Math 210C Algebra: Homework 5

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Exercise 1. *Show that $G \circ L_i F \cong L_i(G \circ F)$ as δ functors for $F : (C) \to (D)$ right exact and $G : (D) \to (E)$ exact.*

Proof. Consider the following diagram of projective resolutions of some short exact sequence (each is truncated noting that this yields a quasi iso to the original short exact sequence embedded in chain complexes concentrated at degree 0):

$$\begin{tikzcd}
 & {} \arrow[d] & {} \arrow[d] & {} \arrow[d] & \\
0 \arrow[r] & P_2 \arrow[r] \arrow[d, "\partial_2^P"] & Q_2 \arrow[r] \arrow[d, "\partial_2^Q"] & R_2 \arrow[r] \arrow[d, "\partial_2^R"] & 0 \\
0 \arrow[r] & P_1 \arrow[r] \arrow[d, "\partial_1^P"] & Q_1 \arrow[r] \arrow[d, "\partial_1^Q"] & R_1 \arrow[r] \arrow[d, "\partial_1^R"] & 0 \\
0 \arrow[r] & P_0 \arrow[r] & Q_0 \arrow[r] & R_0 \arrow[r] & 0
\end{tikzcd}$$

Applying F, we get a long exact sequence of the form:

$$... \longrightarrow H_1(F(P)) \longrightarrow H_1(F(Q)) \longrightarrow H_1(F(R)) \xrightarrow{\delta} H_0(F(P)) \longrightarrow H_0(F(Q)) \longrightarrow ...$$

Now consider the sequence:

$$... \longrightarrow H_1(GF(P)) \longrightarrow H_1(GF(Q)) \longrightarrow H_1(GF(R)) \xrightarrow{\delta_G} ...$$

$$... \xrightarrow{\delta_G} H_0(GF(P)) \longrightarrow H_0(GF(Q)) \longrightarrow H_0(GF(R)) \longrightarrow 0$$

Now all we need is the fact that exact functors naturally commute with the homology functor: this is clearly by considering the following diagram:

$$0 \longrightarrow im\ \partial_i^C \longrightarrow ker\ \partial_{i-1}^C \longrightarrow H_{i-1}(C) \longrightarrow 0$$

Applying G to the above diagram yields the easy result that $H_{i-1}(GC) = GH_{i-1}(C)$ by exactness of G on the above diagram, and noting that $im\ G\partial_i^{GC} = G(im\partial_i^C)$, and likewise for kernel; naturality follows immediately by taking two separate diagrams of the above form.. Seeing this, we have that both connecting homomorphisms fit into the following exact sequence:

$$H_1(GF(Q)) \xrightarrow{Q \to R} H_1(GF(R)) \xrightarrow{A} H_0(GF(P)) \xrightarrow{P \to Q} H_0(GF(Q))$$

With $A = G(\delta), \delta_G$. Thus, as both maps have naturally isomorphic kernels (from the inclusions $Q \to R$ and images (as the kernel of the map induced by $P \to Q$), they must be the same map, and thus G must commute with application of the connecting homomorphism on homology as well. As this computation did not depend on the indices used, we may apply this for arbitrary $i, i-1$. This yields the result. □

Exercise 2. *Show that an R-module A is flat ⇔ $\mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z})$ is an injective object in $R^{op} - mod$ with $(r \cdot f)(m) = f(mr)$.*

Proof. Note that $R^{op} - mod \cong mod - R$ so we will carry out the proof in the latter category. Suppose A is flat. Note that for any ideal $I \subset R$, $\mathrm{Hom}_R(I, \mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z})) \cong \mathrm{Hom}_{\mathbb{Z}}(I \otimes_R A, \mathbb{Q}/\mathbb{Z})$ naturally as abelian groups by the tensor-hom adjunction, where we may view A as a bimodule ${}_RA_{\mathbb{Z}}$, so $- \otimes_R A : mod - R \to \mathbb{Z} - mod = mod - \mathbb{Z}$. Regard the following diagram:

$$\begin{tikzcd}
\mathrm{Hom}_R(I, \mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z})) \arrow[r, "\cong"] & \mathrm{Hom}_{\mathbb{Z}}(I \otimes_{\mathbb{Z}} A, \mathbb{Q}/\mathbb{Z}) \\
\mathrm{Hom}_R(R, \mathrm{Hom}_{\mathbb{Z}}(R, \mathbb{Q}/\mathbb{Z})) \arrow[r, "\cong"] \arrow[u, "i_1^*"] & \mathrm{Hom}_{\mathbb{Z}}(R \otimes_{\mathbb{Z}} A, \mathbb{Q}/\mathbb{Z}) \arrow[u, "i_2^*"']
\end{tikzcd}$$

As $\mathbb{Z}$-modules, $I \otimes_R A \to R \otimes_R A$ is still an injection (as A is flat) and $\mathbb{Q}/\mathbb{Z}$ is an injective object, so i_2^* must be an isomorphism as every morphism in $\mathrm{Hom}_{\mathbb{Z}}(I \otimes_R A, \mathbb{Q}/\mathbb{Z})$ must be the restriction of some morphism in $\mathrm{Hom}_{\mathbb{Z}}(R \otimes_R A, \mathbb{Q}/\mathbb{Z})$ by injectiveness of $\mathbb{Q}/\mathbb{Z}$. The naturality of the tensor-hom isomorphism implies that i_1^* must also be an isomorphism, implying that every morphism in $\mathrm{Hom}_R(I, \mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z})$ must arise from restriction in R, so every morphism from an ideal must lift $\implies$ $\mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z})$ satisfies Baer's criterion and thus must be injective (Note that this is the same proof as for R).

Now suppose that $\mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z})$ is injective in $mod{-}R$. We have that the functor $\mathrm{Hom}_R(-, \mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z}))$ must be an exact functor, being both right exact and preserving injections. Via the tensor-hom adjunction, this is naturally isomorphic to $\mathrm{Hom}_{\mathbb{Z}}(- \otimes_R A, \mathbb{Q}/\mathbb{Z})$, which is the composition of functors $h^{\mathbb{Q}/\mathbb{Z}} \circ t_A$. We have that the composition $h^{\mathbb{Q}/\mathbb{Z}} \circ t_A$ is exact via the natural isomorphism, and that $h^{\mathbb{Q}/\mathbb{Z}}$ is both faithful and exact in $\mathbb{Z} - mod$ as the module is injective; from problem 1, we have the natural comparison map $h^{\mathbb{Q}/\mathbb{Z}} \circ L_i t_A \cong L_i(h^{\mathbb{Q}/\mathbb{Z}} \circ t_A)$, implying that for any R-module M, $h^{\mathbb{Q}/\mathbb{Z}} \circ L_i t_A(M) \cong 0 \implies L_i t_A(M) \cong 0$ as $h^{\mathbb{Q}/\mathbb{Z}}$ is faithful and thus does not send nonzero objects to zero, which is the terminal object (as this would send the identity map to the zero map). Thus, as all the left derived functors are zero, t_A must be exact and so A is flat. $\square$

Exercise 3. *Let R be commutative. Show that TFAE:*
i) A is flat and t_A is faithful.
ii) A is flat and $\mathfrak{m}A \neq A$ for all maximal ideals $\mathfrak{m}$ of R.
iii) A complex $0 \to B \to C \to D \to 0$ is exact iff $0 \to A \otimes_R B \to A \otimes_R C \to A \otimes_R D \to 0$ is exact.

Proof. "i) $\implies$ iii)" Suppose A is flat. Then a complex $0 \to B \to C \to D \to 0$ is exact implies the complex $0 \to A \otimes_R B \to A \otimes_R C \to A \otimes_R D \to 0$ is exact. If t_A is faithful, then if $0 \to A \otimes_R B \to A \otimes_R C \to A \otimes_R D \to 0$ is exact, consider the following complexes:

$$0 \longrightarrow ker(B \to C) \longrightarrow B \longrightarrow C$$

$$0 \longrightarrow coker(B \to C) \longrightarrow D \longrightarrow coker(coker B \to C) \to D) \longrightarrow 0$$

We have that $A \otimes ker(B \to C) \cong 0 \implies ker(B \to C) \cong 0$. In similar fashion, $coker(coker B \to C) \to D$ must also be zero, and thus $0 \to B \to C \to D \to 0$ must necessarily be exact.

