# Math 225B Differential Geometry: Homework 3

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**Exercise 1** (ACITDG Spivak, Chapter 4 Exercise 5)**.**

*Proof.* i) Suppose the quantity $\lambda^i\mu_i$ is an invariant, and let $\lambda^i$ by the component of any arbitrary covariant vector field. We have that:

$$\lambda^j\mu_j = \lambda'^i\mu'_i \implies \lambda^j\mu_j = \lambda^j\frac{\partial x'_i}{\partial x'_j}\mu'_i \implies \lambda^j\left(\mu_j - \frac{\partial x'_i}{\partial x_j}\mu'_i\right) = 0$$

And as $\lambda^j$ was assumed to be the components of any arbitrary vector it is clear that $\mu_j$ satisfies the contravariant property symmetric logic applies for when $\mu_i$ is assumed to be the components of an arbitrary contravariant vector.
ii) We have that locally, $n$ independent vector fields yield a basis for the vector space at each trivialization and the result clearly holds in a neighborhood with coordinates $(x, U)$ by previous homework. It suffices to show that this expression is invariant of parametrization. Let $x'$ be a new system of coordinates for the neighborhood. We have:

$$a^\alpha\lambda'^i_{\alpha|} = \frac{\partial x'_i}{\partial x_j}(a^\alpha\lambda^i_{\alpha|}) = \lambda'^i$$

And thus $a^\alpha$ is an invariant.
iii) This is only true as a local condition: from earlier, we
iv) Suppose $a^{ij} = a^{ji}$ are the components of a vector field. For any other coordinate system, we have the associated equations:

$$a'^{ij}T = a'^{ji}T$$

For $T$ an associated transformation for a vector field from one coordinate to another for $a^{ij}$ tensor field. Cancelling out the $T$'s yields the result.
v) Suppose $a^{ij}$ and $b^{ij}$ are tensor fields. Then both satisfy the same property $a'^{kl}T = a^{ij}$ for $T$associated transformation from old coordinates to new; furthermore, both have the same transformation matrix. Thus, $(a'^{kl} + b'^{kl})T = a'^{kl}T + b'^{kl}T = a^{ij} + b^{ij}$.
vi) Applying similar logic as in part $i$) yields that $a^{ij}$ and $a^{ji}$ must be tensors, and part $v$) yields the property that $a^{ij} + a^{ji}$ must be a tensor. We have that $b_{ij} = a_{ij} + a_{ji}$ is such that $b_{ij} = b_{ji}$.
viii) If $ba_{ss} + ca_{ss} = 0$ then either $b = -c$ in which case $a_{rs} = a_{sr}$ and $a$ is symmetric, or $a_{ss} = 0$ and $ba_{sr} = -ca_{rs} \implies -\frac{b}{c} = \frac{a_{rs}}{a_{sr}}$ which implies $-\frac{b}{c} = -\frac{c}{b}$ and thus $b = \pm c$, so $a$ is either symmetric or skew-symmetric.
ix) Under transformations of coordinates we have $A_{ij} = A'_{kl}T$ with $T$ being a scaling matrix, which preserves rank of $A = (a_{ij})$, and thus rank is invariant under all transformations of coordinates.
x) We have that the tensor $a_ib_j$ with components two vectors $a_i$ and $b_j$ has rank 1 as any column vector in the mamtrix $a_ib_j$ is proportional to the the first column vector by multiplying by $\frac{a_1}{a_i}$. For the symmetric tensor $a_ib_j + a_jb_i$ we have that for $a_i \neq xb_j$ for $x$ a constant there are at least two columns $k, l$ s.t. $a_ib_k + a_kb_i \neq x(a_ib_l + a_lb_i)$ for some $i$ (as if this were not true then $b_i$ would necessarily have to be a multiple of $a_i$. Thus, there are at least two linearly independent columns, $A + A^T$ can only have a maximum rank equal to the sum of the ranks, i.e. the rank is 2.
xi) $a^i_j\lambda_i = \alpha\lambda_j \implies a^i_j\lambda_j - \alpha\lambda_j = 0 \implies \lambda_j - \alpha\delta^i_j\lambda_i = 0$ and thus the result, as $d^i_j$ represents the change of basis matrix from $i$ to $j$ via $V^* \otimes V \cong End(V)$. If the result holds for arbitrary vectors we have that in particular it holds for each coordinate and thus $a^i_j = \alpha\delta^i_j$.
xii) Suppose $a^i_j = \alpha\delta^i_j$ for $\lambda_i$ s.t. $\mu^i\lambda_i = 0$; we may look at each vector $\lambda_i$ and decompose it into some sum $v + c_0y$ for $v \in ker\mu^i$. Thus, by linearity, $a^i_j$ is completely determined by the value it takes on the unit vector $\vec{u}$ not in the kernel of $\mu_i$, which can be given by some vector $\eta_j = a^i_j\vec{u}$. Writing $\sigma_j = a^i_j\vec{u} - \eta_j$, the result follows.
xiii) Consider the multilinear map given by $\det(\mu'^{i_m}(\lambda'_{j_n}))$ for the $m, n$ matrix with $m, n = 1, ..., p$ for $\mu'^{i_m}$ any contravariant and $\lambda'_{j_n}$ any covariant vector. We know that under a coordinate transformation, we have

that

$$\det(\mu^{k_m}(\lambda_{l_n})) = \det\left(\frac{\partial x_{l_m}}{\partial x'_{j_n}}\frac{\partial x'_i}{\partial x_k}\mu'^{i_m}\lambda'_{j_n}\right)$$

Which, expanding out the determinant using the generalized Kronecker delta to keep track of the sign of each pattern, we have the following:

$$\sum \delta^{i_1,\ldots,i_p}_{j_1,\ldots,j_p}(\mu^{i_1}(\lambda_{j_1}),\ldots,\mu^{i_p}(\lambda_{j_p})) = \sum \delta^{k_1,\ldots,k_p}_{l_1,\ldots,l_p}(\mu^{i_1}(\lambda_{j_1}),\ldots,\mu^{i_p}(\lambda_{j_p}))\cdot\frac{\partial x_{l_1}}{\partial x'_{j_1}},\ldots,\frac{\partial x_{l_p}}{\partial x'_{j_p}}\frac{\partial x'_i}{\partial x_k},\ldots,\frac{\partial x'_p}{\partial x_p}$$

Which gives us the result. □

**Exercise 2** (Exercise 6 Chapter 1V Spivak)**.**