Generalizations of Roth's criteria for solvability of matrix equations

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Semilinear mappings

A mapping $\mathcal{A}$ from a complex vector space U to a complex vector space V is semilinear if

$$\mathcal{A}(u + u') = \mathcal{A}u + \mathcal{A}u', \qquad \mathcal{A}(\alpha u) = \bar{\alpha}\mathcal{A}u$$

for all $u, u' \in U$ and $\alpha \in \mathbb{C}$.

We write

- $\mathcal{A} : U \longrightarrow V$ if $\mathcal{A}$ is a linear mapping, and
- $\mathcal{A} : U \dashrightarrow V$ if $\mathcal{A}$ is a semilinear mapping.

Cycles of linear and semilinear mappings

We give a canonical form of matrices of a cycle of linear and semilinear mappings

$$\begin{tikzcd} V_1 \arrow[r, dash, "\mathcal{A}_1"] \arrow[rrrr, dash, bend right, "\mathcal{A}_t"'] & V_2 \arrow[r, dash, "\mathcal{A}_2"] & \dots \arrow[r, dash, "\mathcal{A}_{t-2}"] & V_{t-1} \arrow[r, dash, "\mathcal{A}_{t-1}"] & V_t \end{tikzcd}$$

in which each line is

- a full arrow $\longrightarrow$, $\longleftarrow$, or
- a dashed arrow $\dashrightarrow$, $\dashleftarrow$.

My talk is based on:

- T. Klimchuk, D. Kovalenko, T. Rybalkina, V.V. Sergeichuk, *Tame systems of linear and semilinear mappings*), Contemp. Math. 658 (2016) 103-114.

- D. Duarte de Oliveira, V. Futorny, T. Klimchuk, D. Kovalenko, V.V. Sergeichuk, *Cycles of linear and semilinear mappings*), Linear Algebra Appl. 438 (2013) 3442-3453.

- D. Duarte de Oliveira, R.A. Horn, T. Klimchuk, V.V. Sergeichuk, *Remarks on the classification of a pair of commuting semilinear operators*, Linear Algebra Appl. 436 (2012) 3362-3372.

Empty matrices

- $\forall\, n = 0, 1, 2, \ldots \;\; \exists!$ matrices of sizes $0 \times n$ and $n \times 0$, which correspond to linear mappings $\mathbb{C}^n \to 0$ and $0 \to \mathbb{C}^n$.
- They are denoted by 0_{0n} and 0_{n0} and are considered as zero matrices
- For every $p \times q$ matrix M_{pq}:

$$M_{pq} \oplus 0_{n0} = \begin{bmatrix} M_{pq} & 0 \\ 0 & 0_{n0} \end{bmatrix} = \begin{bmatrix} M_{pq} & 0_{p0} \\ 0_{nq} & 0_{n0} \end{bmatrix} = \begin{bmatrix} M_{pq} \\ 0_{nq} \end{bmatrix}$$

$$M_{pq} \oplus 0_{0n} = \begin{bmatrix} M_{pq} & 0 \\ 0 & 0_{0n} \end{bmatrix} = \begin{bmatrix} M_{pq} & 0_{pn} \\ 0_{0q} & 0_{0n} \end{bmatrix} = \begin{bmatrix} M_{pq} & 0_{pn} \end{bmatrix}$$

Soooooo baaad without empty matrices

$\forall A \ \exists$ nonsingular R, S: $RAS = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$

Observation

- Each matrix is equivalent to a direct sum of indecomposable matrices of the form

$$[1],\ [1\,0],\ [1\,0\,0],\ \ldots,\ \begin{bmatrix}1\\0\end{bmatrix},\ \begin{bmatrix}1\\0\\0\end{bmatrix},\ \ldots$$

- This direct sum is not uniquely determined:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \left[\begin{array}{cc|c} 1 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right] = [1\,0] \oplus [0]$$

$$= \left[\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right] = [1] \oplus [0\,0]$$

Gooood with empty matrices

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = [1\,0] \oplus [0] = [1] \oplus 0_{01} \oplus 0_{01} \oplus 0_{10}$$

$$= [1] \oplus [0\,0] = [1] \oplus 0_{01} \oplus 0_{01} \oplus 0_{10}$$

A la Jordan Theorem

- *Each matrix is equivalent to a direct sum of indecomposable matrices of the form* $[1]$, 0_{01}, 0_{10}.
- *This direct sum is* uniquely determined, *up to permutation of summands.*

Matrix pairs

First consider a very special case of cycles of linear and semilinear mappings: pairs of linear mappings

$$V_1 \underset{\mathcal{B}}{\overset{\mathcal{A}}{\rightrightarrows}} V_2$$

Their matrices are reduced by equivalence transformations:

pairs of $m \times n$ matrices (A, B) and (C, D) are
equivalent, $(A, B) \sim (C, D)$,
if $\exists$ nonsingular $R, S : (RAS,\ RBS) = (C,\ D)$.

