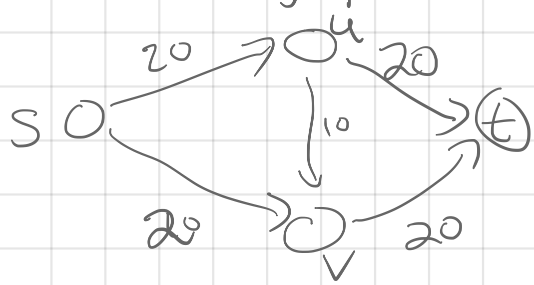
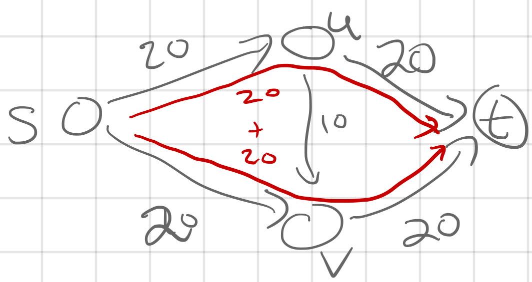
5 March 2018 Flow Networks (§7.1)

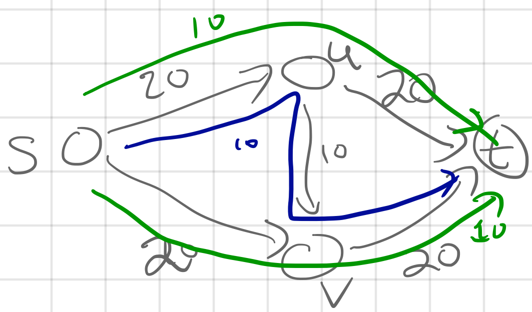
Here's a directed graph with "capacity limits" on edges.



(Think of these as pipes that can transport fluid. Capacity limit of 20, means ≤ 20 units/sec can flow thru pipe.)

What's the maximum rate at which you can send flow from s to t in this network?
E.g. the answer to the flow problem above is 40.





← If you had started solving the problem "greedily" and got into this state... how would you continue making progress toward the optimal solution?

"Augmenting paths" will be the answer.

Definitions. A flow network is a directed graph with

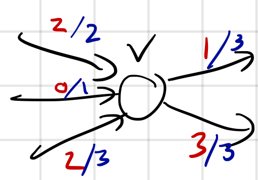
* source node s, sink node t
* capacities $c(e) > 0$ for all edges e.

Remarks. Unless otherwise specified, assume $c(e)$ is an integer $\forall e$. Also assume s has no incoming edges, t has no outgoing edges.

Def. 2. A flow in a flow network is a function $f : E \to \mathbb{R}_{\geq 0}$ assigning a "flow value" to each edge, satisfying

* capacity constraints. $0 \leq f(e) \leq c(e)$
* flow conservation. $\forall\, v \neq s,t, \quad \sum_{e \text{ to } v} f(e) = \sum_{e \text{ from } v} f(e)$

E.g.



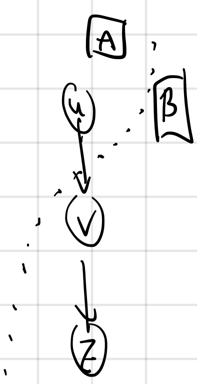
red numbers = flow values
blue numbers = capacities

The value of a flow, $v(f)$, is $\sum_{e \text{ from } s} f(e)$.

The maximum flow problem is to compute, given a flow network, a flow of maximum value.

Lemma. For two sets $A, B \subseteq V$ let $E(A,B) := \{(u,v) \in E \mid u \in A,\ v \in B\}$.
If f is any flow, and we partition V into two disjoint sets A, B such that $s \in A$, $t \in B$ then

$$v(f) = \sum_{e \in E(A,B)} f(e) - \sum_{e \in E(B,A)} f(e)$$



$(u,v) \in E(A,B)$
$(v,z) \in E(B,B)$

Proof.

$$\sum_{e \in E(A,B)} f(e) - \sum_{e \in E(B,A)} f(e) =$$

$$\left[\sum_{e \in E(A,B)} f(e) + \sum_{e \in E(A,A)} f(e)\right] - \left[\sum_{e \in E(A,A)} f(e) + \sum_{e \in E(B,A)} f(e)\right] =$$

$$\sum_{e \in E(A,V)} f(e) - \sum_{e \in E(V,A)} f(e) = \sum_{a \in A} \left[\sum_{e \text{ from } a} f(e) - \sum_{e \text{ to } a} f(e)\right]$$

$\emptyset$ except when $a = s$

$$= \sum_{e \text{ from } s} f(e) - \sum_{e \text{ to } s} f(e) = v(f)$$

$$v(f) = \sum_{e \in E(A,B)} f(e) - \sum_{e \in E(B,A)} f(e)$$

Def. An s-t cut in a flow network is a partition of the vertex set into disjoint A, B such that $s \in A$, $t \in B$. Its capacity $c(A,B)$ is defined by $c(A,B) := \sum_{e \in E(A,B)} c(e)$.

Lemma. If f is a flow and (A,B) is an s-t cut,

$$v(f) \leq c(A,B) \quad \text{"flow-cut inequality"}$$

and the two sides are equal if and only if $f(e) = c(e)$ ("f saturates e") for all $e \in E(A,B)$ and $f(e) = 0$ for all $e \in E(B,A)$.

Proof. Use $v(f) = \sum_{e \in E(A,B)} f(e) - \sum_{e \in E(B,A)} f(e)$ and $f(e) \leq c(e)$ for $E(A,B)$ and $f(e) \geq 0$ for $E(B,A)$

$$\leq \sum_{e \in E(A,B)} c(e) - \emptyset$$

$$= c(A,B)$$

Equality holds only if it holds "term by term" ie. $f(e) = c(e)$ for all $e \in E(A,B)$ and $f(e) = 0$ for all $e \in E(B,A)$.