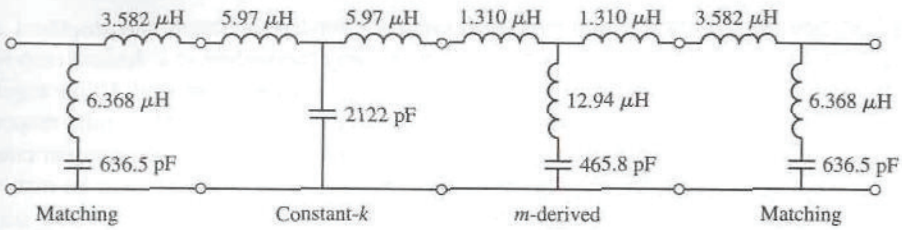


**FIGURE 8.19** Low-pass composite filter for Example 8.2.

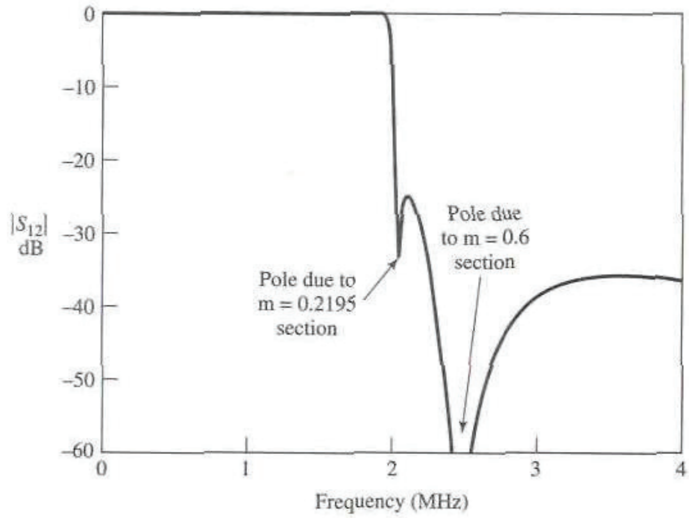


**FIGURE 8.20** Frequency response for the low-pass filter of Example 8.2.

The completed filter circuit is shown in Figure 8.19; the series pairs of inductors between the sections can be combined. Figure 8.20 shows the resulting frequency response for $|S_{12}|$. Note the sharp dip at $f = 2.05$ MHz due to the $m = 0.2195$ section, and the pole at 2.56 MHz, which is due to the $m = 0.6$ matching sections. ○

## 8.3 FILTER DESIGN BY THE INSERTION LOSS METHOD

The perfect filter would have zero insertion loss in the passband, infinite attenuation in the stopband, and a linear phase response (to avoid signal distortion) in the passband. Of course, such filters do not exist in practice, so compromises must be made; herein lies the art of filter design.

The image parameter method of the previous section may yield a usable filter response, but if not there is no clear-cut way to improve the design. The insertion loss method, however, allows a high degree of control over the passband and stopband amplitude and phase characteristics, with a systematic way to synthesize a desired response. The necessary design trade-offs can be evaluated to best meet the application requirements. If, for example, a minimum insertion loss is most important, a binomial response could be used; a Chebyshev response would satisfy a requirement for the sharpest cutoff. If it is possible to sacrifice the attenuation rate, a better phase response can be obtained by using a linear phase filter design. And in all cases, the insertion loss method allows filter performance to be improved in a straightforward manner, at the expense of a higher order filter. For the filter prototypes to be discussed below, the order of the filter is equal to the number of reactive elements.

### Characterization by Power Loss Ratio

In the insertion loss method a filter response is defined by its insertion loss, or *power loss ratio*, $P_{\text{LR}}$:

$$P_{\text{LR}} = \frac{\text{Power available from source}}{\text{Power delivered to load}} = \frac{P_{\text{inc}}}{P_{\text{load}}} = \frac{1}{1 - |\Gamma(\omega)|^2}. \tag{8.49}$$

Observe that this quantity is the reciprocal of $|S_{12}|^2$ if both load and source are matched. The insertion loss (IL) in dB is

$$\text{IL} = 10 \log P_{\text{LR}}. \tag{8.50}$$

From Section 4.1 we know that $|\Gamma(\omega)|^2$ is an even function of $\omega$; therefore it can be expressed as a polynomial in $\omega^2$. Thus we can write

$$|\Gamma(\omega)|^2 = \frac{M(\omega^2)}{M(\omega^2) + N(\omega^2)}, \tag{8.51}$$

where $M$ and $N$ are real polynomials in $\omega^2$. Substituting this form in (8.49) gives the following:

$$P_{\text{LR}} = 1 + \frac{M(\omega^2)}{N(\omega^2)}. \tag{8.52}$$

Thus, for a filter to be physically realizable its power loss ratio must be of the form in (8.52). Notice that specifying the power loss ratio simultaneously constrains the reflection coefficient, $\Gamma(\omega)$. We now discuss some practical filter responses.

*Maximally flat.* This characteristic is also called the binomial or Butterworth response, and is optimum in the sense that it provides the flattest possible passband response for a given filter complexity, or order. For a low-pass filter, it is specified by

$$P_{\text{LR}} = 1 + k^2 \left(\frac{\omega}{\omega_c}\right)^{2N}, \tag{8.53}$$

where $N$ is the order of the filter, and $\omega_c$ is the cutoff frequency. The passband extends from $\omega = 0$ to $\omega = \omega_c$; at the band edge the power loss ratio is $1 + k^2$. If we choose this as the $-3$ dB point, as is common, we have $k = 1$, which we will assume from now on. For $\omega > \omega_c$, the attenuation increases monotonically with frequency, as shown in Figure 8.21. For $\omega >> \omega_c$, $P_{LR} \simeq k^2(\omega/\omega_c)^{2N}$, which shows that the insertion loss increases at the rate of $20N$ dB/decade. Like the binomial response for multisection quarter-wave matching transformers, the first $(2N - 1)$ derivatives of (8.53) are zero at $\omega = 0$.

*Equal ripple.* If a Chebyshev polynomial is used to specify the insertion loss of an $N$-order low-pass filter as

$$P_{LR} = 1 + k^2 T_N^2 \left(\frac{\omega}{\omega_c}\right), \tag{8.54}$$

then a sharper cutoff will result, although the passband response will have ripples of amplitude $1+k^2$, as shown in Figure 8.21, since $T_N(x)$ oscillates between $\pm 1$ for $|x| \leq 1$. Thus, $k^2$ determines the passband ripple level. For large $x$, $T_N(x) \simeq 1/2(2x)^N$, so for $\omega >> \omega_c$ the insertion loss becomes

$$P_{LR} \simeq \frac{k^2}{4}\left(\frac{2\omega}{\omega_c}\right)^{2N},$$

which also increases at the rate of $20N$ dB/decade. But the insertion loss for the Chebyshev case is $(2^{2N})/4$ greater than the binomial response, at any given frequency where $\omega >> \omega_c$.

*Elliptic function.* The maximally flat and equal-ripple responses both have monotonically increasing attenuation in the stopband. In many applications it is adequate to specify a minimum stopband attenuation, in which case a better cutoff rate can be obtained. Such filters are called elliptic function filters [3], and have equal-ripple responses in the passband as well as the stopband, as shown in Figure 8.22. The maximum attenuation in the passband, $A_{max}$, can be specified, as well as the minimum attenuation

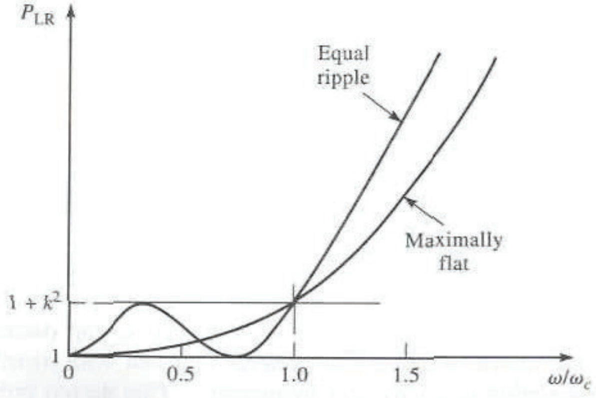


**FIGURE 8.21** Maximally flat and equal-ripple low-pass filter responses ($N = 3$).

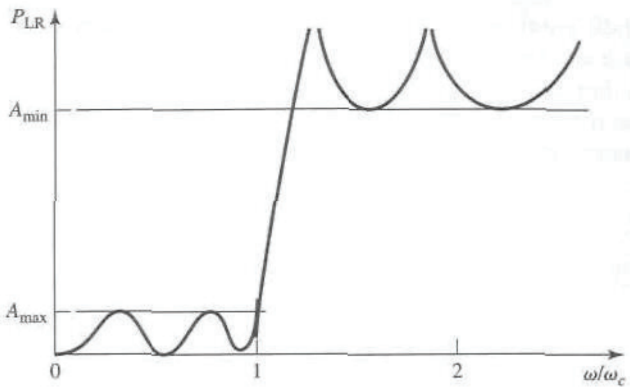


**FIGURE 8.22** Elliptic function low-pass filter response.

in the stopband, $A_{\min}$. Elliptic function filters are difficult to synthesize, so we will not consider them further; the interested reader is referred to reference [3].

