Math 225A Differential Topology: Homework 5

November 2nd, 2018

Professor Peter Petersen

Anish Chedalavada

Exercise 1. *Find maps of the solid torus into itself having no fixed points. Where does the proof of the Brouwer theorem fail?*

Proof. As the torus is given by $S^1 \times S^1$, we may assume the solid torus may be written as $S^1 \times D^2$, where D^2 is the closed ball of radius 1 in $\mathbb{R}^2$. Let $f : S^1 \times D^2 \to S^1 \times D^2$ via $f : (\theta, x) \mapsto (\theta + \pi, x)$ where we parametrize S^1 by $\mathbb{R}/2\pi\mathbb{Z}$. Assume x is a fixed point of this map. Then we have that $\exists x \in [0, 2\pi)$ s.t. $x \equiv x \mod \pi$, which is not possible as $0 \not\equiv 0 \mod \pi$. Thus, this map has no fixed points. This is possible as the proof of the Brouwer theorem aassumes that every line from x to $f(x)$ (viewed via the parametrization to Euclidean space) must eventually intersect the boundary of the manifold, resulting in a retract to the boundary. However, for the parametrization of $S^1 \times D^2$, we have that this map yields a line from (θ, x) to $(\theta + \pi, x)$ lying entirely in $S^1 \times \{x\}$, which never leaves this subspace and thus never passes through the boundary. Thus, we cannot extend this map to a retract onto the boundary of a compact manifold and the proof fails. □

Exercise 2. *Prove that if the entries of an $n \times n$ matrix A are all nonnegative, then A has a real nonnegative eigenvalue.*

Proof. We have that as all the entries of A are nonnegative, for any x s.t. $x_1, ..., x_n \geq 0$ we have that for $Ax = b$, $b_1, ..., b_n \geq 0$. Thus, we may view the map $f : x \mapsto \frac{Ax}{|Ax|}$ as a smooth map, normalizing the image, as a map $f : S^{n-1} \to S^{n-1}$ mapping the compact submanifold with boundary $M = \{x \in S^{n-1} | x_1, ..., x_n \geq 0\}$ to itself. We have that the manifold M is homeomorphic to the closed ball $B^{n-1} \subset \mathbb{R}^{n-1}$. We now prove the following general result: If $f : B^{n-1} \to B^{n-1}$ is a continuous map, f has a fixed point.
If f is a continuous map from $\mathbb{R}^n \to \mathbb{R}^n$, we have that in a compact set it can be coordinatewise approximated by polynomials p_n s.t. $\forall \epsilon > 0 \; \exists \; N \in \mathbb{N}$ s.t. $|p_n(x) - f(x)| < \epsilon \; \forall \; n > N$. As polynomials are smooth, for each map p_n we have an associated fixed point c_n s.t. the c_n form a convergent sequence. We have that as B^{n-1} is compact, the function $|f(x) - x|$ must be bounded from below by some value $c > 0$ (as no fixed points by assumption. However, we have that $\exists p_n$ s.t. $|p_n(x) - f(x)| < \frac{c}{2}$ for all $x \in B^{n-1}$. Thus, for associated fixed point $c_n \in B^{n-1}$, we have that $|c_n - f(c_N)| < \frac{c}{2}$, a contradiction. Thus, we have the lemma. Using the lemma, we have a map from a set homeomorphic to B^{n-1} to itself, resulting in the existence of a fixed point. Thus, the map $\frac{Ax}{|Ax|}$ must have a fixed point x, or that $Ax = |Ax|x$ for some vector $x \in M$. Thus, A has a positive real eigenvalue. □

Exercise 3. *Let Y be a compact submanifold of $\mathbb{R}^M$, and let $w \in]mathbbR^M$. Show that there exists a closest point $y \in Y$, and $w - y \in N_y(Y)$.*

Proof. We have that the function $|w - y|$ is continuous and thus must attain a minimum at some point $y \in Y$. Let $c : [0, 1] \to Y$ be an arbitrary curve s.t. $c(0.5) = y$ (can select this by taking the straight line through 0 in the local parametrization of y, assuming Y is not a 0-manifold in which case the result clearly holds. The function $|c(t) - w|^2$ attains a minimum at 0.5, and so taking the derivative must yield 0 at 0.5 (local minimum). Thus, we take the derivative of $|c(t) - w|^2$, given by $2(w_1 - c_1(t))c_1'(t) + ... + 2(w_n - c_n(t))c_n'(t)\big|_{t=0.5}$. Thus, at 0.5, we have that $w - c(0.5)$ is perpendicular to $c'(0.5) \in T_y(Y)$, and thus $w - y \in N_y(Y)$. □

Exercise 4. *Prove that if $w \in Y^\epsilon$, then $\pi(w)$ is the unique point closest to w in Y.*