Math 225B Differential Geometry: Homework 4

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Professor Peter Petersen

Anish Chedalavada

Exercise 1 (Problem 10).

Proof.

$$a.i)\quad \lim_{h\to 0}\frac{1}{h}[\Phi_t^*(f\cdot w|_{\Phi_t}) - f\cdot w)] \quad = \lim_{h\to 0}\frac{1}{h}\ [f(\Phi_t(p))\cdot \Phi_t^*(w|_{\Phi_t}) - f(\Phi_t(p))\cdot w(p)) - f(\Phi_t(p)\cdot w(p)) - f(p)\cdot w(p)]$$
$$= (D_X f)\cdot w + (L_x w)\cdot f$$

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$$a.ii)\quad (L_X(w))(Y) = \lim_{t\to 0}\frac{1}{t}[\Phi_t^*(w|_{\Phi_t})(Y) - w(Y)] = \lim_{t\to 0}\frac{1}{t}[w|_{\Phi_t}(D\Phi^t Y) - w(Y)]$$
$$= \lim_{t\to 0}\frac{1}{t}[w|_{\Phi_t}(Y|_{\Phi^t}) - w(tD\Phi_t(L_X Y)) - w(Y)] = \lim_{t\to 0}\frac{1}{t}[w(Y)\circ \Phi^t - w(tD\Phi_t(L_X Y)) - w(Y)]$$
$$= \lim_{t\to 0}\frac{1}{h}[w(Y) + tD_X(w(Y)) - w(tD\Phi_t(L_X Y)) - w(Y)] = D_X w(Y) - w(L_X Y)$$
$$\implies D_X w(Y) = (L_X w)(Y) + w(L_X Y)$$

b) The new definition would yield the following changes in sign:

$$L_X(f\cdot W) = f\cdot L_X Y - Xf\cdot Y L_X(w(Y)) = (L_X w)(Y) - w(L_X Y)$$

For parts 4) and 5). ☐

Exercise 2 (Problem 11).

Proof. a) $\phi^*(df)(Y) = (d(f\circ\phi))(Y) = Y(f\circ\phi)$

b) $[L_X df(p)](Y_p) = \lim_{t\to 0}\frac{1}{t}[\Phi_t^*(df|_{\Phi^t(p)}) - (df)(p)](Y_p) = \lim_{t\to 0}\frac{1}{t}[Y_p(f\circ\Phi_t) - Y(f)] =$
$\lim_{t\to 0}\frac{1}{t}[Y_p[(f\circ\Phi^t - f)] = Y_p(L_X f)$. Thus, $L_X df = Y_p(L_X f) = d(L_X f)$.

c) $D_1\alpha(0,0) = \lim_{t\to 0}\frac{1}{t}[Y(f\circ\Phi_{-t}) - Y(f)] = -X_p(Yf)$
$D_2\alpha(0,0) = \lim_{h\to 0}\frac{1}{h}[Y(f\circ\Phi_h) - Yf] = Y_p(Xf)$.
Thus, for $c(h) = \alpha(h,h)$ we have $L_X Y(p)(f) = -c'(0) = -\alpha'(h,h) = X_p(Yf) - Y_p(Xf) = [X,Y]f$. ☐

Exercise 3 (Problem 13).

Proof. a) It suffices to show X, Y, Z are linearly independent at any point p as this must then yield a linear isomorphism. Suppose $X_p = aY_p + bZ_p$ for some $a, b \neq 0$. This implies:

$$X_p = p_z\partial_y - p_y\partial_z = bp_x\partial_y - bp_y\partial_x + ap_x\partial_z - ap_z\partial_x$$

