# Macroeconomics 1 First assignment

Dario Alfredo De Falco

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## 1 Exercise 1 - RCK

*Consider an RCK economy with constant intertemporal elasticity of substitution, as described in class, which is at its Balanced Growth Path.*

### 1.1 Point A: transfer scheme

***a)*** *Assume that the government runs a transfer scheme (with a balanced budget).($\tau$). According to the scheme, in each instant in time, the representative household receives a lump-sum transfer which is financed by a constant tax rate on returns to capital. The after tax real interest rate thus becomes*

$$r(t) = (1 - \tau)[f'(\hat{k}(t)) - \delta]$$

*How, if at all, does this scheme change the household's asset per capita accumulation equation, relative to the no-government case?*
In the absence of the intervention of the government we have that the household's asset per capita accumulation equation is:

$$\dot{a} = w + [f'(\hat{k}_t) - \delta]a - c - na$$

With the government intervention we can rewrite said equation as:

$$\begin{aligned}\dot{a} &= w + (1 - \tau)[f'(\hat{k}(t)) - \delta]a + \tau[f'(\hat{k}(t)) - \delta]a - c - na \\ &= w + [f'(\hat{k}(t)) - \delta]a - \tau a[f'(\hat{k}(t)) - \delta] + \tau[f'(\hat{k}(t)) - \delta]a - c - na \\ &= w + [f'(\hat{k}(t)) - \delta]a - c - na\end{aligned}$$

Where the fact that the economy is closed, implying $k = a$ was exploited, and it is assumed that the government lump sum transfer is equal to $G = \tau[f'(\hat{k}(t)) - \delta]k$, one could furthermore rewrite $\dot{k} = \dot{a}$. Note also that loans are generated in the model only when an other agent, different from the one that lends, has a negative value for assets and is thus in debt; in the representative agent framework all agents make the same choices, therefore if that were to happen this would mean that the whole economy is in debt to someone else; which can't happen because the economy is closed. If one were to assume that $k = a$ is false then one could assume that $a = k + l$ where l denotes loans, and come up with this equation:

$$\dot{a} = w + (1 - \tau)[f'(\hat{k}(t)) - \delta](k + l) + \tau[f'(\hat{k}(t)) - \delta]k - c - na$$

Where the variation with respect to the original equation can be seen as the difference between this last one and the first asset accumulation equation introduced and is as follows:

$$\begin{aligned}\Delta a &= \tau[f'(\hat{k}(t)) - \delta](k - a) \\ &= \tau[f'(\hat{k}(t)) - \delta](k - k - l) \\ &= \tau[f'(\hat{k}(t)) - \delta](-l)\end{aligned}$$

The previous equality implies that the tax would have a negative effect on the per capita asset accumulation through the lowering of the interest rate on loans due to the no arbitrage condition, thus making households that hold loans poorer. The resulting dead-weight loss is due to the fact that the government doesn't compensate the households since the tax is only on capital, if the tax on capital was also on the loans, the transfer would augment and no effect would occur. This line of reasoning doesn't apply to our economy for the aforementioned reasons that lead to assume that $k = a$, but it's discussed for transparency.

*-How, if at all, does this scheme change the economy's resource constraint? How, if at all, does this scheme affect the Euler equation?*

Before the scheme the economy's resource constraint is:

$$\dot{\hat{k}} = f(\hat{k}) - \hat{c} - (x + n + \delta)\hat{k}$$

With the government's intervention the resource constraint remains the same. This is because the asset per capita accumulation equation didn't change, alongside all the other equations needed to derive the resource constraint, thus the derivation is the same as in the normal model. Alternatively, one could leave the government transfer implicit in the asset accumulation equation and only simplify at the end, this procedure takes longer and yields the same result. Intuitively, the resource constraint doesn't change due to the fact that the negative impact of the tax is balanced by an equal positive impact in the form of a lump sum transfer that is added to the resource constraint; since the effects balance out it is immediate to realize that indeed the constraint is unaffected by the policy.