The direct sum of matrix pairs:

$$(A, B) \oplus (C, D) = \left(\begin{bmatrix} A & 0 \\ 0 & C \end{bmatrix}, \begin{bmatrix} B & 0 \\ 0 & D \end{bmatrix} \right).$$

Kronecker's theorem

Write

$$R_n := \begin{bmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ 0 & & 0 & 1 \end{bmatrix}, \quad L_n := \begin{bmatrix} 1 & 0 & & 0 \\ & \ddots & \ddots & \\ 0 & & 1 & 0 \end{bmatrix}, \quad n \geqslant 1.$$

$$(R_1, L_1) = (0_{01}, 0_{01}).$$

Theorem (Kronecker, 1890)

Each pair of matrices of the same size is equivalent to a direct sum, determined uniquely up to permutation of summands, of pairs of the following form:

$$(I_n, J_n(\lambda)), \quad (J_n(0), I_n), \quad (R_n, L_n), \quad (R_n^T, L_n^T).$$

Isomorphism of cycles

$\mathcal{A}$ and $\mathcal{B}$ are isomorphic if there exist linear bijections

$$\varphi_1 : V_1 \to W_1, \ \ldots, \ \varphi_t : V_t \to W_t$$

that transform $\mathcal{A}$ to $\mathcal{B}$:

$$\begin{tikzcd}
V_1 \arrow[r, dash, "\mathcal{A}_1"] \arrow[rrr, dash, bend right, "\mathcal{A}_t"] \arrow[d, "\varphi_1"] & V_2 \arrow[r, dash, "\mathcal{A}_2"] \arrow[d, "\varphi_2"] & \cdots \arrow[r, dash, "\mathcal{A}_{t-1}"] & V_t \arrow[d, "\varphi_t"] \\
W_1 \arrow[r, dash, "\mathcal{B}_1"] \arrow[rrr, dash, bend right, "\mathcal{B}_t"] & W_2 \arrow[r, dash, "\mathcal{B}_2"] & \cdots \arrow[r, dash, "\mathcal{B}_{t-1}"] & W_t
\end{tikzcd}$$

The direct sum of cycles

$\mathcal{A} \oplus \mathcal{B}$:

$$\begin{tikzcd}
V_1 \oplus W_1 \arrow[r, dash, "\mathcal{A}_1 \oplus \mathcal{B}_1"] \arrow[rrrr, dash, bend right, "\mathcal{A}_t \oplus \mathcal{B}_t"'] & V_2 \oplus W_2 \arrow[r, dash, "\mathcal{A}_2 \oplus \mathcal{B}_2"] & \dots \arrow[r, dash, "\mathcal{A}_{t-1} \oplus \mathcal{B}_{t-1}"] & V_t \oplus W_t
\end{tikzcd}$$

The direct sum of cycles

$\mathcal{A} \oplus \mathcal{B}:$

$$
\begin{tikzcd}
V_1 \oplus W_1 \arrow[r, dash, "\mathcal{A}_1 \oplus \mathcal{B}_1"] \arrow[rrrr, dash, bend right, "\mathcal{A}_t \oplus \mathcal{B}_t"'] & V_2 \oplus W_2 \arrow[r, dash, "\mathcal{A}_2 \oplus \mathcal{B}_2"] & \ \dots \ \arrow[r, dash, "\mathcal{A}_{t-1} \oplus \mathcal{B}_{t-1}"] & V_t \oplus W_t
\end{tikzcd}
$$

The Krull–Schmidt theorem holds for cycles of linear and semilinear mappings: *each cycle is isomorphic to a direct sum of indecomposables, which is uniquely determined, up to permutation and isomorphisms of direct summands*.

Thus, it suffices do classify all indecomposable cycles.

Let $\mathcal{A}$ be indecomposable into a direct sum and let all $\mathcal{A}_i$ be nonsingular

Then $\mathcal{A}$ is isomorphic to a cycle

$$\mathcal{B}: \quad \begin{tikzcd} V \arrow[r, dash, "\mathbb{1}"] \arrow[rrr, dash, bend right, "\mathcal{B}_t"'] & V \arrow[r, dash, "\mathbb{1}"] & \dots \arrow[r, dash, "\mathbb{1}"] & V \end{tikzcd}$$

Changing the basis in V, we can reduce the matrix of $\mathcal{B}_t$ by

- similarity transformations (i.e., $V \circlearrowleft \mathcal{B}_t$) if the number of dashed arrows in $\mathcal{A}$ is even;
- consimilarity transformations (i.e., $V \circlearrowleft \mathcal{B}_t$) if the number of dashed arrows in $\mathcal{A}$ is odd.