*Linear phase.* The above filters specify the amplitude response, but in some applications (such as multiplexing filters for communication systems) it is important to have a linear phase response in the passband to avoid signal distortion. It turns out that a sharp-cutoff response is generally incompatible with a good phase response, so the phase response of the filter must be deliberately synthesized, usually resulting in an inferior amplitude cutoff characteristic. A linear phase characteristic can be achieved with the following phase response:

$$\phi(\omega) = A\omega\left[1 + p\left(\frac{\omega}{\omega_c}\right)^{2N}\right], \tag{8.55}$$

where $\phi(\omega)$ is the phase of the voltage transfer function of the filter, and $p$ is a constant. A related quantity is the group delay, defined as

$$\tau_d = \frac{d\phi}{d\omega} = A\left[1 + p(2N+1)\left(\frac{\omega}{\omega_c}\right)^{2N}\right], \tag{8.56}$$

which shows that the group delay for a linear phase filter is a maximally flat function.

More general filter specifications can be obtained, but the above cases are the most common. We will next discuss the design of low-pass filter prototypes which are normalized in terms of impedance and frequency; this normalization simplifies the design of filters for arbitrary frequency, impedance, and type (low-pass, high-pass, bandpass, or bandstop). The low-pass prototypes are then scaled to the desired frequency and impedance; and the lumped-element components replaced with distributed circuit elements for implementation at microwave frequencies. This design process is illustrated in Figure 8.23.

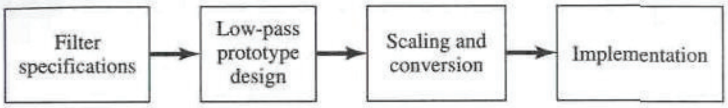


**FIGURE 8.23** The process of filter design by the insertion loss method.

### Maximally Flat Low-Pass Filter Prototype

Consider the two-element low-pass filter prototype shown in Figure 8.24; we will derive the normalized element values, $L$ and $C$, for a maximally flat response. We assume a source impedance of 1 $\Omega$, and a cutoff frequency $\omega_c = 1$. From (8.53), the desired power loss ratio will be, for $N = 2$,

$$P_{\mathrm{LR}} = 1 + \omega^4. \tag{8.57}$$

The input impedance of this filter is

$$Z_{\mathrm{in}} = j\omega L + \frac{R(1 - j\omega RC)}{1 + \omega^2 R^2 C^2}. \tag{8.58}$$

Since
$$\Gamma = \frac{Z_{\mathrm{in}} - 1}{Z_{\mathrm{in}} + 1},$$

the power loss ratio can be written as

$$P_{\mathrm{LR}} = \frac{1}{1 - |\Gamma|^2} = \frac{1}{1 - [(Z_{\mathrm{in}} - 1)/(Z_{\mathrm{in}} + 1)][(Z_{\mathrm{in}}^* - 1)/(Z_{\mathrm{in}}^* + 1)]} = \frac{|Z_{\mathrm{in}} + 1|^2}{2(Z_{\mathrm{in}} + Z_{\mathrm{in}}^*)}. \tag{8.59}$$

Now,
$$Z_{\mathrm{in}} + Z_{\mathrm{in}}^* = \frac{2R}{1 + \omega^2 R^2 C^2},$$

and
$$|Z_{\mathrm{in}} + 1|^2 = \left(\frac{R}{1 + \omega^2 R^2 C^2} + 1\right)^2 + \left(\omega L - \frac{\omega C R^2}{1 + \omega^2 R^2 C^2}\right)^2,$$

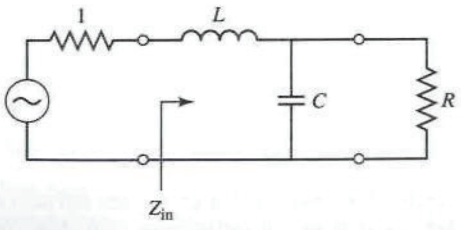


**FIGURE 8.24** Low-pass filter prototype, $N = 2$.

so (8.59) becomes

$$P_{\mathrm{LR}} = \frac{1+\omega^2R^2C^2}{4R}\left[\left(\frac{R}{1+\omega^2R^2C^2}+1\right)^2 + \left(\omega L - \frac{\omega CR^2}{1+\omega^2R^2C^2}\right)^2\right]$$
$$= \frac{1}{4R}(R^2+2R+1+R^2\omega^2C^2+\omega^2L^2+\omega^4L^2C^2R^2-2\omega^2LCR^2)$$
$$= 1+\frac{1}{4R}\left[(1-R)^2+(R^2C^2+L^2-2LCR^2)\omega^2+L^2C^2R^2\omega^4\right]. \quad 8.60$$

Notice that this expression is a polynomial in $\omega^2$. Comparing to the desired response of (8.57) shows that $R = 1$, since $P_{\mathrm{LR}} = 1$ for $\omega = 0$. In addition, the coefficient of $\omega^2$ must vanish, so

$$C^2 + L^2 - 2LC = (C-L)^2 = 0,$$

or $L = C$. Then for the coefficient of $\omega^4$ to be unity we must have

$$\frac{1}{4}L^2C^2 = \frac{1}{4}L^4 = 1,$$

or

$$L = C = \sqrt{2}.$$

In principle, this procedure can be extended to find the element values for filters with an arbitrary number of elements, $N$, but clearly this is not practical for large $N$. For a normalized low-pass design where the source impedance is 1 Ω and the cutoff frequency is $\omega_c = 1$, however, the element values for the ladder-type circuits of Figure 8.25 can

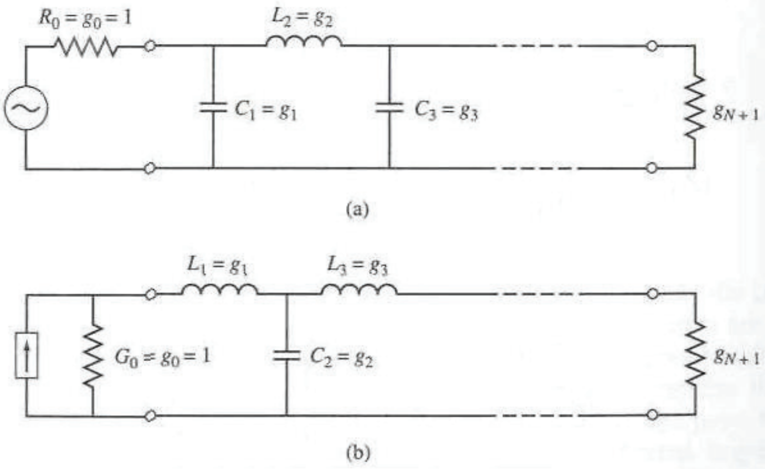


**FIGURE 8.25** Ladder circuits for low-pass filter prototypes and their element definitions. (a) Prototype beginning with a shunt element. (b) Prototype beginning with a series element.

