Vector Equations (Workshop)

Geometric Algorithms
Lecture 4

CAS CS 132

Keywords

vector

vector addition

vector scaling/multiplication

the zero vector

vector equations

linear combinations

span

Vectors

Column Vectors

Definition. a ***column vector*** is a matrix with a single column, e.g.,

A Note on Matrix Size

an $(m \times n)$ matrix is a matrix with m rows and n columns

$$\begin{array}{cc} & n \\ m & \begin{bmatrix} * & * & \cdots & * & * \\ * & * & \cdots & * & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ * & * & \cdots & * & * \\ * & * & \cdots & * & * \end{bmatrix} \end{array} \qquad \begin{array}{cc} & 1 \\ 4 & \begin{bmatrix} 2 \\ 3 \\ 0.1 \\ -2 \end{bmatrix} \end{array}$$

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the number of rows of a vectors is called its **dimension**

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Vector Operations

Vector "Interface"

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addition — what does $\mathbf{u} + \mathbf{v}$ (adding two vectors mean?

scaling — what does $a\mathbf{v}$ (multiplying a vector by a real number) mean?

Vector Equality

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \quad \texttt{is the same as} \quad \begin{matrix} a_1 = b_1 \\ a_2 = b_2 \\ \vdots \\ a_n = b_n \end{matrix}$$

Vector Addition

adding two vectors means adding their entries

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \vdots \\ a_n + b_n \end{bmatrix}$$

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!!IMPORTANT!!
WE CAN ONLY ADD VECTORS OF THE SAME SIZE

Vector Scaling

scaling/multiplying a vector by a number means multiplying each of it's elements

$$a \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} ab_1 \\ ab_2 \\ \vdots \\ ab_n \end{bmatrix}$$

Algebraic Properties

For any vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ and any real numbers c, d:

$$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$$

$$(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$$

$$\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$$

$$\mathbf{u} + (-\mathbf{u}) = -\mathbf{u} + \mathbf{u} = \mathbf{0}$$

$$c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$$

$$(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$$

$$c(d\mathbf{u}) = (cd)\mathbf{u}$$

$$1\mathbf{u} = \mathbf{u}$$

demo
(from ILA)

Linear Combinations

Linear Combinations

Definition. a ***linear combination*** of vectors

$$\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$$

is a vector of the form

$$\alpha_1 \mathbf{v}_1 + \alpha_1 \mathbf{v}_2 + \ldots + \alpha_n \mathbf{v}_n$$

where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are in $\mathbb{R}$

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Looks suspiciously like
a linear equation...

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demo

(from ILA)

Vector Equations and Linear Systems

The Fundamental Connection

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$$x_1 \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$$

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Some Symbol Pushing...

$$x_1 + 2x_2 = 7$$
$$(-2)x_1 + 5x_2 = 4$$
$$-5x_1 + 6x_2 = -3$$

we get a system of linear equations we know how to solve

General Vector Equations

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{1m} \end{bmatrix} + x_2 \begin{bmatrix} a_{21} \\ a_{21} \\ \vdots \\ a_{2m} \end{bmatrix} + \ldots + x_n \begin{bmatrix} a_{n1} \\ a_{n2} \\ \vdots \\ a_{nm} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

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by vector scaling

General Vector Equations

$$\begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

`by vector addition`

General Vector Equations

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\
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&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m
\end{aligned}
$$

by vector equality

The Fundamental Connection

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{bmatrix}$$

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this is notation for building a matrix out of column vectors

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read: $\mathbf{u}$ is an element of $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$

$\mathbf{u} \in \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ **exactly when $\mathbf{u}$ can be expressed as a linear combination of those vectors** demo

(from ILA)

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you know how to do this now

demo

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workshop