Quaternion

Related Topics:

$$q = s + ix + jy + kz$$

Quaternion is a geometrical operator to represent the relationship (relative length and relative orientation) between two vectors in 3D space. William Hamilton invented Quaternion and completed the calculus of Quaternions to generalize complex numbers in 4 dimension (one real part and 3 imaginary numbers). In this article, we focus on rotations of 3D vectors because Quaternion implementation for 3D rotation is usually simpler, cheaper and better behaved than other methods.

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Background of Quaternion

[Euler's equation (formula)](#) can be used to represent a 2D point with a length and angle on a complex plane. Multiplication of 2 complex numbers implies a [rotation in 2D](#). One may think instantly it can be extended to 3D rotation by adding additional dimension. William Hamilton initially studied it by adding an additional imaginary number, *j* to generalize complex numbers to 3D.

$$z = a + ib + jc \quad \text{where } i^2 = j^2 = -1$$

However, the set of 3 dimensional complex numbers is not closed under multiplication. For example, the multiplication of *i* and *j* cannot be represented as a form of *a+ib+jc*. If multiplication is closed, then there exist *a, b, c* ∈ **R** that satisfies *ij=a+ib+jc*.

$$ij = a + ib + jc$$

$$i^2j = ia + i^2b + ijc \quad \text{(multiply } i \text{ on both sides)}$$

$$-j = ia - b + ijc$$

$$-j = -b + ia + (a + ib + jc)c \quad (\because ij = a + ib + jc)$$

$$-j = -b + ia + ac + ibc + jc^2$$

$$-j = (ac - b) + i(a + bc) + jc^2$$

$$0 = (ac - b) + i(a + bc) + j(c^2 + 1)$$

$$\therefore ac - b = a + bc = c^2 + 1 = 0$$

$$\text{but } c^2 + 1 \neq 0$$

The equation $c^2+1=0$ gives the contradiction. There is no real number *c* to satisfy $c^2+1=0$ (*It should be an imaginary number*).

Later, Hamilton realized 4 dimensional complex numbers are required for multiplication to be closed by adding an additional imaginary part, *k*. And, he denoted this 4D complex number set as Quaternion. *(In mathematics, each algebra has twice the dimension of the previous one. Therefore, the higher number set of complex number is quaternion, and the next number set is octonion.)*

$$q = s + ix + jy + kz$$

$$\text{where } i^2 = j^2 = k^2 = ijk = -1$$

Now, the above example satisfies as *ij = 0+i0+j0+k1*, which *ij* is another imaginary number, *ij = k*. The condition of these 3 imaginary numbers is explained later, in [Quadrantal vectors](#) section.
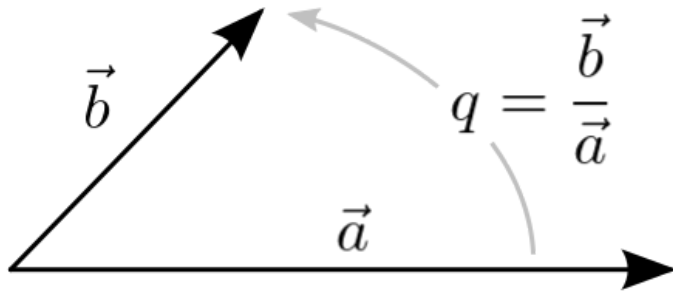
Understanding of Quaternion

Definition

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Definition of Quaternion operator

Hamilton's motivation was to create a geometrical operator to transform from a vector to the other in 3D space. This operator is the geometric quotient (ratio) between two vectors that changes the length and the orientation, and it is called a Quaternion because the operation is required 4 parameters.

$$q = \frac{\vec{b}}{\vec{a}}, \text{ or, } \vec{b} = q\vec{a}$$

(Note that this notation is not same as numerical division nor multiplication. Instead, it is translated as "the quaternion operator q on $\vec{a}$ to produce $\vec{b}$, or the operator q to convert $\vec{a}$ into $\vec{b}$".)