The impact of the government's policy comes from the fact that the Euler equation varies. In particular, households will save less under the policy than without it. This is due to the fact that the real interest rate is effectively lower, meaning households have a lower incentive to save. From another perspective: the relative price of consumption today in terms of future consumption is lower. The Euler equation, both in per capita and in per effective labor terms varies as follows:

$$\frac{\dot{c}}{c} = \frac{1}{\theta}(r - \rho) \tag{Euler before}$$

$$\frac{\dot{c}}{c} = \frac{1}{\theta}\{(1 - \tau)[f'(\hat{k}(t)) - \delta] - \rho\} \tag{Euler after}$$

$$\frac{\dot{\hat{c}}}{\hat{c}} = \frac{1}{\theta}(f'(\hat{k})(\hat{k}_t - \delta) - \rho - \theta x) \tag{Effective labor Euler Before}$$

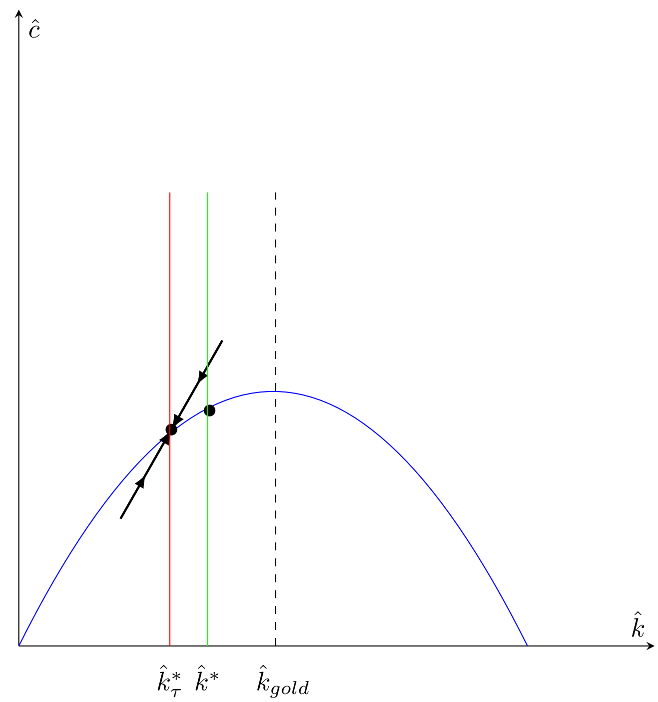
$$\frac{\dot{\hat{c}}}{\hat{c}} = \frac{1}{\theta}((1 - \tau)(\hat{k}_t - \delta) - \rho - \theta x) \tag{Effective labor Euler after}$$

Where the first and the third equations are the Euler equations before the policy, whilst the second and the fourth are immediately obtained by substituting the new returns to capital that the households face.

*- Set up the phase diagram and represent the BGP in the presence of the scheme. Do you think that the same policy would affect the steady-state level of $\hat{k}(t)$ in a Solow economy? Explain.*

In a Solow economy such policy wouldn't have had any effect as the savings rate is assumed to be exogenous. Indeed, it is precisely by varying the economic incentives to save faced by the households that this policy reaches its effect. To have an effect in the Solow model, the policy should change the resource constraint of the economy, that as previously discussed isn't affected by the scheme.

The Phase diagram for the RCK economy is represented in the graph sketched below. As can be clearly seen from the graph, the lower amount of savings results in a steady state that has lower consumption and capital in effective labor terms than before the policy scheme was introduced by the government.