"iii) $\implies$ ii)" Consider the complex $0 \to R/\mathfrak{m} \to 0$. We must have that $0 \to A \otimes R/\mathfrak{m} \to 0$ cannot be exact, and so $A/\mathfrak{m}A \neq 0$.
"ii) *implies* i)" As t_A is additive, it suffices to show the induced map on Hom sets is an abelian group monomorphism to imply faithfulness. In particular, we must show that $A \otimes_R M \neq 0$ for any $M \neq 0$, as this implies t_A cannot send a nonzero image to 0, i.e. cannot send a nonzero morphism to the 0 morphism. However, for any module M, we may select some nonzero element $f : R \to M$ via $1 \mapsto m$. This extends into an exact sequence $0 \to R/\mathrm{ann}(m) \to M$, and as t_A preserves monomorphisms and $t_A(R/\mathrm{ann}(m))$ is nonzero, this must be a nonzero morphism and thus M is nonzero. Thus, t_A must be faithful. $\square$

Exercise 4. *Let R $F[x, y, z]$ with F a field, and consider the R-module $F \cong R/(x, y, z)$. Find all Ext and Tor groups of F with coefficients in F.*

Proof. Consider the following projective resolution:

$$\begin{tikzcd} 0 \arrow[r] & R^2 \arrow[r, "A"] & R^3 \arrow[r, "{(x,y,z)}"] & R \arrow[d, "\pi"] \\ & & & F \end{tikzcd}$$

Where $A = \begin{pmatrix} y & z \\ -x & 0 \\ 0 & -x \end{pmatrix}$, and $\pi : R \to F$ is the canonical projection.

This complex is clearly exact, as the kernel of (x, y, z) is generated by $(xe_2 - ye_1, xe_3 - ze_1, ye_3 - ze_2)$ and taking determinants shows that only two out of three are linearly independent, and thus A is monic.
We first compute $\mathrm{Tor}_i^R(F, F)$. Tensoring the above complex with F, we get:

$$\begin{tikzcd} 0 \arrow[r] & F^2 \arrow[r, "0"] & F^3 \arrow[r, "0"] & F \arrow[d, "Id"] \\ & & & F \end{tikzcd}$$

Where all maps disappear as their images lie in $(x, y, z)P_i$ for the corresponding projective P_i. Thus, the resultant Tor values are

$$\mathrm{Tor}_i^R(F, F) \cong \begin{cases} F & i = 0 \\ F^3 & i = 1 \\ F^2 & i = 2 \\ 0 & i \geq 3 \end{cases}$$

Now applying $\mathrm{Hom}_R(-, F)$ to the above complexes, we have:

$$\begin{tikzcd} 0 & \mathrm{Hom}_R(R, F)^2 \arrow[l] & \mathrm{Hom}_R(R, F)^3 \arrow[l, "A^T"'] & \mathrm{Hom}_R(R, F) \arrow[l, "\begin{pmatrix} x \\ y \\ z \end{pmatrix}"'] \\ & & & F \arrow[u, "\pi^*"'] \end{tikzcd}$$

We have that any $\phi \in \mathrm{Hom}_R(R, F)$ is uniquely determined by the value of $\phi(1)$. Furthermore, it is an $R^{op}/(x^{op}, y^{op}, z^{op}) \cong F^{op} \cong F$ module of dimension on under the natural multiplication on maps.
Thus, we end up with the complex:

$$\begin{tikzcd} 0 & F^2 \arrow[l, "0"'] & F^3 \arrow[l, "0"'] & F \arrow[l, "0"'] \\ & & & F \arrow[u, "\pi^*"'] \end{tikzcd}$$

