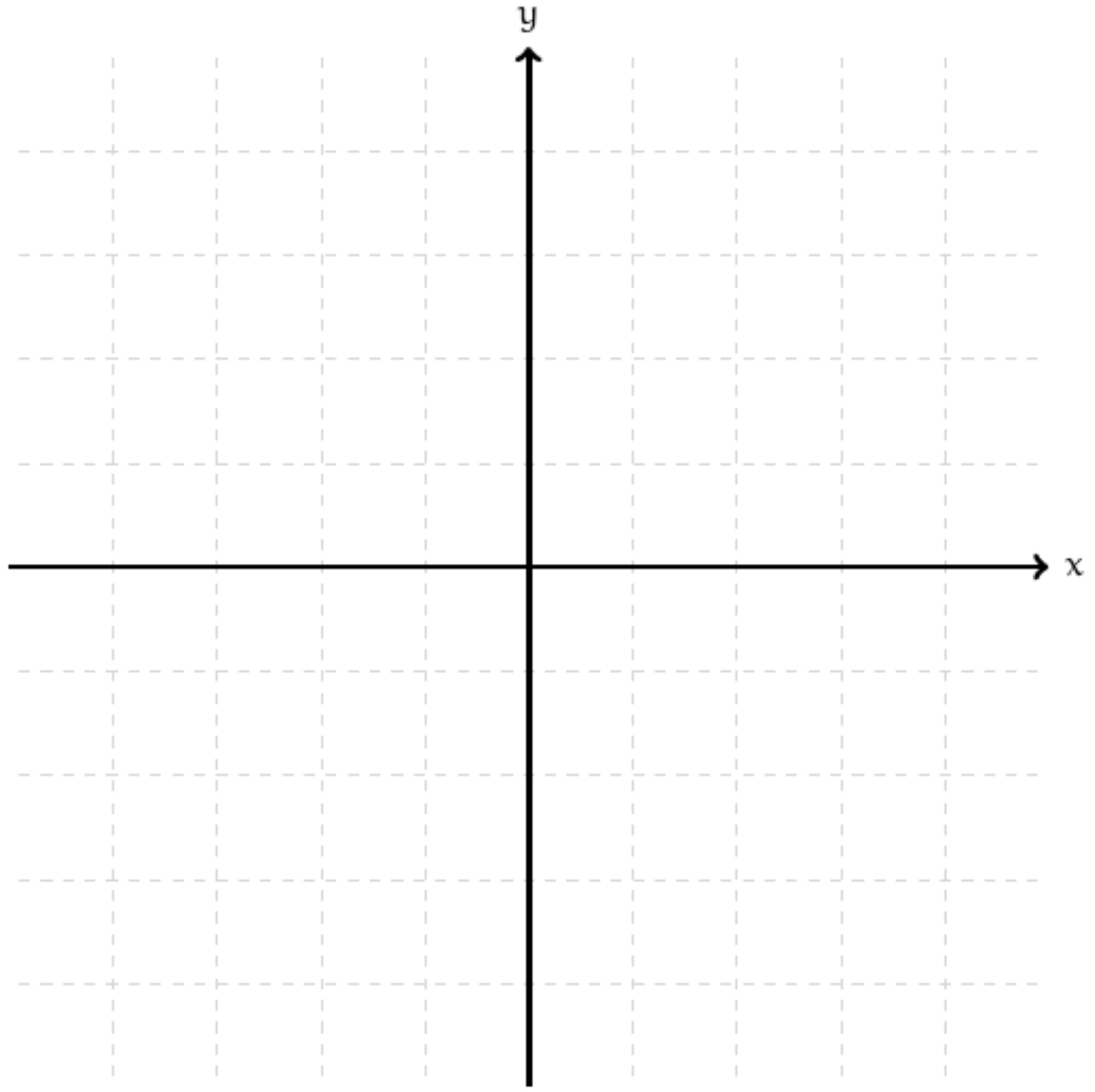



Draw how this matrix transforms points. What kind of transformation does it represent?

# Answer: Reflection




$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \mathbf{x}$$



demo

# Summary

Matrices can be viewed as **linear transformations**

Matrix transformations change the **shape** of points sets

Linear transformations behave well with respect to **linear combinations**