Efficient Test-Time Scaling with Large Language Models

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Certificate

This is to certify that the thesis titled **Efficient Test-time Scaling with Large Language Models** being submitted by **Aradhye Agarwal** for the award of **Dual Degree** in **Computer Science & Engineering** is a record of bona fide work carried out by him under my guidance and supervision at the **Department of Computer Science & Engineering**. The work presented in this thesis has not been submitted elsewhere either in part or full, for the award of any other degree or diploma.

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Abstract

Test-time scaling (TTS), which involves dynamic allocation of compute during inference, offers a promising way to improve reasoning in large language models. While existing TTS methods work well, they often rely on long decoding paths or require a large number of samples to be generated, increasing the token usage and inference latency. We observe the surprising fact that for reasoning tasks, shorter traces are much more likely to be correct than longer ones. Motivated by this, we introduce First Finish Search (FFS), a training-free parallel decoding strategy that launches n independent samples and returns as soon as any one completes. We evaluate FFS alongside simple decoding, beam search, majority voting, and budget forcing on four reasoning models (DeepSeek-R1, R1-Distill-Qwen-32B, QwQ32B and Phi-4-Reasoning-Plus) and across four datasets (AIME24, AIME25-I, AIME25-II and GPQA Diamond). With DeepSeek-R1, FFS achieves 82.23% accuracy on the AIME datasets, a 15% improvement over DeepSeek-R1's standalone accuracy, nearly matching OpenAI's o4-mini performance. Our theoretical analysis explains why stopping at the shortest trace is likely to yield a correct answer and identifies the conditions under which early stopping may be suboptimal. The elegance and simplicity of FFS demonstrate that straightforward TTS strategies can perform remarkably well, revealing the untapped potential of simple approaches at inference time.

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Chapter 1

Introduction

Large language models (LLMs) [32, 8, 20] have achieved impressive performance on complex reasoning tasks [10], predominantly due to larger model sizes, greater pretraining compute, and vast training corpora [15, 3]. Further improvements, however, are proving increasingly difficult to achieve as the benefits of increasing model size diminish and the cost of further pretraining rises [35]. Recent research has, instead, started exploring the possibility of test-time intervention [29] for enhancing LLM performance. This approach, known as **test-time scaling (TTS)** [24], dynamically allocates compute during decoding in order to improve accuracy and consistency without any additional training. TTS is especially effective on complex and reasoning-heavy benchmarks, where greater compute often translates into more reliable and precise outputs. OpenAI's o1 [20] and DeepSeek-R1 [8] are notable examples of models which utilize this approach.

According to the taxonomy proposed by [35], TTS strategies fall into three main categories: parallel scaling [24, 4], sequential scaling [13, 14], and hybrid approaches [31, 28]. Parallel scaling methods, such as beam search (BS), diverse beam search (DVBS) [26], and majority voting (MV), involve generating multiple candidate outputs concurrently and selecting the best response using heuristics or scoring functions. While these approaches excel in throughput and leverage computational parallelism effectively, they suffer from high token usage and require sophisticated mechanisms for evaluating and comparing candidate responses. For instance, MV assumes that the generated outputs can be easily compared using string equality or semantic similarity, which is often not the case in open-ended tasks. Sequential scaling techniques, on the other hand, extend the reasoning path deliberately with methods like budget forcing (BF) [19] introducing artificial delay tokens such as *"Wait"* to encourage deeper reasoning, while others like thought switching penalty (TIP) [27] discouraging premature shifts in reasoning direction.

While these methods are powerful in improving accuracy, especially in tasks where extended deliberation leads to better answers, they are computationally expensive and introduce latency, making them less suitable for real-time or API-constrained deployments. Hybrid strategies, another class of TTS methods, combine elements of both parallel and sequential scaling to adapt to task difficulty and budget constraints. While combining diverse strategies does allow test-time compute to be allocated more judiciously, hybrid methods, which mix sequential and parallel strategies, still suffer from the same weaknesses to a greater or lesser extent.

To address these challenges, we introduce **F**irst **F**inish **S**earch **(FFS)**, a training-free test-time scaling method that launches n samples in parallel and selects the output trace that completes first as the final answer. FFS is motivated by a surprising empirical finding (Figure 1.1) – *for reasoning tasks, shorter traces are significantly more likely to be correct.*[1] FFS, unlike MV, needs no equality check between generated outputs, and unlike BF or TIP, it requires no special tokens or branching during decoding. It also uses far fewer tokens, and requires lesser sequential computation. We validate FFS with experiments on four benchmarks (GPQA Diamond, AIME24 and AIME25-I and AIME25-II) using four models: DeepSeek-R1, its distilled variant R1-Distill-Qwen-32B, QwQ-32B and Phi-4-Reasoning-Plus. Across all settings, FFS matches or exceeds the accuracy of strong baselines such as MV and BF while reducing token usage by up to 45%.

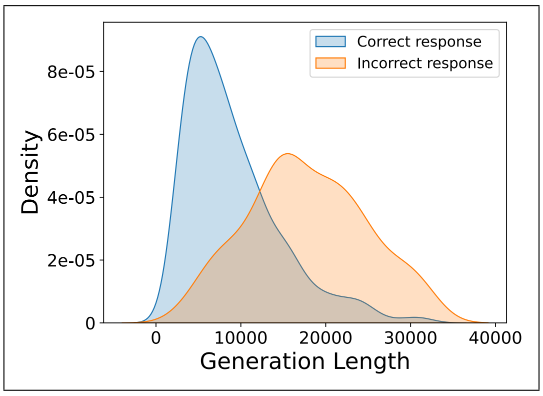


Figure 1.1: Distribution of trace lengths for correct and incorrect responses generated by QwQ [33] and R1-Distill-Qwen models [8]. The distributions show that shorter generation length and correctness are correlated, providing empirical justification for FFS.

[1]Note that Figure 1.1 actually implies the *reverse*, i.e., correct traces are more likely to be short. We analyze in Appendix 8.1 as to why this also means that shorter traces are more likely to be correct.

On AIME24 and AIME25 (AIME25-I and AIME25-II), with DeepSeek-R1, FFS attains an accuracy of 82.23% on average, nearly reaching OpenAI's o4-mini (83.7%) accuracy.[2] Our experiments show that FFS gains more from increased model capacity than any competing method, enabling it to outperform every baseline on DeepSeek-R1. Our theoretical results show that FFS's expected sequential cost decreases with increasing sample count, demonstrating its compute efficiency for inference-time scaling. These findings establish FFS as a scalable, efficient and effective alternative for boosting LLM reasoning at test-time.[3]

[2]o4-mini accuracy is obtained from `https://www.vals.ai/benchmarks/aime-2025-05-09`
[3]The source code is available at `https://github.com/Aradhye2002/reasoning_exps`

Chapter 2

Related Work

TTS techniques aim to improve LLM reasoning by dynamically adjusting the compute budget [24] during inference, without needing expensive model fine-tuning. Existing TTS methods can be categorized into two broad categories:

1. Training-based strategies. In these methods, the base model is fine-tuned in order to *follow* a particular compute-allocation policy at inference time. For instance, Inference-Aware Fine-Tuning (IA-FT) [5] optimizes a supervised or RL objective directly targeting best-of-N accuracy: during pretraining, the model learns to generate diverse candidates so that pass@N is optimized. Similarly, L1 [2] trains the model to obey an explicit "think for L tokens" instruction in the prompt, granting the user direct control over the length of the chain-of-thought. While powerful, these methods require access to model weights, substantial compute for fine-tuning, and are not applicable to closed-source or API-only LLMs.

2. Training-free strategies. Instead of fine-tuning the existing model weights, these methods work by adapting additional compute-allocation at decoding time. Based on the decoding strategies, these methods can be further classified into: **parallel scaling**, where decoding threads are launched in parallel from the same prompt, and a selection rule picks one final output. Examples include BS, which maintains a fixed-width beam of top-scoring partial sequences; DVBS, which imposes diversity penalties to encourage variation before ranking; and MV/self-consistency, which samples N full answers and returns the one with highest agreement. Other notable parallel scaling strategies [12, 22, 35] utilize the best response based on an external reward models. In contrary to parallel scaling methods, in another class of training-free TTS approach **sequential scaling**, a single decoding thread deliberately extends model's reasoning chain to consume more tokens before yielding an answer [9, 11]. Budget forcing interposes special "wait" tokens to delay termination artificially, while the TIP applies dynamic logit penalties to discourage

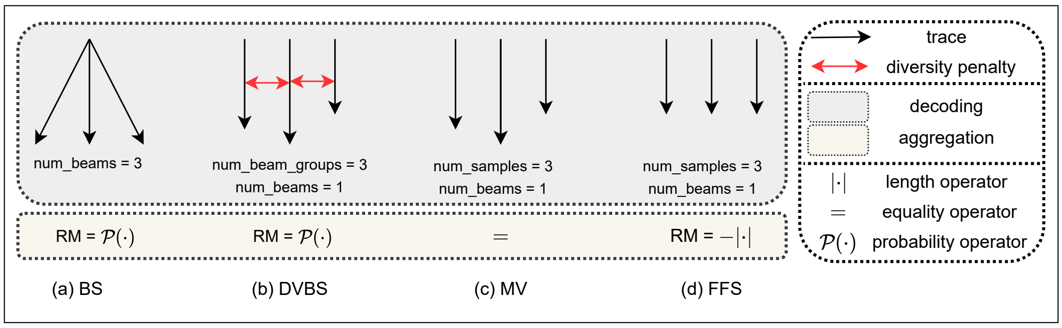


Figure 2.1: Sampling-based TTS strategies. (a) BS: expands k partial hypotheses synchronously, ranking both intermediate and final beams with the model probability $\mathcal{P}(\cdot)$. Beams share prefixes, so their lengths tend to remain aligned. (b) DVBS: launches g independent single-beam groups; a diversity term keeps the groups apart, so they may hit EOS at different steps. After all groups finish, a verifier (again $\mathcal{P}(\cdot)$ here) selects the best trace. (c) MV/self-consistency: draws n complete, independent samples and picks the modal answer using a string equality test "=". (d) FFS: starts the same n stochastic samples but terminates the batch as soon as the first trace reaches EOS; all displayed arrows are therefore cut to that minimal length. The selection rule is equivalent to maximising the negative length reward $R(T) = -|T|$.

early topic shifts. Recently proposed tree-based TTS methods [36, 30, 7, 34] draw balance between sequential and parallel scaling strategies. **Hybrid scaling** methods like Monte Carlo Tree Search (MCTS) [17, 21] and self-backtracking [34] aim to recognize and correct suboptimal reasoning paths through simultaneous parallel and sequential decoding.