And so we have:

$$\mathrm{Ext}_i^R(F, F) \cong \begin{cases} F & i = 0 \\ F^3 & i = 1 \\ F^2 & i = 2 \\ 0 & i \geq 3 \end{cases}$$

And furthermore, in particular, that $\mathrm{Ext}_i^R(F, F)$ and $\mathrm{Tor}_i^R(F, F)$ are dual vector spaces over F (resp. F^{op}) actions for each i. $\square$

Exercise 5. *Let G be a finite group and A be a finite $\mathbb{Z}[G]$ module. Show that*

$$H^i(G, A) = \begin{cases} A^G / N_G A & i = 2n \\ (ker N_G)/I_G A & i = 2n - 1 \end{cases}$$

Proof. Consider the following resolution of $\mathbb{Z}$:

$$... \xrightarrow{I_G} \mathbb{Z}[G] \xrightarrow{N_G} \mathbb{Z}[G] \xrightarrow{I_G} \mathbb{Z}[G] \xrightarrow{N_G} \mathbb{Z}$$

Where N_G is multiplication by the norm elements and I_G is multiplication by $\sigma - 1$ for $G =< \sigma >$. This is clearly exact by inspection, and periodic as $kerI_G$ is generated by the norm element while $kerN_G$ is generated by I_G. Applying $\text{Hom}_{\mathbb{Z}[G]}(-, A)$ we have:

$$... \xleftarrow{I_G} A \xleftarrow{N_G} A \xleftarrow{I_G} A \xleftarrow{N_G} A^G$$

Now it is clear that $kerI_G = \{a \in A \mid \sigma \cdot a - a = 0\} = A^G$. Thus, at even indices we have that $H^i(G, A) = kerI_G/N_GA = A^G/N_GA$. At the odd spots, by definition, we have $H^i(G, A) = (kerN_G)/I_GA$. □

Exercise 6. *Show $H^1(G, L^\times) = 0$ for $G = Gal(L/K)$ a Galois extension.*

Proof. Consider the standard bar complex $C^i(G, L^\times$ with the standard coboundaries. Let $\phi \in C^1(G, L^\times)$, and using the known condition for the value of δ^1 on the bar complex, we have:

$$\delta^1(\phi)((g_1, g_2)) = g_1\phi(g_2) - \phi(g_1g_2) + \phi(g_1)$$

Denoting $\phi(g_2) = x_{g_2} \in L^\times$ and switching to multiplicative notation for $L^\times$ we have that

$$\phi \text{ cocycle } \implies g_1(x_{g_2}).x_{g_1g_2}^{-1}x_{g_1} = 1 \implies x_{g_1}g_1(x_{g_2}) = x_{g_1g_2}$$

Suppose ϕ is a cocycle. Then for some $a \in L^\times$ we have that $x_{g_1}g_1(a) + ... + x_{g_n}g_n(a) = k \neq 0$ for some $a \in L^\times$ by the linear independence of characters (here $G(L/K)$ is a linearly set of characters $L^\times \to L^\times$). Now, note:

$$\begin{aligned}
&g_1(k) = \sum_{g_i \in G} g_1(x_{g_i})g_1g_i(a) \\
&\implies x_{g_1}g_1(k) = \sum_{g_i \in G} x_{g_1}g_1(x_{g_i})g_1g_i(a) = \sum_{g_i \in G} x_{g_1g_i}g_1g_i(a) = x_{g_1}g_1(a) + ... + x_{g_n}g_n(a) = k \\
&\implies x_{g_1} = \frac{k}{g_1(k)}
\end{aligned}$$

Noting that in the last step translating by g_1 still ranged over all elements in the group. Note that this method yields the same k for all g_i. However, this implies that the morphism ϕ above must have been of the form $g_1\psi(e_G) - \psi(e_G)$ for ψ selecting an element of $L^\times$, which is precisely of the form $\delta^0(\psi)$ for $\psi \in C^0(G, L^\times)$. Thus, any cocycle is a coboundary and the first cohomology group is trivial. □