Math 215A Commutative Algebra: Homework 3

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Exercise 1. *Suppose that $0 \to L \to M \to N \to 0$ is an exact sequence of R-modules. Show that if L and N are flat, or if M and N are flat, then so is the third.*

Proof. Let $0 \to L \to M \to N \to 0$ be an exact sequence. Let A be an arbitrary R-module: then we have a long exact sequence given by:

$$\mathrm{Tor}^1_R(L,A) \longrightarrow \mathrm{Tor}^1_R(M,A) \longrightarrow \mathrm{Tor}^1_R(N,A) \longrightarrow L \otimes_R A \longrightarrow M \otimes_R A \longrightarrow N \otimes_R A \longrightarrow 0$$

Now suppose L and N are flat. We have then that $\mathrm{Tor}^1_R(L,-) = 0 = \mathrm{Tor}^1_R(N,-)$. Thus, by the exactness of the sequence above, $\mathrm{Tor}^1_R(M,-) = 0$ as it is 0 for A any arbitrary R-module. Thus, given any short exact sequence $0 \to A\ toB \to C \to 0$, we have the following sequence:

$$\mathrm{Tor}^1_R(M,C) \cong 0 \longrightarrow M \otimes_R A \longrightarrow M \otimes_R B \longrightarrow M \otimes_R C \longrightarrow 0$$

And so tensoring with M preserves short exact sequences, yielding that M is flat. Now suppose M and N are flat. Given any monomorphism $f : A \to B$, we have the following diagram where rows and columns are exact:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] \\
\ker(A \otimes f) \arrow[r] & A \otimes_R L \arrow[r, "A\otimes f"] \arrow[d] & B \otimes_R L \arrow[d] \\
0 \arrow[r] & A \otimes_R M \arrow[r] & B \otimes_R M
\end{tikzcd}$$

where the exactness of the second row uses the flatness of M and the exactness of the columns comes from $\ker(A \otimes_R L \to A \otimes_R M) \cong \mathrm{Tor}^1_R(N,A) = 0$ and similarly for B. Thus, we have that the composite map $\ker(A \otimes f) \to A \otimes_R L \to A \otimes_R M \to B \otimes_R M$ is equal to the composite map $\ker(A \otimes f) \to B \otimes_R L \to B \otimes_R M$ which is the 0 map by definition. As the first map is a composition of monomorphisms, we have that the 0 map on $\ker(A \otimes f)$ is a monomorphism, implying that is is 0. Thus, $- \otimes_R L$ must preserve monomorphisms, implying that L is flat. $\square$

Exercise 2. *1) Show that an R-module X is flat if and only if any map from a finitely presented R-module to X factors through a free module. Conclude that finitely presented free modules are projective.*
2) Show that for M a finitely presented module over a local ring, TFAE: a) M is free, b) M is projective, c) M is flat.
3) Show that a finitely presented R-module is projective if and only if it is locally free.

Proof. 1) Let M be a module with finite presentation $R^m \to R^n \to M \to 0$. We have that any map from $M \to X$ may be written as a map $f : R^n \to X$ such that the composite $R^m \to R^n \to X$ is the zero map. Now, we have that $\mathrm{Im}(R^m \to R^n)$ is finitely generated, say by elements $q_1, ..., q_m$ the image of basis elements. For any given q_1, we have that $q_1 \in \ker f$. By the equational criterion for flatness, we have that we may factor the map $f : R^n \to X$ as a map $g_1 : R^n \to R^{k_1}$ followed by a map $h_1 : R^{k_1} \to X$ such that $f = h_1 g_1$ and $q_1 \in \ker g_1$. We may repeat this process in R^{k_1} by noting that $f = h_1 g_1 \implies g_1(q_1) \in \ker h_1$, and so we may factor the map $h_1 : R^{k_1} \to X$ into maps $g_2 : R^{k_1} \to R^{k_2}$ and $h_2 : R^{k_2} \to X$ such that $h_1 = h_2 g_2$ and $g_1(q_1) \in \ker g_2$, yielding a map $g_2 g_1 : R^n \to R^{k_2}$, $h_2 : R^{k_2} \to X$ such that $h_2 g_2 g_1 = f$ and $q_1, q_2 \in \ker g_2 g_1$.

