Math 210B Algebra: Homework 3

Feb 8th, 2019

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Exercise 1. *Let G be a profinite group.*
a. Show that if a subgroup of G is open, then every subgroup containing it is open,
b. Show that G is topologically isomorphic to $\hat{G}$ via the map to $\hat{G}$ as a limit if and only if every subgroup of G is open.

Proof. Let $G = \lim G_i$ be the profinite group in consideration for G_i finite, discrete. a. Let $H \subset G$ be an open subgroup. We have that for any $H \subset K \subset G$ subgroup, $K = \bigcup_{k \in K} kH$ cosets of $H \subset K$. However, as left multiplication is a homeomorphism, it is an open map, and thus kH is open and K is a union of open sets, and is thus open.
b. Suppose that $G \cong \hat{G}$ topologically via the limit map. We thus have continuous maps $\pi_N : G \mapsto G/N$ equipped with the discrete topology for any N a normal subgroup of finite index. We have that $\pi_N^{-1}\{e\} = N$, and as the preimage of an open set is open for a continuous map, we have that N must be open for any normal subgroup of finite index. Furthermore, given any arbitrary subgroup H of finite index, we have that the Cayley Theorem map $\lambda : G \to \Sigma_{[G:H]}$ via left multiplication by g must have a kernel of finite index properly contained in H, implying that H has an open subgroup and is thus open by part a.
Conversely, suppose every subgroup of finite index in G is open. In particular, we have that the surjection $G \to G/N$ must be a continuous surjection of topological groups for N finite index, as the preimage of any $gN \in G/N$ open is $gN \in G$ which is an open coset of G as N is open. Thus, as we have topological homomorphisms $G \to G/N$ for any normal subgroup of finite index, we have an induced map $G \mapsto \hat{G}$ given by $g \mapsto \prod_{G/N} gN$. This map is injective as the kernel must be contained in $\bigcap_{N \text{normal}} N \subset \bigcap_{i \in I} \ker \pi_i = \{e\}$ for I indexing G_i, $\pi_i : G \to G_i$ projection onto the ith coordinate. We have that for cosets $xN \in G/N$, the preimage $\pi_N^{-1}\{xN\}$ for $\pi_N : \hat{G} \to G/N$ forms a basis for the topology of $\hat{G}$. Under the canonical map $G \to \hat{G}$ given above, we have that $xN \subset \pi_N^{-1}\{xN\}$ for $xN \in G$ identified with its image in $\hat{G}$. Thus, we have that any element of the basis of $\hat{G}$ must intersect G, implying that G is a compact, dense subset of $\hat{G} \implies G \to \hat{G}$ surjective, and thus this is a bijective continuous homomorphism from a compact space to a Hausdorff space, yielding the claim. □

Exercise 2. *Show that $\hat{\mathbb{Z}} \cong \prod_p \mathbb{Z}_p$ as topological rings.*

Proof. Let S be an arbitrary ring. We have the natural isomorphism $\mathrm{Hom}(S, \hat{\mathbb{Z}}) \cong \lim_n \mathrm{Hom}(S, \mathbb{Z}/m\mathbb{Z}) \cong \lim_n \prod_{p \text{ prime}} \mathrm{Hom}(S, \mathbb{Z}/p_i^{v_i(n)}\mathbb{Z})$ where $v_i(n)$ is the largest power of p_i that can divide n. The above limit can be rewritten as: $\prod_{p \text{ prime}} \lim_n \mathrm{Hom}(S, \mathbb{Z}/p_i^{v_i(n)}\mathbb{Z}) \cong \prod_{p \text{ prime}} Hom(S, \mathbb{Z}_p) \cong Hom(S, \prod_{p \text{ prime}} \mathbb{Z}_p)$. By the Yoneda Lemma, as the Hom functors are naturally isomorphic and h_A is a fully faithful embedding into the category of functors $Rng \to Set$ we have that $\hat{\mathbb{Z}} \cong \prod_p \mathbb{Z}_p$. □

Exercise 3. *Show that $R[G] \otimes_{\mathbb{R}} R[H] \cong R[G \times H]$ as R algebras for G and H groups.*

Proof. Consider the map $f : R[G] \times R[H] \to R[G \times H]$ via $f : (\sum_I r_i g_i, \sum_J r_j g_j) \mapsto \sum_{I,J} r_i r_j (g_i, h_j)$ for I, J finite indexing sets. This map is clearly bilinear, (which also yields well-definedness) and is also surjective, and thus must factor through a surjective map $\overline{f} : R[G] \otimes_{\mathbb{R}} R[H] \to R[G \times H]$. As R-modules, we may exhibit an explicit inverse via the map $k : (g, h) \mapsto g \otimes h$. This map is well defined as $g' \otimes h' = g \otimes h$ for $g = g', h = h'$. As both algebras are free R-modules, it suffices to check the map is an inverse on the bases for both. Furthermore, we have that $k\overline{f}(g \otimes h) = g \otimes h$ and that $fk(g, h) = (g, h)$; thus, the map agrees on bases for both R-modules. Furthermore, we have that the map extends to multiplicative maps, as $\overline{f}((g, h) \cdot (g', h')) = gg' \otimes hh' = g \otimes h \cdot g' \otimes h' = \overline{f}(g, h) \cdot f(g', h')$. Similarly, $k(g \otimes h \cdot g' \otimes h') = (gg', hh') = (g, h)(g', h') = k(g \otimes h) \cdot k(g' \otimes h')$. Thus, we have inverse R algebra homomorphisms given by $\overline{f}$ and k, yielding the result. □

Exercise 4. *a. Show that tensor products commute with colimits.*
b. Give an example to show that tensor products in general do not commute with limits.