A canonical form for similarity is given by Jordan's theorem.

A canonical form for consimilarity is given in

- Y.P. Hong, R.A. Horn, *A canonical form for matrices under consimilarity*, Linear Algebra Appl. 102 (1988) 143–168.
- R.A. Horn, C.R. Johnson, *Matrix Analysis, 2nd ed.*, Cambridge University Press, New York, 2012.

Each square complex matrix is consimilar to a direct sum, uniquely determined up to permutation of direct summands, of matrices of the following types:

- $J_k(\lambda)$, *in which* $\lambda \geqslant 0$, *and*
- $\begin{bmatrix} 0 & 1 \\ \mu & 0 \end{bmatrix}$, *in which* $\mu \notin \mathbb{R}$ *or* $\mu < 0$.

Let $\mathcal{A}$ be indecomposable into a direct sum and let there exist a singular $\mathcal{A}_i$

Lemma

There are bases of $V_1, \ldots, V_t$ in which the matrix A_i of each $\mathcal{A}_i$ possesses the property: all entries are only 0's and 1's with at most one 1 in each row and each column.

Corollary

- *Each $\mathcal{A}_i$ maps a basis vector to a basis vector or* 0.
- *Each $\mathcal{A}_i$ cannot map distinct basis vectors to the same basis vector:*

$$\begin{tikzcd} e_1 \arrow[dr, "\mathcal{A}_i"] & \\ & f \\ e_2 \arrow[ur, "\mathcal{A}_i"] & \end{tikzcd} \qquad \text{— } \textit{impossible}$$

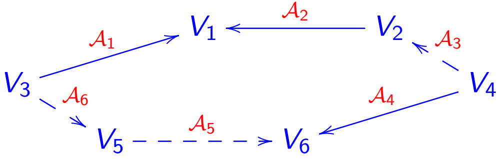
Let there exist a singular $\mathcal{A}_i$

Construct the directed graph,

- its vertices are basis vectors of $V_1, \ldots, V_t$ from Lemma,
- there is an arrow from u to v if and only if $\exists \mathcal{A}_i : u \mapsto v$.

By Lemma, the graph is a DISJOINT UNION OF CHAINS!!!. Since $\mathcal{A}$ is indecomposable, the graph is connected, and so it is a chain.

Consider chains for the cycle



Type I: A chain stops exactly before the starting point; an example

$$
\begin{tikzcd}
 & & e_{21} & & e_{31} & \\
e_{11} \arrow[urr] \arrow[dr, dashed] & & e_{22} & & e_{32} \arrow[ll] & \\
e_{12} \arrow[urr] \arrow[dr, dashed] & e_{61} & & & & e_{41} \arrow[uul, dashed] \arrow[dll] \\
 & e_{62} \arrow[rr, dashed] & & e_{51} & & e_{42} \arrow[uul, dashed] \arrow[dll] \\
 & & & e_{52} & &
\end{tikzcd}
$$

Then

$$
\begin{tikzcd}
 & & \langle e_{21}, e_{22} \rangle & & \langle e_{31}, e_{32} \rangle \arrow[ll, "I_2"'] & \\
\langle e_{11}, e_{12} \rangle \arrow[urr, "I_2"] \arrow[dr, dashed, "I_2"] & & & & & \langle e_{41}, e_{42} \rangle \arrow[ul, dashed, "I_2"'] \arrow[dll, "I_2"'] \\
 & \langle e_{61}, e_{62} \rangle \arrow[rr, dashed, "\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}"] & & \langle e_{51}, e_{52} \rangle & &
\end{tikzcd}
$$

Type I: A chain stops exactly before the starting point; the general case

$$
\begin{tikzcd}
 & \mathbb{C}^n & \mathbb{C}^n \arrow[l, "I_n"'] & \\
\mathbb{C}^n \arrow[ur, "I_n"] \arrow[dr, dashed, "I_n"] & & & \mathbb{C}^n \arrow[ul, dashed, "I_n"'] \arrow[dl, "I_n"'] \\
 & \mathbb{C}^n \arrow[r, dashed, "J_n(0)"] & \mathbb{C}^n &
\end{tikzcd}
$$

The singular Jordan can be over any arrow.