Tensor and Versor

When $\vec{a}$ is transformed to vector $\vec{b}$, quaternion operator performs 2 distinct operations:

1. **Tensor**: scaling the length of $\vec{a}$, so as to make it of the same length as $\vec{b}$.
2. **Versor**: rotating $\vec{a}$, so as to cause it to coincide with $\vec{b}$ in direction.

These 2 operations can be symbolically represented $\mathbf{Tensor}(q)$ (tensor of q) and $\mathbf{Versor}(q)$ (versor of q) respectively.

$$q = \mathbf{Tensor}(q)\mathbf{Versor}(q)$$

The order of these two operations does not make any difference of the result. The combined tensor and versor operation requires 4 numbers; 1 for scaling, 1 for rotation angle, 2 additional angles to determine the orientation of the rotation plane. For example, a xy-plane is rotated about x and y axis. But, rotation along z axis does not change the orientation of the xy plane.

$$\mathbf{Tensor}(q) = \mathbf{Tensor}(\frac{\vec{b}}{\vec{a}})$$
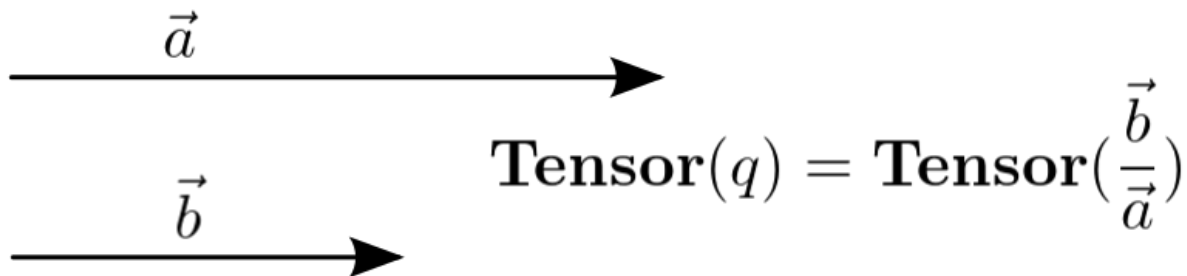
Tensor operator

Tensor
Tensor of q is the geometric quotient (ratio) between the lengths of 2 parallel vectors. It changes the scale of the vector, but, keeps the orientation of the vector unchanged.

$$\mathbf{Versor}(q) = \mathbf{Versor}(\frac{\vec{b}}{\vec{a}})$$

Versor

Versor of *q* is the geometric quotient of 2 non-parallel vectors of equal length. It represents the relative orientation of one vector with respect to the other vector, but, it does not change the length of the vector.

