Math 246A Complex Analysis: Homework 0

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Exercise 1. *a) Fix $\lambda \in \mathbb{R}$ and $a, b \in \mathbb{C}$ with $\lambda > 0,\ \lambda \neq 1,\ a \neq b$. Use algebraic manipulations to identify*

$$\left\{ z \in \mathbb{C} : \left| \frac{z-a}{z-b} \right| = \lambda \right\}$$

as a circle.
b) Show that every circle can be realized in this manner.
c) Give analogues of a) and b) when $\lambda = 1$.

Proof. a) Consider $\frac{z-a}{z-b}$ in the set given above. We have that $|\frac{z-a}{z-b}| = \lambda \implies |z-a| = \lambda|z-b|$. Set $z = x+iy,\ a = a_1 + ia_2,\ b = b_1 + ib_2$. We have from the formula above the following derivation:

$$\begin{aligned}
|z-a| = \lambda|z-b| &\implies (x-a_1)^2 + (y-a_2)^2 = \lambda^2(x-b_1)^2 + \lambda^2(y-b_2)^2 \\
&\to (x-a_1)^2 - \lambda^2(x-b_1)^2 - x^2(1-\lambda^2) - 2x(a_1 - \lambda^2 b_1) + a_1^2 - \lambda^2 b_1 = 0 \ — \ 1 \\
&\to x^2 - \frac{2x(a_1 - \lambda^2 b_1)}{(1-\lambda^2)} + \frac{(a_1^2 - \lambda^2 b_1)}{(1-\lambda^2)} \\
&= x^2 - \frac{2x(a_1 - \lambda^2 b_1)}{(1-\lambda^2)} + \frac{a_1^2 - \lambda^2(a_1 + b_1) + \lambda^4 b_1^2 - 2\lambda^2 a_1 b_1 + 2\lambda^2 a_1 b_1}{(1-\lambda^2)^2} \\
&= x^2 - \frac{2x(a_1 - \lambda^2 b_1)}{(1-\lambda^2)} + \frac{(a_1 - \lambda^2 b_1)^2}{(1-\lambda^2)^2} - \frac{\lambda^2(a_1 - b_1)^2}{(1-\lambda^2)^2} = \left(x - \frac{a_1 + \lambda^2 b_1}{1-\lambda^2}\right)^2 - \left(\frac{\lambda(a_1 - b_1)}{1-\lambda^2}\right)^2
\end{aligned}$$

Similarly simplifying for the parts of the equation involving y, we have by rewriting 1 that :

$$\begin{aligned}
&\to \left(x + \frac{a_1 + \lambda^2 b_1}{1-\lambda^2}\right)^2 - \left(\frac{\lambda(a_1 - b_1)}{1-\lambda^2}\right)^2 + \left(y - \frac{a_2 + \lambda^2 b_2}{1-\lambda^2}\right)^2 - \left(\frac{\lambda(a_2 - b_2)}{1-\lambda^2}\right)^2 = 0 \\
&\implies \left(x - \frac{a_1 + \lambda^2 b_1}{1-\lambda^2}\right)^2 + \left(y - \frac{a_2 + \lambda^2 b_2}{1-\lambda^2}\right)^2 = \left(\frac{\lambda(a_1 - b_1)}{1-\lambda^2}\right)^2 + \left(\frac{\lambda(a_2 - b_2)}{1-\lambda^2}\right)^2
\end{aligned}$$

Which is of the form a circle centered at the point

$$\left(\frac{a_1 + \lambda^2 b_1}{1-\lambda^2}, \frac{a_1 + \lambda^2 b_1}{1-\lambda^2} \right)$$

with radius

$$\left(\frac{\lambda(a_1 - b_1)}{1-\lambda^2}\right)^2 + \left(\frac{\lambda(a_2 - b_2)}{1-\lambda^2}\right)^2$$

b) Given any circle C in the plane, we may translate, rotate, and scale the plane to make the circle above in a) coincide with the circle C: as these are all isometries except for scaling, which does not affect ratios of distance the image of the points above yield points such that C can be realized as the set above for some complex numbers a', b' which are the image of the points a, b under this transformation.
c) When $\lambda = 1$ we may repeat the derivation above to yield:

$$\begin{aligned}
|z-a| = |z-b| &\implies (x-a_1)^2 + (y-a_2)^2 = (x-b_1)^2 + (y-b_2)^2 \\
&\implies 2x(a_1 + b_1) + a_1^2 + b_1^2 + 2y(a_2 + b_2) + a_2^2 + b_2^2 = 0 \\
&\implies y = \frac{2x(a_1 + b_1) + |a|^2 + |b|^2}{2(a_2 + b_2)}
\end{aligned}$$