Iterating this process for all m we have the following diagram:

$$\begin{tikzcd}
R^m \arrow[r] & R^n \arrow[r, "f"] \arrow[d, "g_1"] & X \\
& R^{k_1} \arrow[d, "g_2"] \arrow[ur, "h_1"] & \\
& \vdots \arrow[d, "g_m"] & \\
& R^{k_m} \arrow[uuur, "h_m"'] &
\end{tikzcd}$$

Such that $h_m g_m ... g_1 = f$ and $q_1, ..., q_m \in \mathrm{ker} g_m ... g_1$. Denote the composite $g_m ... g_1 := g$. We have the following diagram:

$$\begin{tikzcd}
R^m \arrow[r] & R^n \arrow[r] \arrow[d, "g"] \arrow[dd, bend right, "f"'] & M \arrow[dl, dotted, "g'"] \arrow[ddl, bend left, "f'"] \\
& R^{k_2} \arrow[d, "h_m"] & \\
& X &
\end{tikzcd}$$

Where any map $f' : M \to X$ corresponds to a map $f : R^n \to X$ such that the composite is 0, which factors as a map $h_m g : R^n \to X$ such that $R^m \to R^n \to R^{k_2}$ is 0, which induces a map $g' : M \to R^{k_2}$ such that the diagram above commutes. Thus, any map from a finitely presented module to a flat module factors through a free module. Now, suppose any map from a finitely presented module to a module X factors through a free module. Given any relation $\sum_{i=1}^n r_i x_i = 0$ in X, we have a map $g : R^n \to X$ via $g : e_i \mapsto x_i$, $f : R \to R^n$ via $1 \mapsto \sum_{i=1}^n r_i e_i$ such that the composite $gf = 0$. By the property above, the map f factors through maps $\alpha : R^j \to X$, $\beta : R^n \to R^j$ such that $g = \alpha\beta$, and $\beta g = 0$: i.e. for $l_1, .., l_k$ a basis for R^k, $\exists \{a_{ij}\}_{i=1...n; j=1...k}$ such that $\beta : e_i \mapsto \sum_{j=1}^k a_{ij} l_j$ such that $\sum_{i=1}^n r_i (\sum_{j=1}^k a_{ij} l_j) = 0 \iff \forall\, j, \sum_{i=1}^n r_i a_{ij} = 0$ by the fact that l_j is a basis: i.e. in X, there exist $y_1, ..., y_j$ such that $x_i = sum_{j=1}^k a_{ij} y_j$ and $\sum_{i=1}^n r_i a_{ij} = 0$, which is exactly the equational criterion for flatness, yielding that X is flat.
In particular, suppose X is finitely presented, flat. We have that the identity map $Id : X \to X$ must factor through maps $h : X \to R^j$ monic, $g : R^j \to X$ epic such that $gh = Id$: this implies that X is a summand of R^j and is thus projective.

2) We have that $b) \iff c)$ for M finitely presented by the result above. We also have that $a) \implies b)$ as free modules are projective. It thus suffices to show that finitely presented projective modules over a local ring are free. Let M be a finitely presented projective module: In particular, M is a summand of some free module R^k. We may pick some set of generators $m_1, ..., m_j$ of M such that $m_1 + \mathfrak{m}M, .., m_j + \mathfrak{m}M$ is a basis of $M/\mathfrak{m}M \subset (R/\mathfrak{m})^k$ over $R/\mathfrak{m}$: we may select this to be a basis as by a corollary of Nakayama's lemma we have that any set of element that generate the vector space $M/\mathfrak{m}$ over the residue field must generate M. We have that this set of elements is linearly independent in $(R/\mathfrak{m})^k$. It thus suffices to show that this finite set of elements that is linearly independent in $(R/\mathfrak{m})^k$ over the residue field must be linearly independent over R. Let $a_i \in R$ not all 0 for $i = 1, ..., j$. Let l minimal such that $a_i \in \mathfrak{m}^l$. We have that the map induced by the R-action below is an isomorphism (as it is an isomorphism for R and tensor products commute with direct sums):

