Math 210C Algebra: Final

June 12th, 2019

Professor Sharifi

Anish Chedalavada

Problem 2

Consider the following diagram where the two longest rows are exact:

$$\begin{tikzcd}
& & & ker\ \overline{h} \arrow[d, hook] & & \\
& & ker\ \beta \arrow[r, "\overline{g}"] \arrow[d, hook] & ker\ \gamma \arrow[r, "\overline{h}"] \arrow[d, hook] & ker\ \delta \arrow[d, hook] & \\
0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[r, "g"] \arrow[d, "\beta"] & C \arrow[r, "h"] \arrow[d, "\gamma"] & D \arrow[r] \arrow[d, "\delta"] & 0 \\
0 \arrow[r] & A' \arrow[r, "f'"] \arrow[d, "\pi_\alpha"] & B' \arrow[r, "g'"] \arrow[d, "\pi_\beta"] & C' \arrow[r, "h'"] & D' \arrow[r] & 0 \\
ker\ \overline{f}' \arrow[r, hook] & coker\ \alpha \arrow[r, "\overline{f}'"] & coker\ \beta & & &
\end{tikzcd}$$

<u>WTS</u> $\exists$ an exact sequence of the form:

$$0 \longrightarrow im\ \overline{g} \longrightarrow ker\ \overline{h} \longrightarrow ker\ \overline{f} \longrightarrow 0$$

Proof. Note that $h \circ g = 0$ by exactness, and thus $h \circ g|_{ker\ \beta} : ker\ \beta \to D = 0$, implying that $\overline{h} \circ \overline{g} : ker\ \beta \to$ ker δ is the zero map, i.e. that $im\ \overline{g} \subset ker\ \overline{h}$. Thus, we have an exact sequence of the form $0 \to im\ \overline{g} \to ker\ \overline{h}$ via the first isomorphism theorem. Now, we define a map $\delta : ker\ \overline{h} \to ker\ \overline{f}'$ in the following manner:
We note firstly that $ker\ \overline{h} \hookrightarrow ker\ h \subset C$ by definition. Let $a \in ker\ \overline{h} \implies a \in im\ g$ by exactness. Thus, we may select a representative in the g-preimage of a, $\widetilde{a} \in B$ (Note: from here onwards the tilde is used as notation for selecting a representative in the g-preimage of some element). Applying β, we have that $\beta(\widetilde{a}) \in ker\ g'$ as $g' \circ \beta(\widetilde{a}) = \gamma \circ g(\widetilde{a}) = \gamma(a) = 0$. By exactness and the fact that f' is monic, we have a unique representative $f'^{-1}(\beta(\widetilde{a})) \in A'$. From here, we may apply π_α, and clearly $\pi(f'^{-1}(\beta(\widetilde{a}))) \in A' \in ker\ \overline{f}'$ as $\overline{f}(\pi(f'^{-1}(\beta(\widetilde{a})))) = \pi_\beta(\beta(\widetilde{a})) = 0$.

We must now show that this is a well-defined process. Let $a = b \in ker\ \overline{h}$. Let $\widetilde{a}, \widetilde{b} \in B$ be lifts in the g-preimage of a, b. We have that $g(\widetilde{a} - \widetilde{b}) = a - b = 0 \implies \widetilde{a} - \widetilde{b} \in ker\ g = im\ f$. Let $k = f^{-1}(\widetilde{a} - \widetilde{b})$, which is well-defined as f monic. W.h.t. $\beta \circ f(k) = f' \circ \alpha(k)$ so $f'^{-1}(\beta(f(k))) = \alpha(k)$. As $\pi_\alpha(f'^{-1}(\beta(f(k)))) = \pi_\alpha(\alpha(k)) = 0 \implies \pi_\alpha(f'^{-1}(\beta(\widetilde{a} - \widetilde{b}))) \implies \pi_\alpha(f'^{-1}(\beta(\widetilde{a}))) - \pi_\alpha(f'^{-1}(\beta(\widetilde{b}))) = 0$ and thus that $a = b \implies \pi_\alpha(f'^{-1}(\beta(\widetilde{a}))) = \pi_\alpha(f'^{-1}(\beta(\widetilde{b})))$ for arbitrary choices of representatives. Note that this map respects the module structure on $ker\ \overline{h}$ as for any $ra + sb \in ker\ \overline{h}$, $a, b \in ker\ \overline{h}$, $r, s \in R$, we may select a representative $r\widetilde{a} + s\widetilde{b} \in B$ in its preimage, and the rest of the maps are compositions of homomorphisms and so respect the module structure on B. Thus, we have a well defined homomorphism $\delta : ker\ \overline{h} \to ker\ \overline{f}'$ via $a \mapsto \pi_\alpha(f'^{-1}(\beta(\widetilde{a})))$.

Now we claim that the following sequence is exact, where as above the left inclusion is induced by the first isomorphism theorem.

$$\begin{tikzcd}
0 \arrow[r] & im\ \overline{g} \arrow[r, hook, dotted] & ker\ \overline{h} \arrow[r, "\delta"] & ker\ \overline{f} \arrow[r] & 0 \\
& ker\ \beta \arrow[u, two heads] \arrow[ur, "\overline{g}"] & & &
\end{tikzcd}$$

Let $\xi \in im\ \overline{g}$. We may choose a lift $\widetilde{\xi} \in ker\ \beta \subset B$ in the preimage $g^{-1}(ker\ \gamma)$, which we earlier showed can be used in the map without altering well-definedness of δ. We have that $\pi_\alpha(f'^{-1}(\beta(\widetilde{\xi}))) = \pi_\alpha(f'^{-1}(0)) = 0$ and so $\xi \in ker\ \delta$.

Now conversely let $\zeta \in ker\ \delta$. We have that $\pi_\alpha(f'^{-1}(\beta(\widetilde{\zeta}))) = 0 \implies f'^{-1}(\beta(\widetilde{\zeta})) \in im\ \alpha$. This implies, in particular, that $\beta(\widetilde{\zeta}) \in f'(im\ \alpha) = \beta(im\ f) \implies \widetilde{\zeta} \in im\ f + ker\ \beta \implies \zeta = g(\widetilde{\zeta}) \in g(ker\ \beta + im\ f) = im\ \overline{g}$ and so we have exactness in the middle.

The last order is to show that δ is surjective. Suppose $u \in ker\ \overline{f}'$. Let $\mu \in A'$ be a representative in the π_α-preimage of u. We have that $f'(\mu) \in \beta(B)$ as $\overline{f}' \circ \pi_\alpha(\mu) = 0 = \pi_\beta \circ f'(\mu)$. Let $\widetilde{\mu}_\beta \in B$ be a representative in the β-preimage of $f'(\mu)$. We have that $\gamma \circ g(\widetilde{\mu}_b) = g' \circ \beta(\widetilde{\mu}_\beta) = g' \circ f(\mu) = 0 \implies g(\widetilde{\mu}_\beta) \in ker\ \gamma$. Thus, $u = \pi_\alpha(f'^{-1}(\beta(\widetilde{\mu}_\beta))) = \delta(g(\mu_\beta))$ by definition $\implies \delta$ is surjective. Thus,

$$0 \longrightarrow im\ \overline{g} \hookrightarrow ker\ \overline{h} \xrightarrow{\delta} ker\ \overline{f} \longrightarrow 0$$

is exact, yielding the result. $\square$