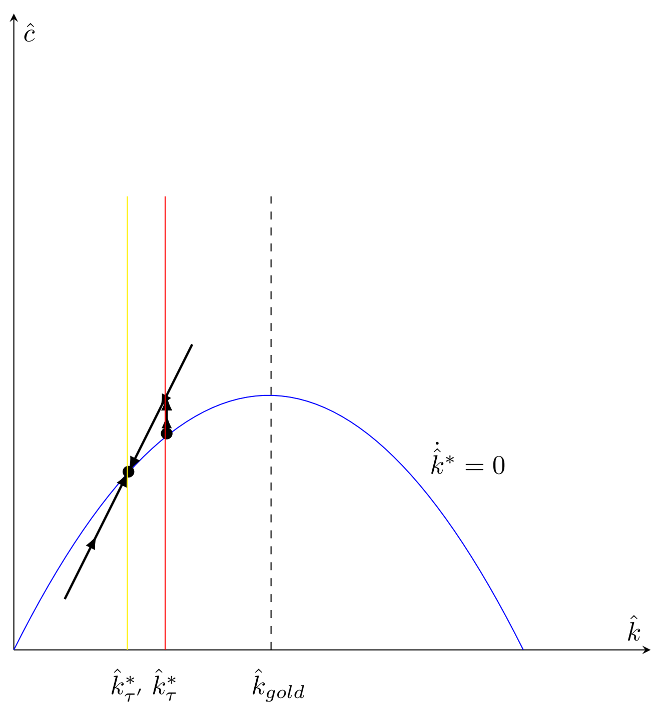

## 1.2 Point B: tax rate

*The government, that always assured it would never raise taxes, completely unexpectedly, raises the tax rate on returns to capital at time $T$ such that $\tau > \tau$ for $t \geq T$. How is the RCK economy affected by this? Describe the dynamics of the economy after the change. Does the intervention make the population better off?*

Assuming that also the lump sum transfer augments, since the governments runs a balanced budget (spending all that it collects), the effect of this augmented taxation is similar to the effect of the introduction of the tax. This implies that savings further reduce and the consumption per capita is at an even lower level at the steady state, whilst the resource constraint and the capital accumulation equation don't vary.

The effects of this second change in the taxation are summarized in the graph below. Due to the fact that the change is unexpected, there is a need for a sudden adjustment of the economy, this , alongside the fact that consumption on the BGP will reach an even lower level than with the previous taxation, will make the households worse off than before. In other words, since the equilibrium consumption and the equilibrium capital are at a lower level in the new equilibrium, the utility level reached by the households shall be lower, thus making the population worse off. Furthermore, one can see that if the population were to be better off from this change due to the fact that consumption temporarily increases and the parameters of the model are such that the households were over-saving, then a contradiction would arise. Indeed, it would imply that households weren't at the BGP before the tax or weren't appropriately maximizing. To see this more clearly, one only needs to notice that given the parameters of the models, and the fact that households could be better off by reducing savings, the household would have simply chosen the level of consumption imposed by the tax before the tax was introduced or a level of consumption in between the two steady states (e.g as their optimal pre tax consumption per effective labor). Thus, it is impossible that the tax makes the population better off unless it is assumed that the population was over-saving, which in turns means households weren't optimizing correctly, which is a contradiction since they were on the BGP obtained through optimization. Note that in the previos discussion sometimes "per effective labor term" is not added only to make the discussion less cumbersome.

Pertaining the transitional dynamics, what happens is that in the moment the tax is raised consumption jumps temporarily upwards to reach the new stable arm of the saddle, afterwards consumption declines until the steady state after the second tax increase is reached. Note that even a social planner wishing to maximize the utility of the agents would choose to bring the economy on the stable arm of the $k^*$ steady state, indeed, such a level of steady state capital is Pareto optimal given the preferences of the individuals and all the other assumptions of the model, therefore this taxation scheme is not pareto optimal. We will derive this result formally in the next point.




## 1.3 Point C: transfer scheme

*Consider a RCK closed economy where consumption and production choices are dictated by a social planner who holds the same preferences as the representative household and is constrained by the economy's resource*

*constraint. Show that the social planner solution is equivalent to the competitive decentralized equilibrium.*
The social maximizer faces a problem similar to the household, the main difference is that the planner faces the resource constraint as a constraint on his maximization problem, mathematically:

$$max \int_0^\infty e^{t(n-\rho)} \frac{\hat{c}^{(1-\theta)} e^{xt(1-\theta)} - 1}{1-\theta}$$
$$s.\ to\ \dot{\hat{k}} = f(\hat{k}) - \hat{c} - (n+x+\delta)\hat{k}$$

Where $\hat{c} = ce^{-xt}$, and the utility function is: $u(\hat{c}) = \frac{\hat{c}^{1-\theta} e^{x(1-\theta)t} - 1}{1-\theta}$.