This taxonomy highlights a clear operational divide: training-based approaches encode compute decisions into the model parameters, while training-free methods manipulate the decoding process itself. Within the latter, one must choose between spawning many concurrent trajectories or elongating a single trajectory, each with distinct implications. We evaluate these methods along five dimensions:

- **Training-free**: Does the method require any fine-tuning or reinforcement learning?
- **API-friendly**: Can the method be implemented using only standard model-serving API?

Table 2.1: Comparison of test-time scaling strategies across five criteria. "Training-free" indicates no additional model training is required; "API-friendly" means the method can be implemented using standard inference APIs without logit edits or repeated calls; "scalable" denotes the ability to improve performance by allocating more compute; "T-parallelizable" means total compute cost decreases when decoding is parallelized across more workers; and "S-parallelizable" means sequential latency decreases as the number of parallel samples increases. **DVBS:** diverse beam search, **BS:** beam search, **TIP:** thought switching penalty, **BF:** budget forcing, **SD:** simple decoding, **MV:** majority voting, **LFS:** last finish search, **FFS:** first finish search.

	DVBS	BS	TIP	BF	SD	MV	LFS	L1	FFS
Training-free	✓	✓	✓	✓	✓	✓	✓	✗	✓
API-friendly	✗	✗	✗	✗	✓	✓	✓	✓	✓
Scalable	✓	✓	✗	✓	✗	✓	✓	✗	✓
T-parallelizable	✓	✓	✗	✗	✗	✓	✓	✗	✓
S-parallelizable	✗	✗	✗	✗	✗	✗	✗	✗	✓

- **Scalable**: Can performance continue to improve as we allocate more compute (more samples or longer generations)?
- **T-parallelizable**: Does total GPU/CPU cost drop when decoding is parallelized across more workers?
- **S-parallelizable**: Does the sequential latency (time to first answer) decrease when we parallelize across more samples?

Table 2.1 summarizes how each method performs across key criteria. BS, DVBS, and MV support parallel scaling by generating multiple outputs and selecting the best (Figure 2.1); they are training-free and T-parallelizable, but BS and DVBS are not API-friendly due to requiring custom diversity penalties or verifier models, while MV is API-compatible but not S-parallelizable, as it must wait for all outputs before voting. BF and TIP scale sequentially by encouraging longer reasoning, are training-free, and potentially API-friendly if token manipulation is allowed, but repeated API calls make BF practically infeasible; neither is parallelizable and both are limited by model context length. Training-based methods like supervised fine-tuning and RL approaches (e.g., IA-FT, L1) improve accuracy but require model access and heavy compute, are API-incompatible, and are not parallelizable beyond the model's sequence limits. FFS combines the best of both: it is training-free, API-friendly, uses standard sampling with a stop-when-finished rule, scales arbitrarily with more samples, is T-parallelizable (lower cost with parallel runs), and uniquely S-parallelizable (latency drops as shorter correct traces finish early), making it a lightweight and effective alternative.

Chapter 3

FFS: Our Proposed Method

Algorithm 1 Synchronous FFS (Sync-FFS)

```
Require: Reasoning model $\mathcal{M}$, prompt $x$, number of traces $n$, max length $L$
 1: Initialize partial traces $T_1, \dots, T_n \leftarrow$ [BOS] $\circ x$
 2: for $\ell = 1$ to $L$ do
 3:     $\mathbf{Y} \leftarrow \mathcal{M}(T_1, \dots, T_n)$                ▷ one batched forward pass
 4:     for $i = 1$ to $n$ do                      ▷ sample next token for each trace
 5:         sample $y_i \sim \text{softmax}(\mathbf{Y}_i)$
 6:         $T_i \leftarrow T_i \circ y_i$
 7:         if $y_i = \text{EOS}$ then
 8:             $T^\star \leftarrow T_i$                          ▷ first finished trace
 9:             return $T^\star$
10:         end if
11:     end for
12: end for
13: return $T_1$                                  ▷ fallback if no trace hit EOS
```

Algorithm 2 Asynchronous FFS (Async-FFS)

```
Require: Reasoning model $\mathcal{M}$, prompt $x$, number of traces $n$, max decode length $L$
 1: Launch $n$ asynchronous decoding jobs $\{J_1, \dots, J_n\}$, each fed with $x$
 2: while true do
 3:     for all running jobs $J_i$ in parallel do
 4:         if $J_i$ produces EOS or $|T_i| = L$ then
 5:             $T^\star \leftarrow T_i$                 ▷ store the first finished trace
 6:             Interrupt all jobs $J_{k \neq i}$ and free their resources
 7:             return $T^\star$
 8:         end if
 9:     end for
10: end while
```

FFS runs n independent decoding operations in parallel and stops as soon as any one trace emits the end-of-sequence token. The winning trace is returned and all others are discarded. In order to ensure diversity among the generations, FFS uses stochastic decoding with beam size 1.

FFS admits two implementations – Sync-FFS and Async-FFS, which constitute the synchronous and asynchronous variants of the high-level algorithm, respectively. Sync-FFS (Algorithm 1) loads a single copy of the model $\mathcal{M}$ and processes a batch of n partial sequences in lock-step. At every decoding step we sample one token for each partial sequence; if any of those tokens is an EOS symbol, we immediately terminate decoding and return that completed trace. By sharing the model across all samples, Sync-FFS minimizes total compute and memory, making it well suited for centralized servers or GPUs. Async-FFS (Algorithm 2) launches n independent decoding jobs, on separate processes or machines, all starting from the same prompt. When any job produces EOS (or reaches the token limit), it interrupts the remaining $n-1$ jobs, frees their resources, and returns the completed trace. This variant naturally fits distributed or multi-worker environments.

While our motivating hypothesis states that shorter reasoning segments are likely to be more accurate, we simplify implementation by measuring the entire trace length (reasoning plus final answer). Since the solution portion is typically small, ranking by full length almost always aligns with ranking by reasoning length. Also, while larger beams can reduce repetition, we deliberately use beam size 1 in order to maximize the number of independent samples, and consequently the chance that a short, correct trace finishes first. In practice, repetitive or degenerate beams rarely finish earliest, so they are automatically filtered out by FFS's "first-to-finish" rule.

To back our claim that shorter traces are likelier to be correct, we derive the following expression for the probability that a randomly drawn trace of length x is correct.

Result 1. Suppose correct and incorrect samples are drawn from distinct normal distributions with means μ_1, μ_2 and standard deviations σ_1, σ_2 respectively. Let α be proportion of samples that are correct. Then the probability that a randomly sampled trace T of a given length x is correct is:

$$\Pr\Big[T \text{ is correct} \mid |T| = x\Big] = \left(1 + \frac{1-\alpha}{\alpha}\frac{\sigma_1}{\sigma_2}e^{-\frac{1}{2}\left[\left(\frac{x-\mu_2}{\sigma_2}\right)^2 - \left(\frac{x-\mu_1}{\sigma_1}\right)^2\right]}\right)^{-1} \tag{3.1}$$

Since FFS always favours the shortest trace, we first examine the limiting case $x = 0$ in Equation (3.1). With comparable variances ($\sigma_1 \approx \sigma_2$), the simplified form shows that the probability of correctness is high only when the mean length of incorrect traces exceeds that of correct ones, i.e., $\mu_2 > \mu_1$. Empirically, reasoning models satisfy this inequality, confirming the link between conciseness and accuracy. A parallel "long-trace" analysis for LFS ($x \to \infty$) is not meaningful. Beyond moderate lengths the empirical trace-length distributions become heavy-tailed (Figure 3.1) and depart from the normal assumption that underlies Equation (3.1). This analysis reveals that correctness of a trace is closely linked to how short it is, making it crucial to devise ways in order to discover increasingly shorter traces, with FFS being one such effective way.

To investigate how FFS's cost reduces with increasing number of samples n, we introduce the following result from extreme value theory [6], which states that the difference between the mean and expected minimum of n i.i.d. normally distributed RVs is $\mathcal{O}(\sqrt{\log n})$.

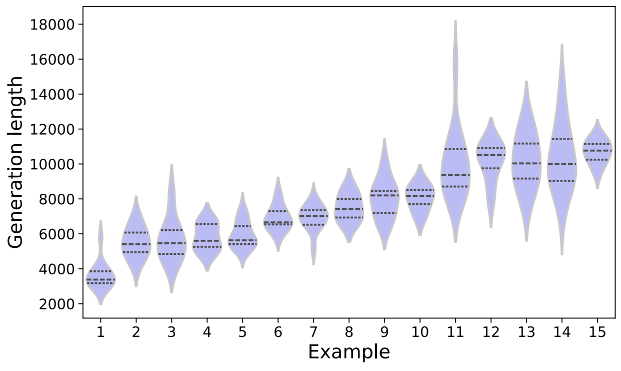
Result 2. Let $Y_1, Y_2, \ldots, Y_n$ be i.i.d. normally distributed random variables with mean μ and standard deviation σ, then as $n \to \infty$, we have[1]

$$\mathbb{E}\big[\min\{Y_1, Y_2, \ldots, Y_n\}\big] = \mu - \sigma\sqrt{2\log n} \tag{3.2}$$

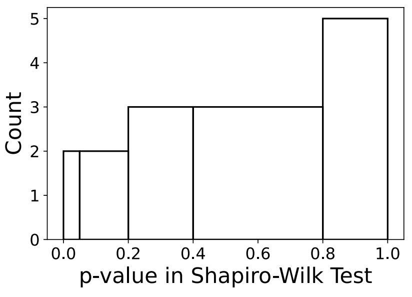
$$\mathbb{E}\big[\max\{Y_1, Y_2, \ldots, Y_n\}\big] = \mu + \sigma\sqrt{2\log n} \tag{3.3}$$

In order to use Result 2 in our analysis of FFS, we define Y_i as $|T_i|$, where T_i is a trace sampled from the model $\mathcal{M}$, and $|\cdot|$ is the usual length operator. Since T_i's are independently and identically sampled, $Y_i = |T_i|$ are i.i.d. random variables. Y_i's are also normally distributed for reasoning models

[1]See Appendices 8.2 and 8.3 for detailed proofs.



(a) Per-example trace length distribution



(b) P-value of normality test

Figure 3.1: Lengths of generated reasoning traces on AIME24. (a) Per-example violin plots of trace length over 8 samples, for each question. (b) Histogram of Shapiro-Wilk p-values for normality across examples, indicating that most trace-length distributions do not reject the gaussian assumption.

as demonstrated through Figure 3.1. On application of Equation (3.2), we obtain that the expectation of the minimum trace length (which is same as FFS's sequential cost) decreases in $\mathcal{O}(\sqrt{\log n})$, with increasing n. Whereas, the maximum trace length (same as MV's sequential cost) increases with n.

