Math 225B Differential Geometry: Homework 2

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Exercise 1. *Show the following:*
a) For any bundle $\pi : E \to B$, the map $s : B \to E$ with $s : p \mapsto 0 \in \pi^{-1}(p)$ is a section.
b) Show that an n-plane bundle is trivial if and only if there exist everywhere linearly independent sections global $s_1, ..., s_n$.
c) Show that locally every n-plane bundle has n linearly independent sections.

Proof. a) It suffices to show that s is continuous, as the composition with the projection is clearly the identity. We have that a base of open sets for a vector bundle locally corresponds with the product topology. Thus, the preimage of any open set in E is determined by local preimages in $U_i \times \mathbb{R}^\urcorner$ for U_i a cover of the manifold, and we have that the preimage of any open set $O \subset U_i \times \mathbb{R}^\urcorner$ is $O \cap U_i \times 0$ which is open in the relative topology on U_i, coinciding with the topology of the manifold and thus the section is continuous.
b) One direction is clear, as if a bundle is trivial then we have the bundle is $M \times \mathbb{R}^k$ and we define n linearly independent sections by $s_i : p \mapsto (p, (0, ..., v_i, ..., 0))$, which is continuous as it corresponds to the zero section followed by translation in the second coordinate. For the backwards direction, we define a homeomorphism from $M \times \mathbb{R}^n$ to E given by $(p, x_1, ..., x_n) \mapsto (p, x_1 s_1(p), ..., x_n s_n(p))$. This map restricts to linear isomorphisms at each point as a linearly independent basis is sent to a linearly independent basis, and is thus bijective, continuous as every open set E is the union of open sets of the form $\bigcup_{x_i \in V_i} U_i \times (s_1 x_1, ..., s_1 x_n)$ as by linear independence, a local trivialization of every open set can be given with the basis of sections, and the preimage is $U_i \times V_i \subset M \times \mathbb{R}^n$ (V_i open by product topology on local trivialization. Clearly, open sets are mapped to open sets and this is a homeomorphism.
c) For an $n-plane$ bundle, around every point we have a local trivialization given by $U \times \mathbb{R}^k$. Selecting any closed subset $K \subset U$ containing p, we define n sections of p by sending p to (p, w_i) for $(w_i)_{i=1,...,n}$ an orthonormal basis for $\pi_2(U \times \mathbb{R}^k)$. This corresponds to the zero section followed by translation in the second coordinate and is clearly smooth. We define this to vanish on $M \setminus U$. This may be extended to a map on all of M by the Tietze extension theorem (the map factors through a map to $1 \in \mathbb{R}$ on K and to 0 on $M \setminus U$. This yields a section of the manifold that locally maps to a basis element. Repeating this process for each basis vector yields n distinct sections that are linearly independent on $p \in K^o$, i.e. satisfying the local condition. □

Exercise 2. *If $g : \mathbb{R} \to \mathbb{R}$ is C^∞ show that $g(x) = g(0) + g'(0)x + x^2 h(x)$ for some C^∞ function $h : \mathbb{R} \to \mathbb{R}$.*

Proof. Define $h(x) = \frac{g(x) - g(0) - g'(0)x}{x^2}$ for $x \neq 0$. At 0 we evaluate the limit:

$$\lim_{x \to 0} \frac{g(x) - g(0) - g'(0)}{x^2} = \lim_{x \to 0} \frac{g'(x) - g'(0)}{2x} = \frac{g''(0)}{2}$$

By l'Höpital's rule. Thus, this function is continuous. Furthermore, evaluating the derivative at 0 yields us:

$$\lim_{x \to 0} \frac{g(x) - g(0) - g'(0)x - \frac{g''(0)}{2}}{x^2} = \lim_{x \to 0} \frac{g'(x) - g'(0)}{x} = \frac{g''(0)}{2}$$

Which agrees on both sides of the limit. Thus, the function is clearly C^∞ everywhere and differentiable at 0, and by similar logic as above we may show that higher derivatives of this function also exist for all nth derivatives. □

