Math 225B Differential Geometry: Homework 5

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Exercise 1 (Problem 6).

Proof. It suffices to show that the distribution Δ is an integrable distribution, as the integral manifold forms the graph of the solution for α. Given arbitrary fields in the distribution, $\sum_{i=1}^m r^i \frac{\partial}{\partial t_i} + \sum_{k=1}^m (\sum_{l=1}^m r^i f_l^k) \frac{\partial}{\partial x^k}$ and $\sum_{i=1}^m s^i \frac{\partial}{\partial t_i} + \sum_{k=1}^m (\sum_{l=1}^m s^i f_l^k) \frac{\partial}{\partial x^k}$, we must show that their Lie Bracket lies in the distribution.

$$\left[\sum_{i=1}^m r^i \frac{\partial}{\partial t_i} + \sum_{k=1}^m (\sum_{l=1}^m r^i f_l^k) \frac{\partial}{\partial x^k}, \sum_{i=1}^m s^i \frac{\partial}{\partial t_i} + \sum_{k=1}^m (\sum_{l=1}^m s^i f_l^k) \frac{\partial}{\partial x^k}\right]$$

$$= \left[\sum_{i=1}^m r^i \frac{\partial}{\partial t_i}, \sum_{i=1}^m s^i \frac{\partial}{\partial t_i}\right] + \left[\sum_{i=1}^m r^i \frac{\partial}{\partial t_i}, \sum_{k=1}^m (\sum_{l=1}^m s^i f_l^k) \frac{\partial}{\partial x^k}\right] + \left[\sum_{k=1}^m (\sum_{l=1}^m r^i f_l^k) \frac{\partial}{\partial x^k}, \sum_{i=1}^m s^i \frac{\partial}{\partial t_i}\right]$$
$$+ \left[\sum_{k=1}^m (\sum_{l=1}^m r^i f_l^k) \frac{\partial}{\partial x^k}, \sum_{k=1}^m (\sum_{l=1}^m s^i f_l^k) \frac{\partial}{\partial x^k}\right]$$

The first and last terms disappear by moving s^i and r^i out of the Lie Brackets of each summand, leaving the sum:

$$\left[\sum_{i=1}^m r^i \frac{\partial}{\partial t_i}, \sum_{k=1}^m (\sum_{l=1}^m s^i f_l^k) \frac{\partial}{\partial x^k}\right] + \left[\sum_{k=1}^m (\sum_{l=1}^m r^i f_l^k) \frac{\partial}{\partial x^k}, \sum_{i=1}^m s^i \frac{\partial}{\partial t_i}\right] = 0$$

Which also results from considering the Lie Brackets in each summand and moving the scalars out. This implies that the Lie Brackets of arbitrary elements in the distribution at any point also belong in the distribution, and thus Δ is integrable and has an integral solution. □

Exercise 2 (Problem 8).

Proof. Consider the function $f = u + iv$ as being a function of $\mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ via u is a function to the first coordinate and iv is a function to the second. Theorem 1 now applies and yields a function $\alpha : (0,0) \in U \to \mathbb{R}^2$ with the conditions of Theorem 1, given by:

$$\frac{\partial \alpha^1}{\partial x_1}(x_1, y_1) = u(x_1, y_1, \alpha(x_1, y_1)) = \frac{\partial \alpha^2}{\partial y_1}$$

$$\frac{\partial \alpha^1}{\partial y_1}(x_1, y_1) = v(x_1, y_2, \alpha(x_1, y_1)) = -\frac{\partial \alpha^2}{\partial x_1}$$

Thus, replacing x_1, y_1 with z, being as the initial two are the coordinate functions, we have that α solves the Cauchy-Riemann equations and must be complex analytic. Thus, $\alpha'(z) = f(z, \alpha(z))$ is a solution by direct application of Theorem 1. □