Proof. a. As the balanced tensor product functor $-\otimes_R N$ is a left adjoint (via the tensor-Hom adjunction), we have that left adjoint functors must preserve colimits and thus this functor must also preserve colimits.
b. Consider the following diagram:

$$\mathbb{Z} \underset{2}{\overset{0}{\rightrightarrows}} \mathbb{Z}$$

The limit of the above diagram (which we will denote N_1) is the equalizer of 0 and 2, i.e. the kernel of this map, given by 0. However, applying the tensor product functor $-\otimes \mathbb{Z}/2$ under the observation that $\mathbb{Z}\otimes_{\mathbb{Z}}\mathbb{Z}/2 \cong \mathbb{Z}/2$ as left $\mathbb{Z}$ modules, we have the diagram:

$$\mathbb{Z}/2 \underset{2}{\overset{0}{\rightrightarrows}} \mathbb{Z}/2$$

The map given by multiplication by 2 coincides with the 0 map for $\mathbb{Z}/2$, and thus the equalizer (kernel) of the above 2-multiplication is the whole module $\mathbb{Z}/2$, which we will denote N_2. However, if this functor commuted with all limits, we have $N_1 \cong N_2$, which is clearly not true. $\square$

Exercise 5 (Problem 6)**.**

Proof. Let $N^{\perp} = \{f \in M^* \mid f(N) = 0\}$. We have the natural pullback map $h^* : (M/N)^* \to M^*$ along the map $h : M \to M/N$ the canonical surjection (this is a map of right R-modules). Given $\phi \in (M/N)^*$ we have $h^*\phi(m) = \phi \circ h(m)$, which is a map that disappears on N as $h(N) = 0$. Thus, $h^* : (M/N)^* \to N^{\perp}$. This map is injective, as if $h^*\varphi(m) = 0$ then $\varphi \circ h(m) = 0\ \forall\ m \in M$, which is online possible if $\varphi \in (M/N)^*$ was itself 0. Similarly, it is surjective, as given a map $f \in N^{\perp}$, we have that $f : M \to R$ s.t. $f(N) = 0$, which, by the first isomorphism theorem, yields a distinct map $\overline{f} : M/N \to R$ s.t. $f = h^*\overline{f}$, and $f \in N^{\perp}$. Thus, $(M/N)^* \cong N^{\perp}$.
Now consider $M^*/N^{\perp}$. We have the natural map $res_N : M^*/N^{\perp} \to N^*$ given by $res_N f = f|_N$. This map is also clearly a right R-module homomorphism. Furthermore, we have that $res_N f = 0 \implies f|_N = 0$, or that $f(N) = 0 \implies f = 0 \in M^*/N^{\perp}$, so res_N is an injection. $\square$

Exercise 6. *Show that* $\mathbb{Q}\otimes_{\mathbb{Z}}\mathbb{Z}[\alpha] \cong \mathbb{Q}(\alpha)$.

Proof. Consider $f : \mathbb{Z}[t]\times\mathbb{Q} \to \mathbb{Q}[t]$ given by $nt^i \times q \mapsto qnt^i$. The map is clearly $\mathbb{Z}$-bilinear, well-defined, and is surjective. We have that it must factor through a surjective map $\overline{f} : \mathbb{Z}[t]\otimes_{\mathbb{Z}}\mathbb{Q} \to \mathbb{Q}[t]$, in fact as a ring homomorphism. We may define an inverse map, given by $k : \mathbb{Q}(t) \to \mathbb{Z}[t]\otimes\mathbb{Q}$ via $k : \sum_I n_i t^i \mapsto \sum_I n_i \otimes t^i$. This map is well defined as if $\sum_I n'_{i'} t^{i'} = \sum_{I'} n_{i'} t^{i'}$ with I indexing set of distinct is and n_i nonzero then $i = i', n = n'$, and $n\otimes t^i = n'\otimes t^{i'}$. It is clearly a ring homomorphism as it preserves multiplicativity and additivity, and is thus an inverse ring homomorphism to $\overline{f}$ as $k\overline{f} = id$, $\overline{f}k = id$. Thus, $\mathbb{Q}\otimes_{\mathbb{Z}}\mathbb{Z} \cong \mathbb{Q}[t]$. In particular, under this isomorphism, we have that the ideal $(f) \subset \mathbb{Q}[t]$ ($f = m_{\mathbb{Q}}(\alpha)$ viewed with coefficients in $\mathbb{Z}$) is the isomorphic image of the ideal $\mathbb{Q}\otimes(f_{\mathbb{Z}})$. We thus have that $\mathbb{Q}[t]/(f) \cong \mathbb{Q}\otimes\mathbb{Z}[t]/(f_{\mathbb{Z}}) \cong \mathbb{Q}\otimes_{\mathbb{Z}}\mathbb{Z}[\alpha]$. $\square$

Exercise 7. *Let K/F be a finite separable extension of fields, E an arbitrary extension of F. Show that $K\otimes_F E$.*

Proof. By the primitive element theorem, $K = F(\alpha) = F[t]/(f)$ for f an irreducible polynomial with coefficients in F. By similar logic as in the previous problem, we have that $K\otimes_F E = E[t]/(f)$. Decomposing f into its coprime factors $h_1, ..., h_n$ in E, we have that by Chinese Remainder Theorem, $E[t]/(f) \cong E[t]/(h_1)\times ...\times E[t]/(h_n)$. As F is separable, we have that $h_1, ..., h_n$ are always irreducible, (as if h_i were not irreducible then $h_i = gh$ for g, h not coprime by assumption, which is only possible if they share a root.). Thus, $K\otimes_F E \cong E[t] \cong E[t]/(h_1)\times ...\times E[t]/(h_n)$ for $h_1, ..., h_n$ irreducible, yielding the result. $\square$