# 7 May 2018. Primal–Dual Approx. Algorithms & Randomized Approx. Algorithms

LP relaxation of Vertex Cover:

minimize $\sum_{v \in V} w_v x_v$

subject to $x_u + x_v \geq 1 \quad \forall e = (u,v) \in E$

$0 \leq x_v \leq 1 \quad \forall v \in V$

Whenever we have a vector $(y_e)_{e \in E}$ that satisfies $y_e \geq 0 \ \forall e$

$$\sum_{e \sim v} y_e \leq w_v \quad \forall v$$

then every solution $\vec{x}$ of the vertex cover LP must satisfy

$$\sum_v w_v x_v \geq \sum_e y_e$$

## Dual LP of vertex cover:

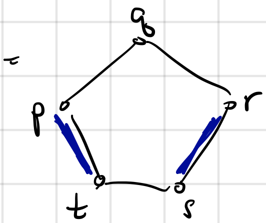
maximize $\sum_{e \in E} y_e$

subject to $\sum_{e \sim v} y_e \leq w_v \quad \forall v \in V$

$y_e \geq 0 \quad \forall e \in E$

Primal-dual algorithm searches simultaneously for dual solution $\vec{y}$ and vertex cover vector $\vec{x}$ such that

$$\sum_{v \in V} w_v x_v \leq 2 \sum_{e \in E} y_e \leq 2 \cdot OPT_{VC}$$

E.g. $G =$ 

$w_v = 1 \ \forall v$

minimize $x_p + x_q + x_r + x_s + x_t$

s.t.

$x_p + x_q \geq 1$

$x_q + x_r \geq 1$

$\boxed{x_r + x_s \geq 1}$

$x_s + x_t \geq 1$

$\boxed{x_t + x_p \geq 1}$

$0 \leq x_v \leq 1 \quad \forall v$

Sum

$x_p + x_r + x_s + x_t \geq 2$

$\therefore x_p + x_q + x_r + x_s + x_t \geq 2$

General case of this form of reasoning:

Multiply the constraint $x_u + x_v \geq 1$ by a coefficient $\boxed{y_{uv} \geq 0}$ on both sides.

Sum up over all edges:

$$\sum_{e=(u,v)} (y_e x_u + y_e x_v) \geq \sum_{e=(u,v)} y_e$$

Let $Y(v) := \sum_{\substack{\text{edges } e \\ \text{incident to } v}} y_e$.

$$\sum_{v \in V} Y(v) x_v \geq \sum_{e \in E} y_e.$$

$$\boxed{w_v \geq Y(v) = \sum_{e \sim v} y_e \quad \forall v}$$

<u>Algorithm:</u> Keep track of the following: $y_e$ := dual var. for edge $e$

$Y(v)$ := sum of dual vars of all edges incident to $v$.

$x_v := \begin{cases} 1 & \text{if } v \text{ is in vtx cover} \\ \emptyset & \text{if not} \end{cases}$

Theme of algorithm. "Increase $y$ variables greedily. Let the $x$ variables come along for the ride."

Recall $\max \sum_{e \in E} y_e$

s.t. $\sum_{e \sim v} y_e \leq w_v \quad \forall v$

$y_e \geq 0 \quad \forall e$

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Initialize $y_e = 0 \ \forall e$, $Y(v) = 0 \ \forall v$, $x_v = 0 \ \forall v$.

while $\exists$ edge $e = (u,v)$ s.t. $x_u + x_v = 0$

// increase $y_e$ as much as possible.

Let $\delta := \min\{w_v - Y(v), w_u - Y(u)\}$

$y_e = \delta$

$Y(u) = Y(u) + \delta$; $Y(v) = Y(v) + \delta$

$x_u = \begin{cases} 1 & \text{if } Y(u) = w_u \\ \emptyset & \text{if not} \end{cases}$; $x_v = \begin{cases} 1 & \text{if } Y(v) = w_v \\ \emptyset & \text{if not} \end{cases}$

endwhile

output $S := \{v \mid x_v = 1\}$.

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<u>Analysis.</u> $O(m)$ running time.

At termination $\vec{y}$ satisfies $\{\sum_{e \sim v} y_e \leq w_v \ \forall v, \ y_e \geq 0 \ \forall e\}$ because it was initialized satisfying these, and no single loop iteration can lead to a constraint violation.

The set $S$ is a vertex cover because otherwise the while-loop wouldn't finish.

Approximation guarantee: the invariant $\sum_{v \in V} Y(v) = 2 \sum_{e \in E} y_e$ holds throughout. (Both sides increase by $2\delta$ during one loop iteration.)

$$\text{Weight}(S) = \sum_{v \in S} w_v = \sum_{v \in S} Y(v) \leq \sum_{v \in V} Y(v) = 2 \sum_{e \in E} y_e \leq 2 \cdot \text{OPT}$$

The algorithm never solves a linear program, but you'd have a hard time inventing it without knowing about the linear program.

Randomized 2-approximation alg for vertex cover

$S = \emptyset$
while $\exists$ an uncovered edge $e=(u,v)$
| pick one of $u$ or $v$ at random (equal probability)
  insert into $S$
endwhile
output $S$

Theorem. If $G$ has a vertex cover of cardinality $k$ then the algorithm above outputs a random set whose expected cardinality is $\le 2k$.

Proof. Let $S_t :=$ set $S$ after $t$ iterations.
Let $S^* :=$ any vertex cover of size $k$.
Claim. $\mathbb{E}|S_t \cap S^*| \ge \mathbb{E}|S_t \setminus S^*|$.
Proof. Induction on $t$. $t=0 \Rightarrow$ trivial.
$t>0$. Consider edge $e=(u,v)$ that was active in iteration $t$. If $u\in S^*, v\notin S^*$ or $u\notin S^*, v\in S^*$ expected change in LHS = expected change in RHS.
If $u,v\in S^*$ then expected change in LHS $= +1$, expected change in RHS $=0$.

After $m$ iterations $S_m = S$. So

$$\mathbb{E}|S| = \mathbb{E}|S_m| = \mathbb{E}|S_m \cap S^*| + \mathbb{E}|S_m \setminus S^*|$$
$$\le 2\cdot \mathbb{E}|S_m\cap S^*| \le 2\cdot|S^*|. \qquad \text{QED.}$$

Extends to weighted vertex cover: pick $u$ or $v$ at random with probabilities

$$\Pr(u) = \frac{w_v}{w_u+w_v} \qquad \Pr(v) = \frac{w_u}{w_u+w_v}.$$