

Generalizations of Roth's criteria for solvability of matrix equations



Tetiana Klymchuk

Universitat Politècnica de Catalunya

XXV CEDYA / XV CMA

June 26–30, 2017

Semilinear mappings

A mapping \mathcal{A} from a complex vector space U to a complex vector space V is **semilinear** if

$$\mathcal{A}(u + u') = \mathcal{A}u + \mathcal{A}u', \quad \mathcal{A}(\alpha u) = \bar{\alpha}\mathcal{A}u$$

for all $u, u' \in U$ and $\alpha \in \mathbb{C}$.

We write

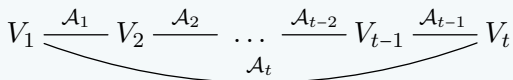
- $\mathcal{A} : U \longrightarrow V$ if \mathcal{A} is a **linear mapping**, and
- $\mathcal{A} : U \dashrightarrow V$ if \mathcal{A} is a **semilinear mapping**.

Cycles of linear and semilinear mappings

We give a canonical form of matrices of a **cycle of linear and semilinear mappings**

$$V_1 \xrightarrow{\mathcal{A}_1} V_2 \xrightarrow{\mathcal{A}_2} \dots \xrightarrow{\mathcal{A}_{t-2}} V_{t-1} \xrightarrow{\mathcal{A}_{t-1}} V_t$$

\mathcal{A}_t



in which **each line is**

- a full arrow \longrightarrow , \longleftarrow , or
- a dashed arrow \dashrightarrow , \dashleftarrow .

My talk is based on:

- T. Klimchuk, D. Kovalenko, T. Rybalkina, V.V. Sergeichuk, *Tame systems of linear and semilinear mappings*, Contemp. Math. 658 (2016) 103-114.
- D. Duarte de Oliveira, V. Futorny, T. Klimchuk, D. Kovalenko, V.V. Sergeichuk, *Cycles of linear and semilinear mappings*, Linear Algebra Appl. 438 (2013) 3442-3453.
- D. Duarte de Oliveira, R.A. Horn, T. Klimchuk, V.V. Sergeichuk, *Remarks on the classification of a pair of commuting semilinear operators*, Linear Algebra Appl. 436 (2012) 3362-3372.

Empty matrices

- $\forall n = 0, 1, 2, \dots \exists!$ matrices of sizes $0 \times n$ and $n \times 0$, which correspond to linear mappings $\mathbb{C}^n \rightarrow 0$ and $0 \rightarrow \mathbb{C}^n$.
- They are denoted by 0_{0n} and 0_{n0} and are considered as zero matrices
- For every $p \times q$ matrix M_{pq} :

$$M_{pq} \oplus 0_{n0} = \begin{bmatrix} M_{pq} & 0 \\ 0 & 0_{n0} \end{bmatrix} = \begin{bmatrix} M_{pq} & 0_{p0} \\ 0_{nq} & 0_{n0} \end{bmatrix} = \begin{bmatrix} M_{pq} \\ 0_{nq} \end{bmatrix}$$

$$M_{pq} \oplus 0_{0n} = \begin{bmatrix} M_{pq} & 0 \\ 0 & 0_{0n} \end{bmatrix} = \begin{bmatrix} M_{pq} & 0_{pn} \\ 0_{0q} & 0_{0n} \end{bmatrix} = \begin{bmatrix} M_{pq} & 0_{pn} \end{bmatrix}$$

Sooooo baaad without empty matrices

$$\forall A \exists \text{ nonsingular } R, S: RAS = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$$

Observation

- Each matrix is equivalent to a direct sum of indecomposable matrices of the form

$$[1], [1\ 0], [1\ 0\ 0], \dots, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \dots$$

- This direct sum **is not uniquely determined**:

$$\begin{aligned} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} &= \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right] = [1\ 0] \oplus [0] \\ &= \left[\begin{array}{c|cc} 1 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right] = [1] \oplus [0\ 0] \end{aligned}$$

Goooooood with empty matrices

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = [1 \ 0] \oplus [0] = [1] \oplus 0_{01} \oplus 0_{01} \oplus 0_{10} \\ = [1] \oplus [0 \ 0] = [1] \oplus 0_{01} \oplus 0_{01} \oplus 0_{10}$$

A la Jordan Theorem

- *Each matrix is equivalent to a direct sum of indecomposable matrices of the form $[1]$, 0_{01} , 0_{10} .*
- *This direct sum is uniquely determined, up to permutation of summands.*