**TABLE 8.3 Element Values for Maximally Flat Low-Pass Filter Prototypes ($g_0 = 1$, $\omega_c = 1$, $N = 1$ to 10)**

| $N$ | $g_1$ | $g_2$ | $g_3$ | $g_4$ | $g_5$ | $g_6$ | $g_7$ | $g_8$ | $g_9$ | $g_{10}$ | $g_{11}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 2.0000 | 1.0000 | | | | | | | | | |
| 2 | 1.4142 | 1.4142 | 1.0000 | | | | | | | | |
| 3 | 1.0000 | 2.0000 | 1.0000 | 1.0000 | | | | | | | |
| 4 | 0.7654 | 1.8478 | 1.8478 | 0.7654 | 1.0000 | | | | | | |
| 5 | 0.6180 | 1.6180 | 2.0000 | 1.6180 | 0.6180 | 1.0000 | | | | | |
| 6 | 0.5176 | 1.4142 | 1.9318 | 1.9318 | 1.4142 | 0.5176 | 1.0000 | | | | |
| 7 | 0.4450 | 1.2470 | 1.8019 | 2.0000 | 1.8019 | 1.2470 | 0.4450 | 1.0000 | | | |
| 8 | 0.3902 | 1.1111 | 1.6629 | 1.9615 | 1.9615 | 1.6629 | 1.1111 | 0.3902 | 1.0000 | | |
| 9 | 0.3473 | 1.0000 | 1.5321 | 1.8794 | 2.0000 | 1.8794 | 1.5321 | 1.0000 | 0.3473 | 1.0000 | |
| 10 | 0.3129 | 0.9080 | 1.4142 | 1.7820 | 1.9754 | 1.9754 | 1.7820 | 1.4142 | 0.9080 | 0.3129 | 1.0000 |

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be tabulated [1]. Table 8.3 gives such element values for maximally flat low-pass filter prototypes for $N = 1$ to 10. (Notice that the values for $N = 2$ agree with the above analytical solution.) This data is used with either of the ladder circuits of Figure 8.25 in the following way. The element values are numbered from $g_0$ at the generator impedance to $g_{N+1}$ at the load impedance, for a filter having $N$ reactive elements. The elements alternate between series and shunt connections, and $g_k$ has the following definition:

$$g_0 = \begin{cases} \text{generator resistance (network of Figure 8.25a)} \\ \text{generator conductance (network of Figure 8.25b)} \end{cases}$$

$$\underset{(k=1 \text{ to } N)}{g_k} = \begin{cases} \text{inductance for series inductors} \\ \text{capacitance for shunt capacitors} \end{cases}$$

$$g_{N+1} = \begin{cases} \text{load resistance if } g_N \text{ is a shunt capacitor} \\ \text{load conductance if } g_N \text{ is a series inductor} \end{cases}$$

Then the circuits of Figure 8.25 can be considered as the dual of each other, and both will give the same response.

Finally, as a matter of practical design procedure, it will be necessary to determine the size, or order, of the filter. This is usually dictated by a specification on the insertion loss at some frequency in the stopband of the filter. Figure 8.26 shows the attenuation characteristics for various $N$, versus normalized frequency. If a filter with $N > 10$ is required, a good result can usually be obtained by cascading two designs of lower order.

**EXAMPLE 8.3 Low-Pass Filter Design**

A maximally flat low-pass filter is to be designed with a cutoff frequency of 8 GHz and a minimum attenuation of 20 dB at 11 GHz. How many filter elements are required?

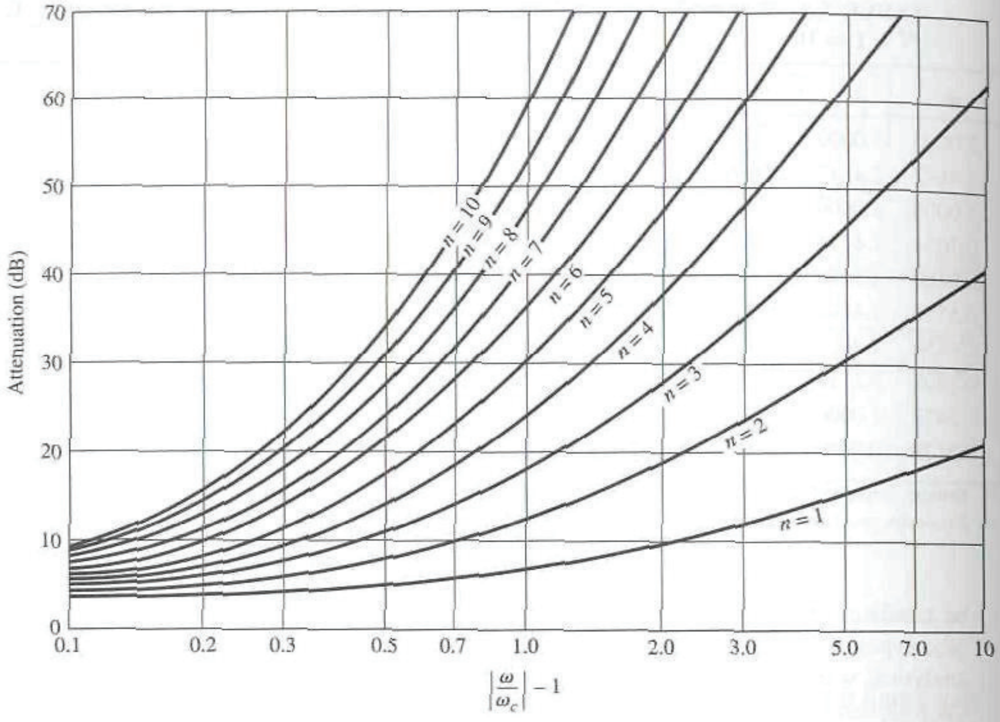


**FIGURE 8.26** Attenuation versus normalized frequency for maximally flat filter prototypes. Adapted from G. L. Matthaei, L. Young, and E. M. T. Jones, *Microwave Filters, Impedance-Matching Networks, and Coupling Structures* (Dedham, Mass.: Artech House, 1980) with permission.

*Solution*

We have $\omega/2\pi = 11$ GHz and $\omega_c/2\pi = 8$ GHz, so

$$\left|\frac{\omega}{\omega_c}\right| - 1 = \frac{11}{8} - 1 = 0.375.$$

Then from Figure 8.26 we see that an attenuation of 20 dB at this frequency requires that $N \geq 8$. Further design details will be discussed in Section 8.4. ◯

## Equal-Ripple Low-Pass Filter Prototype

For an equal-ripple low-pass filter with a cutoff frequency $\omega_c = 1$, the power loss ratio from (8.54) is

$$P_{LR} = 1 + k^2 T_N^2(\omega), \tag{8.61}$$

where $1 + k^2$ is the ripple level in the passband. Since the Chebyshev polynomials have the property that

$$T_N(0) = \begin{cases} 0 & \text{for } N \text{ odd,} \\ 1 & \text{for } N \text{ even,} \end{cases}$$

equation (8.61) shows that the filter will have a unity power loss ratio at $\omega = 0$ for $N$ odd, but a power loss ratio of $1 + k^2$ at $\omega = 0$ for $N$ even. Thus, there are two cases to consider, depending on $N$.

For the two-element filter of Figure 8.24, the power loss ratio is given in terms of the component values in (8.60). From (5.56b), we see that $T_2(x) = 2x^2 - 1$, so equating (8.61) to (8.60) gives

$$1+k^2(4\omega^4-4\omega^2+1) = 1+\frac{1}{4R}[(1-R)^2+(R^2C^2+L^2-2LCR^2)\omega^2+L^2C^2R^2\omega^4], \qquad 8.62$$

which can be solved for $R, L$, and $C$ if the ripple level (as determined by $k^2$) is known. Thus, at $\omega = 0$ we have that

$$k^2 = \frac{(1-R)^2}{4R},$$

or

$$R = 1 + 2k^2 \pm 2k\sqrt{1+k^2} \qquad \text{(for } N \text{ even).} \qquad 8.63$$

Equating coefficients of $\omega^2$ and $\omega^4$ yields the additional relations,

$$4k^2 = \frac{1}{4R}L^2C^2R^2,$$

$$-4k^2 = \frac{1}{4R}(R^2C^2 + L^2 - 2LCR^2),$$

which can be used to find $L$ and $C$. Note that (8.63) gives a value for $R$ that is not unity, so there will be an impedance mismatch if the load actually has a unity (normalized) impedance; this can be corrected with a quarter-wave transformer, or by using an additional filter element to make $N$ odd. For odd $N$, it can be shown that $R = 1$. (This is because there is a unity power loss ratio at $\omega = 0$ for $N$ odd.)

Tables exist for designing equal-ripple low-pass filters with a normalized source impedance and cutoff frequency ($\omega_c' = 1$) [1], and can be applied to either of the ladder circuits of Figure 8.25. This design data depends on the specified passband ripple level; Table 8.4 lists element values for normalized low-pass filter prototypes having 0.5 dB or 3.0 dB ripple, for $N = 1$ to 10. Notice that the load impedance $g_{N+1} \neq 1$ for even $N$. If the stopband attenuation is specified, the curves in Figures 8.27a,b can be used to determine the necessary value of $N$ for these ripple values.