Type II: A chain does not stop exactly before the starting point; an example

The chain

$$
\begin{tikzcd}
 & V_2 & V_3 \arrow[l, "\mathcal{A}_2"'] & \\
V_1 \arrow[ur, "\mathcal{A}_1"] \arrow[dr, dashed, "\mathcal{A}_6"] & & & V_4 \arrow[ul, dashed, "\mathcal{A}_3"'] \arrow[dl, "\mathcal{A}_4"'] \\
 & V_6 \arrow[r, dashed, "\mathcal{A}_5"] & V_5 &
\end{tikzcd}
$$

gives the cycle

$$
\begin{tikzcd}
 & \langle e_{21} \rangle & \langle e_{31}, e_{32} \rangle \arrow[l, "R_2=\begin{bmatrix} 0 & 1 \end{bmatrix}"'] & \\
\langle e_{11} \rangle \arrow[ur, "{[1]}"] \arrow[dr, dashed, "{[1]}"] & & & \langle e_{41}, e_{42} \rangle \arrow[ul, dashed, "\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}"'] \arrow[dl, "\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}"] \\
 & \langle e_{61} \rangle \arrow[r, dashed, "L_2^T=\begin{bmatrix} 1 \\ 0 \end{bmatrix}"] & \langle e_{51}, e_{52} \rangle &
\end{tikzcd}
$$

Type II: A chain does not stop exactly before the starting point; the general case

Two arrows are assigned by M and N; the others by I:

$$
\begin{tikzcd}
 & V & & V \arrow[ll, "M"'] & \\
V \arrow[ur, "I"] \arrow[dr, dashed, "I"] & & & & W \arrow[ul, dashed, "I"'] \arrow[dl, "I"] \\
 & W \arrow[rr, dashed, "N"] & & W &
\end{tikzcd}
$$

$$
(M, N) = \begin{cases} (R_n, L_n) \text{ or } (R_n^T, L_n^T), & \text{if } V \underset{N}{\overset{M}{\rightrightarrows}} W \\ (R_n, L_n^T) \text{ or } (R_n, L_n^T), & \text{if } V \underset{N}{\overset{M}{\rightleftarrows}} W \end{cases},
$$

where

$$
R_n := \begin{bmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ 0 & & 0 & 1 \end{bmatrix}, \quad L_n := \begin{bmatrix} 1 & 0 & & 0 \\ & \ddots & \ddots & \\ 0 & & 1 & 0 \end{bmatrix}
$$

The classification of cycles

Theorem

Each cycle of linear and semilinear mappings is isomorphic to a direct sum, determined uniquely up to isomorphisms of summands, of indecomposable cycles of the following types:

- $$\begin{tikzcd} V \arrow[r, dash, "\mathbb{1}"] \arrow[rrr, dash, bend right, "\mathcal{B}_t"'] & V \arrow[r, dash, "\mathbb{1}"] & \dots \arrow[r, dash, "\mathbb{1}"] & V \end{tikzcd}$$

  in which $\mathcal{B}_t$ is given by a Jordan block or an indecomposable canonical block under consimilarity if the number of dashed arrows is even or odd, respectively.

- *Cycles that are given by chains.*

Known cases: the classification of

- $V \rightleftarrows W$ was given in
  - *N.M. Dobrovol'skaya, V.A. Ponomarev, A pair of counter-operators (in Russian), Uspehi Mat. Nauk 20 (no. 6) (1965) 80–86;*
  - *R.A. Horn, D.I. Merino, Contragredient equivalence: a canonical form and some applications, Linear Algebra Appl. 214 (1995) 43–92.*

- arbitrary cycles of linear mappings is well known in the theory of representations of quivers.

- $V \rightrightarrows W$ was given in
  - *D.Ž. Djoković, Classification of pairs consisting of a linear and a semilinear map, Linear Algebra Appl. 20 (1978) 147–165.*

Future research: regularizing algorithm for cycles of linear and semilinear mappings

Paul Van Dooren in the article

- The computation of Kroneckers canonical form of a singular pencil, Linear Algebra Appl. 27 (1979) 103–140.

gave an algorithm that for each matrix pencil constructs its regularizing decomposition into a direct sum of

- a nonsingular pencil;
- Kronecker's singular indecomposable canonical pencils.

The algorithm uses only unitary transformations, which improves its computational stability.

Future research: regularizing algorithm for cycles of linear and semilinear mappings

Van Dooren's algorithm was extended by Sergeichuk to cycles of linear mappings:

$$\begin{tikzcd} V_1 \arrow[r, dash] \arrow[rrrr, dash, bend right] & V_2 \arrow[r, dash] & \dots \arrow[r, dash] & V_{t-1} \arrow[r, dash] & V_t \end{tikzcd}$$

in which $V_1 \dots V_t$ are complex vector spaces and each line is

- a full arrow $\longrightarrow$ or $\longleftarrow$.

I will give an analogous algorithm and construct a regularizing decomposition for cycles of linear and semilinear mappings:

$$\begin{tikzcd} V_1 \arrow[r, dash] \arrow[rrrr, dash, bend right] & V_2 \arrow[r, dash] & \dots \arrow[r, dash] & V_{t-1} \arrow[r, dash] & V_t \end{tikzcd}$$

in which each line is

- a full arrow $\longrightarrow$, $\longleftarrow$, or

Thank you
for your attention!