Chapter 4

Experimental Setup

We evaluate FFS with four reasoning models: Phi-4-Reasoning [1], DeepSeek-R1 [8], QwQ-32B [25] and R1-Distill-Qwen [33] and one non-reasoning model: DeepSeek-V3 [16]. On DeepSeek-R1 and Phi-4-Reasoning, we evaluate only MV, LFS, and FFS, omitting BF and BS due to API constraints. On QwQ-32B and R1-Distill-Qwen, we report results for all five methods: MV, LFS, FFS, BF, and BS. Last finish search (LFS) is an additional baseline added by us for comparison with FFS, where we take the last finished trace. We conduct evaluations on four datasets: GPQA Diamond [23], AIME24 [18], AIME25-I and AIME25-II. The AIME datasets (AIME24, AIME25-I, and AIME25-II) totaling 60 problems, are derived from the American Invitational Mathematics Examination, consisting of challenging high school-level competition math problems. Each question has a numeric answer between 000 and 999. For consistent evaluation, we strip visual figures and convert all problems into text, consistent with the approach of [19]. The GPQA Diamond dataset comprises 198 graduate-level multiple-choice science questions spanning physics, chemistry, and biology. Each question is designed to be google-proof and tests high-level conceptual reasoning. The task templates and output parsing logic for these datasets are provided in Appendix 8.4.

We evaluate each method using three metrics: (1) **accuracy**, measured as the proportion of exact matches with ground-truth answers; (2) **total compute**, defined as the total number of generated tokens across all parallel traces for a given question; and (3) **sequential compute**, which is the minimum number of sequential tokens required to be produced to obtain the final answer. For GPQA, we consider a response correct only if the selected option exactly matches the correct choice while for the three AIME datasets, only the last integer matches are considered.

All models are evaluated in a zero-shot setting with no additional fine-tuning.[1] We use `top_p` $= 0.95$ and `temperature` $= 0.6$ for all methods across all datasets. For all sampling based methods (FFS, LFS, and MV) we take the number of samples `n` $= 4$. Since AIME problems require deeper reasoning, we set a maximum generation length of 32K tokens; on GPQA Diamond, 16K tokens suffice. Appendix 8.4.3 provides the complete hyperparameter list.

[1]We use `deepinfra.com` API for all our evaluations.

Chapter 5

Results

Token-efficient accuracy gains. Across all evaluated settings FFS either matches or improves on baseline accuracy while reducing token usage (Tables 5.1-5.2). With DeepSeek-R1 it scores 86.7% on AIME24 and 93.3% on AIME25-II while averaging only 31.1K total tokens and 7.8K sequential tokens per query, about 26% less compute than MV's 42.2 K-token budget. On the smaller R1-Distill-Qwen-32B model, FFS reaches 80.0% on AIME24 and 62.6% on GPQA using the same 7.8 K/31.3K token budget, whereas MV consumes 15.7K/45.8K tokens for slightly lower scores. BS can equal FFS on a few cases (e.g. 66.7% on AIME25-I) but does so at roughly 1.4 times the compute cost.

Consistent improvements across models and datasets. FFS remains competitive even on larger models. On QwQ-32B it reaches 78.0% on AIME25-II, just 3.9 points less than MV's top score while using 25% fewer tokens (47.2K vs. 59.7K). For Phi-4-Reasoning-Plus, FFS increases GPQA accuracy to 67.2%, beating MV by 4.5 points while reducing the total token budget by 24% (44.8K vs. 58.8K). On math-heavy AIME25-II it still attains 86.7%, matching MV's peak performance but at a lower cost. In the most compute-intensive setting of DeepSeek-R1 on AIME25-II FFS outperforms LFS by 33% and beats MV by 13% while remaining the cheapest of the multi-sample strategies.

Linear scalability and reduced latency. Since FFS halts decoding once the shortest trace finishes, its cost scales linearly with the number of parallel samples and never exceeds a single-trace budget. Concretely, FFS reduces the sequential budget down to 7.8K tokens for DeepSeek-R1 and 11.8K for QwQ-32B, compared with 13-19K for MV and LFS. These savings translate

Table 5.1: Accuracy (%) and compute cost ($\times 10^3$ tokens) for R1-Distill-Qwen and QwQ-32B. Rows list metrics; columns list decoding methods. For each method, token counts are averages over the datasets. Bold, gray cells mark the best value in each row.

(a) R1-Distill-Qwen

Metric	SD	BF	BS	MV	LFS	FFS
Seq. tokens	11.4	25.7	11.2	15.7	15.7	**7.8**
Total tokens	**11.4**	25.7	44.8	45.8	45.8	31.3
GPQA	60.7	58.6	62.6	62.1	60.1	**62.6**
AIME24	68.3	60.0	66.7	77.9	60.6	**80.0**
AIME25-I	51.7	53.3	46.7	53.3	49.7	**59.6**
AIME25-II	46.7	57.1	57.1	**60.0**	**60.0**	46.7

(b) QwQ-32B

Metric	SD	BF	BS	MV	LFS	FFS
Seq. tokens	14.9	23.7	12.8	18.8	18.8	**11.8**
Total tokens	**14.9**	23.7	51.2	59.7	59.7	47.2
GPQA	–	60.1	57.1	64.7	57.6	**65.2**
AIME24	79.2	**86.7**	80.0	83.5	80.1	81.2
AIME25-I	60.0	60.0	66.7	**69.9**	53.5	59.9
AIME25-II	69.2	71.4	78.6	**81.9**	70.4	78.0

Table 5.2: Accuracy (%) and compute cost ($\times 10^3$ tokens) for DeepSeek-R1, Phi-4-Reasoning-Plus, and DeepSeek-V3. For each method, token counts represent averages over all datasets. Bold, gray cells mark the best value per row.

(a) DeepSeek-R1

Metric	SD	MV	LFS	FFS
Seq. tokens	10.6	13.8	13.8	**7.8**
Total tokens	**10.6**	42.2	42.2	31.1
GPQA	72.0	73.2	72.2	**74.2**
AIME24	75.0	83.3	70.0	**86.7**
AIME25-I	51.7	60.0	53.3	**66.7**
AIME25-II	75.0	80.0	60.0	**93.3**

(b) Phi-4-Reasoning-Plus

Metric	SD	MV	LFS	FFS
Seq. tokens	14.7	18.7	18.7	**11.2**
Total tokens	**14.7**	58.8	58.8	44.8
GPQA	62.9	67.7	**70.2**	67.2
AIME24	71.7	**80.0**	**80.0**	76.7
AIME25-I	66.7	73.3	**80.0**	66.7
AIME25-II	76.7	**93.3**	86.7	86.7

(c) DeepSeek-V3

Metric	SD	MV	LFS	FFS
Seq. tokens	2.7	4.4	4.4	**1.4**
Total tokens	**2.7**	10.8	10.8	5.5
GPQA	53.3	**54.0**	52.5	50.0
AIME24	30.0	**40.0**	**40.0**	23.3
AIME25-I	33.3	**46.7**	33.3	**46.7**
AIME25-II	16.7	**20.0**	**20.0**	**20.0**

directly into lower end-to-end latency in throughput-bound or API-metered deployments.

Behaviour on a non-reasoning model. On DeepSeek-V3, a model without explicit chain-of-thought supervision, we observe a reverse trend: MV surpasses both LFS and FFS on every dataset, and FFS records the lowest scores (e.g. 50% on GPQA and 20% on AIME25-II). This suggests that the "shortest correct trace" bias exploited by FFS is specific to models that already internalise multi-step reasoning.

FFS benefits from model scaling. The advantage of FFS widens with capacity. Upgrading from R1-Qwen-32B to DeepSeek-R1 boosts the method's

AIME accuracy from 80.0% to 86.7% on AIME24 and from 62.6% to 74.2% on GPQA, while preserving its compute edge. FFS attains 93.3% on AIME25-II, over 14 points higher than the strongest baseline, showing that it scales more favourably with model size than competing approaches.

Chapter 6

Analysis

Scaling behaviour of TTS methods. Figure 6.1 plots accuracy versus total token budget, averaged over R1-Distill-Qwen and QwQ-32B on the AIME24 and AIME25-I tasks. BF attains roughly 30% accuracy at the smallest budget and then plateaus once the generation reaches the 32K token limit. Its final point drops slightly because traces that hit this limit generally yield no answer and are marked incorrect. MV and FFS coincide at the single-sample setting, both achieving about 60% accuracy, since FFS degenerates to MV when $n = 1$. As the number of samples increases, MV's accuracy improves slowly but at the cost of a steep rise in tokens, proportional to the mean trace length times the sample count. FFS, in contrast, halts decoding as soon as the first trace finishes, reaching comparable or higher accuracy with far fewer tokens; its curve therefore rises more sharply. Across the entire budget range, FFS matches or surpasses MV while operating at a lower cost.

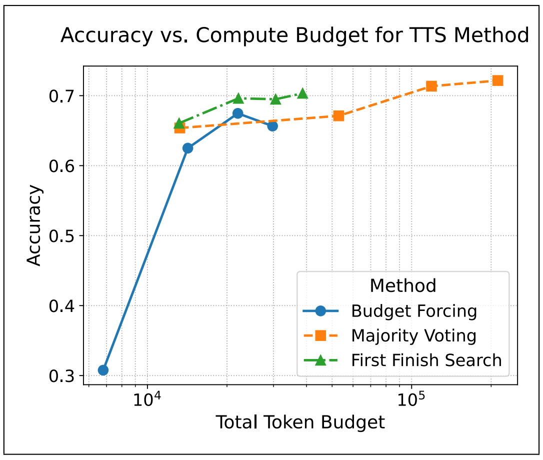
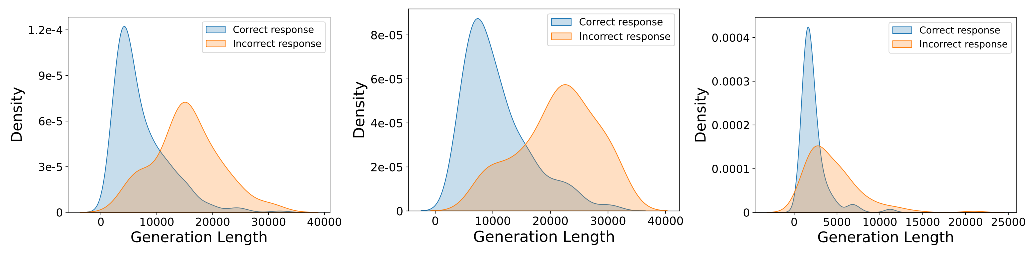


Figure 6.1: Accuracy versus total token budget for three test-time scaling methods averaged over R1-Distill-Qwen and QwQ-32B on AIME24 and AIME25-I. FFS attains higher accuracy at lower budgets; MV improves more slowly and consumes more tokens; BF plateaus near the 32K token limit.

