Math 225A Differential Topology: Homework 5

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Exercise 1. *Prove that $H_x(X)$ does not depend on the choice of local parametrization.*

Proof. Suppose there are two local parametrizations of $x \in \partial X$, $\phi : U \to X$ and $\psi : V \to X$. By shrinking, we may assume $\phi(U) = \psi(V)$, where $U, V \subset H^k$ open. We have that $h : \psi^{-1} \circ \phi$ is a diffeomorphism from U to V. Then $\phi = \psi \circ h$. Let $w \in H^k$. We have that diffeomorphisms must take boundaries to boundaries: thus, that $h(IntH^k) \subset IntH^k$. Thus, the map sends the tangent space $T_x(\partial X)$ to $T_x(\partial X)$, and w to a vector in the upper half plane. At 0, we have that $dh(w) = \lim_{t\to 0} \frac{h(tw)}{t} \in H^k$ as all values $h(w)$ lie in H^k. Thus, by chain rule, we have that $d\phi(H^k) \subset d\psi(H^k$. Symmetric reasoning by reversing the order of h yields the result. □

Exercise 2. *Show that there are precisely two vectors perpendicular to $T_x(\partial X)$ in $T_x(X)$: one inside H^k and one outside H^k*

Proof. We have that if a vector is perpendicular to $T_x(\partial X)$ then it is linearly independent from $T_x(\partial X)$. As any parametrization ϕ induces an isomorphism from $\mathbb{R}^k \to T_x(X)$ for X locally a k-dimensional manifold. we have that the preimage of any vector linearly independent from $T_x(\partial X)$ must be linearly independent from the plane $\mathbb{R}^{k-1}$, yielding that it lies in either the upper or lower half space. As codimension of (∂X) in X is 1, we have that the space of vectors perpendicular to that submanifold tangent space is one-dimensional, from linear algebra. Thus, there are precisely two unit vectors perpendicular, one in the upper and one in the lower half space. □

Exercise 3. *Show that for any $x \in \partial X$, there is a smooth nonnegative function f on $x \in U \subset X$ open s.t. $f(z) = 0$ iff $z \in \partial U$, and if $z \in \partial U$, then $df_z(\vec{n}(z)) > 0$.*

Proof. We have that the function exists for any open set $U \subset H^k$ open, namely given by the projection onto the k^{th} coordinate, or the one-dimensional linear subspace perpendicular to the boundary hyperplane $\mathbb{R}^{k-1}$. Thus, the function exists by composition with any coordinate system $\phi : N(x) \to H^l$ on an open neighborhood $N(x)$, which exists by assumption. It is clear that all the conditions must hold: if $z \in \partial U$, then $z \in \mathbb{R}^{k-1}$ and thus the projection is zero, and this is clearly an if and only if. If $z \in \partial U$, we have from before that $\vec{n}(z)$ belongs in the upper half space of the coordinate system, and thus has a nonzero projection onto the k^{th} coordinate. Thus, $df_x\vec{n}(z) > 0$. □

Exercise 4. *Show that there exists a smooth nonnegative function f on X with a regular value at 0 s.t. $\partial X = f^{-1}(0)$.*

Proof. We have that X is a manifold and thus has a compact exhaustion. We have a covering of countably many compact sets covering ∂X with a refinement s.t. every compact set is diffeomorphic to a compact set in the half space H^k. We then union this with a countable compact exhaustion of the interior of X Using partitions of unity, we may construct a smooth proper nonzero function p from X to $\mathbb{R}$ subordinate to the covering above. For each compact set homeomorphic to the half space H^k, we may multiply this proper function with the local smooth functions on each compact set covering ∂X, given by f_n, from the previous exercise to yield a function that is identically 0 on all points on the boundary, and nonnegative elsewhere. This function has a regular value at 0 as each local function f_n has a regular value at 0, and the proper function constructed via the partitions of unity yields a nonzero vector in $\mathbb{R}$ as the differential locally is given by $\frac{df_n}{dx}p + \frac{dp}{dx}f_n$, by the product rule, with $f_n = 0$ at all points on the boundary, guaranteeing regularity at all points on the preimage. □

Exercise 5. *Find maps of the solid torus into itself having no fixed points. Where does the proof of the Brouwer theorem fail?*