### Linear Phase Low-Pass Filter Prototypes

Filters having a maximally flat time delay, or a linear phase response, can be designed in the same way, but things are somewhat more complicated because the phase of the voltage transfer function is not as simply expressed as is its amplitude. Design

**TABLE 8.4 Element Values for Equal-Ripple Low-Pass Filter Prototypes ($g_0 = 1$, $\omega_c = 1$, $N$ = 1 to 10, 0.5 dB and 3.0 dB ripple)**

| | | | | | 0.5 dB Ripple | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $N$ | $g_1$ | $g_2$ | $g_3$ | $g_4$ | $g_5$ | $g_6$ | $g_7$ | $g_8$ | $g_9$ | $g_{10}$ | $g_{11}$ |
| 1 | 0.6986 | 1.0000 | | | | | | | | | |
| 2 | 1.4029 | 0.7071 | 1.9841 | | | | | | | | |
| 3 | 1.5963 | 1.0967 | 1.5963 | 1.0000 | | | | | | | |
| 4 | 1.6703 | 1.1926 | 2.3661 | 0.8419 | 1.9841 | | | | | | |
| 5 | 1.7058 | 1.2296 | 2.5408 | 1.2296 | 1.7058 | 1.0000 | | | | | |
| 6 | 1.7254 | 1.2479 | 2.6064 | 1.3137 | 2.4758 | 0.8696 | 1.9841 | | | | |
| 7 | 1.7372 | 1.2583 | 2.6381 | 1.3444 | 2.6381 | 1.2583 | 1.7372 | 1.000 | | | |
| 8 | 1.7451 | 1.2647 | 2.6564 | 1.3590 | 2.6964 | 1.3389 | 2.5093 | 0.8796 | 1.9841 | | |
| 9 | 1.7504 | 1.2690 | 2.6678 | 1.3673 | 2.7239 | 1.3673 | 2.6678 | 1.2690 | 1.7504 | 1.0000 | |
| 10 | 1.7543 | 1.2721 | 2.6754 | 1.3725 | 2.7392 | 1.3806 | 2.7231 | 1.3485 | 2.5239 | 0.8842 | 1.9841 |
| | | | | | 3.0 dB Ripple | | | | | | |
| $N$ | $g_1$ | $g_2$ | $g_3$ | $g_4$ | $g_5$ | $g_6$ | $g_7$ | $g_8$ | $g_9$ | $g_{10}$ | $g_{11}$ |
| 1 | 1.9953 | 1.0000 | | | | | | | | | |
| 2 | 3.1013 | 0.5339 | 5.8095 | | | | | | | | |
| 3 | 3.3487 | 0.7117 | 3.3487 | 1.0000 | | | | | | | |
| 4 | 3.4389 | 0.7483 | 4.3471 | 0.5920 | 5.8095 | | | | | | |
| 5 | 3.4817 | 0.7618 | 4.5381 | 0.7618 | 3.4817 | 1.0000 | | | | | |
| 6 | 3.5045 | 0.7685 | 4.6061 | 0.7929 | 4.4641 | 0.6033 | 5.8095 | | | | |
| 7 | 3.5182 | 0.7723 | 4.6386 | 0.8039 | 4.6386 | 0.7723 | 3.5182 | 1.0000 | | | |
| 8 | 3.5277 | 0.7745 | 4.6575 | 0.8089 | 4.6990 | 0.8018 | 4.4990 | 0.6073 | 5.8095 | | |
| 9 | 3.5340 | 0.7760 | 4.6692 | 0.8118 | 4.7272 | 0.8118 | 4.6692 | 0.7760 | 3.5340 | 1.0000 | |
| 10 | 3.5384 | 0.7771 | 4.6768 | 0.8136 | 4.7425 | 0.8164 | 4.7260 | 0.8051 | 4.5142 | 0.6091 | 5.8095 |

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values have been derived for such filters [1], however, again for the ladder circuits of Figure 8.25, and are given in Table 8.5 for a normalized source impedance and cutoff frequency ($\omega_c' = 1$). The resulting group delay in the passband will be $\tau_d = 1/\omega_c' = 1$.

## 8.4 FILTER TRANSFORMATIONS

The low-pass filter prototypes of the previous section were normalized designs having a source impedance of $R_s = 1\ \Omega$ and a cutoff frequency of $\omega_c = 1$. Here we show how these designs can be scaled in terms of impedance and frequency, and converted to give high-pass, bandpass, or bandstop characteristics. Several examples will be presented to illustrate the design procedure.

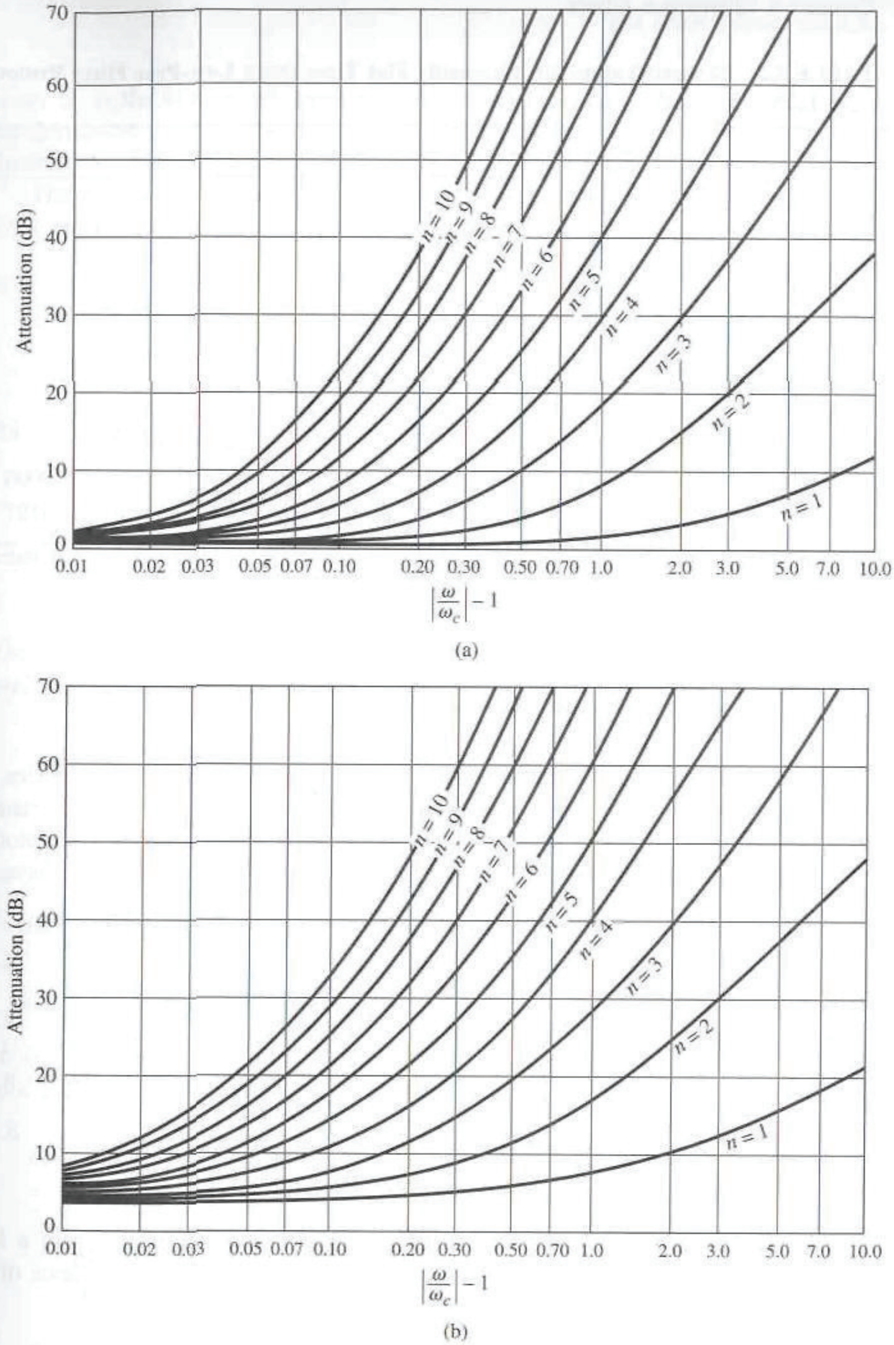


**FIGURE 8.27** Attenuation versus normalized frequency for equal-ripple filter prototypes. (a) 0.5 dB ripple level. (b) 3.0 dB ripple level.