Which is a linear relation between x and y, implying that z lies along a line. For the line $y = 0$ we may select the points $i, -i$ in the complex plane. Given any line $y = mx + c$ we may thus rotate and translate the entire plane to make $y = 0$ coincide with the line, yielding points which satisfy the condition that every line may be realized as such. $\square$

Exercise 2. *Show algebraically for every triple a, b, c of distinct unimodular complex numbers,*

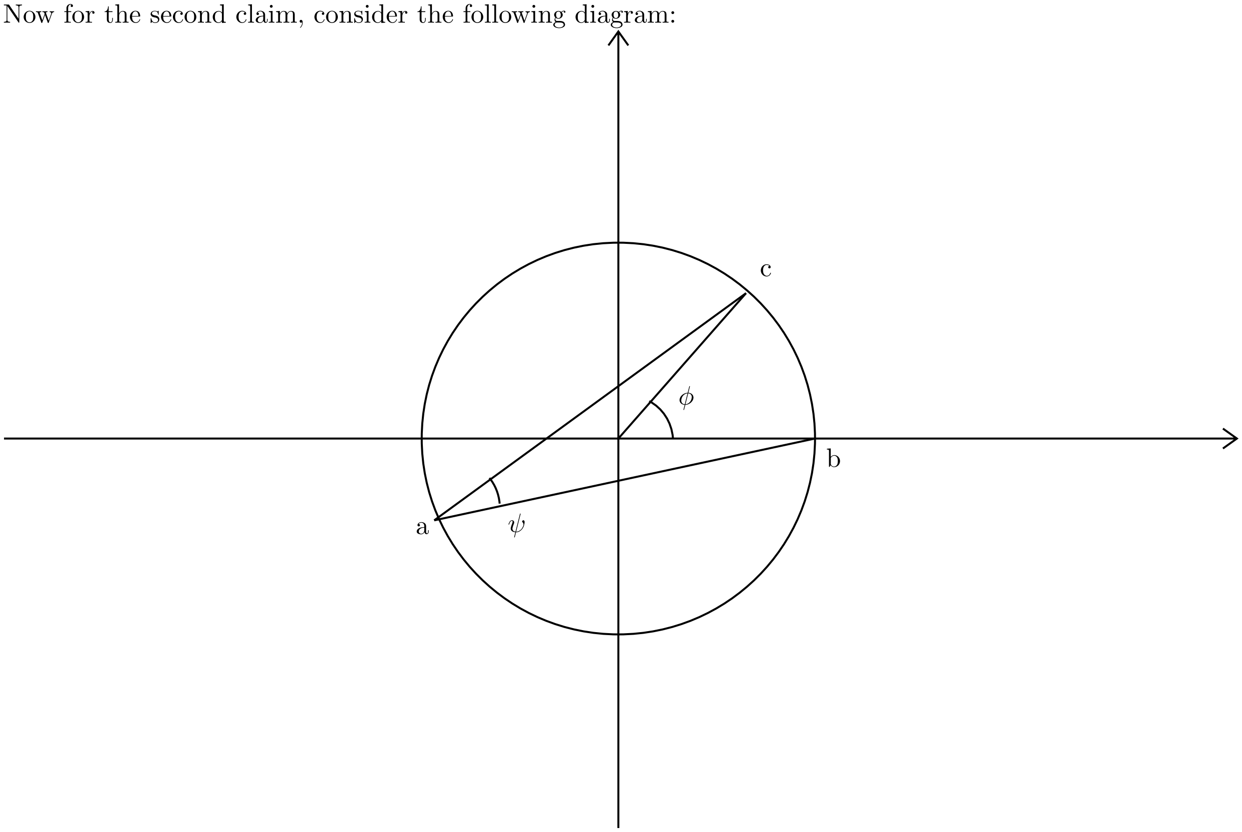
$$\frac{b-a}{1-\bar{a}b} = \frac{c-a}{1-\bar{a}c}$$

Show that with a little further manipulation this expresses the inscribed angle theorem.

