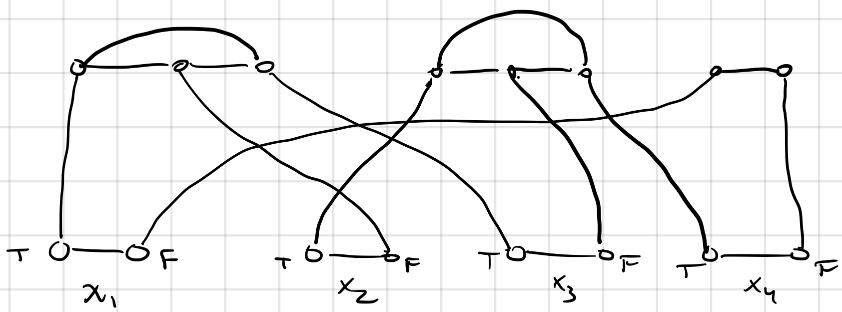
23 March 2018

Reducing 3SAT to IND SET.

$$(\overline{x_1} \vee x_2 \vee \overline{x_3}) \wedge (\overline{x_2} \vee x_3 \vee \overline{x_4}) \wedge (x_1 \vee x_4)$$

"Gadgets": translate each piece of the problem (3SAT in this case) into pieces of ~~a~~ graph.

"Clause gadgets"



"Variable gadgets"

Reduction constructs graph G as above and sets $k = n + m = (\#\text{ variables}) + (\#\text{ clauses})$.

What do we need to show about this reduction?

1. Poly-time: obvious.
2. If 3SAT formula has a satisfying truth assignment, the graph has an indep set of size k.

"Show that your gadgets work as intended."

Given a truth assignment pick the corresponding vertex in each variable gadget (n vertices) and for each clause pick one of its vertices whose variable-gadget neighbor was not picked (m vertices). Such a vertex (at least one) is guaranteed to exist because at least one literal of the clause is satisfied.

This set of $n+m$ vertices is an independent set.

- the graph contains no edge between 2 vertices in different variable gadgets.
- the graph contains no edge between 2 vertices in different clause gadgets.
- our set has no "var-gadget vertex" and "clause-gadget vertex" joined by an edge, because of the way we constructed it.

(3) If the 3SAT formula has no satisfying assignment, the graph has no independent set of size k.
"The gadgets have no unintended uses."
Usually easiest to prove the contrapositive:
If the graph has an indep set of size k, the 3SAT formula has a satisfying assignment.

Given an indep set S of size $k = n+m$, S cannot contain more than one vertex from any gadget (variable or clause). There are $n+m$ gadgets in the graph so S must contain exactly one vertex from each.
Extract a truth assignment from S: for each variable, set it to TRUE if S contains the "T node" of its gadget and FALSE if S contains the "F node."
This truth assignment satisfies the formula because in each clause, there is a satisfied literal corresponding to the node of S in that clause gadget.

Other NP-Complete graph problems:
① CLIQUE: given graph G and integer k. Does G contain k vertices such that every two of them are joined by an edge?
② VERTEX COVER: given graph G and integer k. Does G have a set of k vertices such that every edge has at least one endpoint belonging to the set?

IND SET $\le_p$ CLIQUE

$$(V, E, k) \longmapsto (V, \overline{E}, k)$$

IND SET $\le_p$ VTX COVER.

$$(V, E, k) \longmapsto (V, E, n-k)$$

S is an independent set in G iff $V \setminus S$ is a vertex cover of G.

HAMILTONIAN CYCLE. Given directed graph G, does it have a cycle that visits every vertex exactly once?

verifier is given graph & cycle, checks that every 2 consecutive vertices of cycle are joined by an edge & that every vertex is used exactly once.

$\therefore$ HAM CYCLE $\in$ NP.

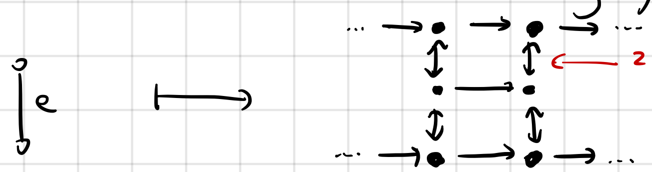
Next step: reduce some other NP-Complete problem to HAM CYCLE.

3SAT $\leq_p$ HAM CYCLE: see book.

VERTEX COVER $\leq_p$ HAM CYCLE.

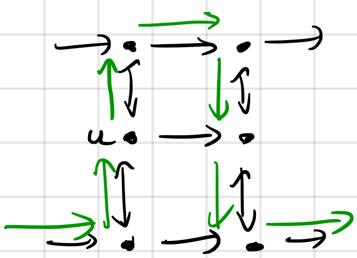
$(V, E, k) \longmapsto$ directed graph.

Each edge e transforms into gadget G_e:



2 dir edges, one up, one down

The 6 nodes in this gadget have no edges other than the ones shown.



what edge of the cycle exits u?