Proof. As the torus is given by $S^1 \times S^1$, we may assume the solid torus may be written as $S^1 \times D^2$, where D^2 is the closed ball of radius 1 in $\mathbb{R}^2$. Let $f : S^1 \times D^2 \to S^1 \times D^2$ via $f : (\theta, x) \mapsto (\theta + \pi, x)$ where we parametrize S^1 by $\mathbb{R}/2\pi\mathbb{Z}$. Assume x is a fixed point of this map. Then we have that $\exists x \in [0, 2\pi)$ s.t. $x \equiv x$

mod π, which is not possible as $0 \not\equiv 0 \mod \pi$. Thus, this map has no fixed points. This is possible as the proof of the Brouwer theorem aassumes that every line from x to $f(x)$ (viewed via the parametrization to Euclidean space) must eventually intersect the boundary of the manifold, resulting in a retract to the boundary. However, for the parametrization of $S^1 \times D^2$, we have that this map yields a line from (θ, x) to $(\theta + \pi, x)$ lying entirely in $S^1 \times \{x\}$, which never leaves this subspace and thus never passes through the boundary. Thus, we cannot extend this map to a retract onto the boundary of a compact manifold and the proof fails. □

Exercise 6. *Prove that if the entries of an $n \times n$ matrix A are all nonnegative, then A has a real nonnegative eigenvalue.*

Proof. We have that as all the entries of A are nonnegative, for any x s.t. $x_1, ..., x_n \geq 0$ we have that for $Ax = b$, $b_1, ..., b_n \geq 0$. Thus, we may view the map $f : x \mapsto \frac{Ax}{|Ax|}$ as a smooth map, normalizing the image, as a map $f : S^{n-1} \to S^{n-1}$ mapping the compact submanifold with boundary $M = \{x \in S^{n-1} | x_1, ..., x_n \geq 0\}$ to itself. We have that the manifold M is homeomorphic to the closed ball $B^{n-1} \subset \mathbb{R}^{n-1}$. We now prove the following general result: If $f : B^{n-1} \to B^{n-1}$ is a continuous map, f has a fixed point.
If f is a continuous map from $\mathbb{R}^n \to \mathbb{R}^n$, we have that in a compact set it can be coordinatewise approximated by polynomials p_n s.t. $\forall \epsilon > 0 \; \exists \; N \in \mathbb{N}$ s.t. $|p_n(x) - f(x)| < \epsilon \; \forall \; n > N$. As polynomials are smooth, for each map p_n we have an associated fixed point c_n s.t. the c_n form a convergent sequence. We have that as B^{n-1} is compact, the function $|f(x) - x|$ must be bounded from below by some value $c > 0$ (as no fixed points by assumption. However, we have that $\exists p_n$ s.t. $|p_n(x) - f(x)| < \frac{c}{2}$ for all $x \in B^{n-1}$. Thus, for associated fixed point $c_n \in B^{n-1}$, we have that $|c_n - f(c_N)| < \frac{c}{2}$, a contradiction. Thus, we have the lemma. Using the lemma, we have a map from a set homeomorphic to B^{n-1} to itself, resulting in the existence of a fixed point. Thus, the map $\frac{Ax}{|Ax|}$ must have a fixed point x, or that $Ax = |Ax|x$ for some vector $x \in M$. Thus, A has a positive real eigenvalue. □

Exercise 7. *Let Y be a compact submanifold of $\mathbb{R}^M$, and let $w \in]mathbbR^M$. Show that there exists a closest point $y \in Y$, and $w - y \in N_y(Y)$.*

Proof. We have that the function $|w - y|$ is continuous and thus must attain a minimum at some point $y \in Y$. Let $c : [0, 1] \to Y$ be an arbitrary curve s.t. $c(0.5) = y$ (can select this by taking the straight line through 0 in the local parametrization of y, assuming Y is not a 0-manifold in which case the result clearly holds. The function $|c(t) - w|^2$ attains a minimum at 0.5, and so taking the derivative must yield 0 at 0.5 (local minimum). Thus, we take the derivative of $|c(t) - w|^2$, given by $2(w_1 - c_1(t))c_1'(t) + ... + 2(w_n - c_n(t))c_n'(t)\big|_{t=0.5}$. Thus, at 0.5, we have that $w - c(0.5)$ is perpendicular to $c'(0.5) \in T_y(Y)$, and thus $w - y \in N_y(Y)$. □

Exercise 8. *Prove that if $w \in Y^\epsilon$, with Y compact, then $\pi(w)$ is the unique point closest to w in Y.*

Proof. We have that if there is a point $y \in Y$ closest to $w \in Y^\epsilon$ then $w - y$ is in the space normal to Y. Given that h is a diffeomorphism, we have that $w - y$ is perpendicular to y and is minimal in $Y^{epsilon}$ if and only if $(y, w - y) \in h^{-1}(w)$, as if it belongs in the ϵ nbhd of some point y then it is in the preimage of the open neighborhood around that y in $N(Y)$: for fixed epsilon, the open nbhd mapped diffeomorphically onto $Y$by h is the neighborhood of all normal vectors with norm less than ϵ around a point y, given that $|y - w| = |h^{-1}(w)|$ evidently from the map. Thus, as the diffeomorphism must be injective, we have each point must have a unique preimage and thus only one point such that it is normal to that point. Thus, thus point must be the unique point closest to it, as all points fulfilling this property have the vector $w - y$ as a normal vector. □