

Linear Algebra 1

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1 Linear equations

1.1 Basics

A linear equation is any equation of form: $a_1x_1 + \dots + a_nx_n = b$ where the a are 'scalars' that belong to a field and the x belong to the vector set.

A *system of linear equations* is simply a collection of linear equations. The solution of this system, if there is any, is an ordered list (s_1, \dots, s_n) where each s_i is the value of each x_i .

A system consisting of 2 unknowns and two linear equations is generally the intersection of two lines on a cartesian plane. Note that the lines may be parallel or even colinear.

In general, a system will have:

- no solution
- unique solution
- infinite solutions

We may choose to represent a system of linear equations as an *augmented matrix*. A matrix is called $n \times m$ if it is of form:

$$\begin{bmatrix} a_{11} & a_{12} & \dots \\ a_{21} & a_{22} & \dots \end{bmatrix}$$

1.2 Solving a linear system

1.2.1 Basics

When solving a system, our goal is to replace each linear equation with an equivalent set (one that has the same solution) and obtain single linear equations which are trivial to solve.

In solving a system, we use the elementary row operations which are:

- Interchange two rows (or columns).
- Multiply each element in a row (or column) by a non-zero number.
- Multiply a row (or column) by a non-zero number and add the result to another row (or column).

Our goal is to transform our matrix into echelon or row reduced echelon form. A matrix is in echelon form if it looks like this:

$$\begin{bmatrix} \blacktriangledown & \blacktriangle & \blacktriangle \\ 0 & \blacktriangledown & \blacktriangle \\ 0 & 0 & \blacktriangledown \end{bmatrix}$$

Here, each ∇ and \blacktriangle may take on any value from the set the vector space is defined on.

To obtain this form, we first arrange our matrix into a form where the row with the least amount of trailing zeros is placed uptop. Then we ensure the *pivot position* (meaning first non-zero entry) has only 0 in its own column. When done, we move on the second row, find the pivot position and repeat. We repeat this process for all rows.

If we have a system where the number of unknowns exceeds the number of equations, we obtain a parametric solution. Consider this:

$$\begin{bmatrix} 1 & 0 & -5 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Which means:

$$x_1 - 5x_3 = 1$$

$$x_2 + x_3 = 4$$

Now, we can express both x_1 and x_2 in terms of x_3 . We call x_1 and x_2 basic variables and x_3 the free variable. We call x_3 a free variable as we are free to choose any value for it.

Most importantly, these are the conditions for row echelon and reduced row echelon form.

Row echelon form if:

- all nonzero rows are above zero rows
- a leading entry is the right column of a leading entry above it
- all entries in a column below a leading entry are 0

Reduced row echelon also if:

- leading entry in each nonzero row is 1
- the entries in the column of the leading entry are 0

To obtain reduced row echelon form, starting from the lowest row, the terms in the column with the leading term are made zero and the leading term scaled to 1. We repeat this process.

As a general remark, we obtain the following.

Theorem 1.2.1. *A system is consistent iff, the rightmost column has no pivot. More simply, iff the lowest row is not of the form: $[0 \quad 0 \dots \quad b]$ where b is nonzero.*

A common confusion when solving a linear system is why the Gaussian algorithm works? To help answer this, the idea is to visualize a Cartesian plane with two random lines intersecting. Now, there can be multiple lines that intersect at the same point. But the intersection of something like $x = 2$ and $y = 5$ is obviously $(2, 5)$ and requires less effort to solve. Thus, we try to replace our equations with those that are easier to solve and have the same solution set.