Expression with unit vectors
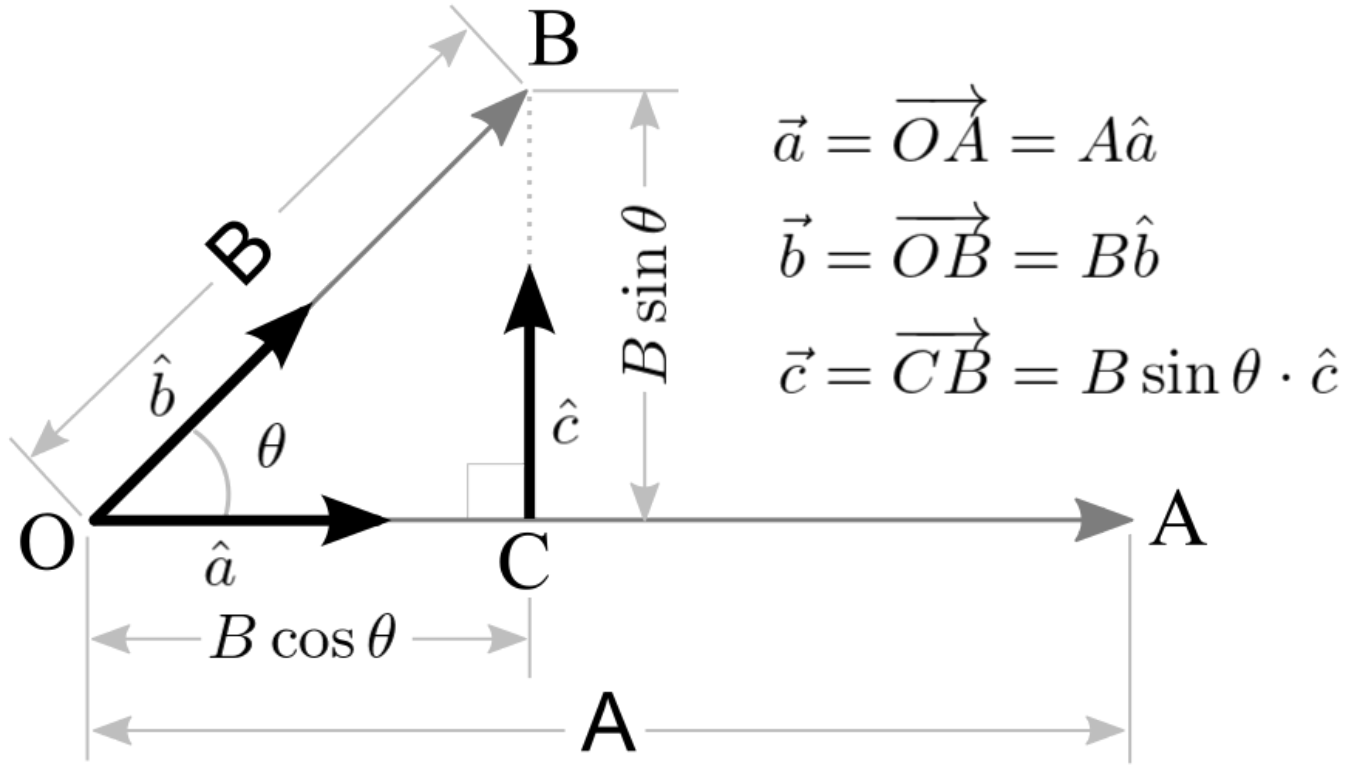
Quaternion with unit vectors

To investigate the quaternion operator deeper, let $\hat{a}$ and $\hat{b}$ as unit vectors along $\vec{a}$ and $\vec{b}$. so that

$$\vec{a} = \overrightarrow{OA} = A\hat{a}$$
$$\vec{b} = \overrightarrow{OB} = B\hat{b}$$

$$\therefore q = \frac{\vec{b}}{\vec{a}} = \frac{B\hat{b}}{A\hat{a}}$$

And, draw BC perpendicular to $\vec{a}$, and let the unit vector along $\overrightarrow{CB}$ be $\hat{c}$. Therefore, $\overrightarrow{OC}$ becomes $\overrightarrow{OC} = B\cos\theta \cdot \hat{a}$ and $\overrightarrow{CB} = B\sin\theta \cdot \hat{c}$.

Since $\overrightarrow{OB} = \overrightarrow{OC} + \overrightarrow{CB}$, we substitute $\overrightarrow{OC}$ and $\overrightarrow{CB}$ in the above equation, then we have a quaternion, $q = \vec{b}/\vec{a}$;

$$\overrightarrow{OB} = \overrightarrow{OC} + \overrightarrow{CB}$$

$$\frac{\overrightarrow{OB}}{\overrightarrow{OA}} = \frac{\overrightarrow{OC} + \overrightarrow{CB}}{\overrightarrow{OA}} \qquad \text{(divide by } \overrightarrow{OA})$$

$$\frac{\overrightarrow{OB}}{\overrightarrow{OA}} = \frac{\overrightarrow{OC}}{\overrightarrow{OA}} + \frac{\overrightarrow{CB}}{\overrightarrow{OA}}$$

$$\frac{\vec{b}}{\vec{a}} = \frac{B\cos\theta \cdot \hat{a}}{A\hat{a}} + \frac{B\sin\theta \cdot \hat{c}}{A\hat{a}}$$