$$\mathfrak{m}^l/\mathfrak{m}^{l+1} \otimes_R R^k \to (\mathfrak{m}^l/\mathfrak{m}^{l+1})^k$$

We have that the left hand side is isomorphic to $\mathfrak{m}^l/\mathfrak{m}^{l+1} \otimes_{R/\mathfrak{m}} (R/\mathfrak{m})^k$. We have that the sum $\sum_{i=1}^j a_i \otimes m_i \neq 0$ as m_i are linearly independent in $(R/\mathfrak{m})^k$. Thus, the sum is nonzero in $(\mathfrak{m}^l/\mathfrak{m}^{l+1})^k$, implying it is nonzero in R^k. Given that the a_i were chosen arbitrarily, this must be true for all collections a_i with all a_i nonzero, and thus m_i must be linearly independent in $R^k \implies$ they are a linearly independent generating set for M and so M is free.

3) We have that if X is a finitely presented projective R-module, then X is a direct summand of some free module R^k. We have thus that $S^{-1}X$ is a direct summand of $(S^{-1}R)^k$ as localization commutes with direct products, and thus $S^{-1}X$ is projective and finitely presented, implying that it is free. Now suppose X is locally free. This yields in particular that X is locally flat. Now, given any short exact sequence $0 \to A \to B \to C \to 0$, we have the following exact sequence:

$$\mathrm{Tor}^1_R(X, A) \longrightarrow X \otimes_R A \longrightarrow X \otimes_R B \longrightarrow X \otimes_R C \longrightarrow 0$$

As localization is flat, we have the following sequence is also exact:

$$S^{-1}R \otimes_R \mathrm{Tor}^1_R(X, C) \longrightarrow S^{-1}R \otimes_R X \otimes_R A \longrightarrow S^{-1}R \otimes_R X \otimes_R B \longrightarrow S^{-1}R \otimes_R X \otimes_R C \longrightarrow 0$$

And the above sequence is isomorphic to the following one:

$$S^{-1}\mathrm{Tor}^1_R(X, C) \longrightarrow S^{-1}X \otimes_{S^{-1}R} S^{-1}A \longrightarrow S^{-1}X \otimes_{S^{-1}R} S^{-1}B \longrightarrow S^{-1}X \otimes_{S^{-1}R} S^{-1}C \longrightarrow 0$$

But we know by flatness of localization that $0 \to S^{-1}A \to S^{-1}B \to S^{-1}C \to 0$ is exact, and by assumption the flatness of $S^{-1}X$ yields that the image of $S^{-1}Tor^1_R(X, C)$ in $S^{-1}X \otimes S^{-1}A$ must be 0. As this holds at every localization, we have that the image of $Tor^1_R(X, A) \to X \otimes_R A$ must be zero as it is zero at every localization $\implies 0 \to X \otimes_R A \to X \otimes_R B \to X \otimes_R C \to 0$ must be exact and so X is flat. As it is finitely presented, it is projective. $\square$

Exercise 3. *Show that for X a flat R-module TFAE:*
1) X is faithfully flat.
2) $L \to M \to N$ is exact $\iff L \otimes_R X \to M \otimes_R X \to N \otimes_R X$ is exact.
3) $N \otimes_R X = 0 \iff N = 0$.
4) For every ideal I of R, $IX \neq X$.
5) For every maximal ideal $\mathfrak{m}$ of R, $\mathfrak{m}X \neq X$.