Elf Game Mathematics

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Abstract

Elf game is a game where throughout the 24 days of advent you allocate elves to collect trees for you to sell and gain money. There are a few events throughout the 24 days which allow you to make extra decisions, but the main actions you can take is how to allocate your elves between 3 locations. The Nearby Forest, The Faraway Forest and The Forest Far Beyond. Each of these locations yields a different amount of money, but come with their own risks. For the first 16 days only the Nearby Forest and the Faraway Forest are available. Every day there is a random chance of $\frac{1}{3}$ that it snows. In the Nearby Forest you will always get £10. In the Faraway Forest You will get £20 if it is sunny and £0 if it snows. In the Forest Far Beyond you will get £50 if it is sunny, but get £0 and lose the elves for the rest of the game if it snows.

1 Introduction

Our aim will be to maximise the chances of winning when playing elf game. This will include calculating the best strategy of allocation of elves and what to do in the events that occur throughout the game. But first we need to understand the game a little better.

2 Game Mechanics

2.1 Events

There are 5 events that happen throughout the 24 days, these are:

- Workers Revolt That occurs on day 7
- The Tax Man on day 10
- A Jackpot on day 11
- The Forest Far Beyond Opening on day 17
- Elf Hiring on day 21

2.1.1 Workers Revolt

https://www.overleaf.com/project/656471dde6dd66d59b6bac62 The workers revolt is an event where the elves believe they are being worked too hard, so they demand a day off, if you give them this day off for day 7, they will not revolt, however if you don't give them a day off there is a $\frac{1}{3}$ chance that they will revolt for days 8-9.

2.1.2 Tax Man

The Tax man arrives on day 10 and demands 10% of your current money, you have a choice, give the tax man your money now, or risk having to pay 20% of your money on day 13. If you do not pay the tax man on day 10 there is a $\frac{1}{2}$ chance you have to pay the 20% on day 13.

2.1.3 Jackpot

This occurs on day 11 where 2 dice are rolled and the teams which guess what the total of the two dice are have a Jackpot of £1000 split between them

2.1.4 Forest Far Beyond Opens

On day 17, the forest far beyond opens, there is now the possibility of losing elves as if it snows you will lose the elves in the Forest Far Beyond.

2.1.5 Buying Elves

On day 21 you get the opportunity to get more elves for £75 each.

2.2 Days

Now this is where the real problem lies, this is where you decide to allocate your elves on each day which will account for most of the reward you will get throughout the game.

3 Objective Functions

A strategy denotes how you will act on all of the turns, and given there is a optimal distribution of elves at each go, the actual amount of elves you have will not affect the strategy, so we can say this is set from the start. It has 3 columns $w_{\mathrm{t_{NF}}}$, $w_{\mathrm{t_{FF}}}$ and $w_{\mathrm{t_{FFB}}}$. Where t is the current day starting from 0 and ranging up to 23. 24 days in total.

$$t = [0, 1, 2 \ldots 23]$$

These are the weights for allocating elves to the 3 locations on the nth day and each row must sum to 1.

$$\text{Let Strategy } S = \begin{pmatrix} S_{0_{\mathrm{NF}}} & S_{0_{\mathrm{FF}}} & S_{0_{\mathrm{FFB}}} \\ S_{1_{\mathrm{NF}}} & S_{1_{\mathrm{FF}}} & S_{1_{\mathrm{FF}}} \\ S_{2_{\mathrm{NF}}} & S_{2_{\mathrm{FF}}} & S_{2_{\mathrm{FF}}} \\ & \vdots & \\ S_{23_{\mathrm{NF}}} & S_{23_{\mathrm{FF}}} & S_{23_{\mathrm{FF}}} \end{pmatrix}$$

Note that as the FFB is not open until day 17, until then that column must be 0.
So an example might be:

$$S = \begin{pmatrix} 0.5 & 0.5 & 0 \\ 0.8 & 0.2 & 0 \\ 0 & 1 & 0 \\ & \vdots & \\ 0.2 & 0.5 & 0.3 \\ 0 & 0.6 & 0.4 \end{pmatrix}$$

These numbers are also not an optimal solution but show how the Stategy could be created.
S_{tL} is the weight allocation for the t^{th} in location L.
S_2 from before would be $\begin{pmatrix} 0.8 & 0.2 & 0 \end{pmatrix}$ so $S_{2_{FF}}$ would be 0.2.

We also have the enumeration of the different locations D.

$$D = \begin{bmatrix} \mathrm{NF} & \mathrm{FF} & \mathrm{FFB} \end{bmatrix}$$

The reward of each location Re(L) is as below, this is the amount of money you get if it your elves succeed.

$$Re(L) = \begin{cases} 10, & L = \text{NF} \\ 20, & L = \text{FF} \\ 50, & L = \text{FFB} \end{cases}$$

For the distribution of elves on each day as given by our strategy we can work out the expected value of a single day, and its variance.
If we define a day as being D_t where D_0 is the first day and corresponds to the weights in S. Then the value of an elf at a certain t is given by $E(D_t)$, and the variance as $Var(D_t)$.