$$q = \frac{B}{A}\cos\theta + \frac{B}{A}\sin\theta \cdot \frac{\hat{c}}{\hat{a}} \qquad (q = \frac{\vec{b}}{\vec{a}})$$

$$q = \frac{B}{A}\left(\cos\theta + \frac{\hat{c}}{\hat{a}}\sin\theta\right)$$

$$\hat{u} = \frac{\hat{c}}{\hat{a}}$$

Vector of Rotation axis

The last term contains the geometric quotient $\hat{c}/\hat{a}$ of two unit vectors at right angles (90 degree) to each other. This quotient represents a unit vector perpendicular to the plane of $\hat{a}$ and $\hat{c}$. (Think of cross product of 2 vectors) This unit vector is indicating the rotation axis of the plane and the direction of rotation. If we define it $\hat{u}$, the above equation becomes

$$q = \frac{\vec{b}}{\vec{a}} = \frac{B}{A}\left(\cos\theta + \hat{u}\sin\theta\right)$$

From this equation, B/A is tensor (scaling) operation of a quaternion, and $\cos\theta + \hat{u}\sin\theta$ is versor (rotation) operator of a quaternion. Note that versor of q is very similar to Euler's equation. Euler's equation contains an imaginary number i, but a quaternion has a vector instead, which is the rotation axis perpendicular to its rotation plane.

Thus, a quaternion is also expressed as the sum of scalar **S**(q) and vector **V**(q) parts;

$$q = s + ix + jy + kz$$
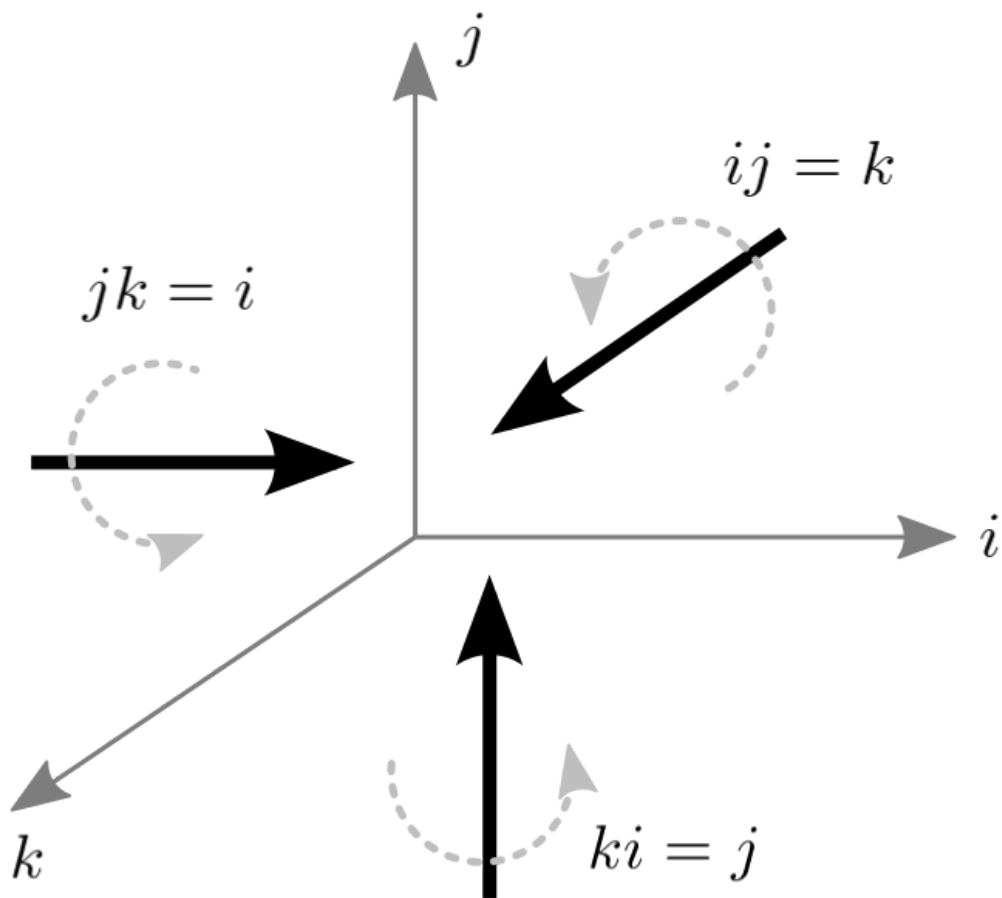
$$q = \frac{B}{A} \left(\cos \theta + \hat{u} \sin \theta \right)$$