A theoretical perspective on FFS's scaling behavior with model size. As demonstrated in Section 5, FFS shows greater relative improvements on the more capable DeepSeek-R1 model compared to other baselines.



(a) R1-Distill-Qwen (b) QwQ-32B (c) DeepSeek-V3

Figure 6.2: Distribution of generated text lengths for different reasoning and non-reasoning models on AIME24 and AIME25 datasets. Welch statistic of 13.53, 13.3 and 6.9 with p-values < 0.001 indicates the statistical significance of the fact that correct traces are more likely to be shorter.

This aligns with our intuition: all traces begin in a correct state by default (no reasoning has occurred yet). Consider now an idealized oracle model that never transitions from a correct reasoning state to an incorrect one. In this case, any trace generated would either be entirely correct or remain incomplete. Therefore, among all correct traces, the shortest one would always be optimal. As model capacity increases and approaches oracle-like behavior, the likelihood of making an incorrect reasoning step diminishes. Consequently, by favoring shorter correct traces, FFS naturally benefits from the improved reliability of larger models, leading to its enhanced performance.

Testing the "shorter-trace" hypothesis. Using Equation (3.1) with a neutral prior ($\alpha = 0.5$), we estimate the probability that FFS returns a correct answer, for each model (distributions of text lengths shown in Figure 6.2). Table 6.1 reports this probability (Pr) together with normal param-

Table 6.1: Trace-length statistics and predicted FFS success rates, measured per 1000 tokens. Averages are computed over the AIME datasets.

Model	Pr	Correct traces		Incorrect traces	
		μ_1	σ_1	μ_2	σ_2
R1-Distill-Qwen	0.92	7.2	4.8	15.4	6.0
QwQ-32B	0.97	10.7	5.7	21.3	6.7
DeepSeek-V3	0.66	2.2	1.6	4.5	3.2

eters for the trace-length distributions of correct and incorrect generations. The two reasoning models (R1-Distill-Qwen and QwQ-32B) show high success probabilities (92% and 97%) since their correct traces are substantially shorter than their incorrect ones. DeepSeek-V3, a non-reasoning model, exhibits a much smaller gap and a lower predicted success of 66%, mirroring the limited empirical gains we observed for FFS on this model (Table 5.2c).

Chapter 7

Conclusion

We introduced FFS, a training-free and compute-efficient TTS strategy. FFS exploits the empirical observation that, in reasoning models, shorter traces are more likely to be correct: it launches multiple stochastic decodings in parallel and returns the first to finish. Our theoretical analysis and experiments on AIME24, AIME25, and GPQA Diamond with DeepSeek-R1, QwQ-32B, R1-Distill-Qwen, and Phi-4-Reasoning-Plus show that FFS matches or surpasses strong baselines like MV, BS, and BF, while reducing token usage by up to 45%. We further demonstrated that increasing the number of parallel samples improves both accuracy and latency, giving FFS favourable scaling properties.

Limitations and future work. FFS assumes that correct traces tend to terminate earlier than incorrect ones. This pattern holds for the reasoning-oriented models we study, but it is weaker for non-reasoning models such as DeepSeek-V3, where the separation between correct and incorrect trace lengths is small. In such settings, FFS yields limited gains and can be outperformed by alternatives like LFS. Future work could combine FFS with deeper or revisional decoding when model confidence is low, producing a hybrid system that adapts to task difficulty and mitigates the above limitation. Overall, our results illustrate how simple inference-time strategies can unlock large efficiency gains without additional training, and we hope this work inspires further research on lightweight test-time scaling for LLM reasoning.

Chapter 8

Theoretical Results and Proofs

8.1 Short traces are more likely to be correct

Let C be the event that a randomly sampled trace is correct, while S be the event that the trace is short, where we call a trace T short if $|T| \leq s$, for some small length s. Let us call $L = \frac{\Pr(C|S)}{\Pr(C)}$ the "relative lift" of C given S, or in other words, the factor by which the probability of C increases if it becomes known that S has occurred. We would like to show that if correct traces are more likely to be short then L should be larger than 1.

$$\Pr(C \mid S) = \frac{\Pr(S \mid C) \cdot \Pr(C)}{\Pr(S \mid C) \cdot \Pr(C) + \Pr(S \mid \neg C) \cdot \Pr(\neg C)}$$

Or,

$$\begin{aligned} L = \frac{\Pr(C \mid S)}{\Pr(C)} &= \frac{\Pr(S \mid C)}{\Pr(S \mid C) \cdot \Pr(C) + \Pr(S \mid \neg C) \cdot \Pr(\neg C)} \\ &= \frac{1}{\Pr(C) + \frac{\Pr(S|\neg C)}{\Pr(S|C)} \cdot \Pr(\neg C)} \end{aligned}$$

But we know $\frac{\Pr(S|\neg C)}{\Pr(S|C)}$ is vanishingly small as correct traces are much more likely to be short than incorrect ones. Hence we have,

$$L = \frac{1}{\Pr(C)}$$

If $\Pr(C)$ is not yet saturated then we get a lift of $\frac{1}{\Pr(C)}$ which is greater than 1. Notice that our assumption of correct traces being more likely to be short relies on a appropriate definition of "shortness," and hence a suitable value of s. A good choice would be an s which is larger than a significant proportion of correct trace lengths, while at the same time smaller than a major portion of incorrect trace lengths.

8.2 Proof of Result 1

Using Bayes' theorem:

$$\Pr\left(T \text{ is correct} \mid |T| = x\right) = \frac{\Pr\left(T \text{ is correct}, |T| = x\right)}{\Pr\left(T \text{ is correct}, |T| = x\right) + \Pr\left(T \text{ is incorrect}, |T| = x\right)}.$$

Applying the definition of conditional probability:

$$= \frac{\Pr\left(|T| = x \mid T \text{ is correct}\right) \cdot \Pr\left(T \text{ is correct}\right)}{\Pr\left(|T| = x \mid T \text{ is correct}\right) \cdot \Pr\left(T \text{ is correct}\right) + \Pr\left(|T| = x \mid T \text{ is incorrect}\right) \cdot \Pr\left(T \text{ is incorrect}\right)}.$$

Letting the accuracy be $\alpha = \Pr[T \text{ is correct}]$, and thus $\Pr(T \text{ is incorrect}) = 1 - \alpha$, we write:

$$= \frac{\alpha \cdot \mathrm{pdf}_{\text{correct}}(x)}{\alpha \cdot \mathrm{pdf}_{\text{correct}}(x) + (1-\alpha) \cdot \mathrm{pdf}_{\text{incorrect}}(x)}.$$

Assuming that $|T|$ is normally distributed for both correct and incorrect cases:

$$\mathrm{pdf}_{\text{correct}}(x) = \frac{1}{\sigma_1\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{x-\mu_1}{\sigma_1}\right)^2\right].$$

$$\mathrm{pdf}_{\text{incorrect}}(x) = \frac{1}{\sigma_2\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{x-\mu_2}{\sigma_2}\right)^2\right].$$

Substituting these into our expression:

$$\Pr\left[T \text{ is correct} \mid |T| = x\right] = \frac{\frac{1}{\sigma_1\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{x-\mu_1}{\sigma_1}\right)^2\right] \cdot \alpha}{\frac{1}{\sigma_1\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{x-\mu_1}{\sigma_1}\right)^2\right] \cdot \alpha + \frac{1}{\sigma_2\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{x-\mu_2}{\sigma_2}\right)^2\right] \cdot (1-\alpha)}.$$

Canceling out the common $\sqrt{2\pi}$ and rearranging:

$$\Pr\left(T \text{ is correct} \mid |T| = x\right) = \left(1 + \frac{1-\alpha}{\alpha} \cdot \frac{\sigma_1}{\sigma_2} \cdot \exp\left[-\frac{1}{2}\left[\left(\frac{x-\mu_2}{\sigma_2}\right)^2 - \left(\frac{x-\mu_1}{\sigma_1}\right)^2\right]\right]\right)^{-1}.$$

8.3 Proof of Result 2

Suppose $Z_1, Z_2, \ldots, Z_k$ are i.i.d standard normal RVs. Then from extreme value theory [6] we have:

$$\lim_{k\to\infty} \frac{\mathbb{E}[\max\{Z_1, Z_2, \ldots, Z_k\}]}{\sqrt{2\log k}} = 1.$$

Or, in other words, in the limit $n \to \infty$, we have

$$\mathbb{E}[\max\{Z_1, Z_2, \ldots, Z_k\}] = \sqrt{2\log k}.$$

It following that with $Y_i \sim \mathcal{N}(\mu, \sigma)$ we have,

$$\boxed{\mathbb{E}\big[\max\{Y_1, Y_2, \ldots, Z_k\}\big] = \mu + \sigma\sqrt{2\log k}}. \tag{8.1}$$

For the expected minimum of the RVs, replace Y_i with $-Y_i$ in the (3):

$$\mathbb{E}\big[\max\{-Y_1, -Y_2, \ldots, -Y_k\}\big] = -\mu + \sigma\sqrt{2\log k},$$

which means

$$-\mathbb{E}\big[\min\{Y_1, Y_2, \ldots, Y_k\}\big] = -\mu + \sigma\sqrt{2\log k},$$

and therefore

$$\boxed{\mathbb{E}\big[\min\{Y_1, Y_2, \dots, Y_k\}\big] = \mu - \sigma\sqrt{2\log k}}.$$

8.4 Experimental Details

8.4.1 Reasoning templates

Table 8.1 and 8.2 highlights the reasoning templates to generate the reasoning traces for AIME and GPQA examples, respectively.

Table 8.1: Reasoning template for AIME24 and AIME25 tasks.

Task Template

{problem}

Please reason step by step, and put your final answer within \boxed{}.

Table 8.2: Reasoning template for GPQA.

Task Template

What is the correct answer to this question:
{problem}

Choices:
(A) {Option 1}
(B) {Option 2}
(C) {Option 3}
(D) {Option 4}

Answer: (A), (B), (C), or (D). Choose the correct option within \boxed{}.

