Math 215A Commutative Algebra: Homework 2

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Exercise 1. *Show that if $M \cong R^n$ then any set of n elements that generate M form a free basis.*

Proof. For $x_1, ..., x_n$ a set of generators, we have a short exact sequence $0 \to \ker f \to R^n \to R^n \to 0$ for $f; R^n \to R^n$ given by sending each element e_i of the standard basis to x_i. This map is surjective as the x_i generate. Now note that for each maximal ideal $\mathfrak{m}$ we have the sequence $0 \to \ker f_{\mathfrak{m}} \to R^n_{\mathfrak{m}} \to R^n_{\mathfrak{m}} \to 0$. Note that $\frac{x_i}{1}$ generate $R^n_{\mathfrak{m}}$ and are a basis as it is a vector space, and so must be linearly independent and thus a basis for $R^n_{\mathfrak{m}}$ as well. Thus, $\ker f_{\mathfrak{m}}$ must be 0, and as $\mathfrak{m}$ was an arbitrary maximal ideal, it must be zero at every maximal ideal.
$\implies \ker f = 0$ and thus $0 \to R^n \to R^n \to 0$ is an isomorphism by projection to the generators, implying they are a basis. □

Exercise 2. *a) For S any multiplicatively closed subset of R show that the map $R \to S^{-1}R$ induces a homeomorphism Spec $S^{-1}R \to \{x \in$ Spec $R : \mathfrak{p}_x \cap S = \emptyset\}$ where the right hand side has the subspace topology.*
b) Given $f \in R$ let $D(f) = \{\mathfrak{p} \in$ Spec $R : f \notin \mathfrak{p}\}$. Show that $R \to R_f$ inducs an open embedding Spec $R_f \to$ Spec R with image $D(f)$.
c) Show that the $D(f)$ form a basis for Spec R

Proof. a) We claim that the map $f : R \to S^{-1}R$ to the localization induces a map $f_\# :$ *Spec* $S^{-1}R \to \{x \in$ *Spec* $R : \mathfrak{p}_x \cap S = \emptyset\} \subset$ *Spec* R. It is clear to see this as no ideals in the localization contain a unit, i.e. no element of S, and the thus the pullback of any prime ideal cannot contain an element of S as then it would have contained a unit in the localization. Thus, the image of $f_\# \subset \{x \in$ *Spec* $R : \mathfrak{p}_x \cap S = \emptyset\}$ and the map is continuous in the subspace topology. Now consider a prime in the set above. We have that $\ker f$ is $r \in R$ s.t. $rs = 0$ for some $s \in S$, and as $rs \in \mathfrak{p}$ we have that $r \in \mathfrak{p}$. Thus, all primes above must contain $\ker f$. This tells us that $f^{-1}(f(\mathfrak{p})) = \mathfrak{p}$. Thus, distinct prime ideals in the localization $S^{-1}R$ must have distinct preimages, as any two prime ideals $\mathfrak{p}_1$, $\mathfrak{p}_2$ are equal in $S^{-1}R$ if $\mathfrak{p}_1 \cap R/\ker f = \mathfrak{p}_2 \cap R/\ker f$, which is true if and only if their R-preimages are equal as both their R preimages contain $\ker f$. Now consider any prime ideal $\mathfrak{p} \in \{x \in$ *Spec* $R : \mathfrak{p}_x \cap S = \emptyset\}$. We have that $S^{-1}\mathfrak{p}$ is a prime ideal in $S^{-1}R$: localization preserves primality and the fact that it is an ideal as it contains no units: (clearing denominators of any linear combinations checks this). Thus, $f^{-1}(S^{-1}\mathfrak{p}) = \mathfrak{p}$ and this yields surjectivity. It now suffices to show this is a closed map in the subspace topology. Let $V(I)$ be a neighborhood in *Spec* $S^{-1}R$. It is clear that $f^{-1}(I) \subset f^{-1}(S^{-1}\mathfrak{p}) = \mathfrak{p}$ and so we have that $f_\#(V(I)) \subset V(f^{-1}I) \cap \{x \in$ *Spec* $R : \mathfrak{p}_x \cap S = \emptyset\}$. Now let $y \in V(f^{-1}I) \cap \{x \in$ *Spec* $R : \mathfrak{p}_x \cap S = \emptyset\}$. We have a corresponding prime ideal $I \subset S^{-1}\mathfrak{p}_y \in$ *Spec* $S^{-1}R$ and by the bijectivity of $f_\#$ we know that $f_\#(S^{-1}\mathfrak{p}_y) = \mathfrak{p}_y \implies y \in f(V(I))$ yielding that $f_\#(V(I))$ is closed in the subspace topology, yielding the homeomorphism.