$$H \equiv e^{t(n-\rho)} \frac{\hat{c}^{1-\theta} e^{tx(1-\theta)} - 1}{1-\theta} + \lambda[f(\hat{k}) - \hat{c} - (n+x+\delta)\hat{k}]$$

Where the trasversality condition is defined as:

$$\lim_{t\to+\infty} \hat{k} \cdot exp[-\int_0^t f'(\hat{k}(s) - \delta - x - nds] = 0$$

The FOC are as follows:

$$\frac{\partial H}{\partial \hat{c}} = 0 \Rightarrow e^{[n-\rho+x(1-\theta)]t}\ \hat{c}^{-\theta} = \lambda$$
$$\frac{\partial H}{\partial \hat{k}} = 0 \Rightarrow \lambda[f'(\hat{k}) - (n+x+\delta)] + \dot{\lambda} = 0$$

From the first condition we obtain:

$$\dot{\lambda} = (n-\rho+x-x\theta)e^{(n-\rho+x-x\theta)t}\hat{c}^{-\theta} + e^{(n-\rho+x-x\theta)t}(-\theta)\hat{c}^{-1-\theta}\dot{\hat{c}}$$

Therefore we have the following relation:

$$\frac{\dot{\lambda}}{\lambda} = (n-\rho+x-x\theta) + (-\theta)\hat{c}^{-1}\dot{\hat{c}} = (n+x+\delta) - f'(\hat{k})$$
$$\frac{\dot{\hat{c}}}{\hat{c}} = \frac{1}{\theta}[f'(\hat{k}) - \delta - \rho - x\theta]$$

Second order conditions are satisfied since the Hamiltonian is concave in both $\hat{c}$ and $\hat{k}$ , thus the Mangasarian sufficiency conditions are satisfied and the homonymous theorem applies. Since we have the same Euler equation and resource constraint in per effective labor terms as with the decentralized equilibrium, the problem is described by the same phase diagram and admits the same solutions, thus proving that the social planner problem is solution-wise equivalent to the one resulting from the decentralized equilibrium.

## 2 Exercise 2- Testing the Solow Model

All the regressions reported in this part are obtained using data from 2017. The reason is that this is the year containing the greatest number of countries without missing values in the variables of interest, thus allowing us to obtain the biggest effective sample. Replicating what is done in the paper by the authors, but using different notation I estimated those two augmented Solow models in order to test the predictions of the augmented solow model with a Cobb Douglas production function

$$ln(y_i) = \beta_0 + \beta_1 ln(sk_i) + \beta_2 ln(sh_i) + \beta_3 ln(n_i + g + \delta_i) \quad \text{(Augmented Solow SH)}$$
$$ln(y_i) = \beta_0 + \beta_1 ln(sk_i) + \beta_2 ln(h_i) + \beta_3 ln(n_i + g + \delta_i) \quad \text{(Augmented Solow HC)}$$

In these two specifications, y represents the output per capita, sk is the fraction of income invested in physical capital, sh is the fraction of income invested in human capital, h is the human capital in level, n is the growth rate of the population, g is the growth rate of technological progress, $\delta$ is the depreciation rate of both forms of capital. Given the fact that human capital investment is unobserved a proxy is used, which in the first regression is computed as the enrollment rate of population aged 15-19. In the second regression a human capital index is used as a proxy for human capital investment. The values for $\delta, n, sk, sh$ are all computed as historical averages on the whole sample prior to 2017, this is done in order to avoid that outlier values of these variables distort the inference. Furthermore, it seems implausible that the current value of these variables determines the level of output per capita. Rather, we assume that coherently with the model, a country must save consistently through time to reach a given steady state level, and thus historical averages differences can explain the gap in gdp per capita levels observed. Pertaining g, the growth of technology, it is set equal to a constant and is 0.02, the reason is that in the Solow framework technology is assumed to be a public good, thus in this approach technological differences are possible in starting levels, (e.g different institutions) but the accumulation of knowledge is seen as following the same process for all countries, since technology is non rival by definition and once a new technology is discoverer it is assumed that said technology immediately spreads and it is impossible to exclude others from accessing it. This is a source of problems for the empirical validity of the model. First, the model doesn't endogenize the main driver of long term growth predicted by the model itself. Second, this assumption doesn't hold well to the real world test since we observe plenty of patents, different growth rates in technology and exclusion from knowledge.