For $E(D_t)$ we find the expected value if it is sunny, which is all of the allocations and the expected value if it is snowy, and then multiply by the respective probabilities.

$$E(D_t) = \frac{2}{3}\sum^{L\in D} S_{tL} \cdot Re(L) + \frac{1}{3} S_{t_{NF}} Re(NF)$$

The variance of a day at day t can be found by finding $E(D_t^2) - E(D_t)^2$. We have defined $E(D_t)$ already so we just need to find $E(D_t^2)$ which we can do by the fact that $E(X^2) = \sum P(X = x)x^2$

$$E(D_t^2) = \frac{2}{3}\left(\sum^{L\in D} S_{tL} \cdot Re(L)\right)^2 + \frac{1}{3}\left(S_{t_{NF}} Re(NF)\right)^2$$

$\therefore$ We can now calculate $Var(D_t)$

$$Var(D_t) = E(D_t^2) - E(D_t)^2 \tag{1}$$

Where

$$E(D_t) = \frac{2}{3}\sum^{L\in D} S_{tL} \cdot Re(L) + \frac{1}{3} S_{t_{NF}} Re(NF) \tag{2}$$

$$E(D_t^2) = \frac{2}{3}\left(\sum^{L\in D} S_{tL} \cdot Re(L)\right)^2 + \frac{1}{3}\left(S_{t_{NF}} Re(NF)\right)^2 \tag{3}$$

If D_{Ct} is all actions from t onward, and all days were independent from each other, this could be written recursively like so:

$$D_{Ct} = D_t + D_{C(t+1)}$$

As there is a possibility to loose elves, we will have to modify this a little, to do this easily we can introduce a new random variable M. M_t will have a correlation of 1 with D_t and is as follows:

$P(M_t = m)$	$2/3$	$1/3$
m	1	$1 - S_{t_{FFB}}$

This is then used as a multiplier in our calculations

$$D_{Ct} = D_t + M_t D_{C(t+1)}$$

This represents the fact that one third of the time you loose a proportion of your elves equal to $S_{t_{FFB}}$, leaving you with a proportion of elves equal to $1 - S_{t_{FFB}}$. The other 2/3 of the time you keep all of your elves. And as this is perfectly correlated with D_t it will match the outcome of D_t to give the correct outcome.

We can work out some values for M_t as we will be needing them later. $E(M_t)$ can be calculated as so:

$$\begin{aligned} E(M_t) &= \sum P(M_t = m)m \\ &= \frac{2}{3}1 + \frac{1}{3}(1 - S_{t_{FFB}}) \\ &= \frac{2 + 1 - S_{t_{FFB}}}{3} \\ &= \frac{3 - S_{t_{FFB}}}{3} \end{aligned} \tag{4}$$

We will also work out $E(M_t^2)$ and $Var(M_t)$ for later.

$$\begin{aligned} E(M_t^2) &= \sum P(M_t = m)m^2 \\ &= \frac{2}{3}1^2 + \frac{1}{3}(1 - S_{t_{FFB}})^2 \\ &= \frac{2 + 1 - 2S_{t_{FFB}} + S_{t_{FFB}}^2}{3} \\ &= \frac{3 - 2S_{t_{FFB}} + S_{t_{FFB}}^2}{3} \end{aligned} \tag{5}$$

$$\begin{aligned} Var(M_t) &= E(M_t^2) - E(M_t)^2 \\ &= \frac{3 - 2S_{t_{FFB}} + S_{t_{FFB}}^2}{3} - \left(\frac{3 - S_{t_{FFB}}}{3}\right)^2 \\ &= \frac{9 - 6S_{t_{FFB}} + 3S_{t_{FFB}}^2}{9} - \frac{9 - 6S_{t_{FFB}} + S_{t_{FFB}}^2}{9} \\ &= \frac{2S_{t_{FFB}}^2}{9} \\ &= \frac{2}{9}S_{t_{FFB}}^2 \end{aligned} \tag{6}$$

This means we can calculate the expected value from any day to the end of the game. This is given by and due to probability laws this is equal to:

$$E(D_{Ct}) = E(D_t + M_t D_{C(t+1)})$$

Some properties of discrete random variables can help simplify this as shown below.

$$E(X + Y) = E(X) + E(Y)\text{always} \tag{7}$$

$$E(XY) = E(X)E(Y) \impliedby \text{X and Y are independent} \tag{8}$$

As M_t and $D_{C(t+1)}$ are not correlated we can deduce

$$\begin{aligned} E(D_{Ct}) &= E(D_t + M_t D_{C(t+1)}) \\ &= E(D_t) + E(M_t D_{C(t+1)}) \\ &= E(D_t) + E(M_t)E(D_{C(t+1)}) \end{aligned} \tag{9}$$

Where $E(D_{C23})$ is the final day and is equal to $E(D_{23})$, terminating the recursion.
The one we will use is $E(D_{C0})$ as this is the total expected value for the strategy.

