Math 215A Commutative Algebra: Homework 3

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Exercise 1. *Show that being reduced is a local property, i.e. R reduced $\iff$ $R_\mathfrak{m}$ reduced for all maximals $\iff$ $R_\mathfrak{p}$ reduced for all primes.*

Proof. Suppose the localization at every prime ideal is reduced, this clearly implies the localization at every maximal ideal is reduced as this is strictly weaker. Now, suppose $R_\mathfrak{m}$ is reduced for all $\mathfrak{m}$. We have that, $R_\mathfrak{m} \otimes_R \sqrt{0} \subset \sqrt{0}_\mathfrak{m}$ where $\sqrt{0}_\mathfrak{m}$ denotes the nilradical of the localization. It is clear to see this as multiplying a nilpotent element by a unit preserves nilpotency in a commutative ring. Thus, by assumption, $R_\mathfrak{m} \otimes_R \sqrt{0}$ is 0, being contained in a 0 module. Furthermore, this is true at all maximal ideals, and so as a module $R_\mathfrak{m} \otimes_R \sqrt{0}$ is trivial, implying that as an R-module and thus an ideal, $\sqrt{0}$ is trivial. Now suppose that $\sqrt{0} \subset R$ is trivial. We have that at any localization $R_\mathfrak{p}$, the nilradical must be generated by the image of elements in $\sqrt{\mathrm{ker} f}$ for f the localization map, as all ideals in the localization are generated by ideals in $R/\mathrm{ker} f \subset R_\mathfrak{p}$, and the result is clear in $R/\mathrm{ker} f$. Thus, it suffices to show that $\sqrt{\mathrm{ker} f}$ is $\mathrm{ker} f$ itself for arbitrary closed subsets excluding a prime. However, we know that $\mathrm{ker} f = \{r \in R \mid \exists\ s \in S \text{ s.t. } sr = 0\}$. Thus, for any element $a \in R$ s.t. $a^n \in \mathrm{ker} f$, we have that $sa^n = 0$ some s, implying that sa is nilpotent, and so $sa = 0$ by triviality of the nilradical, implying $a \in \mathrm{ker} f$. Thus, the radical of $\mathrm{ker} f$ is itself, and so the nilradical of $R_\mathfrak{p}$ must be trivial. As $\mathfrak{p}$ was arbitrary, this must be true at all primes, concluding the proof. $\square$

Exercise 2. *Show that $k[x,y]/(x^2 - xy)$ is not flat as a $k[x]$−module.*

Proof. Consider the short exact sequence given by:

$$0 \longrightarrow k[x] \xrightarrow{x} k[x] \longrightarrow k \longrightarrow 0$$

Upon tensoring with $k[x,y]/(x^2 - xy)$ this becomes the following right exact sequence:

$$k[x,y]/(x^2 - xy) \xrightarrow{x} k[x,y]/(x^2 - xy) \longrightarrow k[y]/(xy) \longrightarrow 0$$

Where the last module is derived from the fact that the tensor product is right exact. It is clear, however, that the map

$$k[x,y]/(x^2 - xy) \xrightarrow{x} k[x,y]/(x^2 - xy)$$

Has the element $(x - y)$ in its kernel, and is thus not injective. Therefore, this module is not flat. $\square$

Exercise 3. *1) Give a construction of pushouts in the category of rings, and show how this gives coproducts in rings when A is chosen appropriately.*
2) Give an example to show that Spec of a pushout is not generally sent to a pullback.

Proof. 1) Given any commuting diagram of the following form:

$$\begin{tikzcd} A \arrow[r, "i_1"] \arrow[d, "i_2"] & B \arrow[d, "f"] \\ C \arrow[r, "g"] & X \end{tikzcd}$$

We have that any map out of the cone $C \leftarrow A \rightarrow B$ into X yields a unique set map from $C \times B$ the cartesian product (i.e. not the ring product) given by $h : (b, c) \mapsto f(b) \cdot g(c)$, and the set map f factors as $f = h\iota_B$ for $\iota_B : B \hookrightarrow B \times C$ via $b \mapsto (b, 1)$: this works similarly for ι_C and g. We have the following diag
Note that the following properties are satisfied:

- h is bilinear in both coordinates: $h(b_1 + b_2, c) = (f(b_1) + f(b_2))g(c) = h(b_1, c) + h(b_2, c)$, and symmetrically for C.
- h is A bilinear by the commutativity of the diagram above: $A \rightarrow B \rightarrow X = A \rightarrow C \rightarrow X$, implying $A \rightarrow B \rightarrow B \times C \rightarrow X = A \rightarrow C \rightarrow B \times C \rightarrow X$, and so $h(i_1(a), 1) = h(1, i_2(a))$. This gives us one better, as $h(i_1(a)b, c) = f(b) \cdot fi_1(a) \cdot g(c) = f(b) \cdot gi_2(a) \cdot g(c) = h(b, i_2(a))$.