$$\mathbf{S}(q) = \frac{B}{A} \cos \theta$$

$$\mathbf{V}(q) = \hat{u} \cdot \frac{B}{A} \sin \theta$$

$$q = \mathbf{S}(q) + \mathbf{V}(q)$$
$$= s + (ix + jy + kz)$$
$$= [s, v]$$

Quaternion can be also written as a 2-tuple form, [*s, v*]. A unit quaternion can be represented as $\left[\cos \theta, \hat{u} \sin \theta \right]$, because the tensor of *q* is 1 (B/A = 1 or no scaling).

Quadrantal vectors: *i, j, k*

Quadrantal vectors

Let *i, j* and *k* represent unit vectors orthogonal (perpendicular) each other. We define multiplications and divisions as rotating a unit vector to another at right angle. Note that, this multiplication and division is not numeric algebra. This kind of multiplication and quotient is called geometric. Thus, production or quotient of two unit vectors at right angles to each other produces a unit vector, perpendicular to their plane. And, it reads as *i* operating on *j* (or, rotating from *i* to *j*) produces *k*.

$$ij = k, \quad j/i = k$$

$\vee$

In same manner, we can write other multiplications and divisions;

$$jk = i, \qquad\qquad\qquad k/j = i$$
$$ki = j, \qquad\qquad\qquad i/k = j$$
$$ji = -k, \qquad\qquad\qquad i/j = -k$$
$$kj = -i, \qquad\qquad\qquad j/k = -i$$
$$ik = -j, \qquad\qquad\qquad k/i = -j$$

The square of a unit vector can be defined from above equations;

$$ij = k$$
$$i(ij) = ik \qquad (\text{multiply } i)$$
$$i(ij) = -j \qquad (ik = -j)$$
$$i^2 j = -j$$
$$i^2 = -1 \qquad (\text{divide by } j)$$

In same manner,
$$j^2 = k^2 = -1$$

$$jk = i \qquad\qquad\qquad\qquad ki = j$$
$$j(jk) = ji \qquad\qquad\qquad\qquad k(ki) = kj$$
$$j(jk) = -k \qquad\qquad\qquad\qquad k(ki) = -i$$
$$j^2 k = -k \qquad\qquad\qquad\qquad k^2 i = -i$$
$$j^2 = -1 \qquad\qquad\qquad\qquad k^2 = -1$$

Also, *ijk* can be defined as -1 using above multiplication table and square of unit vector.

$$ij = k, \quad k^2 = -1$$
$$kk = -1$$
$$(ij)k = -1$$
$$\therefore ijk = -1$$

Quaternion Algebra

Here, the basic quaternion mathematics is described. These algebraic definitions and properties are specially required for rotation in 3D space, which describe in the next section. You may skip this section and move on the next section. And, come back later if you need to review a specific definition or property.