We now need to work out the variance of the strategy and we can use similar reasoning.

$$Var(D_{Ct}) = Var(D_t + M_t D_{C(t+1)}))$$

Because of variance laws this can be expressed as a sum of variances, but because D_t and M_t are correlated we need to include a covariance term.

$$Var(X+Y) = Var(X) + Var(Y) + 2Cov(X,Y) \tag{10}$$

$$\implies Var(D_t + M_t D_{C(t+1)})) = Var(D_t) + Var(M_t D_{C(t+1)})) + 2Cov(D_t, M_t D_{C(t+1)})) \tag{11}$$

We now need to use some variance definitions to break this down further. [con23a]

$$Var(XY) = E(X^2)E(Y^2) - E(X)^2E(Y)^2$$
$$Cov(X,Y) = E(XY) - E(X)E(Y)$$

So we can rewrite the equation above as:

$$Var(D_{Ct}) = Var(D_t) + E(M_t^2)E(D^2_{C(t+1)}) - E(M_t)^2E(D_{C(t+1)})^2 + \tag{12}$$

$$2\left[E(D_t M_t D_{C(t+1)}) - E(D_t)E(M_t D_{C(t+1)}))\right] \tag{13}$$

The only one we can't separate from here is $E(D_t M_t)$ as D_t and M_t are correlated, but due to the fact their correlation coefficient is 1, it means we can rewrite D_t as $a_t M_t + b_t$, so
$E(D_t M_t)$ becomes $E((a_t M_t + b_t)M_t) = E(a_t M_t^2 + b_t M_t) = a_t E(M_t^2) + b_t E(M_t)$

$$\begin{aligned} Var(D_{Ct}) &= Var(D_t) + E(M_t^2)E(D^2_{C(t+1)}) - E(M_t)^2E(D_{C(t+1)})^2 + && (14) \\ & \quad 2\left[E(D_t M_t D_{C(t+1)}) - E(D_t)E(M_t D_{C(t+1)}))\right] && (15) \\ &= Var(D_t) + E(M_t^2)E(D^2_{C(t+1)}) - E(M_t)^2E(D_{C(t+1)})^2 + && (16) \\ & \quad 2\left[E(D_t M_t)E(D_{C(t+1)}) - E(D_t)E(M_t)E(D_{C(t+1)}))\right] && (17) \\ &= Var(D_t) + E(M_t^2)E(D^2_{C(t+1)}) - E(M_t)^2E(D_{C(t+1)})^2 + && (18) \\ & \quad 2\left[\left(a_t E(M_t^2) + b_t E(M_t)\right) E(D_{C(t+1)}) - E(D_t)E(M_t)E(D_{C(t+1)}))\right] && (19) \\ & && (20) \\ &= Var(D_t) + E(M_t^2)E(D^2_{C(t+1)}) - E(M_t)^2E(D_{C(t+1)})^2 + && (21) \\ & \quad 2E(D_{C(t+1)})\left[a_t E(M_t^2) + b_t E(M_t) - E(a_t M_t + b_t)E(M_t))\right] && (22) \\ & && (23) \end{aligned}$$

Note:

$$a_t E(M_t^2) + b_t E(M_t) - E(a_t M_t + b_t) E(M_t) \tag{24}$$
$$= a_t E(M_t^2) + b_t E(M_t) - (a_t E(M_t) + b_t) E(M_t) \tag{25}$$
$$= a_t E(M_t^2) + b_t E(M_t) - (a_t E(M_t)^2 + b_t E(M_t)) \tag{26}$$
$$= a_t E(M_t^2) - a_t E(M_t)^2 \tag{27}$$
$$= a_t (E(M_t^2) - E(M_t)^2) \tag{28}$$
$$= a_t Var(M_t) \tag{29}$$

$$\therefore Var(D_{Ct}) = Var(D_t) + E(M_t^2)\left(Var(D_{C(t+1)}) + E(D_{C(t+1)})^2\right) - E(M_t)^2 E(D_{C(t+1)})^2 + \tag{30}$$
$$2E(D_{C(t+1)})\left[aVar(M_t)\right] \tag{31}$$

And From before we have the equations:

$$E(M_t) = \frac{3 - S_{t_{FFB}}}{3} \tag{32}$$
$$E(M_t^2) = \frac{3 - 2S_{t_{FFB}} + S_{t_{FFB}}^2}{3} \tag{33}$$
$$Var(M_t) = \frac{2}{9} S_{t_{FFB}}^2 \tag{34}$$

