Math 210C Algebra: Homework 4

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Exercise 1. *For $f : A. \to B.$ morphism of chain complexes if $H_i(kerf) = H_i(cokerf) = 0$ then f is a quasi-isomorphism.*

Proof. We know $\partial f_{i-1} = f_i|_{B_i}$. Regard the following LE sequences restricted to the kernel (which works by naturality of chain maps, exact by assumptions of the problem):

$$... \longrightarrow kerf_{i-1} \xrightarrow{\partial} kerf_i \xrightarrow{\partial} kerf_{i+1} \longrightarrow ...$$

$$... \longrightarrow cokerf_{i-1} \xrightarrow{\partial} cokerf_i \xrightarrow{\partial} cokerf_{i+1} \longrightarrow ...$$

In particular, the exactness of the above sequence implies that $ker(\partial|_{kerf}) = \partial(kerf)$, or the the inclusion of the boundary $B_i(A) \to Z_i(A)$ induces an isomorphism $\delta(kerf_{i-1}) = kerf_i \cap Z_i(A)$, and likewise for the case of the $cokerf$ sequence.

$$\begin{tikzcd}
 & \partial(kerf_{i-1}) \arrow[r, "\cong", "i"'] \arrow[d] & kerf_i \cap Z_i(A) \arrow[r] \arrow[d] & ker\overline{f} \arrow[r, hook, "\delta"] \arrow[d] & coker\ \partial f_i \\
0 \arrow[ur] \arrow[r] & B_i(A) \arrow[r, "i"] \arrow[d, "f_i|B_i"] & Z_i(A) \arrow[r] \arrow[d, "f"] & H_i(A) \arrow[r] \arrow[d, "\overline{f}"] & 0 \\
0 \arrow[r] & B_i(B) \arrow[r, "i"] \arrow[d] & Z_i(B) \arrow[r] \arrow[d] & H_i(A) \arrow[r] \arrow[d] & 0 \\
ker\overline{f} \arrow[r, hook, "\delta"] & \partial(cokerf_{i-1}) \arrow[r, "\cong", "i"'] & cokerf_i \cap Z_i(A) \arrow[r, two heads] & coker\overline{f} \arrow[ur] &
\end{tikzcd}$$

Chasing through the fact that this diagram is exact, we see that $ker\overline{f} = 0 = coker\overline{f}$ and thus the induced map on homology $\overline{f}$ is an isomorphism. □

Exercise 2.

Proof. We know the claim about $0 \to B. \to C. \to A[-1]. \to 0$ as a result of problem 8 on Homework 3, where the short exact sequence obtained there had the projection onto $A.$ as opposed to the negative projection, which is obtained here due to the sign change in the definition of $C.$; however, the argument is exactly analogous. Regard the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & B_{i+1} \arrow[r, "\iota_B"] \arrow[d, "d_i^B"] & B_{i+1} \oplus A_i \arrow[r, "-\pi_A"] \arrow[d, "(d_i^B - f_i, -d_i^A)"] & A_i \arrow[r] \arrow[d, "Id"] & 0 \\
0 \arrow[r] & B_i \arrow[r, "\iota_B"] & B_i \oplus A_{i-1} \arrow[r, "-\pi_A"] & A_{i-1} \arrow[r] & 0
\end{tikzcd}$$

We construct the connecting homomorphism δ by taking a cohomology class $[a] \in H_i(A[-1]) = H_{i+1}(A)$ and lift it to a class in $H_i(A)$: as this is levelwise split, we have that the lift is given by a splitting of $-\pi_A$ at each grade, which is $-\iota_A$ levelwise. We then applying the differential in $C_.$ and then pull back to a class in $H_i(B)$. This is given by the map induced on homology by $\pi_B \circ (d_i^B - f_i, -d_i^A) \circ (-\iota_A)$, which passing to homology is equivalent to the map induced by $\pi_B \circ (-f_i, 0) \circ (-\iota_A)$ as differentials induce trivial maps on homology (by definition). Clearly, the map above must be the map inducedby f_i, and thus we have the result. □

Exercise 3. *Let D be a divisible group, R a ring. Show that* $\operatorname{Hom}_{\mathbb{Z}}(R, D)$ *is an injective object in $R - mod$ with* $(r \cdot f)(m) = f(mr)$.

Proof. Note that for any ideal $I \subset R$, $\operatorname{Hom}_R(I, \operatorname{Hom}_{\mathbb{Z}}(R, D)) \cong \operatorname{Hom}_{\mathbb{Z}}(R \otimes_R I, D)$ naturally as abelian groups by the tensor-hom adjunction, where we view R as a bimodule ${}_{\mathbb{Z}}R_R$, so $R \otimes_R - : R - mod \to \mathbb{Z} - mod$. Regard the following diagram: z

$$\begin{tikzcd}
\operatorname{Hom}_R(I, \operatorname{Hom}_{\mathbb{Z}}(R, D)) \arrow[r, "\cong"] & \operatorname{Hom}_{\mathbb{Z}}(R \otimes_{\mathbb{Z}} I, D) \\
\operatorname{Hom}_R(R, \operatorname{Hom}_{\mathbb{Z}}(R, D)) \arrow[r, "\cong"] \arrow[u, "i_1^*"] & \operatorname{Hom}_{\mathbb{Z}}(R \otimes_{\mathbb{Z}} R, D) \arrow[u, "i_2^*"']
\end{tikzcd}$$