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**TABLE 8.5 Element Values for Maximally Flat Time Delay Low-Pass Filter Prototypes ($g_0 = 1$, $\omega_c = 1$, $N = 1$ to 10)**

| $N$ | $g_1$ | $g_2$ | $g_3$ | $g_4$ | $g_5$ | $g_6$ | $g_7$ | $g_8$ | $g_9$ | $g_{10}$ | $g_{11}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 2.0000 | 1.0000 | | | | | | | | | |
| 2 | 1.5774 | 0.4226 | 1.0000 | | | | | | | | |
| 3 | 1.2550 | 0.5528 | 0.1922 | 1.0000 | | | | | | | |
| 4 | 1.0598 | 0.5116 | 0.3181 | 0.1104 | 1.0000 | | | | | | |
| 5 | 0.9303 | 0.4577 | 0.3312 | 0.2090 | 0.0718 | 1.0000 | | | | | |
| 6 | 0.8377 | 0.4116 | 0.3158 | 0.2364 | 0.1480 | 0.0505 | 1.0000 | | | | |
| 7 | 0.7677 | 0.3744 | 0.2944 | 0.2378 | 0.1778 | 0.1104 | 0.0375 | 1.0000 | | | |
| 8 | 0.7125 | 0.3446 | 0.2735 | 0.2297 | 0.1867 | 0.1387 | 0.0855 | 0.0289 | 1.0000 | | |
| 9 | 0.6678 | 0.3203 | 0.2547 | 0.2184 | 0.1859 | 0.1506 | 0.1111 | 0.0682 | 0.0230 | 1.0000 | |
| 10 | 0.6305 | 0.3002 | 0.2384 | 0.2066 | 0.1808 | 0.1539 | 0.1240 | 0.0911 | 0.0557 | 0.0187 | 1.0000 |

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### Impedance and Frequency Scaling

*Impedance scaling.* In the prototype design, the source and load resistances are unity (except for equal-ripple filters with even $N$, which have nonunity load resistance). A source resistance of $R_0$ can be obtained by multiplying the impedances of the prototype design by $R_0$. Then, if we let primes denote impedance scaled quantities, we have the new filter component values given by

$$L' = R_0 L, \tag{8.64a}$$

$$C' = \frac{C}{R_0}, \tag{8.64b}$$

$$R'_s = R_0, \tag{8.64c}$$

$$R'_L = R_0 R_L, \tag{8.64d}$$

where $L, C$, and $R_L$ are the component values for the original prototype.

*Frequency scaling for low-pass filters.* To change the cutoff frequency of a low-pass prototype from unity to $\omega_c$ requires that we scale the frequency dependence of the filter by the factor $1/\omega_c$, which is accomplished by replacing $\omega$ by $\omega/\omega_c$:

$$\omega \leftarrow \frac{\omega}{\omega_c}. \tag{8.65}$$

Then the new power loss ratio will be

$$P'_{LR}(\omega) = P_{LR}\left(\frac{\omega}{\omega_c}\right),$$

where $\omega_c$ is the new cutoff frequency; cutoff occurs when $\omega/\omega_c = 1$, or $\omega = \omega_c$. This transformation can be viewed as a stretching, or expansion, of the original passband, as illustrated in Figure 8.28a,b.

The new element values are determined by applying the substitution of (8.65) to the series reactances, $j\omega L_k$, and shunt susceptances, $j\omega C_k$, of the prototype filter. Thus,

$$jX_k = j\frac{\omega}{\omega_c}L_k = j\omega L'_k,$$

$$jB_k = j\frac{\omega}{\omega_c}C_k = j\omega C'_k,$$

which shows that the new element values are given by

$$L'_k = \frac{L_k}{\omega_c}, \tag{8.66a}$$

$$C'_k = \frac{C_k}{\omega_c}. \tag{8.66b}$$

When both impedance and frequency scaling are required, the results of (8.64) can be combined with (8.66) to give

$$L'_k = \frac{R_0 L_k}{\omega_c}, \tag{8.67a}$$

$$C'_k = \frac{C_k}{R_0 \omega_c}. \tag{8.67b}$$

*Low-pass to high-pass transformation.* The frequency substitution where,

$$\omega \leftarrow -\frac{\omega_c}{\omega}, \tag{8.68}$$

can be used to convert a low-pass response to a high-pass response, as shown in Figure 8.28c. This substitution maps $\omega = 0$ to $\omega = \pm\infty$, and vice versa; cutoff occurs when

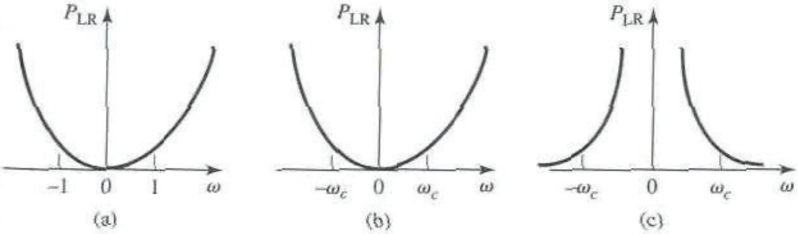



**FIGURE 8.28** Frequency scaling for low-pass filters and transformation to a high-pass response. (a) Low pass filter prototype response for $\omega_c = 1$. (b) Frequency scaling for low-pass response. (c) Transformation to high-pass response.

$\omega = \pm\omega_c$. The negative sign is needed to convert inductors (and capacitors) to realizable capacitors (and inductors). Applying (8.68) to the series reactances, $j\omega L_k$, and the shunt susceptances, $j\omega C_k$, of the prototype filter gives

$$jX_k = -j\frac{\omega_c}{\omega}L_k = \frac{1}{j\omega C'_k},$$

$$jB_k = -j\frac{\omega_c}{\omega}C_k = \frac{1}{j\omega L'_k},$$

which shows that series inductors $L_k$ must be replaced with capacitors $C'_k$, and shunt capacitors $C_k$ must be replaced with inductors $L'_k$. The new component values are given by

$$C'_k = \frac{1}{\omega_c L_k}, \tag{8.69a}$$

$$L'_k = \frac{1}{\omega_c C_k}. \tag{8.69b}$$

Impedance scaling can be included by using (8.64) to give

$$C'_k = \frac{1}{R_0\omega_c L_k}, \tag{8.70a}$$

$$L'_k = \frac{R_0}{\omega_c C_k}. \tag{8.70b}$$

**EXAMPLE 8.4 Low-Pass Filter Design Comparison**

*Design a maximally flat low-pass filter with a cutoff frequency of 2 GHz, impedance of 50 Ω, and at least 15 dB insertion loss at 3 GHz. Compute and plot the amplitude response and group delay for $f = 0$ to 4 GHz, and compare with an equal-ripple (3.0 dB ripple) and linear phase filter having the same order.*

*Solution*

First find the required order of the maximally flat filter to satisfy the insertion loss specification at 3 GHz. We have that $|\omega/\omega_c| - 1 = 0.5$; from Figure 8.26 we see that $N = 5$ will be sufficient. Then Table 8.3 gives the prototype element values as

$$g_1 = 0.618,$$
$$g_2 = 1.618,$$
$$g_3 = 2.000,$$
$$g_4 = 1.618,$$
$$g_5 = 0.618.$$

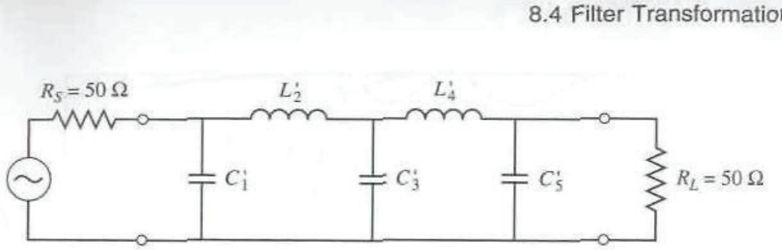



**FIGURE 8.29** Low-pass maximally flat filter circuit for Example 8.4.

Then (8.67) can be used to obtain the scaled element values:

$$C_1' = 0.984 \text{ pF},$$
$$L_2' = 6.438 \text{ nH},$$
$$C_3' = 3.183 \text{ pF},$$
$$L_4' = 6.438 \text{ nH},$$
$$C_5' = 0.984 \text{ pF}.$$

The final filter circuit is shown in Figure 8.29; the ladder circuit of Figure 8.25a was used, but that of Figure 8.25b could have been used just as well.

