Generalized Disjunctive Programming

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Overview

- **Introduction**
- **Disjunctive Programming**
- **GDP Solving Approaches**
- **GDP Programming Tools**

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Optimization

- The process of maximizing/minimizing a real function considering some constraints on decision variables
- Optimization Problem Components:
  - ➢ Objective Function
  - ➢ Decision Variables
  - ➢ Constraints

Popular Optimization Approaches

- Linear Programming
- Integer Programming
- Quadratic Programming
- Semi-Definite Programming
- Vector Programming
- **Disjunctive Programming**

Optimization Problem Example

- Graph Coloring (ILP):

$$\min \sum_j w_j$$

$$\sum_j x_{ij} = 1 \quad \forall i \in V$$

$$x_{uj} + x_{vj} \leq 1 \quad \forall u, v \in E, j \in C$$

$$x_{ij} \leq w_j \quad \forall i \in V, j \in C$$

$$x_{ij} \in \{0, 1\} \quad \forall i \in V, j \in C$$

$$w_j \in \{0, 1\} \quad \forall j \in C$$

Motivation

- Common optimization approaches like LP and ILP are based on conjunction of some constraints.
- In many theoretical and real world problems we are struggling with **disjunction forms**.
- We need a framework which handles disjunctive constraints and proposes a systematic approach to solve such problems.
- Here is where **Disjunctive Programming** comes in handy.

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Disjunctive Programming

- Using disjunction form of constraints instead of conjunctive constraints
- In the other word, disjunctive programming is optimization over unions of polyhedra.
- So useful in modeling of problems containing discrete decisions

Disjunctive Programming Form

$$max/min \quad f(x)$$

$$s.t: \quad \vee[g_{ki}(x) \leq b_{ki}] \quad i \in D_k, k \in K$$

$$x \in R^n, \quad x_i^{lb} \leq x_i \leq x_i^{ub}$$

Disjunctive Programming Example

$$\boldsymbol{max} \quad \mathbf{2}\boldsymbol{x} + \mathbf{3}\boldsymbol{y}$$

$$s.t: \quad [x + 2y \leq -2] \vee [-3x - y \geq 1]$$

$$\mathbf{2} \leq \boldsymbol{x} \leq \mathbf{3}$$

$$\boldsymbol{y} \leq \mathbf{1}$$

Generalized Disjunctive Programming (GDP)

- GDP consists of:
  - ➢ Objective Function
  - ➢ Algebraic Constraints
  - ➢ Disjunction Sets
  - ➢ Logic Propositions
  - ➢ Continuous Variables
  - ➢ Discrete Variables

GDP Form

$$
\begin{aligned}
\min \quad & f(x) \\
\text{s.t.} \quad & g(x) \le 0 \\
& \bigvee_{i \in D_k} \begin{bmatrix} Y_{ki} \\ r_{ki}(x) \le 0 \end{bmatrix} && k \in K \\
& \underline{\bigvee}_{i \in D_k} Y_{ki} && k \in K \\
& \Omega(Y) = True \\
& x^{lo} \le x \le x^{up} \\
& x \in \mathbb{R}^n \\
& Y_{ki} \in \{True, False\} && k \in K,\, i \in D_k
\end{aligned}
$$

GDP Example

$$
\begin{array}{lll}
\textbf{Minimize} & c + 2x_1 + x_2 & \textbf{Objective Function} \\
\textbf{subject to} & & \\
& \begin{bmatrix} Y_1 \\ -x_1 + x_2 + 2 \leq 0 \\ c \leq 5 \end{bmatrix} \vee \begin{bmatrix} Y_2 \\ 2 - x_2 \leq 0 \\ c \leq 7 \end{bmatrix} & \textbf{Disjunctions} \\
& \begin{bmatrix} Y_3 \\ x_1 - x_2 \leq 0 \end{bmatrix} \vee \begin{bmatrix} \neg Y_3 \\ x_1 \leq 1 \end{bmatrix} & \\
& Y_1 \wedge \neg Y_2 \Rightarrow \neg Y_3 & \textbf{Logic Propositions} \\
& Y_2 \Rightarrow \neg Y_3 & \\
& Y_3 \Rightarrow \neg Y_2 & \\
& 0 \leq x_1 \leq 5 & \textbf{Continuous Variables} \\
& 0 \leq x_2 \leq 5 & \\
& c \geq 0 & \\
& Y_j \in \{\text{True}, \text{False}\},\ j = 1, 2, 3 & \textbf{Boolean Variables}
\end{array}
$$

GDP Applications

- Modeling real word problems in GDP framework:
    - ➢ Process Networks
    - ➢ Open Dimension Problems (Strip Packing)
    - ➢ Routing Problems (Robotic Arm Routing)

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GDP Form

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& \Omega(Y) = True \\
& x^{lo} \le x \le x^{up} \\
& x \in \mathbb{R}^n \\
& Y_{ki} \in \{True, False\} && k \in K, i \in D_k
\end{aligned}
$$

GDP Different Types

- Three main functions of a GDP form, ie **f(x)**, **g(x)** and **r(x)** can be:
  - ➢ Linear
  - ➢ Non-Linear but Convex
  - ➢ Non-Convex

- In this project we've focused on the first case.

GDP Solving Approaches

- Main Idea: Reformulation a GDP problem as a MILP problem using one of the following methods
  - ➢ Big-M
  - ➢ Improved Big-M
  - ➢ Convex Hull

Big-M Reformulation

$$
\begin{aligned}
\min \quad & f(x) \\
\text{s.t.} \quad & g(x) \le 0 \\
& r_{ki}(x) \le M^{ki}(1 - y_{ki}) && k \in K, i \in D_k \\
& \sum_{i \in D_k} y_{ki} = 1 && k \in K \\
& Hy \ge h \\
& x^{lo} \le x \le x^{up} \\
& x \in \mathbb{R}^n \\
& y_{ki} \in \{0, 1\} && k \in K, i \in D_k
\end{aligned}
$$

Convex Hull Reformulation

$$
\begin{aligned}
& \min \ z = f(x) & & \\
\text{s.t.} \quad & g(x) \leq 0 & & \\
& x = \sum_{i \in D_k} \nu^{ki} & & k \in K \\
& y_{ki} r_{ki} \left(\nu^{ki} / y_{ki} \right) \leq 0 & & k \in K, i \in D_k \\
& \sum_{i \in D_k} y_{ki} = 1 & & k \in K \\
& Hx \geq h & & \\
& x^{lo} y_{ki} \leq \nu^{ki} \leq x^{up} y_{ki} & & k \in K, i \in D_k \\
& x^{lo} \leq x \leq x^{up} & & \\
& x \in \mathbb{R}^n & & \\
& y_{ki} \in \{0, 1\} & & k \in K, i \in D_k
\end{aligned}
$$

Big-M vs Convex Hull

- Big-M method is simpler to implement
- Big-M method results in a more compact model
- Convex Hull method construct a tighter model

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GDP Programming Tools

- Most popular GDP programming tools:
  - ➢ EMP GAMS
  - ➢ Pyomo (Python Module)

- Mentioned tools first reformulate the GDP problem to a MILP problem and then solve it using common MILP solvers.

Thanks :)

Any Question?

Contact me by **arashmarioriyad@gmail.com**