PROGRAMMING WITH LINEAR FRACTIONAL FUNCTIONALS

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INTRODUCTION

The problems that we shall deal with may be called by the one name, "programming with linear fractional functionals." Members of this class have been encountered in a variety of contexts. One such occurrence [3] involved situations in which the more usual sensitivity analyses were extended to problems involving plans for optimal data changes. In these instances, linear programming inequalities were to be considered relative to a functional formulated as a ratio of two variables wherein one variable, in the numerator, represented the change in total cost and the other variable, in the denominator, represented the volume changes that might attend the possible variations of a particular cost coefficient. Another example was dealt with by M. Klein in [6]. There a problem in optimal maintenance and repair policies was encountered in the context of a Markoff process formulation (see [3] also). In that context a ratio of homogeneous linear functions was to be maximized subject to a single homogeneous linear equation and a norming condition on non-negative variables. To handle this problem, Klein applied a square-root transformation (which he attributed to C. Derman [4]) in order to effect a reduction to an equivalent linear programming problem. Finally, a special instance of our general case was treated by J. R. Isbell and W. H. Marlow in their article on "Attrition Games,"[5]. In considering a ratio of (possibly) nonhomogeneous linear forms subject to general linear inequality constraints, Isbell and Marlow were able to establish a convergent iterative process which involved replacing the ratio by the problem of optimizing a sequence of different linear functionals. The linear functional at any stage in the iterations was determined by optimization of the linear functional at the preceding stage.

The objective of the present paper is to replace any "linear fractional programming problem" with, at most, two straightforward linear programming problems that differ from each other by only a change in sign in the functional and in one constraint. Also the variable transformations to be utilized will be simpler than the square-root transformations employed in [6]. Our transformations are also homeomorphisms; from which follows the globality of local optima with linear fractional functionals.

GENERAL LINEAR FRACTIONAL MODELS

The general class of linear fractional models is conveniently rendered in the following form:

$$\text{(1.1)} \qquad \begin{aligned} &\text{maximize} && \frac{c^T x + \alpha}{d^T x + \beta} \equiv R(x) \\ &\text{subject to} && Ax \leq b \\ & && x \geq 0, \end{aligned}$$

where A is an $m \times n$ matrix and b is an $m \times 1$ vector so that the two sets of constants for the constraints are related by the $n \times 1$ vector of variables, x. Similarly, c^T and d^T are transposes of the $n \times 1$ vectors of coefficients, c and d, respectively, while α and β are arbitrary scalar constants.

It is assumed, unless otherwise noted, that the constraints of (1.1) are regular[1] so that the solution set

$$\text{(1.2)} \qquad X \equiv \{x; Ax \leq b, \ x \geq 0\}$$

is nonempty and bounded.

The following transformation of variables is now introduced:

$$\text{(2.1)} \qquad y \equiv t x$$

where $t \geq 0$ is to be chosen so that

$$\text{(2.2)} \qquad d^T y + \beta t = \gamma$$

where $\gamma \neq 0$ is a specified number. On multiplying numerator and denominator and the system of inequalities in (1.1) by t and taking (2.2) into account, we obtain the linear programming problem

$$\text{(3)} \qquad \begin{aligned} &\text{maximize} && c^T y + \alpha t \equiv L(y,t) \\ &\text{subject to} && Ay - bt \leq 0 \\ & && d^T y + \beta t = \gamma \\ & && y, t \geq 0. \end{aligned}$$

We now proceed to prove

LEMMA 1: Every (y,t) satisfying the constraints of (3) has $t > 0$,

PROOF: Suppose $(\hat{y}, 0)$ satisfied the constraints of (3). Let $\hat{x}$ be any element of X. Then $x_\mu \equiv \hat{x} + \mu \hat{y}$ is in X for $\mu > 0$ since $A\hat{y} \leq 0$, $\hat{y} \geq 0$. But then X is unbounded contrary to the regularity hypothesis imposed on X. Q.E.D.

[1] If necessary, regularization procedures are available to bring about the indicated conditions. See [6].

THEOREM 1:

If (i) $0 < \operatorname{sgn}(\gamma) = \operatorname{sgn}(d^T x^* + \beta)$ for x^* an optimal solution of (1.1), and

(ii) (y^*, t^*) is an optimal solution of (3),

then y^*/t^* is an optimal solution of (1.1).

PROOF: Suppose the theorem were false, i.e., assume that there exists an optimal $x^* \in X$ such that

$$\frac{c^T x^* + \alpha}{d^T x^* + \beta} > \frac{c^T(y^*/t^*) + \alpha}{d^T(y^*/t^*) + \beta}.$$

By condition (i)

$$d^T x^* + \beta = \theta\gamma$$

for some $\theta > 0$. Consider

$$\hat{y} = \theta^{-1} x^*, \hat{t} = \theta^{-1}.$$

Then

$$\theta^{-1}(d^T x^* + \beta) = d^T\hat{y} + \hat{t}\beta = \gamma$$

and $(\hat{y}, \hat{t})$ also satisfies $A\hat{y} - b\hat{t} \leq 0$, $\hat{y}, \hat{t} \geq 0$. But

$$\frac{c^T x^* + \alpha}{d^T x^* + \beta} = \frac{\theta^{-1}(c^T x^* + \alpha)}{\theta^{-1}(d^T x^* + \beta)} = \frac{c^T\hat{y} + \alpha\hat{t}}{d^T\hat{y} + \beta\hat{t}} = \frac{c^T\hat{y} + \alpha\hat{t}}{\gamma}.$$