8.4.2 Output parsing logic

We provide the Python implementation for parsing the final response from the reasoning traces for the AIME and GPQA datasets in Listings 1 and 2, respectively.

```python
DIGITS = ["1", "2", "3", "4", "5", "6", "7", "8", "9", "0"]

import itertools
import re

def parse_output(output):
    last = output.rfind("\\boxed{")
    if last == -1:
        return "NO_ANSWER"
    cnt = 0
    start = last + len("\\boxed{")
    curr = start
    while(cnt >= 0 and curr < len(output)):
        if (output[curr] == "{"):
            cnt += 1
        elif (output[curr] == "}"):
            cnt -= 1
        curr += 1
    if cnt >= 0:
        return "NO_ANSWER"
    answer = output[start:curr-1]
    return answer

def convert_answer_to_aime_format(trace):
    answer = parse_output(trace)
    length = len(answer)
    char_is_digit = True
    for char in answer:
        if char not in DIGITS:
            char_is_digit = False

    if not char_is_digit or length > 3:
        return  "NO_ANSWER"
    return "0"*(3-length) + answer
```

Listing 8.1: Parsing logic for traces for AIME24 and AIME25 examples.

```python
pattern = r'(\\boxed\{.*?\})'
match = re.findall(pattern, text)
if len(match) > 0:
    return match[-1]
```

Listing 8.2: Parsing logic for traces for GPQA examples.

8.4.3 Hyperparameters

All hyperparameters for our experiments are given in Table 8.3.

Table 8.3: Decoding hyper-parameters used in all experiments. Values that are identical across datasets are shown once.

Hyper-parameter	**GPQA**	**AIME24**	**AIME25-I**	**AIME25-II**
Max generation tokens	16K	32K	32K	32K
BF answer-reserve	3K	3K	3K	3K
Global settings (shared by every dataset)				
Top-p			0.95	
Temperature			0.6	
Samples n (MV/LFS/FFS)			4	
Beam width (Beam Search)			4	

8.5 Reasoning Trace Samples

Incorrect trace of DeepSeek-R1 on AIME24 problem 5 [longer]

Problem

Let p be the least prime number for which there exists a positive integer n such that $n^4 + 1$ is divisible by p^2. Find the least positive integer m such that m^4+1 is divisible by p^2. Please reason step by step, and put your final answer within $\boxed{}$.

Thinking

Okay, so I need to find the least prime number p such that there's a positive integer n where n^4+1 is divisible by p^2. Then, find the smallest m where m^4+1 is divisible by p^2. Hmm, let me start by understanding

the problem step by step.
First, primes p where $n^4 \equiv -1 \mod p$. So, $n^4 \equiv -1 \mod p$. That equation suggests that the order of n modulo p divides 8, because $n^8 \equiv 1 \mod p$. Since $n^4 \equiv -1 \mod p$, squaring both sides gives $n^8 \equiv 1 \mod p$, so the order is 8. Therefore, 8 divides $p-1$ because the multiplicative group modulo p is cyclic of order $p-1$. Hence, $p \equiv 1 \mod 8$. Wait, but is that the only possibility?
Wait, if p is an odd prime, then the multiplicative group $\mod p$ has order $p-1$. So, for there to be an element of order 8, 8 must divide $p-1$, so $p \equiv 1 \mod 8$. So primes $p \equiv 1 \mod 8$. But the problem says "the least prime p". The primes congruent to $1 \mod 8$ start at 17, 97, etc. Wait, no, 17 is $1 \mod 8$? Let me check: 17 divided by 8 is $2 * 8 = 16$, so remainder 1. Yes, $17 \equiv 1 \mod 8$. Then 97 is $12 * 8 + 1 = 97$. But hold on, maybe there are smaller primes where $n^4 \equiv -1 \mod p$? Wait, but according to the logic above, if $n^4 \equiv -1 \mod p$, then n has order $8 \mod p$, so 8 divides $p-1$, so $p \equiv 1 \mod 8$. Therefore, primes $p \equiv 1 \mod 8$. The smallest such primes are $17, 97, 113, 193$, etc. Wait, but 17 is the first prime $\equiv 1 \mod 8$. Then 97 is next.
But wait, the problem is not just asking for primes where $n^4 \equiv -1 \mod p$, but where p^2 divides $n^4 + 1$. So, primes p where p^2 divides $n^4 + 1$. So, we need to find primes p such that $n^4 \equiv -1 \mod p^2$. Then, the least such prime p, and then the least m for that p.
I remember that sometimes primes that satisfy a congruence modulo p can be lifted to modulo p^2 using Hensel's lemma, but only if certain conditions are met. For example, if $f(n) \equiv 0 \mod p$ and $f'(n) \not\equiv 0 \mod p$, then there is a unique lift to $\mod p^2$.
So, if we have $n^4 \equiv -1 \mod p$, and the derivative $4n^3 \not\equiv 0 \mod p$, then Hensel's lemma says we can lift n to a solution modulo p^2. So, if $p \neq 2$, and n is not $0 \mod p$, which it can't be since $n^4 \equiv -1 \mod p$. So, $4n^3 \equiv 0 \mod p$ would require p divides $4n^3$, but since p is prime and n is not $0 \mod p$, p must divide 4. So, $p = 2$. But if p is 2, then $n^4 + 1$. Let's check $p = 2$. Let's see $n = 1 : 1 + 1 = 2$, which is divisible by 2, but not 4. $n = 3 : 81 + 1 = 82$, which is $2 * 41$, not divisible

by 4. So, $2^2 = 4$ does not divide $n^4 + 1$ for any n. Because n odd: $n^4 \equiv 1 \mod 4$, so $1 + 1 = 2 \mod 4$. So 2 isn't going to work. So, for primes $p \equiv 1 \mod 8$, we can have solutions to $n^4 \equiv -1 \mod p$, and if the derivative $4n^3$ is not $0 \mod p$, which since $p \equiv 1 \mod 8$, p is at least 17, so p doesn't divide 4, and since $n \not\equiv 0 \mod p$, derivative non-zero. Therefore, Hensel's lemma applies, so solutions can be lifted to $\mod p^2$.
Therefore, primes $p \equiv 1 \mod 8$ will have solutions $\mod p^2$. So, the smallest such prime is 17.
Wait, but before assuming that, maybe check smaller primes. For example, $p = 5$. Let's see if there's an n with $n^4 \equiv -1 \mod 5$. Let's compute $n^4 \mod 5$. Since $1^4 = 1, 2^4 = 16 \equiv 1, 3^4 = 81 \equiv 1, 4^4 = 256 \equiv 1$. So $n^4 \equiv 1 \mod 5$ for n not divisible by 5. Therefore, $-1 \equiv 4 \mod 5$, but $n^4 \equiv 1 \mod 5$. So no solutions. Similarly, $p = 13$. Let's check $n^4 \mod 13$. Let's compute $1^4 = 1, 2^4 = 16 \equiv 3, 3^4 = 81 \equiv 3, 4^4 = 256 \equiv 9, 5^4 = 625 \equiv 625 - 48 * 13 = 625 - 624 = 1, 6^4 = 1296 \equiv 1296 - 99 * 13 = 1296 - 1287 = 9$, etc. So $n^4 \mod 13$ is $1, 3, 9$. So $-1 \mod 13$ is 12. So no n with $n^4 \equiv 12 \mod 13$. So, $p = 13$ is out.
Similarly, $p = 17$. Let's check if there is an n with $n^4 \equiv -1 \mod 17$. $-1 \mod 17$ is 16. So, $n^4 \equiv 16 \mod 17$. Let's see. Let's compute n^4 for $n = 1$ to 16.
$1^4 = 1$
$2^4 = 16$.
Oh, so $2^4 = 16 \equiv -1 \mod 17$. So $n = 2$ is a solution. So $p = 17$ is a prime where $n = 2$, $2^4 + 1 = 17$, which is divisible by 17. Now, can we check if 17^2 divides $2^4 + 1$? $2^4 + 1 = 17$, which is not divisible by $17^2 = 289$. So, $2^4 + 1 = 17$. So, we need to find some n where $n^4 + 1$ is divisible by 17^2. So, $n^4 \equiv -1 \mod 17^2$. Since $n = 2$ is a solution mod 17, and the derivative $4n^3 = 4 * 8 = 32 \equiv 15 \mod 17$, which is not $0 \mod 17$. So, Hensel's lemma tells us that there is a unique lift mod 17^2. So, there exists some $n \equiv 2 \mod 17$ such that $n^4 \equiv -1 \mod 17^2$. So, the minimal solution m would be $2 + 17k$ for some k. Let's compute $(2 + 17k)^4 \mod 17^2$.

First, compute $(2+17k)^4$. Let's expand using the binomial theorem:
$= 2^4 + 4*2^3*(17k) + 6*2^2*(17k)^2 + 4*2*(17k)^3 + (17k)^4$
Modulo 17^2, terms with $(17k)^2$ or higher will be multiples of 17^2, so they can be ignored. Therefore, modulo 17^2, this is:
$2^4 + 4*2^4*(17k) \mod 17^2$
Compute $2^4 = 16$.
Then, $4*8*17k = 32*17k$. $32 \equiv 15 \mod 17$, so $15*17k = 255k$. But $255 = 15*17$, so $255k = 15*17k$. Thus, $\mod 17^2$, $255k \equiv 15*17k \mod 17^2$. So, the entire expression is $16 + 15*17k \mod 17^2$.
We need this to be $\equiv -1 \mod 17^2$. Since $-1 \mod 17^2$ is $289 - 1 = 288$. So, $16 + 15*17k \equiv 288 \mod 289$.
Subtract 16: $15*17k \equiv 272 \mod 289$.
Compute $15*17 = 255$. So, $255k \equiv 272 \mod 289$.
Solve $255k \equiv 272 \mod 289$.
First, note that 255 and 289: 289 is 17^2, and $255 = 15*17$. So $\gcd(255, 289) = 17$. Let's check if 17 divides 272. 272 divided by 17 is 16. So, $17*16 = 272$. So, the equation is $15*17k \equiv 17*16 \mod 17^2$.
Divide both sides by 17: $15k \equiv 16 \mod 17$.
Solve $15k \equiv 16 \mod 17. 15^-1 \mod 17$. $15*? \equiv 1 \mod 17$. $15*15 = 225 \equiv 225-13*17 = 225-221 = 4 \mod 17$. $15*4 = 60 \equiv 60-3*17 = 60-51 = 9 \mod 17$. $15*7 = 105 \equiv 105-6*17 = 105-102 = 3 \mod 17. 15*12 = 180 \equiv 180-10*17 = 180-170 = 10 \mod 17$. Wait, maybe better way: $17 = 15+2, 15 = 2*7+1$,so $1 = 15-2*7$ $= 15-(17-15)*7 = 15-7*17+7*15 = 8*15-7*17$. So inverse of $15 \mod 17$ is 8. Therefore, $15*8 = 120 \equiv 120-7*17 = 120-119$ $= 1 \mod 17$. So inverse of 15 is 8. Therefore, $k \equiv 16*8 = 128 \equiv 128-7*17 = 128-119 = 9 \mod 17$.
Therefore, $k \equiv 9 \mod 17$. So the minimal positive k is 9. Therefore, $n = 2+17*9 = 2+153 = 155$.
Therefore, $m = 155$. But wait, let's check if 155^4+1 is divisible by $17^2 = 289$.
But $155 \mod 17$: $17*9 = 153$, so $155 \equiv 2 \mod 17$. Then $2^4+1 = 17$, which is $0 \mod 17$, as expected. Now, to check $\mod 289$. Let's

