## Math 215A Commutative Algebra: Homework 3

Oct 30th, 2019

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**Exercise 1.** Suppose that  $0 \to L \to M \to N \to 0$  is an exact sequence of R-modules. Show that if L and N are flat, or if M and N are flat, then so is the third.

*Proof.* Let  $0 \to L \to M \to N \to 0$  be an exact sequence. Let A be an arbitrary R-module: then we have a long exact sequence given by:

$$\operatorname{Tor}^1_R(L,A) \longrightarrow \operatorname{Tor}^1_R(M,A) \longrightarrow \operatorname{Tor}^1_R(N,A) \longrightarrow L \otimes_R A \longrightarrow M \otimes_R A \longrightarrow N \otimes_R A \longrightarrow 0$$

Now suppose L and N are flat. We have then that  $\operatorname{Tor}_R^1(L,-)=0=\operatorname{Tor}_R^1(N,-)$ . Thus, by the exactness of the sequence above,  $\operatorname{Tor}_R^1(M,-)=0$  as it is 0 for A any arbitrary R-module. Thus, given any short exact sequence  $0 \to A \ toB \to C \to 0$ , we have the following sequence:

$$\operatorname{Tor}_R^1(M,C) \cong 0 \longrightarrow M \otimes_R A \longrightarrow M \otimes_R B \longrightarrow M \otimes_R C \longrightarrow 0$$

And so tensoring with M preserves short exact sequences, yielding that M is flat. Now suppose M and N are flat. Given any monomorphism  $f: A \to B$ , we have the following diagram where rows and columns are exact:

$$\ker(A\otimes f) \longrightarrow A\otimes_R L \xrightarrow{A\otimes f} B\otimes_R L$$

$$\downarrow \qquad \qquad \downarrow$$

$$0 \longrightarrow A\otimes_R M \longrightarrow B\otimes_R M$$

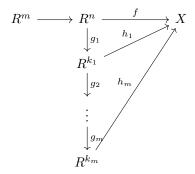
where the exactness of the second row uses the flatness of M and the exactness of the columns comes from  $\ker(A\otimes_R L\to A\otimes_R M)\cong \operatorname{Tor}^1_R(N,A)=0$  and similarly for B. Thus, we have that the composite map  $\ker(A\otimes f)\to A\otimes_R L\to A\otimes_R M\to B\otimes_R M$  is equal to the composite map  $\ker(A\otimes f)\to B\otimes_R L\to B\otimes_R M$  which is the 0 map by definition. As the first map is a composition of monomorphisms, we have that the 0 map on  $\ker(A\otimes f)$  is a monomorphism, implying that is is 0. Thus,  $-\otimes_R L$  must preserve monomorphisms, implying that L is flat.

Exercise 2. 1) Show that an R-module X is flat if and only if any map from a finitely presented R-module to X factors through a free module. Conclude that finitely presented free modules are projective.

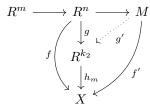
- 2) Show that for M a finitely presented module over a local ring, TFAE: a) M is free, b) M is projective, c) M is flat.
- 3) Show that a finitely presented R-module is projective if and only if it is locally free.

Proof. 1) Let M be a module with finite presentation  $R^m \to R^n \to M \to 0$ . We have that any map from  $M \to X$  may be written as a map  $f: R^n \to X$  such that the composite  $R^m \to R^n \to X$  is the zero map. Now, we have that  $\operatorname{Im}(R^m \to R^n)$  is finitely generated, say by elements  $q_1, ..., q_m$  the image of basis elements. For any given  $q_1$ , we have that  $q_1 \in \ker f$ . By the equational criterion for flatness, we have that we may factor the map  $f: R^n \to X$  as a map  $g_1: R^n \to R^{k_1}$  followed by a map  $h_1: R^{k_1} \to X$  such that  $f = h_1g_1$  and  $q_1 \in \ker g_1$ . We may repeat this process in  $R^{k_1}$  by noting that  $f = h_1g_1 \Longrightarrow g_1(q_1) \in \ker h_1$ , and so we may factor the map  $h_1: R^{k_1} \to X$  into maps  $g_2: R^{k_1} \to R^{k_2}$  and  $h_2: R^{k_2} \to X$  such that  $h_1 = h_2g_2$  and  $g_1(q_1) \in \ker g_2$ , yielding a map  $g_2g_1: R^n \to R^{k_2}$ ,  $h_2: R^{k_2} \to X$  such that  $h_2g_2g_1 = f$  and  $q_1, q_2 \in \ker g_2g_1$ .

Iterating this process for all m we have the following diagram:



Such that  $h_m g_m ... g_1 = f$  and  $q_1, ..., q_m \in \ker g_m ... g_1$ . Denote the composite  $g_m ... g_1 := g$ . We have the following diagram:



Where any map  $f': M \to X$  corresponds to a map  $f: R^n \to X$  such that the composite is 0, which factors as a map  $h_m g: R^n \to X$  such that  $R^m \to R^n \to R^{k_2}$  is 0, which induces a map  $g': M \to R^{k_2}$  such that the diagram above commutes. Thus, any map from a finitely presented module to a flat module factors through a free module. Now, suppose any map from a finitely presented module to a module X factors through a free module. Given any relation  $\sum_{i=1}^n r_i x_i = 0$  in X, we have a map  $g: R^n \to X$  via  $g: e_i \mapsto x_i$ ,  $f: R \to R^n$  via  $1 \mapsto \sum_{i=1}^n r_i e_i$  such that the composite gf = 0. By the property above, the map f factors through maps  $\alpha: R^j \to X$ ,  $\beta: R^n \to R^j$  such that  $g = \alpha \beta$ , and g = 0: i.e. for g = 1, ..., g = 1, and g = 1 is a basis for g = 1, and g = 1 is a basis i.e. in g = 1, there exist g = 1, ..., g = 1, such that g = 1 is a basis: i.e. in g = 1, there exist g = 1, ..., g = 1 such that g = 1 is a basis: i.e. in g = 1, there exist g = 1, ..., g = 1 such that g = 1 is flat.