Also

$$\frac{c^T\left(\frac{y^*}{t^*}\right) + \alpha}{d^T\left(\frac{y^*}{t^*}\right) + \beta} = \frac{c^T y^* + \alpha t^*}{d^T y^* + \beta t^*} = \frac{c^T y^* + \alpha t^*}{\gamma}.$$

But now

$$\frac{c^T x^* + \alpha}{d^T x^* + \beta} > \frac{c^T\left(\frac{y^*}{t^*}\right) + \alpha}{d^T\left(\frac{y^*}{t^*}\right) + \beta}.$$

Since, by hypothesis (i), $\gamma \neq 0$ we have

$$c^T\hat{y} + \alpha\hat{t} > c^T y^* + t^*,$$

a contradiction to (y^*, t^*) optimal for (3). Q.E.D.

If $\operatorname{sgn}(d^T x^* + \beta) < 0$ for x^* an optimal solution (1.1), then replacing (c^T, α) and (d^T, β) by their negatives, the functional is unaltered and for the new (d^T, β) we would have $\operatorname{sgn}(d^T x^* + \beta) > 0$. Thus, we may state

THEOREM 2: For any X regular, to solve the problem (1.1) it suffices to solve the two ordinary linear programming problems,

$$(4.1)\qquad \begin{aligned} &\text{maximize} && c^T y + \alpha t \\ &\text{subject to} && Ay - bt \leq 0 \\ &&& d^T y + \beta t = 1 \\ &&& y, t \geq 0 \end{aligned}$$

and

$$(4.2)\qquad \begin{aligned} &\text{maximize} && -c^T y - \alpha t \\ &\text{subject to} && Ay - bt \leq 0 \\ &&& -d^T y - \beta t = 1 \\ &&& y, t \geq 0. \end{aligned}$$

REMARK 1: It should be observed that the same reduction can be made using the numerator instead of the denominator since

$$(5)\qquad \max \frac{c^T x + \alpha}{d^T x + \beta} \approx \max\ (-1)\ \frac{d^T x + \beta}{c^T x + \alpha}.$$

REMARK 2: Thus, if one knows the sign of either the numerator or the denominator for the functional, at an optimum, one need only solve a single ordinary linear programming problem, i.e., one of either (4.1) or (4.2), in place of the linear fractional programming problem (1.1). (In particular, non-vanishing of the denominator over the set X implies unisignance there, hence only one problem to solve.)

Now we proceed to exhaust all of the remaining possibilities in order to show that the linear programming problems (4.1) and (4.2), continue to correctly characterize the situations for the linear fractional problem. First, we prove

THEOREM 3: If for all $x \in X$, $d^T x + \beta = 0$ then problems (4.1) and (4.2) are both inconsistent.

PROOF: If $d^T x + \beta = 0$ it is impossible to obtain

$$\pm t(d^T x + \beta) = \pm(d^T y + \beta t) = 1. \qquad \text{Q.E.D.}$$

Next, we observe that if there are points in both $d^T x + \beta = 0$ and $d^T x + \beta \neq 0$ then, by convexity, any point in $d^T x + \beta = 0$ is a limit of a sequence of points $\{x^n\}$ for which $d^T x^n + \beta = \epsilon_n \neq 0$ and $\epsilon_n \longrightarrow 0$. Observe further, then, what must be happening on the corresponding sequence $\{y^n, t_n\}$:

(5.1) $$\pm t_n(d^T x^n + \beta) = \pm d^T y^n \pm \beta t_n = t_n \epsilon_n = 1.^2$$

Thus $t_n = 1/\epsilon_n \longrightarrow \infty$. If an optimum of R(x) (here intended to include also max R(x) $= \infty$) is approached by approaching a point of $d^T x + \beta = 0$, then the corresponding sequence involves $t_n \longrightarrow \infty$. Since the linear programming problem (4.1) or (4.2), being computationally solved will have been regularized, this behavior will be evidenced in the attainment of an optimum that involves the artificial bound, U.[3] Thus, we may extend the previous developments to include every possibility as follows:

THEOREM 4: The following corresponding statements are equivalent:

Linear Fractional	Linear Programming
(i) All $x \in X$ satisfy $c^T x + \alpha = d^T x + \beta = 0$.	(4.1) and (4.2) are inconsistent.
(ii) There exists x^n such that $R(x^n) \longrightarrow \max R(x)$ with $d^T x^n + \beta = \epsilon_n \neq 0$, $\epsilon_n \longrightarrow 0$.	t^* in (4.1) or (4.2) involves the artificial bound U.
(iii) $R(x^*) = \max R(x)$ with $d^T x^* + \beta \neq 0$.	$x^* = y^*/t^*$ from (4.1) or (4.2).

CONCLUSION

It may be noted that the interesting situations of Derman, Klein, et al., involve a fortiori only (4.1) in (iii). There are also further extensions which we shall treat elsewhere. The most important case is one which involves a separable concave function for the numerator and a separable convex function for the denominator. If these are also piecewise linear, we can reduce them to the linear fractional programming analysis that has just been concluded.

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[2] The "$\pm$" are entered in lieu of the rather obvious, but extended, verbalizations that would be needed to deal with all of the sign possibilities and cross references to (4.1) and (4.2).

[3] Cf. [6].

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[4]Referee's note.

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