Before looking at the results it is important to notice that since the models are estimated through OLS, the results are valid only given some assumptions. The first assumption is that the models are not misspecified and thus all the regressors are exogenous and there is no omitted variable bias. The second assumption is that there is no reverse causality, whilst the third is technical and pertains the absence of multicollinearity. The fourth assumption is that the asymptotic properties of the estimators are valid, which could be reasonably questioned.

The results of the regressions are reported in "Table 1". For the first model, all the coefficients are statistically significant and have the expected sign which is positive for all coefficients but the one representing total depreciation which is negative as expected. The $R^2$ amounts to 0.5191 and implies that this model is capable of explaining a significant portion of the variability between countries, although much is left unexplained. These results are consistent with the predictions of the augmented Solow model, since countries that have invested more in human and physical capital over time benefit from a higher level of per capita output, whilst the effect of the growth of population and the depreciation rate is negative.

In the second model only the coefficient for human capital is statistically significant, remarkably the model has an even higher $R^2$ of 0.6548, however this specification is shaky on theoretical grounds and doesn't confirm the predictions of the Solow model. The positive, large and significant effect of human capital is coherent with the Solow framework. Due to the non significance of two of the three coefficients the model is not used for further analysis since it wouldn't allow us to get implied $\alpha$ and $\beta$ as factor shares of income.

The models presented thus fare have some limitations, there might be problems related to externalities of capital accumulation of any kind that influence the main omitted variable, that is, technology, thus generating a problem of endogenous regressor. A similar problem pertains the relation between $s_h$ and h itself. As it's pointed out by the authors it is reasonable to think that countries with higher level of human capital also invest in it more. From this fact follows the possibility of another source of regressor's endogeneity. Furthermore, more omitted variables might be present and the functional specification of the model might be wrong. To conclude the discussions of the model's limitation, it is worth noticing that reverse causality cannot be a priori excluded.

To conclude this section a restricted regression is run in order to estimate the values of $\alpha$ and $\beta$ implied by the augmented Solow "SH" model. The values we are referring to are those of the underlying Cobb-Douglas production function defined as follows:

$$Y = K^{\alpha} H^{\beta} (AL)^{1-\alpha-\beta}$$

The restricted regression is specified this way:

$$lny = \delta_0 + \delta_1[ln(s_k) - ln(n+g+\delta)] + \delta_2[ln(sh) - ln(n+g+\delta)]$$

From the paper, we can see that in order to get the implied values for $\beta$ and $\alpha$ from the regression we need to solve the following system:

$$\begin{cases} \delta_1 = \frac{\alpha}{1-\alpha-\beta} \\ \delta_2 = \frac{\beta}{1-\alpha-\beta} \end{cases}$$

By running the aforementioned partial regression and then solving the system one can obtained the implied values of $\beta$ and $\alpha$, regression results are omitted for brevity. This procedure yields an implied $\alpha = 0.18$ and an implied $\beta = 0.47$. In the paper authors argue that the theoretical prediction of the model are of $\alpha \approx 0.33$ and of $\beta \in [\frac{1}{3}; \frac{1}{2}]$, thus only the factor share of income that goes to human capital is in line with the theoretical prediction, whilst the one pertaining to physical capital is implied to be smaller than the one theory predicts.

To conclude, the most recent data for the estimation of the Augmented Solow model with the proxy for human capital savings seem to confirm the results found by Mankiw, et al (1992). However, as previously exposed this model suffers from significant problems from an econometric perspective, thus a more thorough analysis with more sophisticated models should be pursued.

Table 1

| | Augmented Solow | Augmented Solow 2: |
|---|---|---|
| ln_sk | 0.721*** | 0.406* |
| | (0.198) | (0.236) |
| ln_sh | 1.057*** | |
| | (0.138) | |
| ln_td | -1.168*** | 0.0102 |
| | (0.294) | (0.324) |
| ln_hc | | 3.025*** |
| | | (0.264) |
| _cons | 10.85*** | 7.243*** |
| | (0.971) | (0.731) |
| N | 153 | 137 |