The component values for the equal-ripple filter and the linear phase filter, for $N = 5$, can be determined from Tables 8.4 and 8.5. The amplitude and group delay results for these three filters are shown in Figure 8.30. These results clearly show the trade-offs involved with the three types of filters. The equal-ripple response has the sharpest cutoff, but the worst group delay characteristics. The maximally flat response has a flatter attenuation characteristic in the passband, but a slightly lower cutoff rate. The linear phase filter has the worst cutoff rate, but a very good group delay characteristic. ○

### Bandpass and Bandstop Transformation

Low-pass prototype filter designs can also be transformed to have the bandpass or bandstop responses illustrated in Figure 8.31. If $\omega_1$ and $\omega_2$ denote the edges of the passband, then a bandpass response can be obtained using the following frequency substitution:

$$\omega \leftarrow \frac{\omega_0}{\omega_2 - \omega_1}\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right) = \frac{1}{\Delta}\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right), \tag{8.71}$$

where

$$\Delta = \frac{\omega_2 - \omega_1}{\omega_0} \tag{8.72}$$

is the fractional bandwidth of the passband. The center frequency, $\omega_0$, could be chosen as the arithmetic mean of $\omega_1$ and $\omega_2$, but the equations are simpler if it is chosen as the

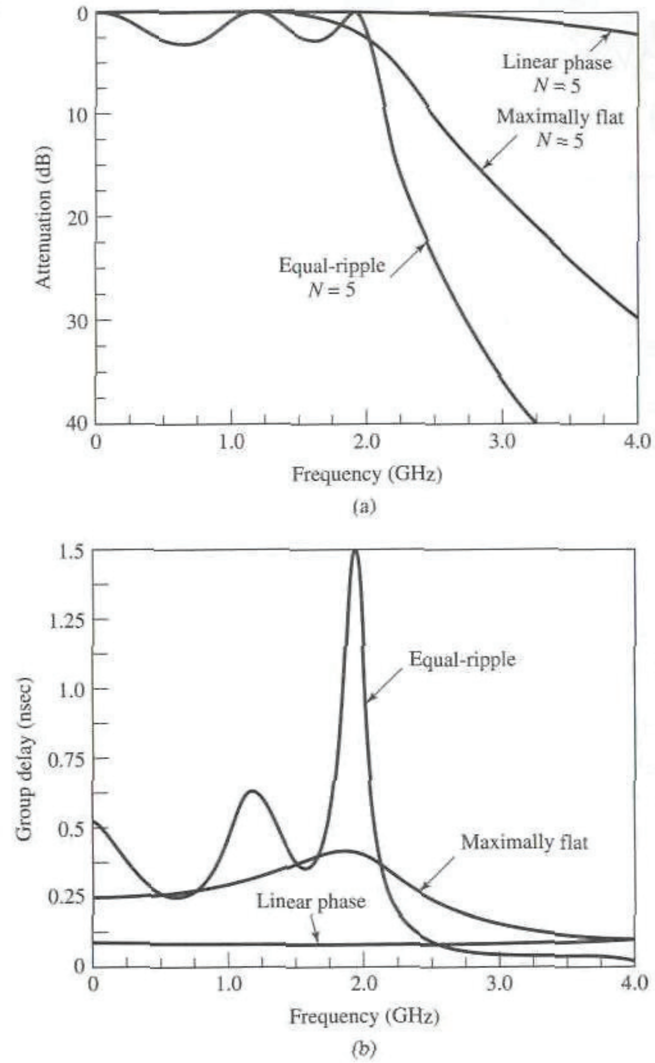



**FIGURE 8.30** Frequency response of the filter design of Example 8.4. (a) Amplitude response. (b) Group delay response.

geometric mean:

$$\omega_0 = \sqrt{\omega_1 \omega_2}. \tag{8.73}$$

Then the transformation of (8.71) maps the bandpass characteristics of Figure 8.31b to the low-pass response of Figure 8.31a as follows:

When $\omega = \omega_0$, $\qquad \dfrac{1}{\Delta}\left(\dfrac{\omega}{\omega_0} - \dfrac{\omega_0}{\omega}\right) = 0;$

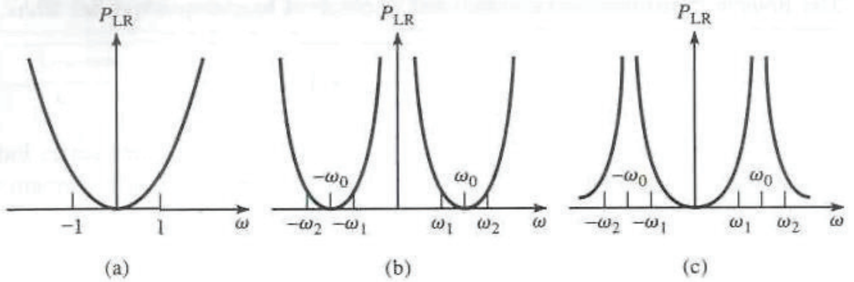



**FIGURE 8.31** *Bandpass and bandstop frequency transformations. (a) Low-pass filter prototype* response for $\omega_c = 1$. (b) Transformation to bandpass response. (c) Transformation to bandstop response.

When $\omega = \omega_1$, $$\frac{1}{\Delta}\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right) = \frac{1}{\Delta}\left(\frac{\omega_1^2 - \omega_0^2}{\omega_0\omega_1}\right) = -1;$$

When $\omega = \omega_2$, $$\frac{1}{\Delta}\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right) = \frac{1}{\Delta}\left(\frac{\omega_2^2 - \omega_0^2}{\omega_0\omega_2}\right) = 1.$$

The new filter elements are determined by using (8.71) in the expressions for the series reactance and shunt susceptances. Thus,

$$jX_k = \frac{j}{\Delta}\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right) L_k = j\frac{\omega L_k}{\Delta\omega_0} - j\frac{\omega_0 L_k}{\Delta\omega} = j\omega L_k' - j\frac{1}{\omega C_k'},$$

which shows that a series inductor, $L_k$, is transformed to a series $LC$ circuit with element values,

$$L_k' = \frac{L_k}{\Delta\omega_0}, \qquad 8.74a$$

$$C_k' = \frac{\Delta}{\omega_0 L_k}. \qquad 8.74b$$

Similarly,

$$jB_k = \frac{j}{\Delta}\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right) C_k = j\frac{\omega C_k}{\Delta\omega_0} - j\frac{\omega_0 C_k}{\Delta\omega} = j\omega C_k' - j\frac{1}{\omega L_k'},$$

which shows that a shunt capacitor, $C_k$, is transformed to a shunt $LC$ circuit with element values,

$$L_k' = \frac{\Delta}{\omega_0 C_k}, \qquad 8.74c$$

$$C_k' = \frac{C_k}{\Delta\omega_0}. \qquad 8.74d$$

The low-pass filter elements are thus converted to series resonant circuits (low impedance at resonance) in the series arms, and to parallel resonant circuits (high impedance at resonance) in the shunt arms. Notice that both series and parallel resonator elements have a resonant frequency of $\omega_0$.

The inverse transformation can be used to obtain a bandstop response. Thus,

$$\omega \longleftarrow \Delta \left( \frac{\omega}{\omega_0} - \frac{\omega_0}{\omega} \right)^{-1}, \tag{8.75}$$

where $\Delta$ and $\omega_0$ have the same definitions as in (8.72) and (8.73). Then series inductors of the low-pass prototype are converted to parallel $LC$ circuits having element values given by

$$L_k' = \frac{\Delta L_k}{\omega_0}, \tag{8.76a}$$

$$C_k' = \frac{1}{\omega_0 \Delta L_k}. \tag{8.76b}$$

The shunt capacitor of the low-pass prototype is converted to series $LC$ circuits having element values given by

$$L_k' = \frac{1}{\omega_0 \Delta C_k}, \tag{8.76c}$$

$$C_k' = \frac{\Delta C_k}{\omega_0}. \tag{8.76d}$$

The element transformations from a low-pass prototype to a highpass, bandpass, or bandstop filter are summarized in Table 8.6. These results do not include impedance scaling, which can be made using (8.64).

**EXAMPLE 8.5 Bandpass Filter Design**

Design a bandpass filter having a 0.5 dB equal-ripple response, with $N = 3$. The center frequency is 1 GHz, the bandwidth is 10%, and the impedance is 50 Ω.