compute $155^4 + 1 \mod 289$.
But calculating 155^4 is tedious. Maybe there's a smarter way. Since we already used Hensel's lemma, the lift is unique, so 155 should work. But just to confirm, perhaps compute $(2 + 17 * 9)^4 + 1 \mod 289$.
Let me compute $(2 + 17 * 9)^4 = (2 + 153)^4 = 155^4$. Let's compute 155 mod 289 is 155.
First, compute $155^2 = 24025$. Then $24025 \mod 289$. Let's compute how many times 289 goes into 24025.
$289*80 = 23120$. $24025-23120 = 905$. $289*3 = 867$. $905-867 = 38$. So, $155^2 \equiv 38 \mod 289$.
Then, compute $38^2 = 1444$. $1444 \mod 289$. $289 * 5 = 1445$. So $1444 = 1445 - 1 \equiv -1 \mod 289$. Therefore, $155^4 = (155^2)^2 \equiv (-1)^2 = 1 \mod 289$. Therefore, $155^4 + 1 \equiv 1 + 1 = 2 \mod 289$. Wait, that's not 0. That contradicts our earlier conclusion. Hmm, so something's wrong here. Where did I go wrong?
Wait, let's retrace. So, using Hensel's lemma, the solution should exist. So why is $155^4 + 1 \equiv 2 \mod 289$? That suggests an error in the calculation.
Wait, first, let me compute 155^2. $155 * 155$. Let's compute $150 * 150 = 22500$, $150 * 5 = 750, 5 * 150 = 750$, $5 * 5 = 25$. So, $(150 + 5)^2 = 150^2 + 2 * 150 * 5 + 5^2 = 22500 + 1500 + 25 = 24025$. Correct. Then 24025 divided by 289. Let's compute $289 * 80 = 23120$, as before. $24025 - 23120 = 9054$. Then, $289 * 3 = 867$. $905 - 867 = 38$. So $155^2 \equiv 38 \mod 289$. Then $38^2 = 1444$. $289 * 5 = 1445$. $1444 = 1445 - 1 \equiv -1 \mod 289$. Therefore, $155^4 = (155^2)^2 \equiv (-1)^2 = 1 \mod 289$. Therefore, $155^4 + 1 \equiv 1 + 1 = 2 \equiv 2 \mod 289$. So $2 \equiv 2 \mod 289$, not 0. So this is a problem. That suggests that my earlier reasoning was wrong.
Hmm, so why? Maybe I made a mistake in Hensel's lemma application. Let's recall Hensel's lemma. If $f(n) \equiv 0 \mod p$ and $f'(n) \not\equiv 0 \mod p$, then there exists a unique lift $\mod p^2$. Here, $f(n) = n^4 + 1$. Then $f'(n) = 4n^3$. So, for $n = 2 \mod 17, f'(2) = 4*8 = 32 \equiv 15 \mod 17 \neq 0$. So, Hensel's lemma says there is a unique solution $\mod 17^2$ lifting

$n \equiv 2 \mod 17$. So, why does 155 not work?
Wait, maybe my calculation is wrong. Let me check $155^4 + 1$ again.
Alternatively, compute $(2 + 17k)^4 + 1 \mod 289$.
We had earlier that $(2 + 17k)^4 \equiv 16 + 15 * 17k \mod 289$. So, setting that equal to$0 \mod 289$, we get $16 + 15 * 17k \equiv 0 \mod 289$.
So $15 * 17k \equiv -16 \mod 289$. Which is $255k \equiv 273 \mod 289$. Wait, earlier I thought it was 272, but $-16 \mod 289$ is 273. Wait, yes, $289 - 16 = 273$. So $255k \equiv 273 \mod 289$. Then, as before, divide both sides by 17: $15k \equiv 273/17 \mod 17$. 273 divided by 17 is 16.058... Wait, $17 * 16 = 272$, so $273 = 17 * 16 + 1$. So $273 = 17 * 16 + 1$. Therefore, $273/17 = 16 + 1/17$. So, $255k \equiv 273 \mod 289$ is equivalent to $15k \equiv 16 + 1/17 \mod 17$. Wait, but dividing both sides by 17 in the congruence $255k \equiv 273 \mod 289$, which is $17 * 15k \equiv 17 * 16 + 1 \mod 17^2$. Therefore, divide both sides by 17: $15k \equiv 16 + 1/17 \mod 17$. Wait, but $1/17 \mod 17$? That doesn't make sense. Hmm, perhaps I made a mistake here.
Alternatively, maybe there's a miscalculation in the earlier steps. Let's start over.
We have $n \equiv 2 \mod 17$, and we want to find k such that $(2 + 17k)^4 \equiv -1 \mod 289$.
Compute $(2 + 17k)^4$:
$= 2^4 + 4 * 2^3 * (17k) + 6 * 2^2 * (17k)^2 + 4 * 2 * (17k)^3 + (17k)^4$
Modulo 289, terms with $(17k)^2, (17k)^3, (17k)^4$ are multiples of 289, so they vanish. So, we have:
$2^4 + 4 * 2^3 * (17k) \mod 289 = 16 + 4 * 8 * 17k$
$= 16 + 32 * 17k$
$32 * 17 = 544$.
So, $16 + 544k \mod 289$.
But $544 \mod 289 = 544 - 2 * 289 = 544 - 578 = -34$. So, $544k \equiv -34k \mod 289$.
Thus, the expression is $16 - 34k \mod 289$.
Set this equal to $-1 \mod 289$:
$16 - 34k \equiv -1 \mod 289$

So, $-34k \equiv -17 \mod 289$
Multiply both sides by -1: $34k \equiv 17 \mod 289$
Divide both sides by 17 : $2k \equiv 1 \mod 17$
So, $2k \equiv 1 \mod 17$. Multiply both sides by inverse of $2 \mod 17$, which is 9 because $2 * 9 = 18 \equiv 1 \mod 17$. So, $k \equiv 9 \mod 17$.
Thus, $k = 9 + 17t$. The minimal positive k is 9. Thus, $n = 2 + 17 * 9 = 155$. Therefore, $m = 155$. But according to our previous calculation, $155^4 + 1 \equiv 2 \mod 289$. Which contradicts. Therefore, my error must be in the calculation of $155^4 + 1 \mod 289$.
Wait, let me compute 155^2 again. $155^2 = 24025$. 24025 divided by 289:
$289 * 80 = 23120$
$24025 - 23120 = 905$
$289 * 3 = 867$
$905 - 867 = 38$
So $155^2 \equiv 38 \mod 289$. Then, $38^2 = 1444$. 1444 divided by 289:
$289 * 5 = 1445$
$1444 = 1445 - 1$, so $38^2 \equiv -1 \mod 289$. Therefore, $155^4 = (38)^2 \equiv (-1) \mod 289$? Wait, no: $38^2 \equiv -1 \mod 289$. Therefore, $155^4 \equiv (-1) \mod 289$. Then $155^4 + 1 \equiv -1 + 1 = 0 \mod 289$. Wait, but then why did I think before that $38^2 = 1444 \equiv -1 \mod 289$? Let's check $1444 \mod 289$.
$289 * 5 = 1445$. So $1444 = 1445 - 1$. Therefore, $1444 \equiv -1 \mod 289$. Therefore, $38^2 \equiv -1 \mod 289$, so $155^4 = (38)^2 \equiv (-1) \mod 289$. Therefore, $155^4 + 1 \equiv -1 + 1 = 0 \mod 289$. So, $155^4 + 1$ is divisible by 289. Wait, then my previous calculation was wrong. How did I get $155^4 + 1 \equiv 2 \mod 289$ before? Maybe miscalculation. Let me check step by step.
$155^2 = 24025$. 24025 divided by 289: $289*80 = 23120$, $24025-23120 = 905$. 905 divided by 289: $289 * 3 = 867$, $905 - 867 = 38$. So $155^2 \equiv 38 \mod 289$. Then $38^2 = 1444$. 1444 divided by 289 : $289 * 5 = 1445$, so $1444 = 1445 - 1 \equiv -1 \mod 289$. So $155^4 = (155^2)^2 \equiv (-1) \mod 289$. Therefore, $155^4 + 1 \equiv -1 + 1 = 0 \mod 289$. Therefore, $155^4 + 1$ is divisible by 289. So my earlier assertion that $155^4{+}1 \equiv 2 \mod 289$ was

incorrect. Probably a miscalculation. Therefore, $m = 155$ is indeed the minimal solution.

But wait, the problem asks for the least positive integer m such that $m^4 + 1$ is divisible by $p^2 = 17^2 = 289$. So, if 155 is the minimal solution, but maybe there is a smaller m. Since we have to find the minimal m, let's check numbers congruent to $2 \mod 17$ less than 155. Because from Hensel's lemma, the solution is unique $\mod 17^2$, so the minimal positive solution is 155. But $155 = 2 + 17 * 9$. Is there a smaller positive integer congruent to $2 \mod 17$ that works? For example, $2 + 17 * 1 = 19, 2 + 17 * 2 = 36, \ldots,$ up to 155.

Wait, but Hensel's lemma gives a unique solution $\mod 17^2$, so the minimal positive solution is 155, but let's verify with $m = 19$. Compute$19^4 + 1$. $19^2 = 361$. $361^2 = 130321$. $130321 + 1 = 130322$. Now divide by 289: $289 * 450 = 130050$. $130322 - 130050 = 272$. 272 is divisible by 17^2? $17^2 = 289$, which is bigger than 272. So no. So, $19^4 + 1 \equiv 272 \mod 289 \neq 0$. Similarly, check $m = 36$. $36^2 = 1296$, $36^4 = 1296^2 = 1679616$. $1679616 + 1 = 1679617$. Divide by 289: $289 * 5800 = 1676200$. $1679617 - 1676200 = 3417$. 3417 divided by 289: $289 * 11 = 3179$, $3417 - 3179 = 238$. 238 is less than 289. So $36^4 + 1 \equiv 238 \mod 289 \neq 0$. So, not divisible.