And we still have the term a, which relates the relationship between D_t and M_t in the form $D_t = a_t M_t + b_t$.
We can find the constant of a_t by scaling the outputs. If the 2/3 result is p and 1/3 result is q then we can say that:

$$D_{tp} = a_t M_{tp} + b_t$$

and

$$D_{tq} = a_t M_{tq} + b_t$$

If we subtract these equations we get:

$$D_{tp} - D_{tq} = a(M_{tp} - M_{tq}) \tag{35}$$
$$\implies a = \frac{D_{tp} - D_{tq}}{M_{tp} - M_{tq}} \tag{36}$$

And we have formulas for p and q for D and M.

$$D_{tp} = \sum^{L \in D} S_{tL} Re(L) \tag{37}$$
$$D_{tq} = S_{t_{NF}} Re(NF) \tag{38}$$
$$M_{tp} = 1 \tag{39}$$
$$M_{tq} = 1 - S_{t_{FFB}} \tag{40}$$

$$\therefore a = \frac{S_{t_{NF}}Re(NF) + S_{t_{FF}}Re(FF) + S_{t_{FFB}}Re(FFB) - S_{t_{NF}}Re(NF)}{1-(1-S_{t_{FFB}})} \tag{41}$$

$$= \frac{S_{t_{FF}}Re(FF) + S_{t_{FFB}}Re(FFB)}{S_{t_{FFB}}} \tag{42}$$

We can see from this equation that a does not exist if $S_{t_{FFB}} = 0$, which it will for the first 16 days, but if this is the case then the next day will be independent from the current day and so the variance can be given by:

$$Var(D_{Ct}) = Var(D_t) + Var(D_{C(t+1)}) \impliedby S_{t_{FFB}} = 0$$

The other problem is $E(D^2_{C(t+1)})$. We can use the link between this and variance to find this in terms we can calculate.

$$Var(D_{C(t+1)}) = E(D^2_{C(t+1)}) - E(D_{C(t+1)})^2 \tag{43}$$

$$\implies E(D^2_{C(t+1)}) = Var(D_{C(t+1)}) + E(D_{C(t+1)})^2 \tag{44}$$

This is then recursive on both the next expected value and the next variance. So now we have all we need to compute the variance and expected value from a day onward.

$$E(D_{Ct}) = E(D_t) + E(M_t)E(D_{C(t+1)})$$

$$Var(D_{Ct}) = Var(D_t) + E(M_t^2)E(D^2_{C(t+1)}) - \left(E(M_t)E(D_{C(t+1)})\right)^2 + 2a_tE(D_{C(t+1)})Var(M_t)$$

Where

$$E(D_t) = \frac{2}{3}\sum^{L\in D} S_{tL}\cdot Re(L) + \frac{1}{3}S_{t_{NF}}Re(NF) \tag{45}$$

$$E(D_t^2) = \frac{2}{3}\left(\sum^{L\in D} S_{tL}\cdot Re(L)\right)^2 + \frac{1}{3}\left(S_{t_{NF}}Re(NF)\right)^2 \tag{46}$$

$$Var(D_t) = E(D_t^2) - E(D_t)^2 \tag{47}$$

$$E(D^2_{C(t+1)}) = Var(D_{C(t+1)}) + E(D_{C(t+1)})^2 \tag{48}$$

$$E(M_t) = \frac{3 - S_{t_{FFB}}}{3} \tag{49}$$

$$E(M_t^2) = \frac{3 - 2S_{t_{FFB}} + S^2_{t_{FFB}}}{3} \tag{50}$$

$$Var(M_t) = \frac{2}{9}S^2_{t_{FFB}} \tag{51}$$

$$a_t = \frac{S_{t_{FF}}Re(FF) + S_{t_{FFB}}Re(FFB)}{S_{t_{FFB}}} \tag{52}$$

$$\tag{53}$$

Up until now the calculations have been based on a split of 1, but you can't split an Elf, you have to use, in this case 12 elves. So we can use the fact that $Var(12X) = 144Var(X)$ and $E(12X) = 12E(X)$ to find the multipliers of 12 for expected value and 144 for variance to make this work for 12 elves.