As $\mathbb{Z}$-modules, $R \otimes_R I \to R$ is still an injection (as tensoring with a free module is flat) and D is an injective object, so i_2^* must be an isomorphism as every morphism in $\operatorname{Hom}_{\mathbb{Z}}(R \otimes_R I, D)$ must be the restriction of some morphism in $\operatorname{Hom}_{\mathbb{Z}}(R \otimes_R R, D)$ by injectiveness of D. The naturality of the tensor-hom isomorphism implies that i_1^* must also be an isomorphism, implying that every morphism in $\operatorname{Hom}_R(I, \operatorname{Hom}_{\mathbb{Z}}(R, D)$ must arise from restriction in R, so every morphism from an ideal must lift $\implies$ $\operatorname{Hom}_{\mathbb{Z}}(R, D)$ satisfies Baer's criterion and thus must be injective. ☐

Exercise 4. *Show that a fractional ideal is projective iff it is invertible*

Exercise 5. *Show that the projective objects in $Ch^{\geq 0}(\mathcal{C})$ are split exact sequences of projective objects in $\mathcal{C}$.*

Exercise 6. *Show that $C.$ is contractible if it is split exact.*

Proof. Regard the following commutative diagram:

$$\begin{tikzcd}
\ldots \arrow[r] & B_{i-1} \oplus B_i \arrow[r] \arrow[d, "Id"] \arrow[dl, "h"] & B_i \oplus B_{i+1} \arrow[r] \arrow[d, "Id"'] \arrow[dl, "h"] & B_{i+1} \oplus B_{i+2} \arrow[r] \arrow[d, "Id"] \arrow[dl, "h"] & \ldots \arrow[dl, "h"] \\
\ldots \arrow[r] & B_{i-1} \oplus B_i \arrow[r] & B_i \oplus B_{i+1} \arrow[r] & B_{i+1} \oplus B_{i+2} \arrow[r] & \ldots
\end{tikzcd}$$

For the above split exact sequence we have that each differential is given by $\iota_{B_{i+1}}\pi_{B_{i+1}}$. Define $h : B_i \oplus B_{i+1} \to B_{i-1} \oplus B_i$ by $h = \iota_{B_i}\pi_{B_i}$ at every level, i.e. projecting onto B_i and including into $B_{i-1} \oplus B_i$. We thus have that $\iota_{B_i}\pi_{B_i} \circ h + h \circ \iota_{B_{i+1}}\pi_{B_{i+1}} = \iota_{B_i}\pi_{B_i} + \iota_{B_{i+1}}\pi_{B_{i+1}} = Id - 0$. Thus, the identity map is homotopic to 0 implying $C.$ is contractible and thus chain homotopy equivalent to its homology complex, which is the zero complex by exactness of the sequence. ☐

Exercise 7. *R a PID means all resolutions of f.g. module by f.g. projectives may be truncated at the second grade to yield another projective resolution*

Proof. Immediate consequence of the structure theorem for PIDs: for P_0 f.g. projective P_0 is free and so $ker(d : P_0 \to M)$ for M f.g. is also f.g. free, implying $0 \to ker\ d \to P_0 \to M$ is a projective resolution. ☐

Exercise 8. *For $F : R - mod \to R/I - mod$ via $M \mapsto M[I]$ show that:*
a. F is left exact

Proof. a. Consider the following short exact sequence in $R - mod$:

$$0 \longrightarrow A \xhookrightarrow{i} B \xtwoheadrightarrow{j} C \longrightarrow 0$$

We have that the monic i is still monic when restricted to $A[I] \subset A$, and that $B[I] \cap ker(j) = B[I] \cap A = A[I]$ (viewing i as inclusion) and so $kerF(j) \cong kerj \cap B[I] = A[I]$ so exact at the left and in the middle $\implies$ left exact.

b. Consider the following short exact sequence of complexes:

$$\begin{tikzcd}
& 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A \arrow[r, "i"] \arrow[d, "a"] & B \arrow[r, "j"] \arrow[d, "a"] & C \arrow[r] \arrow[d, "a"] & 0 \\
0 \arrow[r] & A \arrow[r, "i"] \arrow[d] & B \arrow[r, "j"] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
& 0 & 0 & 0 &
\end{tikzcd}$$

Where $(a) = I$ and a represents the multiplication by a map. By the long exact sequence in homology, we have the sequence:

$$0 \longrightarrow ker_A a = A[I] \longrightarrow ker_B a = B[I] \longrightarrow ker_C a = C[I] \longrightarrow ...$$

$$... \longrightarrow A/aA = A/IA \longrightarrow A/aA = A/IA \longrightarrow A/aA = A/IA \longrightarrow 0$$

Which is precisely the desired result. $\square$