Exercise 3. *Show the following:*
a) Let $p_0 \in S^{n-1}$ be the point $(0, ..., 1)$. For $n \geq 2$ define $f : SO(n) \to S^{n-1}$ by $f(A) = A(p_0)$. Show that f is continuous and open. Show that $f^{-1}(p_0)$ is homeomorphic to $SO(n-1)$, and then show that $f^{-1}(p)$ is homeomorphic to $SO(n-1)$ for all $p \in S^{n-1}$.
b) SO(1) is a point, so it is connected. Using part a) and induction on n show that SO(n) is connected for all n.
c) Show that O(n) has exactly two components.

Proof. f is clearly continuous, as under the normal metric topology on $\mathbb{R}^{n\times n}$ we have that for small perturbations in the entries of an $n \times n$ matrix that the resultant image points of a fixed vector are also perturbed by small amounts: thus, metric space continuity holds.

Claim: This map is open.

Let $A \in U$ open. We have some $\epsilon > 0$ s.t. $B_\epsilon(A) \subset U$ under the Euclidean metric. Thus, there is a $\delta > 0$ s.t. $\forall\ w \in B_\delta(v), \exists\ B \in B_\epsilon(A)$ s.t. $B(v) = w$. Thus, the map is open and we have the claim. The preimage of p_0 is all maps that fix the nth vector, which are of the form: $\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$ with A orthogonal $n-1$ dimensional square matrix with determinant 1; i.e. isomorphic to $SO(n-1)$ via the map $A \mapsto B$ by projecting onto the first $(n-1)^2$ coordinates. This is clearly continuous (it is a projection), injective as the last $2n-1$ coordinates are constant, and bijective as every $n-1$ dimension orthgonal matrix can be augmented to an n dimensional one as above. Finally, it is a bijective continuous map from a compact space to a Hausdorff space, and is thus a homeomorphism. Finally, the preimage of any point p is the left coset of the stabilizer of subgroup of p_0, which is homeomorphic to $SO(n-1)$.

For part b), assume that $SO(n-1)$ is connected for some n. We have that S^{n-1} is connected so the image of any two disjoint clopen sets must overlap at some point p. However, the preimage of p is connected by assumption, yielding a contradiction. Thus, $SO(n)$ is connected.

c) We know that $O(n)$ has matrices either of determinant positive or negative 1. $SO(n)$ is connected from b), and there exists a homeomorphism from one to the other via multiplication of -1 in the first column. They are disjoint, and thus there are two connected components. ☐

Exercise 4. *a) Show that the matrix of the adjoint is the transpose matrix.*
b) Show that a symmetric matrix can be orthogonally diagonalized if we assume that it may be diagonalized.
c) Show that a positive definite matrix is nonsingular.
d) Show that $A^T \cdot A$ is positive semi-definite.
e) Show that a positive semi-definite A can be written as $A = B^2$ for some B.
f) Prove polar decomposition.
g) Show that O_1 and P_1 are continuous functions of A.
h) Show that $GL(n, \mathbb{R}) \cong O(n, \mathbb{R}) \times P(n, \mathbb{R})$

Proof. a) We have that if $\langle T^*v, w\rangle = \langle v, Tw\rangle$ then $T^*v^T \cdot w = v^T \cdot Tw$. Let $\{e_i\}_{i=1,...,n}$ be a basis for $\mathbb{R}^n$. For each basis element, we have that $e_i^T \cdot Ae_j = A^T e_i^T e_j$ as the first inner product represents the ith entry in the jth column, while the second one represents the jth entry in the ith column, and when $A^*_{ij} = A_{ji}$ we have that the adjoint must be the transpose.
b) Selecting the first eigenvector w_1, we may generate an orthogonal basis (by Gram-Schmidt) with orthogonal change of basis matrix O s.t.:

$$OAO^T = \begin{pmatrix} \lambda_1 & ... & 0 \\ \vdots & & \\ 0 & & \text{B} \end{pmatrix}$$