These were programmed up and tested against a simulation, Figure 1 which showed its validity.

```
4186.666666666666 557155.555555555
predicted in 1.008ms
##### t=0 #####
predicted:
 [  4186.6667 557155.5556]
simulated:
 [  4187.7672 556558.6786]
absolue error:
 [  1.1005 596.877 ]
percentage error:
 [0.0263 0.1072]
```

Figure 1: Objective Function Testing

4 Optimisation of Objective Functions

Now we have these values, we can compute the expected value and variance for a strategy S using $E(D_{C0})$ and $Var(D_{C0})$. Now we have these equations the aim is to find the minimum Variance of a strategy for a specific expected value. This is a multi-objective optimization problem. [con23b]

I will minimise the variance: $Var(D_{C0})$ and absolute error against V^{opt} which is a set expected value aim: $|E(D_{C0}) - V^{\text{opt}}|$. This will ensure that the optimisation will find strategies for minimum variance at a set expected value. The algorithm I will use is called NSGA-II (Non-Dominated Sorting Genetic Algorithm II). [DPAM00]

As There are multiple variables to minimise, there is not one correct solution, but a frontier. This set of optimal outcomes is known as the Pareto Frontier, and anything on this frontier is Pareto efficient and NSGA-II optimises to find points along this Pareto frontier.This means that I will get a good spread for each inputted V^{opt}, minimising the number of runs that I need to run in order to find the full Expected-Value Variance Front. I will use a variety of V^{opt} values to gain a full frontier.

I need to make the input variables which will be optimised. an example of S is below

$$S = \begin{pmatrix} 0.5 & 0.5 & 0 \\ 0.8 & 0.2 & 0 \\ 0 & 1 & 0 \\ & \vdots & \\ 0.2 & 0.5 & 0.3 \\ 0 & 0.6 & 0.4 \end{pmatrix}$$

This is a 24x3 matrix, where all values must be between 0 and 1, which can be easily implemented as a constraint. But not all of these values are needed to construct the strategy. For the first 16 days you cannot send elves to the Forest Far Beyond (FFB), and so the last column will be empty. As each row must sum to 1, the first 2 must sum to 1 which allows you to just provide one value for each of the first 16 days.

If $w_{1_{NF}} = 0.2$, then as $w_{1_{NF}} + w_{1_{FF}}$ must sum to 1, $w_{1_{FF}} = 1 - w_{1_{NF}} \implies w_{1_{FF}} = 1 - 0.2 = 0.8$
So for the first 16 days, only one variable is needed with no extra constrains.
From days 17 to 24 however all 3 locations are available, so we need to provide 2 values to calculate the third and there is an extra constraint that the sum of these 2 provided weights sum to less than one which is more difficult to implement. i.e.
$w_{1_{NF}} + w_{1_{FF}} \leq 1$
Where $w_{1_{FFB}}$ is then calculated by $w_{1_{FFB}} = 1 - (w_{1_{NF}} + w_{1_{FF}})$

However we can do this by adding what is called a delta penalty [Coe02] which punishes the optimisation by providing a value for being outside of the constraint, in my case I don't want impossible strategies, so I set this to be a number outside the range of either variable being optimised: 1e+10, this discourages incorrect answers entirely.

Optimising for V^{opt} in steps of 100 from 3600 to 4600, with 1000 generations of optimisation per V^{opt} yields Figure 2.

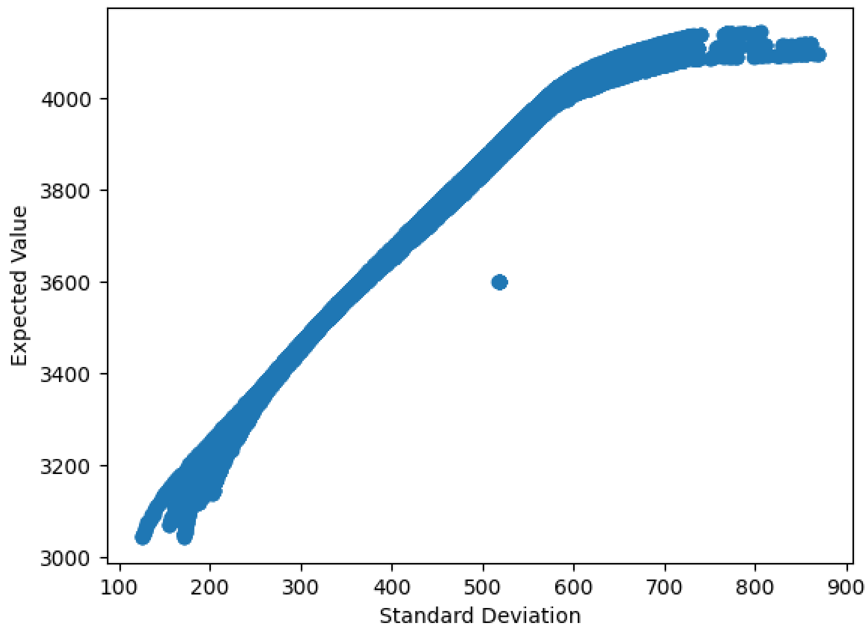

Figure 2: Pareto Front For Elf Game With Extra points

This however includes some non Pareto efficient points so we can slim this down to just the efficient points. Figure 3

From this you can then choose your desired Expected value and from that, get from the program the corresponding strategy.

