Generalizations of Roth's criteria for solvability of matrix equations

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Semilinear mappings

A mapping $\mathcal{A}$ from a complex vector space U to a complex vector space V is semilinear if

$$\mathcal{A}(u + u') = \mathcal{A}u + \mathcal{A}u', \qquad \mathcal{A}(\alpha u) = \bar{\alpha}\mathcal{A}u$$

for all $u, u' \in U$ and $\alpha \in \mathbb{C}$.

We write

- $\mathcal{A} : U \longrightarrow V$ if $\mathcal{A}$ is a linear mapping, and
- $\mathcal{A} : U \dashrightarrow V$ if $\mathcal{A}$ is a semilinear mapping.

Cycles of linear and semilinear mappings

We give a canonical form of matrices of a cycle of linear and semilinear mappings

$$
\begin{tikzcd}
V_1 \arrow[r, dash, "\mathcal{A}_1"] \arrow[rrrrr, dash, bend right, "\mathcal{A}_t"'] & V_2 \arrow[r, dash, "\mathcal{A}_2"] & \dots \arrow[r, dash, "\mathcal{A}_{t-2}"] & V_{t-1} \arrow[r, dash, "\mathcal{A}_{t-1}"] & V_t
\end{tikzcd}
$$

in which each line is

- a full arrow $\longrightarrow$, $\longleftarrow$, or
- a dashed arrow $\dashrightarrow$, $\dashleftarrow$.

My talk is based on:

- T. Klimchuk, D. Kovalenko, T. Rybalkina, V.V. Sergeichuk, *Tame systems of linear and semilinear mappings*), Contemp. Math. 658 (2016) 103-114.

- D. Duarte de Oliveira, V. Futorny, T. Klimchuk, D. Kovalenko, V.V. Sergeichuk, *Cycles of linear and semilinear mappings*), Linear Algebra Appl. 438 (2013) 3442-3453.

- D. Duarte de Oliveira, R.A. Horn, T. Klimchuk, V.V. Sergeichuk, *Remarks on the classification of a pair of commuting semilinear operators*, Linear Algebra Appl. 436 (2012) 3362-3372.

Empty matrices

- $\forall n = 0, 1, 2, \ldots$ $\exists!$ matrices of sizes $0 \times n$ and $n \times 0$, which correspond to linear mappings $\mathbb{C}^n \to 0$ and $0 \to \mathbb{C}^n$.
- They are denoted by 0_{0n} and 0_{n0} and are considered as zero matrices
- For every $p \times q$ matrix M_{pq}:

$$M_{pq} \oplus 0_{n0} = \begin{bmatrix} M_{pq} & 0 \\ 0 & 0_{n0} \end{bmatrix} = \begin{bmatrix} M_{pq} & 0_{p0} \\ 0_{nq} & 0_{n0} \end{bmatrix} = \begin{bmatrix} M_{pq} \\ 0_{nq} \end{bmatrix}$$

$$M_{pq} \oplus 0_{0n} = \begin{bmatrix} M_{pq} & 0 \\ 0 & 0_{0n} \end{bmatrix} = \begin{bmatrix} M_{pq} & 0_{pn} \\ 0_{0q} & 0_{0n} \end{bmatrix} = \begin{bmatrix} M_{pq} & 0_{pn} \end{bmatrix}$$

Soooooo baaad without empty matrices

$\forall A \; \exists$ nonsingular R, S: $RAS = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$

Observation

- Each matrix is equivalent to a direct sum of indecomposable matrices of the form

$$[1],\ [1\,0],\ [1\,0\,0],\ \ldots,\ \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},\ \ldots$$

- This direct sum is not uniquely determined:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \left[\begin{array}{cc|c} 1 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right] = [1\,0] \oplus [0]$$

$$= \left[\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right] = [1] \oplus [0\,0]$$

Gooood with empty matrices

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = [1\,0] \oplus [0] = [1] \oplus 0_{01} \oplus 0_{01} \oplus 0_{10}$$

$$= [1] \oplus [0\,0] = [1] \oplus 0_{01} \oplus 0_{01} \oplus 0_{10}$$

A la Jordan Theorem

- *Each matrix is equivalent to a direct sum of indecomposable matrices of the form* $[1]$, 0_{01}, 0_{10}.
- *This direct sum is* uniquely determined, *up to permutation of summands.*