1. Addition

If $p = [s, v], \quad q = [s', v'], \quad$ then $p + q = [s + s', v + v']$

$$
\begin{aligned}
p + q &= [s, v] + [s', v'] \\
&= (s + ix + jy + kz) + (s' + ix' + jy' + kz') \\
&= (s + s') + i(x + x') + j(y + y') + k(z + z') \\
&= [s + s', v + v']
\end{aligned}
$$

2. Multiplication
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If $p = [s, v],\quad q = [s', v'],\quad$ then $pq = [ss' - v \cdot v',\quad v \times v' + sv' + s'v]$

$$pq = [s, v][s', v']$$
$$= (s + ix + jy + kz)(s' + ix' + jy' + kz')$$
$$= ss' + isx' + jsy' + ksz' + is'x - xx' + kxy' - jxz' +$$
$$\quad js'y - kyx' - yy' + iyz' + ks'z + jzx' - izy' - zz'$$
$$= ss' - (xx' + yy' + zz') + i(yz' - zy') + isx' + is'x +$$
$$\quad j(zx' - xz') + jsy' + js'y + k(xy' - yx') + ksz' + ks'z$$
$$= ss' - v \cdot v' + i(yz' - zy') + j(zx' - xz') + k(xy' - yx') +$$
$$\quad s(ix' + jy' + kz') + s'(ix + jy + kz)$$
$$= [ss' - v \cdot v',\quad v \times v' + sv' + s'v]$$

Note that quaternion multiplication is not commutative, however, multiplication is associative and distributive across addition.

$$pq \neq qp \qquad\qquad \text{(NOT commutative)}$$
$$(op)q = o(pq) \qquad\quad \text{(associative)}$$
$$o(p + q) = op + oq \qquad \text{(distributive across addition)}$$

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3. Scalar Multiplication

If $q = [s, v]$ and $r \in \mathbb{R},\quad$ then $rq = qr = [rs, rv]$

$$rq = [r, 0][s, v]$$
$$= r(s + ix + jy + kz)$$
$$= rs + r(ix + jy + kz)$$
$$= [rs, rv]$$
$$= qr$$

Unlike quaternion multiplication, scalar multiplication is commutative.

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4. Subtraction

Quaternion subtraction can be derived from scalar multiplication and quaternion addition.

If $p = [s, v],\ q = [s', v'],\quad$ then $p - q = [s - s', v - v']$

$$p - q = [s, v] - [s', v']$$
$$= [s, v] + (-1)[s', v']$$
$$= [s, v] + [-s', -v'] \quad \text{(scalar multiplication)}$$
$$= [s - s', v - v']$$

---

5. Conjugate

Quaternion conjugate is defined by negating the vector part of the quaternion.

If $q = [s, v],\quad$ then $q^* = [s, v]^* = [s, -v]$

$\vee\ (q^*)^* = q$

6. Norm (Length)

The norm of a quaternion is defined by;

If $q = [s, v]$, then $|q| = \sqrt{qq^*}$, $|q|^2 = qq^*$

$$
\begin{aligned}
|q| &= \sqrt{[s, v][s, -v]} \\
&= \sqrt{[ss - v \cdot (-v), \ v \times (-v) + s(-v) + sv]} \quad \text{(quaternion multiplication)} \\
&= \sqrt{[ss + v \cdot v, \ 0 - sv + sv]} \quad\quad\quad\quad\quad (v \times (-v) = 0) \\
&= \sqrt{[ss + v \cdot v, \ 0]} \quad\quad\quad\quad\quad\quad\quad \text{(only scalar part)} \\
&= \sqrt{s^2 + x^2 + y^2 + z^2} \\
&= |q^*|
\end{aligned}
$$

The norm of quaternion is multiplicative meaning that the norm of the multiplication of multiple quaternions equals to the multiplication of the norms of quaternions.

$$|pq| = |p||q|$$

$$
\begin{aligned}
|pq| &= \sqrt{pq(pq)^*} \\
&= \sqrt{pqq^*p^*} \quad \text{(conjugate rule)} \\
&= \sqrt{p|q|^2 p^*} \quad \text{(definition of norm)} \\
&= \sqrt{pp^*|q|^2} \quad \text{(scalar is commutative)} \\
&= \sqrt{|p|^2|q|^2} \\
&= |p||q|
\end{aligned}
$$

7. Inverse

The inverse of a quaternion is defined to be;

If $qq^{-1} = q^{-1}q = 1$, then $q^{-1} = \dfrac{q^*}{|q|^2}$

$$
\begin{aligned}
qq^{-1} &= q\frac{q^*}{|q|^2} = \frac{qq^*}{|q|^2} \\
&= \frac{|q|^2}{|q|^2} \quad \text{(definition of norm)} \\
&= 1
\end{aligned}
$$

The quaternion inverse makes it possible to divide two quaternions.