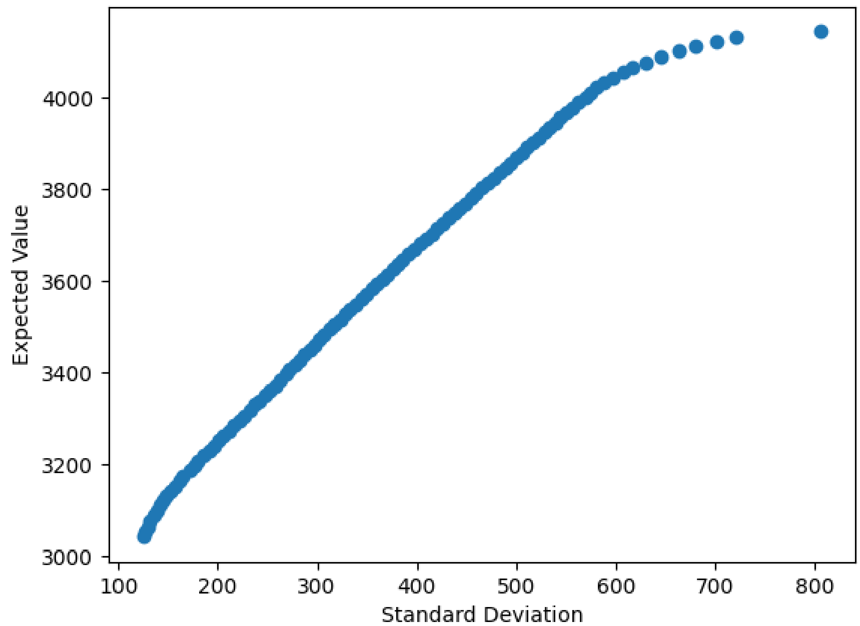

Figure 3: Pareto Front For Elf Game

5 Considering Events

Now having this strategy is good, but we still can't use this, as the events may affect what the optimal distribution of elves is, so we also need to consider all of the different Events as mentioned in Section 2.1.
We will work through these in the order they were mentioned except for the forest far beyond opening as that is accounted for.

5.1 Workers Revolt

As a reminder, the workers revolt is an event where the elves believe they are being worked too hard, so they demand a day off, if you give them this day off for day 7, they will not revolt, however if you don't give them a day off there is a 1/3 chance that they will revolt for days 8-9.

The potential losses are as follows:

1. $E(D_7)$, p = 1
2. $E(D_8) + E(D_9)$, p = $\frac{1}{3}$

So what you will lose will be one of the following:

$$\begin{array}{c} 1 \\ \hline -E(D_7) \end{array} \text{ OR } \begin{array}{c|c} \frac{2}{3} & \frac{1}{3} \\ \hline 0 & -(E(D_8) + E(D_9)) \end{array}$$

So for option 1. we will have an expected value of $-E(D_7)$ and a variance of 0.
And for option 2. we will have an expected value of $-\frac{1}{3}(E(D_8) + E(D_9))$ and a variance given by $E(X^2) - E(X)^2 = \frac{1}{3}v^2 - (\frac{1}{3}v)^2 = \frac{2}{9}(E(D_8) + E(D_9))^2$. So this is one more thing to consider in the weigh up, if we make the assumption that $E(D_7) \approx E(D_8) \approx E(D_9)$ [TODO make it so it is the same, it should be shouldnt it?], then we can call $E(D_7) = V$, so it therefore becomes these 2 options:

1. E(X) = $-V$, Var(X) = 0
2. E(X) = $-\frac{2}{3}V$, Var(X) = $\frac{8}{9}V^2$

So as each has a different benefit, 1. for lower variance and 2. for higher expected value, this is now something that we need to consider when optimising our function, so we will include this as a parameter in our optimisation which NSGA-II will be able to handle for us.

5.2 Tax Man

The Tax man arrives on day 10 and demands 10% of your current money, you have a choice, give the tax man your money now, or risk having to pay 20% of your money on day 13. If you do not pay the tax man on day 10 then the probability you have to pay on day 13 is $\frac{1}{2}$.