Proof. We have that for $x = a, b, c$, $|x|^2 = 1$. Consider the following string of manipulations:

$$\frac{1-\bar{a}b}{1-\bar{a}c} = \frac{1-\bar{a}b}{1-\bar{a}c} \cdot \frac{a}{a} = \frac{a-b}{a-c} = \frac{b-a}{c-a} \implies \frac{b-a}{1-\bar{a}b} = \frac{c-a}{1-\bar{a}c}$$

Now for the second claim, consider the following diagram:



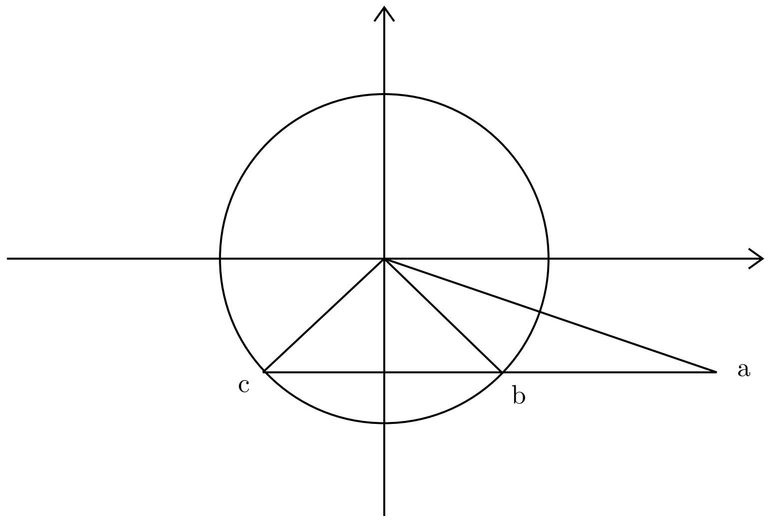
Our objective is to show that $2\psi = \phi$. Note that ψ is the argument of the complex number given by $\frac{c-a}{1-a}$ $(b = 1)$. Dividing this complex number by its conjugate will yield $e^{i2\psi}$, i.e. a unimodular complex number with twice the argument. We have that following set of manipulations:

$$e^{i2\psi} = \frac{c-a}{1-a} \cdot \frac{1-\bar{a}}{\bar{c}-\bar{a}} \cdot \frac{c}{c} = \frac{c-a}{1-\bar{a}c} \cdot \frac{c(1-\bar{a})}{1-a} = \frac{1-a}{1-\bar{a}} \cdot \frac{c(1-\bar{a})}{1-a} = c = e^{i\phi}$$

And thus $2\psi = \phi$. $\square$

Exercise 3. *Give a proof of the Intersecting Cord Theorem of Jakob Steiner.*

Proof. To any diagram of chord passing through an external point we may draw a diagram of the following nature:



The power of the complex number a with respect to this circle may be written as $|b-a| \cdot |c-b|$. Note that the complex numbers a, b, c all share an imaginart part. From basic geometry we have $Re(b) = -Re(c)$. Therefore, we may compute the product above strictly in terms of $|a|$ and r where $r = |b| = |c|$ the radius. We have that $b - c = Re(a) - Re(b) = Re(a) - \sqrt{|b|^2 - Im(a)^2}$ by the Pythagoras Theorem. Similarly, $c - b = Re(a) + \sqrt{|b|^2 - Im(a)^2}$.Furthermore, both $(c-b)$ and $(b-c)$ are real, so the product of their norms is the norm of their product. The product $|b-a| \cdot |c-b|$ is thus explicitly given by:

$$|(Re(a) - \sqrt{|b| - Im(a)^2}) \cdot (Re(a) + \sqrt{|b| - Im(a)^2}| = |Re(a)^2 - |b|^2 + Im(a)^2| = ||a|^2 - |b|^2|$$

Which is entirely independent of the chosen intersecting chord. Thus, the power of a point is independent of the chord chosen to compute it. ☐

Exercise 4. *Any mapping that can be represented in the form*

$$z \mapsto \frac{az+b}{cz+d}$$

with $ad - bc \neq 0$ is called a Mobius transformation.
a) Show that every such mapping can be realised by coefficients satisfying $ad - bc = 1$ and determine the number of such representations.
b) Show that every such map is a bijection of the Riemann sphere.
c) Show that $SL_2(\mathbb{C})$ maps into the complex numbers.
d) Show that Mobius transformations map every circle and every line into another circle or line.