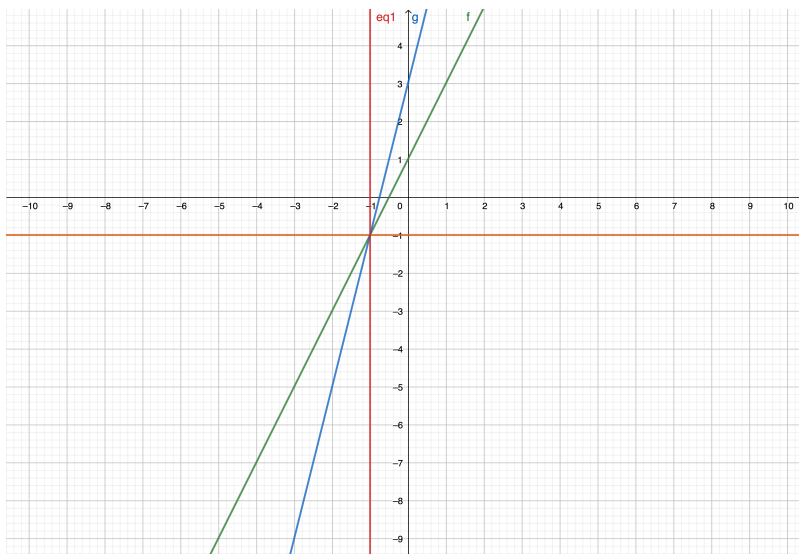


Figure 1: Equivalent systems

1.2.2 Reducing a system to reduced row echelon form

$$\text{Given } \begin{pmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{pmatrix}$$

$$\text{We first realign to get } \begin{pmatrix} 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{pmatrix}$$

After applying elementary row operations, we get the reduced row echelon form of

$$\begin{pmatrix} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{pmatrix}$$

Now anything which is not a pivot column is called a free variable.

$$x_5 = 4$$

$$x_2 = 2x_3 - 2x_4 - 7$$

$$x_1 = 2x_3 + 2x_4 - 24$$

1.2.3 High school basics

When solving systems of 2 unknowns or 3 unknowns, we simply are considering the system of lines or planes. Sticking to 3 dimensions, let's quickly recall how we obtain the formula's of a line and plane in 3-d.

A line in 3-d may be described in Vector form as:

$$l[1] : \begin{bmatrix} A \\ B \\ C \end{bmatrix} + \lambda \begin{bmatrix} D \\ E \\ F \end{bmatrix} + \psi \begin{bmatrix} G \\ H \\ I \end{bmatrix}$$

or may be simplified to cartesian form as:

$$k = \frac{x + A}{B} = \dots$$

Now, the equation of a plane is most commonly written in the form:

$$Ax + By + Cz \text{ where } (A,B,C) \text{ is the normal vector to the plane.}$$

2 Introducing vector spaces

2.1 Basics

A vector space is an algebraic structure consisting of two sets, a scalar set and the vector set. The scalar set happens to be field and the vector set an additive abelian group. We also define a 'scalar multiplication' between the scalars and vectors to satisfy the following:

- $r_1 r_2 (v_1) = (r_1 r_2) v_1$ **associativity of scalar multiplication**
- $r_1 (v_1 + v_2) = r_1 v_1 + r_1 v_2$ and the opposite as well, thus multiplication distributes over addition.
- $1_r v_1 = v_1$

Property 1. *From the above properties, we can neatly deduce also that $0_r v_1 = 0$*

$$\begin{aligned} 0_r v_1 &= (1 - 1) v_1 \\ 1 v_1 - 1 v_1 &= 0 \end{aligned}$$

The most common vector space is that of \mathbb{R}^n . It's vector and scalar set are the same.

2.2 Linear combinations, span and matrix theorems

A **linear combination** of vectors is simply a sum of vectors multiplied by scalars.

$$v_i = a_1 v_1 + \dots + a_k v_k$$

A **span** for given vector set $\{v_1 \dots v_k\}$ is simply a set that consists of all the possible outputs of $\sum_{i=1}^k a_i v_i$

A common idea in linear algebra is to view a linear combination as the product of two matrices. We can't help but notice that:

$$a_1 v_1 + \dots + a_k v_k = [v_1 \dots v_k] \begin{bmatrix} a_1 \\ \vdots \\ a_k \end{bmatrix}$$

An important theorem that follows from the fact that $Ax = b$ is a set of equivalent statements (meaning they always have the same truth value.)

Theorem 2.2.1. *The following are equivalent:*

1. $\forall b \in \mathbb{R}^m, Ax = b$ has a solution
2. $\forall b \in \mathbb{R}^m$ b is a linear combination of the columns of A .
3. columns of A span \mathbb{R}^m
4. A (the coefficient matrix!) has a pivot position in each row.