There are 2 options for the tax man, On day 10 you will have the accumulated value from t=0 to t=9, which can be represented as $V_1 = E(D_{C0}) - E(D_{C10})$.
Or on day 13 where you will have $V_2 = E(D_{C0}) - E(D_{C13})$
So the options are:

1. 10% of V_1, p = 1
2. 20% of V_2, p = 0.5

So what you will pay will be one of the following:

$$\begin{array}{c} 1 \\ \hline -0.1V_1 \end{array} \text{ OR } \begin{array}{c|c} 0.5 & 0.5 \\ \hline 0 & -0.2V_2 \end{array}$$

So for option 1. we will have an expected value of $-0.1V_1$ and a variance of 0.
And for option 2. we will have an expected value of $0.5(0) + 0.5(-0.2V_2) = -0.1(V_2)$ and a variance given by $E(X^2) - E(X)^2 = 0.02V_2^2 - 0.01V_2^2 = 0.01V_2^2$. But as the amount you pay as tax on average is the same percentage 10%, and for option 2, the amount of money you have is more, $V_2 = V_1 + E(D_{10}) + E(D_{11}) + E(D_{12})$. (We can say this without the continued notation as the Forest Far Beyond is not open, so these events are independent.)
It is therefore better to take option one and loose less money with no variance than to take the risk of paying more on the 13th as this minimises the variance and maximises the expected value, our 2 objective values.

You should therefore take the 10% tax on day 10.

5.3 Jackpot

This occurs on day 11 where 2 dice are rolled and the teams which guess what the total of the two dice are have a Jackpot of £1000 split between them.
This is seemingly the easiest event as we can see the results from the rolls of 2 dice very easily in a table.

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

And we can turn this into a distribution X.

$P(X = x)$	1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36
x	2	3	4	5	6	7	8	9	10	11	12

From this we can see that the most likely outcome is 7, so we should choose this. This does not take into account the actions of other players which would make it more complex. [TODO]

You should therefore guess 7.

5.4 Buying Elves

On day 21 you get the opportunity to get more elves for £75 each.

We can pose this problem by looking at the expected value and variance of buying an elf vs not buying an elf.

If the expected value of an elf at t=20, i.e. $E(D_{C20})$ is less than 75, then you shouldn't buy any elves, but there is a high chance that it will be above 75. So then it becomes a weigh up of variance vs expected value. If the number of elves at t=20 is given by T.
$T = 12M_{16}M_{17}M_{18}M_{19}$
The value if no elves are bought is $E(TD_{C20})$.
The value if elves are bought is $E((T+n)D_{C20}) - 75n = E(TD_{C20}) + E(nD_{C20}) - 75n$ $\therefore$ the Δ Expected Value is $E(TD_{C20}) + nE(D_{C20}) - 75n - E(TD_{C20}) = (E(D_{C20}) - 75)n$ The variance is no elves are bought is $Var(TD_{C20})$
The variance if elves are bought is given by:

$$\begin{aligned} Var((T+n)D_{C20}) &= Var(TD_{C20} + nD_{C20}) && (54) \\ &= Var(TD_{C20}) + Var(nD_{C20}) + 2Cov(TD_{C20}, nD_{C20}) && (55) \\ &= Var(TD_{C20}) + n^2Var(D_{C20}) + E(TD_{C20}nD_{C20}) - E((TD_{C20})E(nD_{C20}) && (56) \end{aligned}$$

$\therefore$ the Δ Variance is:
$Var(TD_{C20}) + n^2Var(D_{C20}) + E(TD_{C20}nD_{C20}) - E((TD_{C20})E(nD_{C20}) - Var(TD_{C20}) = n^2Var(D_{C20}) + E(TD_{C20}nD_{C20}) - E((TD_{C20})E(nD_{C20})$
This can be simplified further

$$\begin{aligned} \Delta\text{Variance} &= n^2Var(D_{C20}) + E(TD_{C20}nD_{C20}) - E((TD_{C20})E(nD_{C20}) && (57) \\ &= n^2Var(D_{C20}) + nE(TD_{C20}^2) - nE(TD_{C20})E(D_{C20}) && (58) \\ &= n^2Var(D_{C20}) + nE(T)E(D_{C20}^2) - nE(T)E(D_{C20})^2 && (59) \\ &= n^2Var(D_{C20}) + nE(T)Var(D_{C20}) && (60) \\ &= nVar(D_{C20})(n + E(T)) && (61) \end{aligned}$$

$$\begin{aligned} E(T) &= E(12M_{16}M_{17}M_{18}M_{19}) && (62) \\ &= 12E(M_{16})E(M_{17})E(M_{18})E(M_{19}) && (63) \\ &= 12(\frac{3 - S_{16_{FFB}}}{3})(\frac{3 - S_{17_{FFB}}}{3})(\frac{3 - S_{18_{FFB}}}{3})(\frac{3 - S_{19_{FFB}}}{3}) && (64) \\ &= \frac{12}{81}\sum_{t=16}^{19}(3 - S_{t_{FFB}}) && (65) \\ &= \frac{4}{27}\sum_{t=16}^{19}(3 - S_{t_{FFB}}) && (66) \\ &= \frac{4}{27}\sum_{t=16}^{19}(3 - S_{t_{FFB}}) && (67) \end{aligned}$$

So we can now express the Δ Expected Value and Δ Variance for buying n elves at t=20. This means we can vary n as a parameter, the only caveat is that it is hard to add a constraint for maximum amount of elves due to the fact the amount of money you have is dependant on luck, but we can play it safe and say the max amount to spend in the expected value - 2 standard deviations, this will account for 97.5% of all games.