Proof. a) Given a transformation of the above form, note that $ad - bc = k \neq 0$. As the complex numbers are algebraically closed, we may select w s.t. $w^2 = \frac{1}{k}$. Multiplying both the numerator and the denominator by w yields the same transformation with coefficients a', b', c', d' s.t. $a'd' - b'c' = \frac{1}{k^2}(ad - bc) = 1$. Now suppose there exists another set of coefficients a'', b'', c'', d'' representing the transformation s.t. $a''d'' - b''c'' = 1$. Note that for $z = \frac{-b'}{a'}$, we have that both transformations must send z to 0, which is only possible if the numerator is sent to 0 and thus $a''z + b'' = 0 \implies z = \frac{-b''}{a''}$ and similarly for $z = \frac{-d'}{c'}$ yielding a pole at z. Thus, the coefficients of the latter transformation must be some linear multiple of the coefficients of the initial transformation by some factor λ. We thus have that $a'd' - b'c' = \lambda^2(a''d'' - b''c'')$. However, as both quantities are 1, $\lambda = 1$ and both transformations must have the same coefficient. Thus, there is a unqiue way to represent each Mobius transformation as stated.

b) Let $f = \frac{az+b}{cz+d}$. We will first prove that the function is injective. Suppose $f(z) = f(wz)$ some $w \in \mathbb{C}$. Then we have:

$$\begin{aligned}\frac{az+b}{cz+d} = \frac{waz+b}{wcz+d} &\implies (az+b)(wcz+d) = (waz+b)(cz+d) \\ &\implies bwcz + daz = dwaz + bcz \\ &\implies w(bcz - daz) = bcz - daz \\ &\implies w = 1\end{aligned}$$

Now consider the Mobius function given by $g = \frac{dz-b}{-cz+a}$. The function $g \circ f$ is given by:

$$\frac{d\left(\frac{az+b}{cz+d}\right) - b}{-c\left(\frac{az+b}{cz+d}\right) + a} = \frac{daz + db - bcz - bd}{-caz - cb + caz + ad} = \frac{z(ad - bc)}{(ad - bc)} = z$$

Thus g being a Mobius transformation is a well-defined one-to-one inverse to a one-to-one function, implying that both f and g must have been onto.

c) The above result immediately yields that the set of Mobius transformations with the identity is a group under functional composition, given that functional composition is associative and that each function has an inverse. Note that given two Mobius transformations f_1 and f_2 with coefficients indexed as below, their composite may be represented by:

$$\frac{a_2\left(\frac{a_1z+b_1}{c_1z+d_1}\right) + b_2}{c_2\left(\frac{a_1z+b_1}{c_1z+d_1}\right) + d_2} = \frac{a_2a_1z + a_2b_1 + b_2c_1z + b_2d_1}{c_2a_1z + c_2b_1 + d_2c_1z + d_2d_1} = \frac{(a_2a_1 + b_2c_1)z + (a_2b_1 + b_2d_1)}{(c_2a_1 + d_2c_1)z + (c_2b_1 + d_2d_1)} \quad \text{— } (1)$$

Given two matrices $A, B \in SL_2(\mathbb{C})$ we may form their composite by:

$$\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} = \begin{pmatrix} a_1a_2 + b_1c_2 & a_2b_1 + b_2d_1 \\ c_2a_1 + d_2c_1 & c_2b_1 + d_2d_1 \end{pmatrix} \quad \text{— } (2)$$

Define a map $\phi : SL_2(\mathbb{C}) \to M$ where M is the group of Mobius transformations via:

$$\phi : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \frac{az+b}{cz+d}$$

We have that the map is clearly well defined given coefficients a, b, c, d. Furthermore, we have that the map is a group homomorphism as the composite of two matrices A, B maps to the functional composite $\phi(A) \circ \phi(B)$, which we see by comparing (1) and (2) .

d) Given $k_1, k_2 \in \mathbb{C}$ and $\lambda \in \mathbb{R}^+$ s.t $|\frac{z-k_1}{z-k_2}| = \lambda$ and f a Mobius transformation, consider the following derivation:

$$\frac{az+b}{cz+d} - \frac{ak_1+b}{cz+d} = \frac{daz + bck_1 + bd - ak_1cz - dak_1 - bcz - bd}{(ck_1+d)(cz+d)} = \frac{da(z-k_1) - bc(z-k_1)}{(ck_1+d)(cz+d)}$$

$$\implies \frac{f(z) - f(k_1)}{f(z) - f(k_2)} = \frac{(ad-bc)(z-k_1)}{(ad-bc)(z-k_2)} \cdot \frac{ck_2+d}{ck_1+d} \implies \left|\frac{f(z) - f(k_1)}{f(z) - f(k_2)}\right| = \lambda\left|\frac{ck_2+d}{ck_1+d}\right|$$