*Solution*

From Table 8.4 the element values for the low-pass prototype circuit of Figure 8.25b are given as

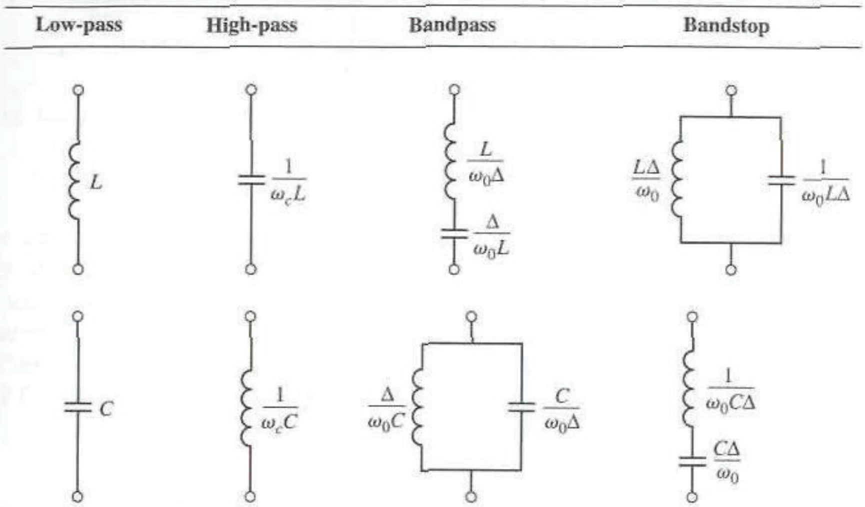
$$g_1 = 1.5963 = L_1,$$
$$g_2 = 1.0967 = C_2,$$
$$g_3 = 1.5963 = L_3,$$
$$g_4 = 1.000 = R_L.$$

Then (8.64) and (8.74) give the impedance-scaled and frequency-transformed element values for the circuit of Figure 8.32 as

$$L_1' = \frac{L_1 Z_o}{\omega_o \Delta} = 127.0 \text{ nH},$$

$$C_1' = \frac{\Delta}{\omega_o L_1 Z_o} = 0.199 \text{ pF},$$

**TABLE 8.6 Summary of Prototype Filter Transformations**




$\Delta = \frac{\omega_2 - \omega_1}{\omega_0}$

$$L_2' = \frac{\Delta Z_o}{\omega_o C_2} = 0.726 \text{ nH},$$

$$C_2' = \frac{C_2}{\omega_o \Delta Z_o} = 34.91 \text{ pF},$$

$$L_3' = \frac{L_3 Z_o}{\omega_o \Delta} = 127.0 \text{ nH},$$

$$C_3' = \frac{\Delta}{\omega_o L_3 Z_o} = 0.199 \text{ pF}.$$

The resulting amplitude response is shown in Figure 8.33. ○

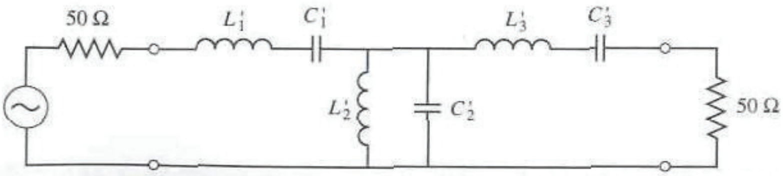



**FIGURE 8.32** Bandpass filter circuit for Example 8.5.

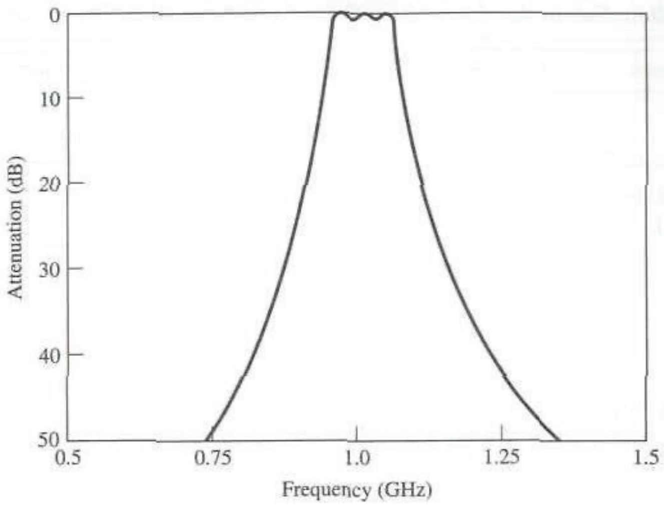



**FIGURE 8.33** Amplitude response for the bandpass filter of Example 8.5.

## 8.5 FILTER IMPLEMENTATION

The lumped-element filter design discussed in the previous sections generally works well at low frequencies, but two problems arise at microwave frequencies. First, lumped elements such as inductors and capacitors are generally available only for a limited range of values and are difficult to implement at microwave frequencies, but must be approximated with distributed components. In addition, at microwave frequencies the distances between filter components is not negligible. Richard's transformation is used to convert lumped elements to transmission line sections, while Kuroda's identities can be used to separate filter elements by using transmission line sections. Because such additional transmission line sections do not affect the filter response, this type of design is called *redundant* filter synthesis. It is possible to design microwave filters that take advantage of these sections to improve the filter response [4]; such *nonredundant* synthesis does not have a lumped-element counterpart.

### Richard's Transformation

The transformation,

$$\Omega = \tan \beta\ell = \tan\left(\frac{\omega\ell}{v_p}\right), \tag{8.77}$$

maps the $\omega$ plane to the $\Omega$ plane, which repeats with a period of $\omega\ell/v_p = 2\pi$. This transformation was introduced by P. Richard [6] to synthesize an $LC$ network using open- and short-circuited transmission lines. Thus, if we replace the frequency variable $\omega$ with $\Omega$, the reactance of an inductor can be written as

$$jX_L = j\Omega L = jL\tan\beta\ell, \tag{8.78a}$$

and the susceptance of a capacitor can be written as

$$jB_C = j\Omega C = jC\tan\beta\ell. \tag{8.78b}$$

These results indicate that an inductor can be replaced with a short-circuited stub of length $\beta\ell$ and characteristic impedance $L$, while a capacitor can be replaced with an open-circuited stub of length $\beta\ell$ and characteristic impedance $1/C$. A unity filter impedance is assumed.

Cutoff occurs at unity frequency for a low-pass filter prototype; to obtain the same cutoff frequency for the Richard's-transformed filter, (8.77) shows that

$$\Omega = 1 = \tan\beta\ell,$$

which gives a stub length of $\ell = \lambda/8$, where $\lambda$ is the wavelength of the line at the cutoff frequency, $\omega_c$. At the frequency $\omega_0 = 2\omega_c$, the lines will be $\lambda/4$ long, and an attenuation pole will occur. At frequencies away from $\omega_c$, the impedances of the stubs will no longer match the original lumped-element impedances, and the filter response will differ from the desired prototype response. Also, the response will be periodic in frequency, repeating every $4\omega_c$.

In principle, then, the inductors and capacitors of a lumped-element filter design can be replaced with short-circuited and open-circuited stubs, as illustrated in Figure 8.34. Since the lengths of all the stubs are the same ($\lambda/8$ at $\omega_c$), these lines are called *commensurate lines*.

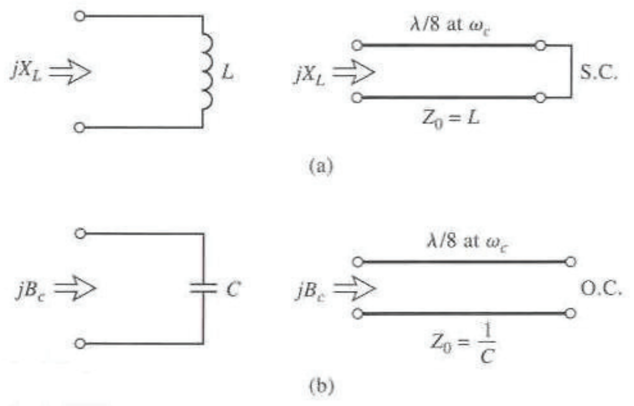



**FIGURE 8.34** Richard's transformation. (a) For an inductor to a short-circuited stub. (b) For a capacitor to an open-circuited stub.

## Kuroda's Identities

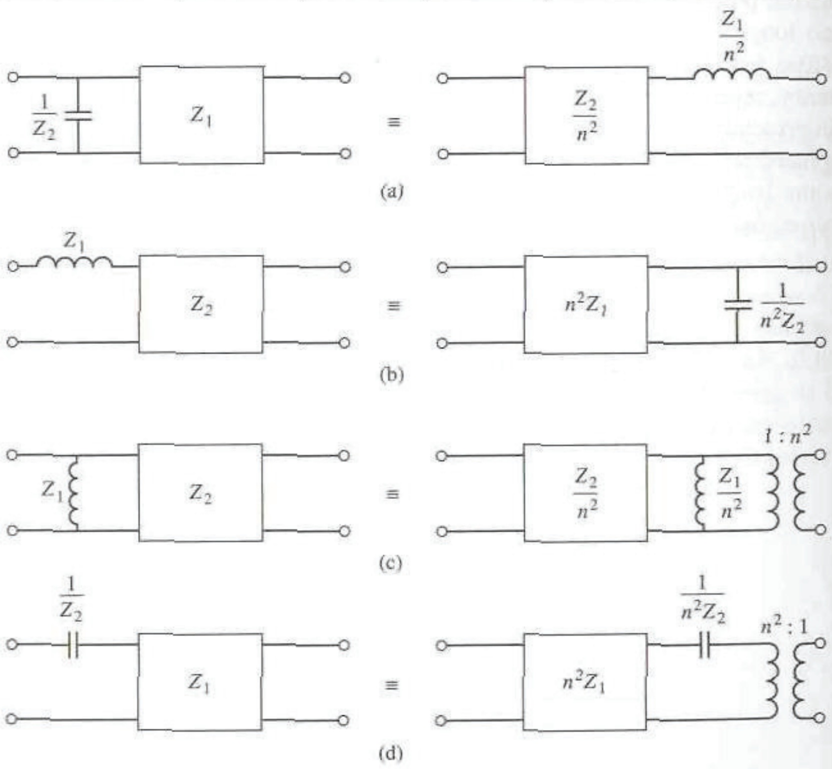
The four Kuroda identities use redundant transmission line sections to achieve a more practical microwave filter implementation by performing any of the following operations:

- Physically separate transmission line stubs
- Transform series stubs into shunt stubs, or vice versa
- Change impractical characteristic impedances into more realizable ones

The additional transmission line sections are called *unit elements* and are $\lambda/8$ long at $\omega_c$; the unit elements are thus commensurate with the stubs used to implement the inductors and capacitors of the prototype design.

