Introduction to Algorithms	**Problem Set 6**
CS 4820, Spring 2018	**Due 11:59pm Thursday, March 15**

Hand in your solution electronically using CMS. Collaboration is encouraged while solving the problems, but:

1. **list the names of those with whom you collaborated;**
2. **you must write up the solutions in your own words.**

Remember that when a problem asks you to design an algorithm, you must also prove the algorithm's correctness and analyze its running time.

(1) *(10 points)* Suppose we are given an $n \times n$ matrix $A = (a_{ij})$ with non-negative entries, and we are interested in the question of whether it can be expressed as a sum of two matrices R, C such that:

1. The entries of R and C are non-negative.
2. The row sums of R are bounded above by 1.
3. The column sums of C are bounded above by 1.

When there exist matrices R, C satisfying these three constraints, such that $R + C = A$, let us call the pair (R, C) a *row-plus-column (RPC) decomposition of A.*

Design a polynomial-time algorithm that takes a non-negative matrix A as input, and outputs either an RPC decomposition or a set of rows, S, and a set of columns, T, such that

$$\sum_{i \in S, j \in T} a_{ij} > |S| + |T|.$$

(In the latter case it follows easily that there is no RPC decomposition of A, since the row set S and column set T identify an $|S| \times |T|$ submatrix of A whose entries sum up to more than $|S| + |T|$, whereas the entries in the corresponding submatrices of R and C must have sums bounded above by $|S|$ and $|T|$, respectively.)

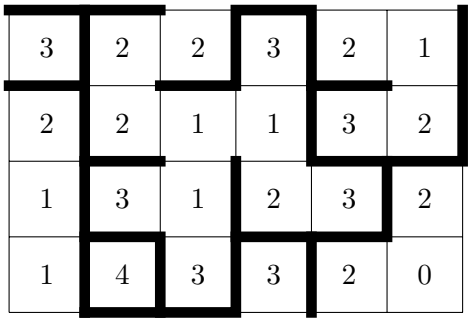
(2) *(10 points)* At Ford-Fulkerson University there are many committees that need to be staffed with professors. The university has n professors organized into d departments; each professor belongs to only one department. There are m committees, and the following constraints must be satisfied when staffing the committees.

1. The required number of professors on committee k is specified by a positive integer r_k.
2. No professor is allowed to serve on more than c committees.
3. No committee is allowed to have more than one professor from the same department.
4. For each professor j, there is a list L_j of the committees on which he or she is qualified to serve. Professor j is not allowed to serve on committee k unless $k \in L_j$.

Design a polynomial-time algorithm to determine whether it is possible to staff each committee without violating any of the constraints listed above. If it is possible to staff the committees, your algorithm should output an assignment of professors to committees that satisfies all of the constraints. The input to the problem is specified by the numbers $n, d, m, r_1, \ldots, r_m$, and the lists $L_1, \ldots, L_n$.

(3) *(10 points)* Consider a puzzle in which you are given an n-by-n square grid, with an integer in the range $\{0, \ldots, 4\}$ written inside each grid cell. You are asked to select a subset F of the edges of the grid, such that for each grid cell the number written inside the cell matches the number of elements of F that belong to the cell's boundary. The following figure shows an example of a puzzle and a valid solution to the puzzle.

3	2	2	3	2	1
2	2	1	1	3	2
1	3	1	2	3	2
1	4	3	3	2	0



Design a polynomial-time algorithm that, when given such a puzzle, either outputs a valid solution or decides (correctly) that there is no valid solution.

HINT: *It might be easiest to start by solving the special case in which each grid cell is labeled with either 0 or 1.*