And thus

$$f : \left\{z \in \mathbb{C} : \left|\frac{z-a}{z-b}\right| = \lambda\right\} \mapsto \left\{z \in \mathbb{C} : \left|\frac{z - f(k_1)}{z - f(k_2)}\right| = \lambda\left|\frac{ck_2+d}{ck_1+d}\right|\right\}$$

Implying that every circle or line is mapped to another circle or line. $\square$

Exercise 5. *a) Show that the usual cross product is not associative, while quaternion multiplication is. b) Show that quaternion multiplication is not commutative. c) Show that $(1,0,0,0)$ is a two-sided identity and that every quaternion has a two-sided inverse. d) Given a quaternion, find its matrix under the left regular representation along with its characterisitc polynomial and minimal polynomial.*

Proof. a) The usual cross product is nonassociative as evidence clearly by the example

$$((1,0,0)\times(0,1,0))\times(0,1,0) = (-1,0,0) \text{ but } (1,0,0)\times((0,1,0)\times(0,1,0)) = 0$$

I'm not verifying that this is associative.

b) Consider the product $(0,1,0,0)\cdot(0,0,1,0) = (0,0,0,1)$, however the product $(0,0,1,)\cdot(0,1,0,0) = (0,0,0,-1)$ coming from the noncommutativity of the cross product, and thus the multiplication is noncommutative.

c) Note that $(1,0,0,0)\cdot(a,b,c,d) = (a-0,(b,c,d)+0-0)$ and that $(a,b,c,d)\cdot(1,0,0,0) = (a,0+(b,c,d)-0)$ and so $(1,0,0,0)$ acts as a two sided identity. We may express every element as a sum of elements in the standard basis for $\mathbb{R}^4$, which we will suggestively write as $1,i,j,k$ for e_1,e_2,e_3,e_4. Note then the relations $i^2=j^2=k^2=ijk=-1$. For any quaternion $a+bi+cj+dk$ consider the product:

$$\begin{aligned}(a+bi+cj+dk)(a-bi-cj-dk) &= a^2 - abi - acj - adk + abi + b^2 - bcij - bdik \\ &\quad + acj + bcij + c^2 - cdjk + adk + bdik + cdjk + d^2 \\ &= a^2+b^2+c^2+d^2\end{aligned}$$

And thus $\frac{1}{a^2+b^2+c^2+d^2}(a-bi-cj-dk)$ is the right inverse to the quaternion above. It is however clear that this is a two sided inverse as $(a+bi+cj+dk)$ is the right inverse to $\frac{1}{a^2+b^2+c^2+d^2}(a-bi-cj-dk)$.

d) Consider the quaternion $w = a+bi+cj+dk$. We may compute its entries by looking at its action on the basis vectors in the standard basis: we have that $w(1+0i+0j+0k) = w$, $w\cdot i = ai - b - ck + dj$, $w\cdot j = aj + bk - c - di$ and $w\cdot k = ak - bj + ci - d$. This gives us that the left multiplication by w map λ_w is:

$$\lambda_w := \begin{pmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{pmatrix}$$

The determinant of the above matrix is $(a^2+b^2+c^2+d^2)^2$. Thus, its characteristic polynomial is

$$((\lambda-a)^2+b^2+c^2+d^2)^2 = (\lambda^2 - 2a\lambda + a^2+b^2+c^2+d^2)^2$$

Which has no real roots, and thus it suffices to check whether or not the minimal polynomial divides the square root of the above polynomial. Applying $\lambda^2 - 2a\lambda + a^2+b^2+c^2+d^2$, we have:

$$\begin{pmatrix} a^2-b^2-c^2-d^2 & -2ab & -2ac & -2ad \\ 2ab & a^2-b^2-c^2-d^2 & -2ad & 2ac \\ 2ac & 2ad & a^2-b^2-c^2-d^2 & -2ab \\ 2ad & -2ac & 2ab & a^2-b^2-c^2-d^2 \end{pmatrix} - \begin{pmatrix} 2a^2 & -2ab & -2ac & -2ad \\ 2ab & 2a^2 & -2ad & 2ac \\ 2ac & 2ad & 2a^2 & -2ab \\ 2ad & -2ac & 2ab & 2a^2 \end{pmatrix}$$
$$+(a^2+b^2+c^2+d^2)I$$

$= 0$, And thus the minimal polynomial is $\lambda^2 - 2a\lambda + a^2+b^2+c^2+d^2$. ∎