# Math 246A Complex Analysis: Homework 0

Oct 4th, 2019

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**Exercise 1.** *a) Fix $\lambda \in \mathbb{R}$ and $a, b \in \mathbb{C}$ with $\lambda > 0,\ \lambda \neq 1,\ a \neq b$. Use algebraic manipulations to identify*

$$\left\{ z \in \mathbb{C} : \left| \frac{z-a}{z-b} \right| = \lambda \right\}$$

*as a circle.*
*b) Show that every circle can be realized in this manner.*
*c) Give analogues of a) and b) when $\lambda = 1$.*

*Proof.* a) Consider $\frac{z-a}{z-b}$ in the set given above. We have that $|\frac{z-a}{z-b}| = \lambda \implies |z-a| = \lambda|z-b|$. Set $z = x+iy,\ a = a_1 + ia_2,\ b = b_1 + ib_2$. We have from the formula above the following derivation:

$$\begin{aligned}
&|z-a| = \lambda|z-b| \implies (x-a_1)^2 + (x-a_2)^2 = \lambda^2(x-b_1)^2 + \lambda^2(x-b_2)^2 \\
&\to (x-a_1)^2 - \lambda^2(x-b_1)^2 - x^2(1-\lambda^2) - 2x(a_1 - \lambda^2 b_1) + a_1^2 - \lambda^2 b_1 = 0 \ — \ 1 \\
&\to x^2 - \frac{2x(a_1-\lambda^2 b_1)}{(1-\lambda^2)} + \frac{(a_1^2 - \lambda^2 b_1)}{(1-\lambda^2)} \\
&= x^2 - \frac{2x(a_1-\lambda^2 b_1)}{(1-\lambda^2)} + \frac{a_1^2 - \lambda^2(a_1+b_1) + \lambda^4 b_1^2 - 2\lambda^2 a_1 b_1 + 2\lambda^2 a_1 b_1}{(1-\lambda^2)^2} \\
&= x^2 - \frac{2x(a_1-\lambda^2 b_1)}{(1-\lambda^2)} + \frac{(a_1-\lambda^2 b_1)^2}{(1-\lambda^2)^2} - \frac{\lambda^2(a_1+b_1)^2}{(1-\lambda^2)^2} = \left(x - \frac{a_1+\lambda^2 b_1}{1-\lambda^2}\right)^2 - \left(\frac{\lambda(a_1+b_1)}{1-\lambda^2}\right)^2
\end{aligned}$$

Similarly simplifying for the parts of the equation involving $y$, we have by rewriting 1 that :

$$\begin{aligned}
&\to \left(x + \frac{a_1-\lambda^2 b_1}{1-\lambda^2}\right)^2 - \left(\frac{\lambda(a_1+b_1)}{1-\lambda^2}\right)^2 + \left(y - \frac{a_2+\lambda^2 b_2}{1-\lambda^2}\right)^2 - \left(\frac{\lambda(a_2+b_2)}{1-\lambda^2}\right)^2 = 0 \\
&\implies \left(x - \frac{a_1+\lambda^2 b_1}{1-\lambda^2}\right)^2 + \left(y - \frac{a_2+\lambda^2 b_2}{1-\lambda^2}\right)^2 = \left(\frac{\lambda(a_1+b_1)}{1-\lambda^2}\right)^2 + \left(\frac{\lambda(a_2+b_2)}{1-\lambda^2}\right)^2
\end{aligned}$$

Which is of the form a circle centered at the point

$$\left(\frac{a_1+\lambda^2 b_1}{1-\lambda^2}, \frac{a_1+\lambda^2 b_1}{1-\lambda^2}\right)$$

with radius

$$\left(\frac{\lambda(a_1+b_1)}{1-\lambda^2}\right)^2 + \left(\frac{\lambda(a_2+b_2)}{1-\lambda^2}\right)^2$$