2.3 Homogeneous system of equations

Any system of linear equations that can be written of the form $Ax = 0$ is called **Homogeneous**. Clearly, this system has at least one solution which is when x is equal to the 0 vector. The non-trivial solution is if x is not equal to 0 in which case the solution is described in terms of a parameter. Now, a theorem that follows is, for a non-homogeneous system $Ax = b$ with solution p , the solution set of $Ax = b$ is the set of vectors of the form $w = p + v_h$ where v_h is any solution to $Ax = 0$.

We now list some useful theorems.

Theorem 2.3.1. *A homogeneous equation $Ax = 0$ has a nontrivial solution iff it has at least one free variable.*

Theorem 2.3.2. *Columns of A are linearly dependent iff $Ax = 0$ has a trivial solution.*

Theorem 2.3.3. *Suppose the equation $Ax = b$ has a solution p . Then the set of possible solutions to $Ax = b$ are of the form $w = p + v_h$ where v_h is a solution of $Ax = 0$. (Ofcourse assuming $Ax = b$ is consistent) That is to say, if we knew two solutions to $Ax = b$, we could find a solution to $Ax = 0$.*

2.4 Linear Independence

A set of vectors is linearly independent if the solution to:

$$a_1v_1 + \dots + a_kv_k = 0$$

is only the trivial solution, that is the $0_v \in \mathbb{R}^k$.

We now present some common statements concerning linear independence.

Theorem 2.4.1. *Whenever $Ax = 0$ has a nontrivial solution, the columns of A are linearly dependent.*

Theorem 2.4.2. *A set of vectors is linearly dependent if at least one of the vectors is a scalar multiple of the other.*

Theorem 2.4.3. *If a set contains the zero vector, then it is linearly dependent.*

Theorem 2.4.4. *If a set of vectors contain more vectors than vector entries, the set is linearly dependent.*

Theorem 2.4.5. *A set of vectors is linearly dependent if the row echelon form of A to $Ax = 0$ has at least one free variable.*

Theorem 2.4.6. *A set of vectors is linearly dependent iff at least one of the vectors can be written as a linear combination of the others.*

Corollary 2.4.1. *Two vectors are linearly dependent iff one is a scalar multiple of the other*

Let's now prove the most useful out of these theorems.

Proof. Let's first prove 2.4.6.

(\rightarrow) Now suppose that for $S = \{v_1, \dots, v_k\}$, $v_1 = a_2v_2 + \dots + a_kv_k$ Then:

$$0 = a_2v_2 + \dots - v_1$$

In which case v_1 has a nonzero coefficient implying a nontrivial solution.

(\leftarrow) Let j represent the vectors that have a nonzero coefficient in $a_1v_1 + \dots + a_jv_j + \dots + a_kv_k$

Now, WLOG, we get for all subscripts less than or equal to j :

$$v_1 = -\frac{a_2}{a_1}v_2 - \dots - \frac{a_k}{a_1}v_k$$

□

2.5 Lec 05 notes

Property 2. A linear T maps linear line segments in \mathbb{R}^n to linear line segments in \mathbb{R}^m

2.5.1 Matrix transformations

In general, the columns of a matrix are the dimensions of the domain set and the rows of the matrix are the codomain.

Theorem 2.5.1. Let T be a square matrix, that is $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$. Then, T is surjective iff it is injective.

Proof. Proving both directions:

(\rightarrow) (T is injective iff T is surjective)

Now whenever T is surjective, we have that there are no rows of 0. It also means that each row has a pivot. Given this, we check the condition that $T(v_1) = T(v_2) \rightarrow v_1 = v_2$. Now we would essentially obtain this in trying to solve it:

$$T \times v = a$$

where $a \in \mathbb{R}^n$. Now because the reduced matrix has no 0 row and as many pivots as the number of variables, we get that there is only one such v resulting in $v_1 = v_2$

(\leftarrow) (T is surjective iff T is injective) From T is injective, we get that there is only one solution to $T \times v = a$ and considering the homogeneous $Tv = 0$ we get that only 0 would be the solution implying columns of T are a basis hence must span \mathbb{R}^n

□

2.6 Linear transformations

Definition 1. A linear map is function between $\mathbb{R}^n \rightarrow \mathbb{R}^m$ such that:

- i) $cT(v) = T(cv) \forall c \in \mathbb{R}, \forall v \in \mathbb{R}^n$
- ii) $T(v + u) = T(v) + T(u) \forall v, u \in \mathbb{R}^n$

A condition equivalent to a linear map is:

[Proposition] 1. Whenever we can show that for some T , $T(cv_1 + dv_2) = cT(v_1) + dT(v_2)$, we have that T is a linear map.