Similarly, check $m = 53$. $53^2 = 2809$, $53^4 = 2809^2 = 7890481$. $7890481+1 = 7890482$. Divide by 289: $289*27300 = 78900*289 =$Wait, maybe better to compute $\mod 289$ directly. $53 \mod 289$ is 53. $53^2 = 2809$. $2809 \mod 289$: $289*9 = 2601$, $2809-2601 = 208$. $208^2 = 43264$. $43264 \mod 289$. $289 * 149 = 289 * (150 - 1) = 289 * 150 - 289 = 43350 - 289 = 43061$. $43264 - 43061 = 203$. So $53^4 + 1 \equiv 203 + 1 = 204 \mod 289 \neq 0$. So, not divisible.

Similarly, $m = 70$. $70 \mod 289$ is 70. $70^2 = 4900$. $4900 \mod 289$: $289 * 16 = 4624$, $4900 - 4624 = 276$. $276^2 = 76176$. $76176 \mod 289$. $289*263 = 289*(260+3) = 289*260+289*3 = 75140+867 = 76007$. $76176 - 76007 = 169$. So $70^4 + 1 \equiv 169 + 1 = 170 \mod 289 \neq 0$.

Continue up to $m = 155$. Since all smaller $m \equiv 2 \mod 17$ don't work, and Hensel's lemma tells us 155 is the unique solution $\mod 289$, so

155 is the minimal solution. Therefore, the answer is $m = 155$.
But let me check another approach. Maybe the minimal m is 2, but $2^4 + 1 = 17$, not 289. 38: wait, $38^4 + 1$. Wait, $38^2 = 1444$, $1444^2 = 2085136$. $2085136 + 1 = 2085137$. Divide by $289 : 289 * 7214 = 289*(7000+214) = 289*7000+289*214 = 2023000+61846 = 2084846$. $2085137 - 2084846 = 291$. $291 \mod 289 = 2$. So not divisible.
Alternatively, since Hensel's lemma gives a unique solution, the minimal m is 155. Therefore, the answer is 155.
But let me check in the literature. I recall that 2 is the smallest solution mod 17, and lifting it gives155. Also, since17 is the smallest prime$\equiv 1 \mod 8$ where $n^4 + 1 \equiv 0 \mod p^2$. Wait, but wait, maybe $p = 2$? But $2^2 = 4$ divides n^4+1? If n is odd, $n^4 \equiv 1 \mod 4$, so $n^4+1 \equiv 2 \mod 4$, so not divisible by 4. So $p = 2$ is invalid. Next prime, $p = 17$. So, the minimal prime p is 17, and minimal m is 155.
Yes, I think that's correct. Therefore, the answer is $\boxed{155}$.

Final Answer

To find the least prime p such that there exists a positive integer n where $n^4 + 1$ is divisible by p^2, we start by noting that $n^4 \equiv -1 \pmod{p}$. This implies that the order of n modulo p is 8, meaning 8 must divide $p - 1$. Therefore, $p \equiv 1 \pmod{8}$. The smallest prime p satisfying this is 17.
Next, we need to find the smallest positive integer m such that $m^4 + 1$ is divisible by $17^2 = 289$. Using Hensel's lemma, we lift the solution from modulo 17 to modulo 289. We start with $n = 2$ since $2^4 \equiv -1 \pmod{17}$.
We expand $(2 + 17k)^4$ using the binomial theorem and consider terms modulo 289. The relevant terms are:

$$(2 + 17k)^4 \equiv 2^4 + 4 \cdot 2^3 \cdot 17k \pmod{289}$$

$$= 16 + 32 \cdot 17k$$

$$= 16 + 544k$$

Since 544 modulo 289 is -34, we have:

$$16 - 34k \equiv -1 \pmod{289}$$

Solving for k:

$$-34k \equiv -17 \pmod{289}$$

$$34k \equiv 17 \pmod{289}$$

$$2k \equiv 1 \pmod{17}$$

The multiplicative inverse of 2 modulo 17 is 9, so:

$$k \equiv 9 \pmod{17}$$

The smallest positive k is 9, giving $m = 2 + 17 \cdot 9 = 155$.
Verification shows $155^4 + 1 \equiv 0 \pmod{289}$, confirming $m = 155$.
Thus, the least positive integer m such that $m^4 + 1$ is divisible by 17^2 is $\boxed{155}$.

Correct trace of DeepSeek-R1 on AIME24 problem 5 [shorter]

Problem

Let p be the least prime number for which there exists a positive integer n such that $n^4 + 1$ is divisible by p^2. Find the least positive integer m such that m^4+1 is divisible by p^2. Please reason step by step, and put your final answer within $\boxed{}$.

Thinking

Okay, so I need to find the least prime number p such that there's a positive integer n where $n^4 + 1$ is divisible by p squared. Then, I have to find the smallest positive integer m where m^4+1 is divisible by that p squared. Hmm, this seems like a number theory problem involving primes and modular arithmetic. Let me think step by step.
First, let me recall that if $n^4 \equiv -1 \mod p^2$, then $n^8 \equiv 1 \mod p^2$. So the order of n modulo p^2 divides 8. But since $n^4 \equiv -1 \mod p^2$, the

order can't be 1, 2, or 4 because squaring -1 gives 1, so maybe the order is 8. Therefore, 8 divides the order of the multiplicative group modulo p^2. The multiplicative group modulo p^2 has order $p(p-1)$. So 8 divides $p(p-1)$. Since p is prime, either $p = 2$ or $p \equiv 1 \mod 8$. Wait, but 2 is a prime. Let me check $p = 2$ first.
If $p = 2$, then $p^2 = 4$. So we need $n^4 + 1 \equiv 0 \mod 4$. Let's check $n = 1$: $1 + 1 = 2 \equiv 2 \mod 4$. Not 0. $n = 2$: $16 + 1 = 17 \equiv 1 \mod 4$. $n = 3{:}81 + 1 = 82 \equiv 2 \mod 4$. So no n exists such that $n^4 + 1 \equiv 0 \mod 4$. Therefore, $p = 2$ is invalid. So p must be $1 \mod 8$. The primes congruent to $1 \mod 8$ are 17, 97, etc. Wait, but maybe smaller primes. Wait, 17 is the next prime after 2, 3, 5, 7, 11, 13. Let me check $p = 5$. Wait, 5 is $5 \mod 8$, right? $5 \mod 8$ is 5. Similarly, 17 is $1 \mod 8$.
Wait, maybe I need to check primes where p divides $n^4 + 1$. So maybe primes where -1 is a fourth power residue modulo p. For $p \equiv 1 \mod 8$, since the multiplicative group is cyclic of order $p-1$, which is divisible by 8, so there are elements of order 8, hence $x^4 = -1$ would have solutions. So primes $p \equiv 1 \mod 8$ might be candidates. But maybe also other primes where $p \equiv 5 \mod 8$? Let me think.
Wait, for primes $p \equiv 1 \mod 4$, -1 is a quadratic residue. For primes $p \equiv 1 \mod 8$, it's also a quartic residue. Hmm. So maybe the primes where -1 is a fourth power residue are primes $p \equiv 1 \mod 8$. Let me check that. Suppose $p \equiv 1 \mod 8$. Then the multiplicative group $\mod p$ has order $p - 1 \equiv 0 \mod 8$, so there exists an element of order 8. Let g be such an element. Then g^4 would have order 2, so $g^4 \equiv -1 \mod p$. Therefore, $x^4 \equiv -1 \mod p$ has solutions. So such primes $p \equiv 1 \mod 8$ would divide $n^4 + 1$ for some n.
But the problem is asking for primes where p^2 divides n^4+1. So I need primes p where the equation $x^4 \equiv -1 \mod p^2$ has solutions. So first, p must divide $n^4 + 1$, so p is as above, $p \equiv 1 \mod 8$. Then, lifting solutions from $\mod p$ to $\mod p^2$. By Hensel's lemma, if $f(n) = n^4 + 1$, and if $f(n) \equiv 0 \mod p$ and $f'(n) \equiv 0 \mod p$, then there's a unique solution $\mod p^2$. Wait, but Hensel's lemma says that if $f(n) \equiv 0 \mod p$ and $f'(n) \not\equiv 0 \mod p$, then there's a unique lift. If

$f'(n) \equiv 0 \mod p$, then maybe multiple solutions or no solutions.
So first, let me suppose that $p \equiv 1 \mod 8$. Let's pick the smallest such prime, which is 17. Then check if there exists n such that $n^4 \equiv -1 \mod 17^2$. Alternatively, maybe check 97, but 17 is smaller.
Wait, maybe even 5? Let me check. For $p = 5$, does there exist n with $n^4 \equiv -1 \mod 25$? Let's compute $n^4 \mod 25$. Since $5 \equiv 5 \mod 8$, maybe $n^4 \equiv -1 \mod 5$is possible?
Wait, for $p = 5$. Let's check $n^4 \mod 5$. The fourth powers $\mod 5$: $0^4 = 0$, $1^4 = 1$, $2^4 = 16 \equiv 1$, $3^4 = 81 \equiv 1$, $4^4 = 256 \equiv 1$. So $n^4 \equiv 0$ or1 $\mod 5$. Therefore, $n^4 \equiv -1 \mod 5$ is equivalent to $1 \equiv -1 \mod 5$, which is $2 \equiv 0 \mod 5$. Not possible. So $p = 5$ is out.
Similarly, check $p = 13$. $13 \equiv 5 \mod 8$. So again, $n^4 \mod 13$: Let's compute. $1^4 = 1$, $2^4 = 16 \equiv 3$, $3^4 = 81 \equiv 3$, $4^4 = 256 \equiv 9$, $5^4 = 625 \equiv 625 - 48 * 13$=$625 - 624 = 1$, $6^4 = 1296 \equiv 1296 - 99 * 13 = 1296 - 1287 = 9$, $7^4 = 2401 \equiv 2401 - 184 * 13 = 2401 - 2392 = 9$, $8^4 = 4096 \equiv 4096 - 315 * 13 = 4096 - 4095 = 1$, etc. So possible residues are 1,3,9. So $-1 \mod 13$ is 12, which is not among them. Hence $p = 13$ is out.
Similarly, $p = 17$, which is $1 \mod 8$. Let's check if there exists n with $n^4 \equiv -1 \mod 17$. Let's compute 4th powers $\mod 17$.
Let's list n from 1 to 16:
$1^4 = 1$
$2^4 = 16$
$3^4 = 81 \equiv 81 - 4 * 17 = 81 - 68 = 13$
$4^4 = 256 \equiv 256 - 15 * 17 = 256 - 255 = 1$
$5^4 = 625 \equiv 625 - 36 * 17 = 625 - 612 = 13$
$6^4 = 1296 \equiv 1296 - 76 * 17 = 1296 - 1292 = 4$
$7^4 = 2401 \equiv 2401 - 141 * 17 = 2401 - 2397 = 4$
$8^4 = 4096 \equiv 4096 - 241 * 17 = 4096 - 4097 = -1 \equiv 16 \mod 17$
Wait, $8^4 \equiv -1 \mod 17$. So $n = 8$ satisfies $n^4 \equiv -1 \mod 17$. Therefore, $p = 17$ is a candidate. Now, we need to check if there exists n such that $n^4 \equiv -1 \mod 17^2$. If so, then $p = 17$ is the minimal prime. Since 17 is the smallest prime $\equiv 1 \mod 8$, and since smaller primes

