Math 225A Differential Topology: Homework 3

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Exercise 1. *Show that if $f : X \to Y$ is a submersion then f maps open sets to open sets.*

Proof. By the local submersion theorem, we have that f is locally equivalent to the canonical submersion for $x \in X$,$y \in Y$ s.t. $f(x) = y$ is a submersion. Let $U \subset X$ be an open set containing $x \in X$. Without loss of generality, we may select a neighborhood $x \in N(x)$ with local coordinates for $N(x), f(N(x))$ fulfilling the local submersion theorem, and shrink $N(x)$ s.t. $N(x) \subset U$. The product topology on R^n for $N(x)$ diffeomorphic to $V \subset R^n$ yields that the projection of any open set onto any subspace yields an open set, i.e. that the projection map is an open map. As f is equivalent to the canonical submersion for $N(x)$, we have that the image of f on $N(x)$ must be an open set in Y. Thus, for any arbitrary $x \in f(U)$ we have a neighborhood $f(x) \in f(N(x)) \subset f(U)$ as constructed above, yielding the claim. □

Exercise 2. *Show that every submersion of a compact space into a connected space is surjective. In particular, show that there exist no submersions of compact manifolds into Euclidean spaces.*

Proof. From the previous exercise, we have that the image of an open set under a submersion is an open set. Thus, for $f : X \to Y$ with X compact, Y connected, we have that the image $f(X)$ must be open as $X \subset X$ open. However, X is compact, f is continuous, and thus $f(X)$ must be closed. As Y is connected, this implies the only clopen set is Y itself, and thus $f(X) = Y$. In particular, we have that as Euclidean space is connected and not compact, there are no continuous surjections from a compact space to all of Euclidean space and therefore no submersions. □