b) Suppose we are given a circle of radius $r$ centered at a point $(a, b)$. □

**Exercise 2.** *Show algebraically for every triple $a, b, c$ of distinct unimodular complex numbers,*

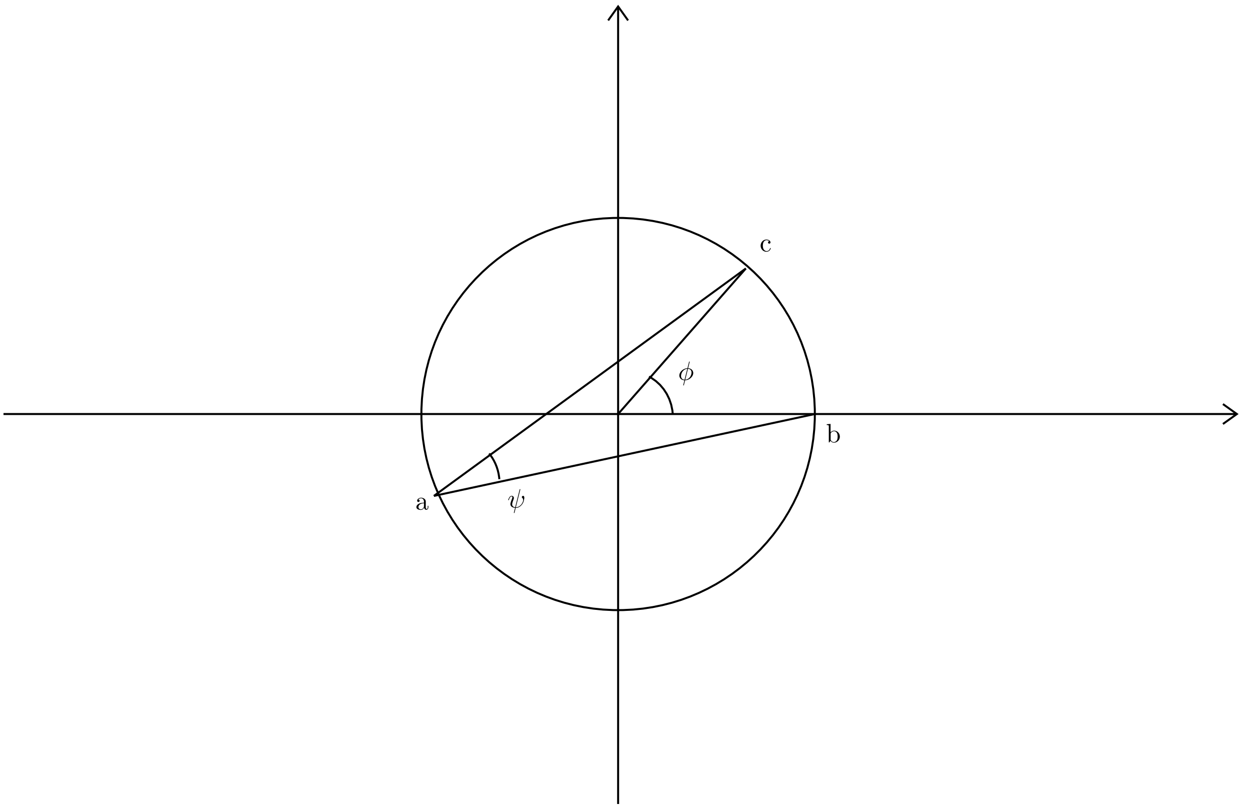
$$\frac{b-a}{1-\bar{a}b} = \frac{c-a}{1-\bar{a}c}$$

*Show that with a little further manipulation this expresses the inscribed angle theorem.*

*Proof.* We have that for $x = a, b, c$, $|x|^2 = 1$. Consider the following string of manipulations:

$$\frac{1-\bar{a}b}{1-\bar{a}c} = \frac{1-\bar{a}b}{1-\bar{a}c} \cdot \frac{a}{a} = \frac{a-b}{a-c} = \frac{b-a}{c-a} \implies \frac{b-a}{1-\bar{a}b} = \frac{c-a}{1-\bar{a}c}$$

Now for the second claim, consider the following diagram:




Our objective is to show that $2\psi = \phi$. Note that $\psi$ is the argument of the complex number given by $\frac{c-a}{1-a}$ ($b = 1$). Dividing this complex number by its conjugate will yield $e^{i2\psi}$, i.e. a unimodular complex number with twice the argument. We have that following set of manipulations:

$$e^{i2\psi} = \frac{c-a}{1-a} \cdot \frac{1-\bar{a}}{\bar{c}-\bar{a}} \cdot \frac{c}{c} = \frac{c-a}{1-\bar{a}c} \cdot \frac{c(1-\bar{a})}{1-a} = \frac{1-a}{1-\bar{a}} \cdot \frac{c(1-\bar{a})}{1-a} = c = e^{i\phi}$$

And thus $2\psi = \phi$. □

**Exercise 3.** *Give a proof of the Intersecting Cord Theorem of Jakob Steiner.*

*Proof.* □

**Exercise 4.** *Any mapping that can be represented in the form*

$$z \mapsto \frac{az+b}{cz+d}$$

*with $ad - bc \neq 0$ is called a Mobius transformation.*
*a) Show that every such mapping can be realised by coefficients satisfying $ad - bc = 1$ and determine the number of such representations.*

*Proof.* a) Given a transformation of the above form, note that $ad - bc = k \neq 0$. As the complex numbers are algebraically closed, we may select $w$ s.t. $w^2 = \frac{1}{k}$. Multiplying both the numerator and the denominator by $w$ yields the same transformation with coefficients $a', b', c', d'$ s.t. $a'd' - b'c' = \frac{1}{k^2}(ad - bc) = 1$. Now suppose there exists another set of coefficients $a'', b'', c'', d''$ representing the transformation s.t. $a''d'' - b''c'' = 1$. Note that for $z = \frac{-b'}{a'}$, we have that both transformations must send $z$ to 0, which is only possible if the numerator is sent to 0 and thus $a''z + b'' = 0 \implies z = \frac{-b''}{a''}$ and similarly for $z = \frac{-d'}{c'}$ yielding a pole at $z$. □