like 2,5,13 don't work, maybe 17 is the answer. Then, we need to find the minimal m such that$m^4 \equiv -1 \mod 289(17^2)$.
So first, let's confirm that 17 is indeed the minimal prime. Next prime after 17 is 97, which is larger, so 17 is the candidate.
So to confirm that 17 is the least prime, we have to check all primes less than 17. The primes less than 17 are 2,3,5,7,11,13. For these primes, as checked above, none of them have $n^4 \equiv -1 \mod p$. Except $p = 17$, which does. So 17 is the least such prime.
Now, need to find the minimal m such that$m^4 \equiv -1 \mod 289$. Let's use Hensel's lemma. We know that $8^4 \equiv -1 \mod 17$. Let's try lifting this solution to $\mod 17^2$.
Let $m = 8+17k$, where k is an integer. Then compute $m^4+1 \mod 17^2$.
First, expand $(8 + 17k)^4$.
Using binomial theorem:
$(8+17k)^4 = 8^4+4*8^3*(17k)+6*8^2*(17k)^2+4*8*(17k)^3+(17k)^4$.
Compute each term modulo $17^2 = 289$.
First term:$8^4 = 4096$. We know $4096 \mod 289$. Let me compute $289 * 14 = 4046$. $4096 - 4046 = 50$. So $4096 \equiv 50 \mod 289$. Then $4096 + 1 = 50 + 1 = 51 \equiv 51 \mod 289$. So we need the sum of all terms to be $\equiv 0 \mod 289$. So:
$(8 + 17k)^4 + 1 \equiv 50 + 1 + 4 * 8^3 * 17k + 6 * 8^2 * (17k)^2 + \dots \mod 289$.
But let me compute each term:
First term: $8^4 = 4096 \equiv 50 \mod 289$.
Second term:$4 * 8^3 * 17k = 4 * 512 * 17k = 2048 * 17k$. $2048 \mod 289$: $289*7 = 2023$, $2048-2023 = 25$. So $25*17k = 425k$. $425 \mod 289 = 425 - 289 = 136$. So$136k \mod 289$.
Third term:$6 * 8^2 * (17k)^2 = 6 * 64 * 289k^2$. Since $289k^2 \equiv 0 \mod 289$, this term is$0 \mod 289$.
Fourth term:$4 * 8 * (17k)^3 = 32 * 4913k^3$. $4913 = 17^3$, so $4913 \equiv 0 \mod 289$. Hence this term is $0 \mod 289$.
Fifth term:$(17k)^4 = 17^4k^4 = 83521k^4 \equiv 0 \mod 289$.
So, combining all terms:
$(8 + 17k)^4 + 1 \equiv 50 + 1 + 136k \mod 289 \equiv 51 + 136k \mod 289$.

We need this $\equiv 0 \mod 289$, so:

$51 + 136k \equiv 0 \mod 289 \implies 136k \equiv -51 \mod 289$.

Let me compute $-51 \mod 289$: $289 - 51 = 238$. So $136k \equiv 238 \mod 289$.

We need to solve for k: $136k \equiv 238 \mod 289$.

First, note that $\gcd(136, 289)$. $289 = 17^2$. $136 = 17 * 8$. So $\gcd(136, 289) = 17$. Since 17 divides 136 and 289. So the equation $136k \equiv 238 \mod 289$ has solutions iff 17 divides 238. Let's check:$238/17 = 14$. So $17 * 14 = 238$. Therefore, 17 divides 238, so solutions exist.

The equation reduces to $(136/17)k \equiv (238/17) mod (289/17)$: $8k \equiv 14 \mod 17$.

So $8k \equiv 14 \mod 17$. Multiply both sides by inverse of $8 \mod 17$. $8 * 2 = 16 \equiv -1$, so $8 * 15 = 120 \equiv 120 - 7 * 17 = 120 - 119 = 1 \mod 17$. So inverse of 8 is $15 \mod 17$.

Thus, $k \equiv 14*15 \mod 17$. $14*15 = 210 \equiv 210-12*17 = 210-204 = 6 \mod 17$. So $k \equiv 6 \mod 17$. So the solutions are $k = 6+17t$. Therefore, the minimal positive k is 6.

Therefore, $m = 8 + 17 * 6 = 8 + 102 = 110$. Therefore, $m = 110$ is the minimal positive integer such that $m^4 + 1 \equiv 0 \mod 289$.

But wait, let me verify this. Let's compute $110^4 + 1 \mod 289$.

But $110 \mod 17$: $17 * 6 = 102$, $110 - 102 = 8$. So $110 \equiv 8 \mod 17$. Then $8^4 + 1 = 4096 + 1 = 4097$. 4097 divided by 17: $17 * 241 = 4097$. Therefore, $4097 = 17 * 241$. Then 241 divided by 17:$17 * 14 = 238$, $241 - 238 = 3$. So $241 = 17 * 14 + 3$. So $4097 = 17^2 * 14 + 17 * 3$. So $4097 = 289*14+51$. Therefore, $4097 \mod 289 = 51$. So $110^4+1 \equiv 51 \mod 289$. But we needed it to be 0. Wait, that contradicts. So what did I do wrong?

Wait, no. Wait, $m = 110$. Let me compute $110^4 + 1$. Alternatively, perhaps my calculation in Hensel's lemma was wrong. Let's go back.

Wait, when I lifted the solution from $\mod 17$ to $\mod 289$, I had $m = 8 + 17k$. Then expanding $(8 + 17k)^4$:

I calculated up to the linear term. The first term was $8^4 = 4096 \equiv 50$

mod 289. Then the linear term was $4 * 8^3 * 17k = 4 * 512 * 17k = 2048 * 17k$. But 2048 mod 289: $289 * 7 = 2023$, $2048 - 2023 = 25$. So $25 * 17k = 425k$. Then 425 $\mod 289 = 425 - 289 = 136$. So $136k$ mod 289. Then total sum was $50 + 1 + 136k$. Wait, but $8^4 + 1 = 4096 + 1 = 4097 \equiv 4097 \mod 289$. 4097 divided by 289: $289 * 14 = 4046$, $4097 - 4046 = 51$. So $8^4 + 1 \equiv 51 \mod 289$. Then the linear term is $136k$. So $(8 + 17k)^4 + 1 \equiv 51 + 136k \mod 289$. So to get this $\equiv 0 \mod 289$, $136k \equiv -51 \mod 289$. But $-51 \mod 289$ is 238. So $136k \equiv 238 \mod 289$.
As $\gcd(136, 289) = 17$, which divides 238, so solutions exist. Dividing equation by 17: $8k \equiv 14 \mod 17$. Then inverse of $8 \mod 17$ is 15, so $k \equiv 14 * 15 = 210 \equiv 6 \mod 17$. So$k = 6 + 17t$. So minimal $k = 6$. Then $m = 8 + 17 * 6 = 8 + 102 = 110$. Then why does $110^4 + 1 \equiv 51 + 136 * 6 = 51 + 816 \mod 289$. 816 divided by 289: $289 * 2 = 578$, $816 - 578 = 238$. So $51 + 238 = 289 \equiv 0 \mod 289$. Ah! So $51 + 816 = 867$. 867 divided by $289 = 3 * 289 = 867$. So $867 \equiv 0 \mod 289$. So $110^4 + 1 \equiv 0 \mod 289$. Therefore, m=110 is indeed the solution. My initial confusion was due to miscalculating $110^4 + 1$ directly, but since $110 \equiv 8 \mod 17$, but in $\mod 289$, the higher power terms do matter. So the calculation via Hensel's lemma was correct. Therefore, the minimal m is 110.
But let me check if there's a smaller m. Suppose we take $k = 6 - 17 = -11$. Then $m = 8 + 17 * (-11) = 8 - 187 = -179 \equiv 110 \mod 289$. So negative solution corresponds to positive one. So the minimal positive solution is 110.
Therefore, the answer is $p = 17$, $m = 110$. Wait, but the problem says "Find the least positive integer m such that $m^4 + 1$ is divisible by p^2." Since $p = 17$, then $m = 110$.
But before finalizing, let me check $m = 110$: compute $110^4 + 1$ divided by $17^2 = 289$. $110^4 = 146410000$. $146410000 + 1 = 146410001$. Divide by 289: Let's see $289 * 500000 = 144500000$. $146410001 - 144500000 = 1910001$. $289 * 6000 = 1734000$. $1910001 - 1734000 = 176001$. $289 * 600 = 173400$. $176001 - 173400 = 2601$. $289 * 9 = 2601$. So total is

$500000+6000+600+9=506609$. So $289*506609=146410001$. Yes, so $110^4+1=289*506609$, so divisible by 289. Therefore, $m=110$ is correct.

But wait, maybe there's a smaller m? Let me check $m=110-289=-179$, which is not positive. The next lower positive would be $m=110-289*0=110$, which is the minimal positive solution. Therefore, the minimal positive integer m is 110.

Therefore, the answer is $\boxed{110}$.

Final Answer

To find the least prime number p such that there exists a positive integer n where n^4+1 is divisible by p^2, we start by considering primes p where -1 is a fourth power residue modulo p. These primes are congruent to $1 \mod 8$. The smallest such prime is 17.

Next, we need to verify if there exists an integer n such that $n^4+1 \equiv 0 \mod 17^2$. We know that $8^4 \equiv -1 \mod 17$. Using Hensel's lemma, we lift this solution modulo 17^2.

Let $m=8+17k$. Expanding $(8+17k)^4$ modulo 289 (which is 17^2) using the binomial theorem, we find the linear term coefficient and solve for k:

$$(8+17k)^4+1 \equiv 51+136k \mod 289$$

We need $51+136k \equiv 0 \mod 289$. Solving $136k \equiv -51 \mod 289$, we find $k \equiv 6 \mod 17$. Thus, the smallest positive k is 6, leading to $m=8+17\cdot 6=110$.

Verification shows 110^4+1 is indeed divisible by 289. Therefore, the least positive integer m such that m^4+1 is divisible by 17^2 is $\boxed{110}$.

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