In particular, suppose X is finitely presented, flat. We have that the identity map  $Id: X \to X$  must factor through maps  $h: X \to R^j$  monic,  $g: R^j \to X$  epic such that gh = Id: this implies that X is a summand of  $R^j$  and is thus projective.

2) We have that  $b) \iff c$  for M finitely presented by the result above. We also have that  $a) \implies b$  as free modules are projective. It thus suffices to show that finitely presented projective modules over a local ring are free. Let M be a finitely presented projective module: In particular, M is a summand of some free module  $R^k$ . We may pick some set of generators  $m_1, ..., m_j$  of M such that  $m_1 + \mathfrak{m}M, ..., m_j + \mathfrak{m}M$  is a basis of  $M/\mathfrak{m}M \subset (R/\mathfrak{m})^k$  over  $R/\mathfrak{m}$ : we may select this to be a basis as by a corollary of Nakayama's lemma we have that any set of element that generate the vector space  $M/\mathfrak{m}$  over the residue field must generate M. We have that this set of elements is linearly independent in  $(R/\mathfrak{m})^k$ . It thus suffices to show that this finite set of elements that is linearly independent in  $(R/\mathfrak{m})^k$  over the residue field must be linearly independent over R. Let  $a_i \in R$  not all 0 for i = 1, ..., j. Let l minimal such that  $a_i \in \mathfrak{m}^l$ . We have that the map induced by the R-action below is an isomorphism (as it is an isomorphism for R and tensor products commute with direct sums):

$$\mathfrak{m}^l/\mathfrak{m}^{l+1}\otimes_R R^k \to (\mathfrak{m}^l/\mathfrak{m}^{l+1})^k$$

We have that the left hand side is isomorphic to  $\mathfrak{m}^l/\mathfrak{m}^{l+1}\otimes_{R/\mathfrak{m}}(R/\mathfrak{m})^k$ . We have that the sum  $\sum_{i=1}^j a_i\otimes m_i\neq 0$  as  $m_i$  are linearly independent in  $(R/\mathfrak{m})^k$ . Thus, the sum is nonzero in  $(\mathfrak{m}^l/\mathfrak{m}^{l+1})^k$ , implying it is nonzero in  $R^k$ . Given that the  $a_i$  were chosen arbitrarily, this must be true for all collections  $a_i$  with all  $a_i$  nonzero,

and thus  $m_i$  must be linearly independent in  $R^k \implies$  they are a linearly independent generating set for M and so M is free.

3) We have that if X is a finitely presented projective R-module, then X is a direct summand of some free module  $R^k$ . We have thus that  $S^{-1}X$  is a direct summand of  $(S^{-1}R)^k$  as localization commutes with direct products, and thus  $S^{-1}X$  is projective and finitely presented, implying that it is free. Now suppose X is locally free. This yields in particular that X is locally flat. Now, given any short exact sequence  $0 \to A \to B \to C \to 0$ , we have the following exact sequence:

$$\operatorname{Tor}_R^1(X,A) \longrightarrow X \otimes_R A \longrightarrow X \otimes_R B \longrightarrow X \otimes_R C \longrightarrow 0$$

As localization is flat, we have the following sequence is also exact:

$$S^{-1}R \otimes_R \operatorname{Tor}^1_R(X,C) \longrightarrow S^{-1}R \otimes_R X \otimes_R A \longrightarrow S^{-1}R \otimes_R X \otimes_R B \longrightarrow S^{-1}R \otimes_R X \otimes_R C \longrightarrow 0$$

And the above sequence is isomorphic to the following one:

$$S^{-1}\operatorname{Tor}_R^1(X,C) \longrightarrow S^{-1}X \otimes_{S^{-1}R} S^{-1}A \longrightarrow S^{-1}X \otimes_{S^{-1}R} S^{-1}B \longrightarrow S^{-1}X \otimes_{S^{-1}R} S^{-1}C \longrightarrow 0$$

But we know by flatness of localization that  $0 \to S^{-1}A \to S^{-1}B \to S^{-1}C \to 0$  is exact, and by assumption the flatness of  $S^{-1}X$  yields that the image of  $S^{-1}Tor^1_R(X,C)$  in  $S^{-1}X \otimes S^{-1}A$  must be 0. As this holds at every localization, we have that the image of  $Tor^1_R(X,A) \to X \otimes_R A$  must be zero as it is zero at every localization  $\implies 0 \to X \otimes_R A \to X \otimes_R B \to X \otimes_R C \to 0$  must be exact and so X is flat. As it is finitely presented, it is projective.

**Exercise 3.** Show that for X a flat R-module TFAE:

- 1) X is faithfully flat.
- 2)  $L \to M \to N$  is exact  $\iff L \otimes_R X \to M \otimes_R X \to N \otimes_R X$  is exact.
- 3)  $N \otimes_R X = 0 \iff N = 0$ .
- 4) For every ideal I of R,  $IX \neq X$ .
- 5) For every maximal ideal  $\mathfrak{m}$  of R,  $\mathfrak{m}X \neq X$ .