This corresponds to: $75n \leq E(TD_{C20}) - 2\sqrt{Var(TD_{C20})}$
So $n \leq \frac{E(T)E(D_{C20}) - 2\sqrt{Var(T) + Var(D_{C20})}}{75}$
The only thing here we don't know is Var(T) which can be worked out easily

$$
\begin{aligned}
Var(T) &= E((12M_{16}M_{17}M_{18}M_{19})^2) - E(T)^2 &(68)\\
&= E(12^2 M_{16}^2 M_{17}^2 M_{18}^2 M_{19}^2) - E(T)^2 &(69)\\
&= E(12^2 M_{16}^2 M_{17}^2 M_{18}^2 M_{19}^2) - E(T)^2 &(70)\\
&= 144E(M_{16}^2)E(M_{17}^2)E(M_{18}^2)E(M_{19}^2) - E(T)^2 &(71)\\
&= 144(\frac{3 - 2S_{16_{FFB}} + S_{16_{FFB}}^2}{3})(\frac{3 - 2S_{17_{FFB}} + S_{17_{FFB}}^2}{3}) \cdot &(72)\\
&\quad (\frac{3 - 2S_{18_{FFB}} + S_{18_{FFB}}^2}{3})(\frac{3 - 2S_{19_{FFB}} + S_{19_{FFB}}^2}{3}) - E(T)^2 &(73)\\
&= \frac{144}{81}\sum_{t=16}^{19}(3 - 2S_{t_{FFB}} + S_{t_{FFB}}^2) - \frac{4}{27}\sum_{t=16}^{19}(3 - S_{t_{FFB}}) &(74)\\
&= \frac{1}{27}\left(48\sum_{t=16}^{19}(3 - 2S_{t_{FFB}} + S_{t_{FFB}}^2) - 4\sum_{t=16}^{19}(3 - S_{t_{FFB}})\right) &(75)\\
&= \frac{1}{27}\left(\sum_{t=16}^{19}(144 - 96S_{t_{FFB}} + 48S_{t_{FFB}}^2) - \sum_{t=16}^{19}(12 - 4S_{t_{FFB}})\right) &(76)\\
&= \frac{1}{27}\sum_{t=16}^{19}(132 - 92S_{t_{FFB}} + 48S_{t_{FFB}}^2) &(77)\\
&= \frac{4}{27}\sum_{t=16}^{19}(33 - 23S_{t_{FFB}} + 12S_{t_{FFB}}^2) &(78)\\
& &(79)
\end{aligned}
$$

So now we have this, we can include this in our optimisation on the strategy. Where NSGA-II can figure out the best number of elves to buy.

6 Coalition Of Events and Objective Functions

We have 2 more things we now must consider in our optimisation of strategy, whether or not to give the workers a day off on day 7 for the Workers Revolt (5.1). And how many elves to buy on day 21 (5.4). Tax max (5.2) and Jackpot (5.3) events are solved in their respective sections.

The parameter for Workers Revolt is a binary value, it can be 1 or 0 which is easy to implement in NSGA-II. When we do this we get the following Pareto front: [TODO]

We can now begin to think about winning, if we assume each of these points is normally distributed which is an ok approximation as we can see in Figure 4. Then we can try find a winning score for someone else playing the game and then finding the distribution with the highest percentage chance of winning.

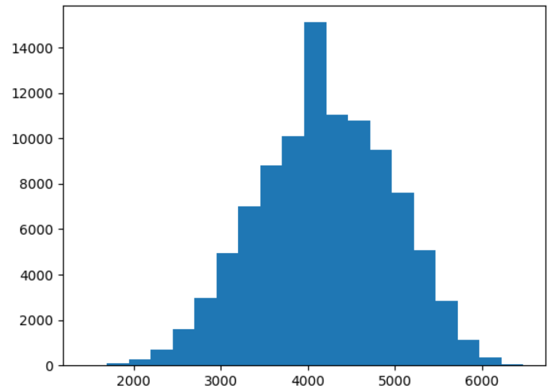

Figure 4: Simulation Distribution

If the distribution R is normally distributed. and W is the winning value.

$$R \sim \mathcal{N}(\mu, \sigma^2)$$

Then $P(R > W) = 1 - \Phi(W)$ So for each of the points calculate this value and then find the argmax. This is the optimal strategy.

7 Conclusion

All code developed can be accessed on GitHub.[1]

References

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[1] https://github.com/AragornOfKebroyd/elf-game