28 March 2018 The Subset Sum Problem

Given list of positive integers $n_1, \dots, n_m$ (represented in binary).
Given "target sum" W.
Does there a subset $S \subseteq \{1, \dots, n\}$ such that

$$\sum_{i \in S} n_i = W \; ?$$

Important Remark: There is a dynamic program that solves SUBSET SUM in time $O(mW)$. Similar to solving knapsack.
Let $T[i,j]$ $(0 \le i \le m,\ 0 \le j \le W)$ denote a table of Boolean values such that

$$T[i,j] = \begin{cases} \text{TRUE} & \text{if } \exists \text{ subset of } \{n_1, \dots, n_i\} \text{ summing up to } j \\ \text{FALSE} & \text{otherwise.} \end{cases}$$

These values can be computed by

$$T[0,j] = \begin{cases} \text{TRUE} & \text{if } j = 0 \\ \text{FALSE} & \text{if } j > 0. \end{cases} \qquad T[i,0] = \text{TRUE} \text{ for all } i.$$

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for i = 1,2,...,m
    for j = 1,...,W
        if n_i > j then T[i,j] = T[i-1,j]
        else T[i,j] = T[i-1,j] v T[i-1, j-n_i]
    endfor
endfor
```

After computing the whole table $T[m,W]$ is the answer to SUBSET SUM.

Reducing other problems to SUBSET SUM involves designing gadgets that represent logical or combinatorial constraints using the digits of the numbers $n_1, \dots, n_m$.

Designing 3SAT $\le_p$ SUBSET SUM reduction.
Idea: figure out one type of gadget that forces the SUBSET SUM solver to make a yes/no decision, and a different gadget that enforces a "three strikes & out" policy if certain triples of "bad decisions" get made.

MSB ← Place values → LSB

Clauses | Variables

n_1

n_2

$\vdots$

n_m ← This value of m need not (and will not) be equal to # of clauses.

W

$(\overline{x_1} \vee x_2 \vee x_n) \wedge (\overline{x_2} \vee x_3 \vee \overline{x_n})$

	0	0	0	0	0		0	1	x_1
	0	0	0	1	0		0	1	$\overline{x_1}$
	0	0	0	1	0		1	0	x_2
	0	1	0	0	0		1	0	$\overline{x_2}$
	0	1	0	0	⋮	…	0	⋮	⋮
	0	0	⋮	⋮	⋮		0	⋮	
	⋮	⋮					⋮	⋮	
	0	0	0	0					
	0	0	0	1	1		0	0	x_n
	0	1	0	0	1		0	0	$\overline{x_n}$
0 — — …	0	0	0	1	0	— — —	0	0	y_1
0 … —	0	0	0	1	0	— —	0	0	z_1
0 — — …	0	1	0	0					y_2
— — …	0	1	0	0					z_2

11 11 1111 . . 1 1 W