b) Using part a) we have that *Spec* R_f embeds into *Spec* R with image $D(f)$ exactly as above. Given that $D(f) =$ *Spec* $R \setminus V(f)$ we have that this is a open embedding.

c) We know that all closed sets in *Spec* R are of the form $V(I)$ for some I ideal of R. Thus, all open sets in *Spec* R are of the form *Spec* $R \setminus V(I) = \{\mathfrak{p} \in$ *Spec* $R : \exists\ f \in I \setminus \mathfrak{p}\}$, i.e. all prime ideals that exclude some element of I. This is exactly of the form $\bigcup_{f \in I} D(f)$ for $D(f)$ as above, yielding that the $D(f)$ must be a basis for the topology.

d) We may show this using the finite intersection property. Suppose $\cap_\alpha V(I_\alpha) = \emptyset$. Then we have that $\sum_\alpha I_\alpha = R$, as it is not contained in any maximal ideal. Thus, $\exists a_{\alpha_1}, ..., a_{\alpha_n}$ with $a_{\alpha_i} \in I_{\alpha_i}$ such that $a_{\alpha_1} + ... + a_{\alpha_n} \in R^\times$. Thus, there is a finite subfamily of closed sets $\{V(I_{\alpha_i})\}_{i=1,...,n}$in the above family with trivial intersection, yielding the claim. □

Exercise 3. *Given a finitely generated module M show that supp$M$ is a closed subset of Spec R. Give an example to show that it is not always closed for arbitrary modules.*

Proof. Consider the closed set $V(ann(M))$. We have that for any nontrivial prime $\mathfrak{p}$ s.t. $\mathfrak{p} \in \text{supp}M$, $\mathfrak{p} \in V(ann(M))$ as the complement of $\mathfrak{p}$ cannot intersect any element in the annihilator (else elements of the annihilator act invertibly, implying the module is 0). Now let $\mathfrak{p} \in V(ann(M))$. We claim that $M_{\mathfrak{p}}$ is nontrivial. Suppose it is trivial, and $m_1, ..., m_n$ are a generating set for M. Then $M = Rm_1 + ... + Rm_n$ and $M_{\mathfrak{p}} = R_{\mathfrak{p}}m_1 + ... + R_{\mathfrak{p}}m_n$, which is 0 iff every summand is 0. If every summand is zero we have that $\exists s_i \notin \mathfrak{p}$ s.t. $s_i m_i = 0$ for every m_i. This in particular implies that $s_1 s_2 ... s_n$ must annihilate every m_i, implying it annihilates M and so $s_1 s_2 ... s_n \in ann(M)$. However by assumption $\forall\imath$, $s_i \notin \mathfrak{p}$ so $s_1 s_2 ... s_n \notin \mathfrak{p}$. Contradiction! Thus, supp$M = V(ann(M))$ closed.
Now consider for example the $\mathbb{Z}$-module $\bigoplus_{p_i \neq p} \mathbb{Z}/p_i\mathbb{Z}$ for some fixed prime p. This is clearly not finitely generated, and has support supp$M = \{p_i : p_i \neq p\}$. Supposing this were a closed set, then there is some ideal $I \subset \bigcap_i (p_i)$ with $V(I)$ the set above. However, it is clear that $\bigcap(p_i) = 0$ as no element $a \in \mathbb{Z}$ with finite prime factorization lies in the above intersection, implying that no $a \in \mathbb{Z}$ lies in the above intersection. As $V(0) = Spec\ R$, we have a contradiction. Thus, this set is not closed. ☐