8. Unit Quaternion

If $|q| = 1$, then q is a unit quaternion.

$$
q^{-1} = \frac{q^*}{|q|^2} = q^* \quad \text{(if } |q| = 1\text{)}
$$

Note that the inverse of a unit quaternion equals to the conjugate of the unit quaternion.

3D Rotation with Quaternion

$$p = i + j + k, \quad \mathbf{V}(p) = i + j + k, \quad |\mathbf{V}(p)| = \sqrt{3}$$
$$q = i \quad (|q| = 1)$$

$$qp = i(i + j + k) = ii + ij + ik = -1 + k - j$$
$$\mathbf{V}(qp) = k - j$$
$$|\mathbf{V}(qp)| = \sqrt{1 + 1} = \sqrt{2}$$

$$\therefore |\mathbf{V}(p)| \neq |\mathbf{V}(qp)|$$

Thus, we need a special multiplication for 3D rotations that is length-conserving transformation. For this reason, we multiply the quaternion q at the front of p and multiplying its inverse q^{-1} at the back of p, in order to cancel out the length changes. This special double multiplication is called "conjugation by q".

If a quaternion is p = [s, v], then the scalar (s) and the length of v, |v| are unchanged after conjugation by q. For the above example;

$$qpq^{-1} = qpq^* \qquad (q^{-1} = q^* \text{ if } |q| = 1)$$
$$= i(i + j + k)(-i)$$
$$= (ii + ij + ik)(-i)$$
$$= (-1 + k - j)(-i)$$
$$= i - ki + ji$$
$$= i - j - k$$

$$\mathbf{V}(qpq^{-1}) = i - j - k$$
$$|\mathbf{V}(qpq^{-1})| = \sqrt{1 + 1 + 1} = \sqrt{3}$$
$$\therefore |\mathbf{V}(p)| = |\mathbf{V}(qpq^{-1})|$$

Proof

If p = [s, v] and p' = qpq^{-1}, then p' = [s, v'] where |v| = |v'|.
(The scalar part of p' is same as p and the length of the vector part is also unchanged.)

The proof takes 4 steps. First, we show the scalar parts of quaternions, $\mathbf{S}(p)$=$\mathbf{S}(p')$ for p=[s, 0], and for p=[0, v]. Then, both results are used to show $\mathbf{S}(p)$=$\mathbf{S}(p')$ for p=[s, v]=[s, 0]+[0, v]. Finally, we show the length of p' is unchanged (conserved), therefore, the length of the vector part is same, |$\mathbf{V}(p)$| = |$\mathbf{V}(p')$|.

Step 1. If p has a scalar part only p = [s, 0], then
$$qpq^{-1} = q[s, 0]q^{-1} = [s, 0]qq^{-1} \quad \text{(scalar is commutative)}$$
$$= [s, 0] \qquad\qquad\qquad (qq^{-1} = 1)$$

Step 2. If p has a vector part only $p = [0, v]$, then the scalar part of qpq^{-1} can be computed by $2\mathbf{S}(p)=p+p^*$.