And B is clearly symmetric as $B^T = OA^TO^T = OAO^T|_{\mathbb{R}^{n-1}}$. Inductively proceeding yields the result.
c) If it were singular then $\langle Tv, v\rangle = 0$ for $v \in \ker T \setminus 0$.
d) $\langle A^T \cdot Av, v\rangle = \langle Av, Av\rangle = ||Av||^2 \geq 0$.
e) A is symmetric and thus orthogonally diagonalizable: positive semi-definite implies all eigenvalues are ≥ 0, so we may define $B = \text{diag}\{\sqrt{\lambda_1}, ..., \sqrt{\lambda_n}\}$ in the diagonalized basis, positive semi-definite is preserved under orthogonal transformations and the diagonalized basis is orthgonal.
f) We have that $A^TA = B^2$ for B constructible as positive definite (can take all roots from e) positive, invertible implies nonzero eigenvalues). We have that $A = (A^T)^{-1}B \cdot B$. $(A^T)^{-1}B \cdot ((A^T)^{-1}B)^T = (A^T)^{-1}A^TAA^{-1} = I$ and so $(A^T)^{-1}B$ is orthogonal. Uniqueness follows as if $O_2^TO_1 = P_1^TP_2$ then $O_2^TO_1$ is orthogonal and diagonalizable for all eigenvalues positive, which is only possible if all eigenvalues are 1 i.e. $O_2^TO_1$ is the identity.
g) If $A^{(n)} \to A$ is a convergent sequence then every subsequence of $A_1^{(n)}$ has a convergent subsequence by compactness of the orthogonal group, and this subsequence converges to the same limit A_2 by uniqueness of the polar decomposition. Given this fact, we have that $(A_1^{(n)})^{-1}A^{(n)} = A_2^{(n)}$, showing that $A_2^{(n)}$ must also converge to the limit A_2, and thus both functions are continuous.
h) We have a bijection from $GL(n,R) to O(n,\mathbb{R}) \times P(n,\mathbb{R})$ via polar decomposition. This is continuous in each coordinate from part g. Furthermore, the inverse, given by multiplication of the two coordinates, must be continuous as multiplication is continuous on a topological group. □

Exercise 5. *a) Show that a nonsingular linear transformation with positive determinant is homotopic to the identity map.*
b) Suppose $f : \mathbb{R}^n \to \mathbb{R}^n$ is C^∞ and $f(0) = 0$, $f(\mathbb{R}^n - 0) \subset \mathbb{R}^n - 0$ then $f : \mathbb{R}^n - 0 \to \mathbb{R}^n - 0$ is homotopic to Df.

Proof. a) Let $A : [0,1] \to GL(n,\mathbb{R})$ be continuous, and define $H : [0,1] \times \mathbb{R}^n \to \mathbb{R}^n$ by $H(x,t) = A(t)(x)$. Equipping a linear transformation with the operator norm (i.e. $||A(t)|| = \sup_{||x||=1}||Ax||$), we have that $||A(t)x|| = ||A(t)|| \cdot ||x||$. Continuity then follows as $||A(t_0)w - A(t_1)v|| < ||A(t_0)w - A(t_1)w|| + ||A(t_1)|| \cdot ||v - w|| < ||A(t_0) - A(t_1)||.||w|| + ||A(t_1)|| \cdot ||v - w||$. We can independently impose restrictions on $t_0 - t_1$ (by continuity) and $v - w$ s.t. the inequality above is less than ϵ for arbitrary $\epsilon > 0$, and thus this function is continuous. For a nonsingular linear transformation, we may define a homotopy to the identity as there is exists path between positive definite matrices by 31 h).

b) We define a homotopy given by $H(x,t) = \frac{f(tx)}{t}$ for $0 < t \leq 1$, and $Df(0)(x)$ for $t = 0$. We have that this is continuous at the origin (as it suffices to show continuity in the second coordinate at $t = 0$, as the directional derivative is defined as the limit as $t \to 0$ for the difference quotient in $H(x,t)$ under the assumption that $f(0) = 0$. □