Thus, X receives a unique map from the tensor product as A algebras, $B \otimes_A C \to X$ such that the following diagram commutes:

$$\begin{tikzcd}
A \arrow[r, "i_1"] \arrow[d, "i_2"] & B \arrow[d, "\iota_1"] \arrow[ddr, bend left] & \\
C \arrow[r, "\iota_2"] \arrow[drr, bend right] & B \otimes_A C \arrow[dr, dotted, "\exists\,!"] & \\
& & X
\end{tikzcd}$$

And thus by universal property $B \otimes_A C$ must be the pushout of this diagram. We may derive coproducts from this construction by noting that any two objects receive unique maps from the initial object in **CRing**, which is $\mathbb{Z}$, and so the coproduct is in fact the pushout of $B \leftarrow \mathbb{Z} \to C$, which from above must be $B \otimes_{\mathbb{Z}} C$.

2) Consider the following diagram:

$$\begin{tikzcd}
\mathbb{C} \arrow[r] \arrow[d] & \mathbb{C}[t] \arrow[d] \\
\mathbb{C}[t^{-1}] \arrow[r] & \mathbb{C}[t^{-1}] \otimes_{\mathbb{C}} \mathbb{C}[t] \cong \mathbb{C}[t, t^{-1}]
\end{tikzcd}$$

We have that the *Spec* of $Spec(\mathbb{C}[t^{-1}]) \cong Spec(\mathbb{C}[t]) = \mathbb{A}^1$ as topological spaces. We also know that $\mathbb{C}[t, t^{-1}]$ is the *Spec* of a localization, is $Spec(\mathbb{C}[t]) \setminus (t) \cong \mathbb{A}^1 \setminus \{0\}$.Thus, under *Spec*, this is sent to the diagram:

$$\begin{tikzcd}
\mathbb{A}^1 \setminus \{0\} \arrow[r] \arrow[d] & \mathbb{A}^1 \arrow[d] \\
\mathbb{A}^1 \arrow[r] & *
\end{tikzcd}$$

In **Top** the pullback of the diagram $\mathbb{A}^1 \to * \leftarrow \mathbb{A}^1$ is $\mathbb{A}^1$ itself mapping homeomorphically onto each space, as there is only one unique map to the point space. Thus, *Spec* does not necessarily send pushouts to pullbacks. $\square$

Exercise 4. *1) Show that if M and N finitely generated modules over a local ring R then $M \otimes_R N = 0$ if and only if either M or N is zero.*
2) Show the following:
a) If $0 \to L \to M \to N \to 0$ is exact then supp $M =$ supp $L \cup$ supp N.
b) If M, N are finitely generated then supp sends the tensor product to an intersection.

1) By Nakayama's lemma, we have that a module L over a local ring if $L/\mathfrak{m}$ is zero for $\mathfrak{m}$ the maximal ideal. Thus, we have that $R/\mathfrak{m} \otimes_R (M \otimes_R N) \cong (R/\mathfrak{m} \otimes_{R/\mathfrak{m}} R/\mathfrak{m}) \otimes_R (M \otimes_R N) \cong (M \otimes_R R/\mathfrak{m}) \otimes_{R/\mathfrak{m}} (R/\mathfrak{m} \otimes_R N) \cong M/\mathfrak{m} \otimes_{R/\mathfrak{m}} N/\mathfrak{m}$ heavily invoking the associativity and commutativity of tensor products over commutative rings. However, $M/\mathfrak{m} \otimes_{R/\mathfrak{m}} N/\mathfrak{m}$ is a tensor product of finite dimensional vector spaces. Let m the dimension of $M/\mathfrak{m}$ and n the dimension of $N/\mathfrak{m}$, with respective bases μ_i, ν_j. Consider the vector space $(R/\mathfrak{m})^{mn}$ with the basis $\{e_{ij}\}$ for $0 \leq \leq m$ and $0 \leq j \leq n$. Define maps $(R/\mathfrak{m})^{mn} \to M/\mathfrak{m} \otimes_{R/\mathfrak{m}} N/\mathfrak{m}$ via $e_{ij} \mapsto \mu_i \otimes \nu_j$ and the inverse defined in the obvious fashion. This implies $M/\mathfrak{m} \otimes_{R/\mathfrak{m}} N/\mathfrak{m}$ has dimension mn, which is 0 if and only if either $m = 0$ or $n = 0$ which, from work above using Nakayama's lemma, implies $M = 0$ or $N = 0$, yielding the result using the same logic.
2) a) We use the flatness of the localization functor to show this claim: We have that supp $L \subset$ supp M, as for any prime in supp L we have that $0 \to L_\mathfrak{p} \to M_\mathfrak{p} \to N_\mathfrak{p} \to 0$ is exact and $L_\mathfrak{p}$ is nonzero and injects into $M_\mathfrak{p}$, which must thus also be nonzero. Symmetric logic yields that supp $N \subset M$. Finally, suppose $\mathfrak{p} \notin$ supp $M \cup$ supp N. Then $0 \to 0 \to M_\mathfrak{p} \to 0 \to 0$ is exact, implying $M_\mathfrak{p} = 0$ and so it is not in supp M. This yields the claim.

b) Using part 1) and the methods of the proof, we have that $(M \otimes_R N)_\mathfrak{p} \cong M_\mathfrak{p} \otimes_R N_\mathfrak{p} = 0 \iff M_\mathfrak{p} = 0$ or $N_\mathfrak{p} = 0$. Thus, $(M \otimes_R N)_\mathfrak{p} \neq 0 \iff M_\mathfrak{p}, N_\mathfrak{p} \neq 0 \iff \mathfrak{p} \in \operatorname{supp} M \cap \operatorname{supp} N$.