$$p + p* = [s, v] + [s, -v] = [s + s, v - v] = [2s, 0]$$
$$= 2\mathbf{S}(p)$$

$$2\mathbf{S}(qpq^{-1}) = qpq^{-1} + (qpq^{-1})^*$$

$$= qp\frac{q^*}{|q|^2} + (qp\frac{q^*}{|q|^2})^* \qquad \text{(inverse definition)}$$

$$= \frac{1}{|q|^2}qpq^* + \frac{1}{|q|^2}(qpq^*)^* \quad \text{(scalar is commutative)}$$

$$= \frac{1}{|q|^2}[qpq^* + (qpq^*)^*] \qquad \text{(scalar is distributive)}$$

$$= \frac{1}{|q|^2}[qpq^* + qp^*q^*] \qquad \text{(conjugate property)}$$

$$= \frac{1}{|q|^2}[q(p + p^*)q^*] \qquad \text{(distributive across addition)}$$

$$= \frac{1}{|q|^2}[q(2\mathbf{S}(p))q^*] \qquad (p + p^* = 2\mathbf{S}(p))$$

$$= \frac{2\mathbf{S}(p)}{|q|^2}[qq^*] \qquad \text{(scalar is commutative)}$$

$$= 2\mathbf{S}(p)\frac{qq^*}{|q|^2}$$

$$= 2\mathbf{S}(p) \qquad (qq^* = |q|^2)$$

$$= 0 \qquad (\mathbf{S}([0, v]) = 0)$$

$$\therefore qpq^{-1} = [0, v'] \qquad \text{if } p = [0, v]$$

Step 3. If $p = [s, v] = [s, 0] + [0, v]$, then $\mathbf{S}(p) = \mathbf{S}(p')$.

$$qpq^{-1} = q([s, 0] + [0, v])q^{-1}$$

$$= q[s, 0]q^{-1} + q[0, v]q^{-1} \quad \text{(multiplication is distributive across addition)}$$

$$= [s, 0] + q[0, v]q^{-1} \qquad \text{(from Step 1)}$$

$$= [s, 0] + [0, v'] \qquad \text{(from Step 2)}$$

$$= [s, v']$$

Step 4. The norm of p' is same as the norm of p;

$$|p'| = |qpq^{-1}|$$

$$= |q||p||q^{-1}| \qquad \text{(norm property)}$$

$$= |p||q||q^{-1}| \qquad \text{(scalar is communative)}$$

$$= |p||qq^{-1}| \qquad \text{(norm property)}$$

$$= |p| \qquad (qq^{-1} = 1)$$

$\vee$

Since the lengths and the scalar parts of *p* and *p'* are same, the lengths of the vector parts are also same;

$$|p| = |p'| \qquad \text{(Step 4)}$$

$$|\mathbf{S}(p)| = |\mathbf{S}(p')| \quad \text{(Step 3)}$$
$$|s| = |s|$$

$$\therefore \ |\mathbf{V}(p)| = |\mathbf{V}(p')|$$
$$|v| = |v'|$$

Quaternion to Matrix

The rotation with quaternion by qpq^{-1} can be expressed with matrix from;

$$qpq^{-1} = \begin{pmatrix} s^2 + x^2 - y^2 - z^2 & 2xy - 2sz & 2xz + 2sy \\ 2xy + 2sz & s^2 - x^2 + y^2 - z^2 & 2yz - 2sx \\ 2xz - 2sy & 2yz + 2sx & s^2 - x^2 - y^2 + z^2 \end{pmatrix} \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix}$$

$$\text{where } p = (0, p_x, p_y, p_z), \ \ q = (s, x, y, z) = s + ix + jy + kz$$

If *q* is a unit quaternion, it can be simplified by;

$$qpq^* = \begin{pmatrix} 1 - 2y^2 - 2z^2 & 2xy - 2sz & 2xz + 2sy \\ 2xy + 2sz & 1 - 2x^2 - 2z^2 & 2yz - 2sx \\ 2xz - 2sy & 2yz + 2sx & 1 - 2x^2 - 2y^2 \end{pmatrix} \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix}$$

$$\text{where } p = (0, p_x, p_y, p_z), \ \ q = s + ix + jy + kz, \ \ |q| = 1$$

The derivation of matrix conversion is explained at Quaternion to OpenGL Matrix.

References

- A. S. Hardy (1881). *Elements of Quaternions*
- E. B. Dam, M. Koch, M. Lillholm (1998). *Quaternions, Interpolation and Animation*

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