CS 4820, Spring 2018	**Homework 11, Problem 1**
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(1) *(10 points)* For each of the following optimization problems, present an integer program whose optimum value matches the optimum value of the given problem. The combined number of variables and constraints in your integer program should be polynomial in the size of the given instance of the optimization problem.

(i). SET COVER. Given a universal set $\mathcal{U}$ and a collection of subsets $S_1, S_2, \ldots, S_m \subseteq \mathcal{U}$, what is the minimum size of a subcollection $\{S_{i_1}, S_{i_2}, \ldots, S_{i_k}\}$ whose union is $\mathcal{U}$?

(ii). INDEPENDENT SET. Given a graph $G = (V, E)$, find an independent set of maximum cardinality.

(iii). MAX-3SAT. Given a set of Boolean variables $x_1, x_2 \ldots, x_n$ and a set of clauses $C_1, C_2, \ldots, C_m$ each consisting of a disjunction of 3 literals from the set $\{x_1, \bar{x}_1, \ldots, x_n, \bar{x}_n\}$, what is the maximum number of clauses that can be satisfied by a truth assignment?

(iv). MAX-CUT. Given an undirected graph $G = (V, E)$, what is the maximum size of a cut? (A cut (A, B) is any partition of the vertex set V into two nonempty subsets. The size of a cut is equal to the number of edges with one endpoint on each side of the partition.)

It is not necessary to prove that your answer is valid. However, you should explain the interpretation of your notation well enough that we completely understand the structure of your integer program.

Example: In the weighted vertex cover problem, one is given a graph $G = (V, E)$ and a non-negative weight w_v for every vertex $v \in V$. One is asked to find the minimum total weight of a vertex cover.

ANSWER: The equivalent integer program is:

$$\begin{aligned} \min \quad & \sum_{v \in V} w_v x_v \\ \text{s.t.} \quad & x_u + x_v \geq 1 \ \text{ for all edges } e = (u, v) \\ & x_v \in \{0, 1\} \ \text{ for all vertices } v \end{aligned}$$

INTERPRETATION: Decision variable x_u equals 1 if vertex u is included in the vertex cover, 0 otherwise.

Solution:

(i) SET COVER

ANSWER: The equivalent integer program is:

$$\begin{aligned} \min \quad & \sum_{s \in S} x_s \\ \text{s.t.} \quad & \sum_{l : u \in S_l} x_l \geq 1 \ \text{ for all } u \in U \\ & x_s \in \{0, 1\} \ \text{ for all subsets } s \end{aligned}$$

INTERPRETATION: Decision variable x_s equals 1 if subset s is included in the set cover, 0 otherwise.

(ii) Independent Set

Answer: The equivalent integer program is:

$$
\begin{aligned}
\min \quad & \sum_{v \in V} x_v \\
\text{s.t.} \quad & x_v + x_u \leq 1 \;\text{ for all edges } e = (u, v) \\
& x_v \in \{0, 1\} \quad \text{for all vertices } v
\end{aligned}
$$

Interpretation: Decision variable x_v equals 1 if vertex v is included in the independent set, 0 otherwise.

(iii) Max-3SAT

Answer: The equivalent integer program is:

$$
\begin{aligned}
\max \quad & \sum_{c \in C} y_c \\
\text{s.t.} \quad & y_c \leq x_1 + x_2 + x_3 \;\text{ for all clauses} \\
& c = (x_1 \cup x_2 \cup x_3), \quad \text{use } (1 - x) \text{ if } \bar{x} \text{ in clause} \\
& y_c \in \{0, 1\} \quad \text{for all clauses } c, \; x_i \in \{0, 1\} \text{ for all variables}
\end{aligned}
$$

Interpretation: Decision variable y_c equals 1 if clause c is true, 0 otherwise. Decision variable x_i equals 1 if the variable is true, 0 otherwise.

(iv) Max-Cut

Answer: The equivalent integer program is:

$$
\begin{aligned}
\max \quad & \sum_{e \in E} y_e \\
\text{s.t.} \quad & y_e = (x_v - x_u)^2 \;\text{ for all edges } e = (u, v) \\
& y_e \in \{0, 1\} \quad \text{for all edges } e, \; x_v \in \{0, 1\} \text{ for all vertices } v
\end{aligned}
$$

Interpretation: Decision variable y_e equals 1 if edge e is saturated, 0 otherwise. Decision variable x_v equals 1 if vertex v is in the max-cut, 0 otherwise.