Proof. (We must show both conditions i) and ii) defined above)

Now showing i) is simple. Simply fix $d = 0$. Then we get that $T(cv_1 + 0) = cT(v_1)$.

To show ii), fix c, d to be 1 and there we have it!

□

Theorem 2.6.1. For a linear transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$, there exists a unique $m \times n$ matrix A such that $T(x) = Ax$

Proof. Now, we can write x as $I_n x$ from which we get $[e_1 \dots e_n] x = x_1 e_1 + \dots + x_n e_n$. Taking $T(x)$ we obtain $x_1 T(e_1) + \dots + x_n T(e_n)$ which is equal to

$$[T(e_1) \dots T(e_n)] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

□

And here are some examples to capturing linear transforms as matrices mainly in \mathbb{R}^2

1. Anti-clockwise rotation by θ degrees. $A = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$
2. $T(x) = 3x$ for $x \in \mathbb{R}^2$, then $A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$
3. Reflection through the y-axis $A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$
4. Contraction or an expansion depending on k $A = \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$
5. Shear $\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$
6. Projection $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

And now we present some theorems relating to surjectivity and injectivity:

Theorem 2.6.2. *Taking $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ we have that T is injective iff the homogeneous $T(x) = 0$ has the trivial solution.*

Proof. (To show this, we show that the two statements always have the same truth value, that they are equivalent)

Now when $T(x)$ is injective, it means that $T(x) = 0$ can have only a unique solution by definition of injectivity. Now when $T(x)$ is not injective, it means that $\exists b$ s.t. $T(u) = b$ and $T(v) = b$ where $u \neq v$. Now $T(u) - T(v) = b - b = 0 = T(u - v)$ and since $u \neq v$ we have that 0 is not the only solution. \square

3 Matrix algebra

Property 3. *Properties of the matrix transpose:*

$$\begin{aligned} (A^T)^T &= A \\ (A + B)^T &= A^T + B^T \\ (rA)^T &= rA^T \\ (AB)^T &= B^T A^T \end{aligned}$$

Property 4.

$$\begin{aligned} A + B &= B + A \\ (A + B) + C &= A + (B + C) \\ A + 0 &= A \\ r(A + B) &= rA + rB \\ (r + s)A &= rA + sA \\ rs(A) &= r(sA) \end{aligned}$$

And here is the general formula for the $i - j^{th}$ entry of AB whenever defined:

$$(AB)_{ij} = a_{i1}b_{1j} + \dots + a_{in}b_{nj}$$

We note that Matrices whose entries belong to \mathbb{R} form a vector space.

And now some warnings about matrix multiplication. We note that in general if $AB = AC$ then it does not hold that $B = C$. And if $AB = 0$ we can not conclude that A or B is 0. We also note that whenever a matrix has entries only along its main diagonal, then to take any power, we simply take the power of each diagonal entry.

A transpose of a matrix is defined as swapping the rows of A with its columns.

Theorem 3.0.1. *A matrix A is invertible if $\exists C$ s.t. $AC = I, CA = I$ meaning that the inverse is unique. Non-invertible matrices are called **singular**. And note that only square matrices are invertible. The reason for this is that for a matrix to be invertible, it must be bijective which can only be the case if it is square*

Proof. (Proving that the inverse matrix is unique) Suppose $\exists B, C$ s.t. $AB = I, AC = I$. We want to show that $B = C$.

$$B = BI = BAC = IC = C$$

□

3.1 Computing inverses

The way we find inverses is best done by Gaussian elimination. We are simply looking for a solution to the equation $Ax = I$ hence we write this as an augmented matrix $[A|I]$ and solve.

Theorem 3.1.1. *Elementary matrices are obtained by only performing one elementary row operation on an identity matrix, thus all elementary matrices are invertible.*

Theorem 3.1.2. *Because inverses are only defined on bijective mappings, whenever a square matrix is invertible, we have that $Ax = b$ has a unique solution $x = A^{-1}b$*