Which implies $p_y = ap_x$, so $p_z\partial_y = bp_x\partial_y - bap_x\partial_x + ap_z\partial_x$, which means $bp_x = p_z$ and $-bap_z = ap_z$, which is only possible if $p = (0,0,0)$ as $a, b \neq 0$. Similar logic shows that Y_p and Z_p are not linear combinations of the other two, implying they are linearly independent at every point except at 0, yielding the isomorphism. We compute the Lie Bracket $[X,Y]$ as an illustrative example: we have that

$$[X,Y] = (p_z\partial_y - p_y\partial_z)(p_x\partial_z - p_z\partial_x) = p_z p_x\partial_y\partial_z - p_z p_z\partial_y\partial_x + p_y p_x\partial_x\partial_z - p_y p_z\partial_z\partial_x = -Z$$

This agrees with the cross product (1,0,0) × (0,1,0) = (0,0,-1). Similarly computing the Lie Brackets for other combinations yields the cross product relations, and linearity yields the result on general vector fields.

b) Let ϕ be the flow along $aX + bY + cZ$. Given an arbitrary point $p \in \mathbb{R}^3$, we have that the derivative with respect to the flow at point p is the Lie Bracket for the vector field given by $\vec{p} : p \mapsto \vec{p}$, given by $L_X\vec{p} = [aX + bY + cZ, p] = (a,b,c)\times(p_1,p_2,p_3)$, the cross product from part a). Thus, the tangent vector of the flow at any point p is always orthogonal to the direction vector $\vec{p}$, which is a rotation. ☐

Exercise 4 (Problem 15). *(Solutions partially adapted from notes by Ian Coley posted on his website at http://www.math.ucla.edu/∼iacoley/hw/diffhwwinter/HW 2.pdf)*

Proof. a) Suppose $D(\partial_i) = \sum a_{ij}\partial_j$. We may extend D to an operator taking (k,l) tensor fields to themselves via defining $D(w) = \sum -a_{ij}dx^j$ for 1-forms, and extending to (k,l)-tensors by $D(A \otimes B) = DA \otimes B + A \otimes DB$, tensored over $\mathbb{R}$. In particular, by the tensor product, this is $\mathbb{R}$-linear. We have that $D(CA) = DC(dx^{i_1} \otimes ... \otimes dx^{i_n} \otimes \partial_{j_1} \otimes ... \otimes \partial_{j_m}) = \delta^{i_1 \dots i_n}_{j_1 \dots j_n} D(B)$ for B some new tensor. Consider $CD(dx^{i_1} \otimes ... \otimes dx^{i_n} \otimes \partial_{j_1} \otimes ... \otimes \partial_{j_m}) = \delta^{i_1 \dots i_n}_{j_1 \dots j_n} D(B)$, as $C(\partial_j \otimes D(dx^i)) = C(\partial_j \otimes \sum a_{im}dx^m) = a_{ij}$, but $C(D(\partial_j) \otimes dx^i) = -a_{ij}$ so application of the contraction cancels out all terms being contracted, yielding that $C(D(A)) = \delta^{i_1 \dots i_n}_{j_1 \dots j_n} = DC(A)$. Uniqueness follows as $D'(dx^i) = D(dx^i)$ via using the contraction to match the coordinates.

b) We have that $D_A f = 0$ so it is linear on both functions and on vector fields, and satifies Liebniz rule as:

$$D_A(fX) = A(fX) = fA(X) = fA(X) + 0 \cdot X = fA(X) + D_A(f)X$$

and thus satisfies the hypotheses of a), yielding the result.

c) From our construction in part a) extending to 1-forms, we have that $(D_A dx^i)(p) = -\sum a_{ij}dx^j(p) = -A^*(p)(dx^i)$,
d) We have $(fL_X - L_{fX})(gY) = fL_X(gY) - L_{fX}(gY) = g.(fL_X - L_{fX}) - (fL_X - L_{fX})(Y)$ as it satifies Liebniz rule in each summand. We have that $(fL_X - L_{fX})(g) = 0$ as both summands evaluate to be equal. This has a unique extension to (1,1) tensors satisfying the properties above, and it this extension is equal to $D_{X \otimes df}$ as they are equal evaluated on ∂_k. □