The four identities are illustrated in Table 8.7, where each box represents a unit element, or transmission line, of the indicated characteristic impedance and length ($\lambda/8$ at $\omega_c$). The inductors and capacitors represent short-circuit and open-circuit stubs, respectively. We will prove the equivalence of the first case, and then show how to use these identities in Example 8.6.

**TABLE 8.7 The Four Kuroda Identities**




where $n^2 = 1 + Z_2/Z_1$

The two circuits of identity (a) in Table 8.7 can be redrawn as shown in Figure 8.35; we will show that these two networks are equivalent by showing that their $ABCD$ matrices are identical. From Table 4.1, the $ABCD$ matrix of a length $\ell$ of transmission line with characteristic impedance $Z_1$ is

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} \cos\beta\ell & jZ_1\sin\beta\ell \\ \dfrac{j}{Z_1}\sin\beta\ell & \cos\beta\ell \end{bmatrix} = \frac{1}{\sqrt{1+\Omega^2}}\begin{bmatrix} 1 & j\Omega Z_1 \\ \dfrac{j\Omega}{Z_1} & 1 \end{bmatrix}, \qquad 8.79$$

where $\Omega = \tan\beta\ell$. Now the open-circuited shunt stub in the first circuit in Figure 8.35 has an impedance of $-jZ_2\cot\beta\ell = -jZ_2/\Omega$, so the $ABCD$ matrix of the entire circuit is

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}_L = \begin{bmatrix} 1 & 0 \\ \dfrac{j\Omega}{Z_2} & 1 \end{bmatrix}\begin{bmatrix} 1 & j\Omega Z_1 \\ \dfrac{j\Omega}{Z_1} & 1 \end{bmatrix}\frac{1}{\sqrt{1+\Omega^2}}$$

$$= \frac{1}{\sqrt{1+\Omega^2}}\begin{bmatrix} 1 & j\Omega Z_1 \\ j\Omega\left(\dfrac{1}{Z_1}+\dfrac{1}{Z_2}\right) & 1-\Omega^2\dfrac{Z_1}{Z_2} \end{bmatrix}. \qquad 8.80a$$

The short-circuited series stub in the second circuit in Figure 8.35 has an impedance of $j(Z_1/n^2)\tan\beta\ell = j(\Omega Z_1/n^2)$, so the $ABCD$ matrix of the entire circuit is

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}_R = \begin{bmatrix} 1 & j\dfrac{\Omega Z_2}{n^2} \\ \dfrac{j\Omega n^2}{Z_2} & 1 \end{bmatrix}\begin{bmatrix} 1 & \dfrac{j\Omega Z_1}{n^2} \\ 0 & 1 \end{bmatrix}\frac{1}{\sqrt{1+\Omega^2}}$$

$$= \frac{1}{\sqrt{1+\Omega^2}}\begin{bmatrix} 1 & \dfrac{j\Omega}{n^2}(Z_1+Z_2) \\ \dfrac{j\Omega n^2}{Z_2} & 1-\Omega^2\dfrac{Z_1}{Z_2} \end{bmatrix}. \qquad 8.80b$$

The results in (8.80a) and (8.80b) are identical if we choose $n^2 = 1 + Z_2/Z_1$. The other identities in Table 8.7 can be proved in the same way.

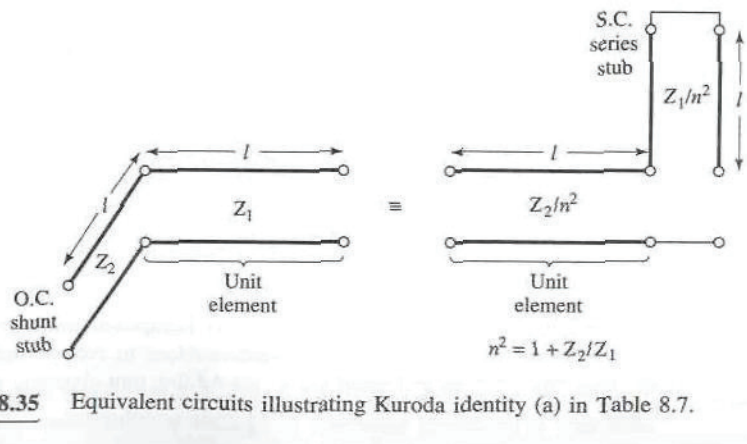



**FIGURE 8.35** Equivalent circuits illustrating Kuroda identity (a) in Table 8.7.

**EXAMPLE 8.6 Low-Pass Filter Design Using Stubs**

Design a low-pass filter for fabrication using microstrip lines. The specifications are: cutoff frequency of 4 GHz, third order, impedance of 50 Ω, and a 3 dB equal-ripple characteristic.

*Solution*

From Table 8.4, the normalized low-pass prototype element values are

$$g_1 = 3.3487 = L_1,$$
$$g_2 = 0.7117 = C_2,$$

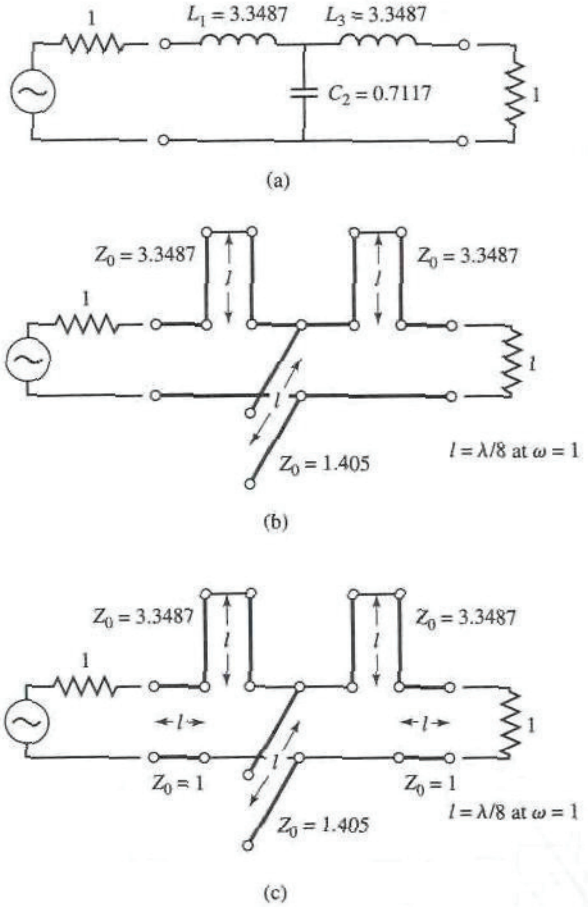



**FIGURE 8.36** Filter design procedure for Example 8.6. (a) Lumped-element low-pass filter prototype. (b) Using Richard's transformations to convert inductors and capacitors to series and shunt stubs. (c) Adding unit elements at ends of filter.

$$g_3 = 3.3487 = L_3,$$
$$g_4 = 1.0000 = R_L,$$

with the lumped-element circuit shown in Figure 8.36a.

The next step is to use Richard's transformations to convert series inductors to series stubs, and shunt capacitors to shunt stubs, as shown in Figure 8.36b. According to (8.78), the characteristic impedance of a series stub (inductor) is $L$, and the characteristic impedance of a shunt stub (capacitor) is $1/C$. For commensurate line synthesis, all stubs are $\lambda/8$ long at $\omega = \omega_c$. (It is usually most convenient to work with normalized quantities until the last step in the design.)

The series stubs of Figure 8.36b would be very difficult to implement in microstrip form, so we will use one of the Kuroda identities to convert these to shunt stubs. First, we must add unit elements at either end of the filter, as shown in Figure 8.36c. These redundant elements do not affect filter performance since

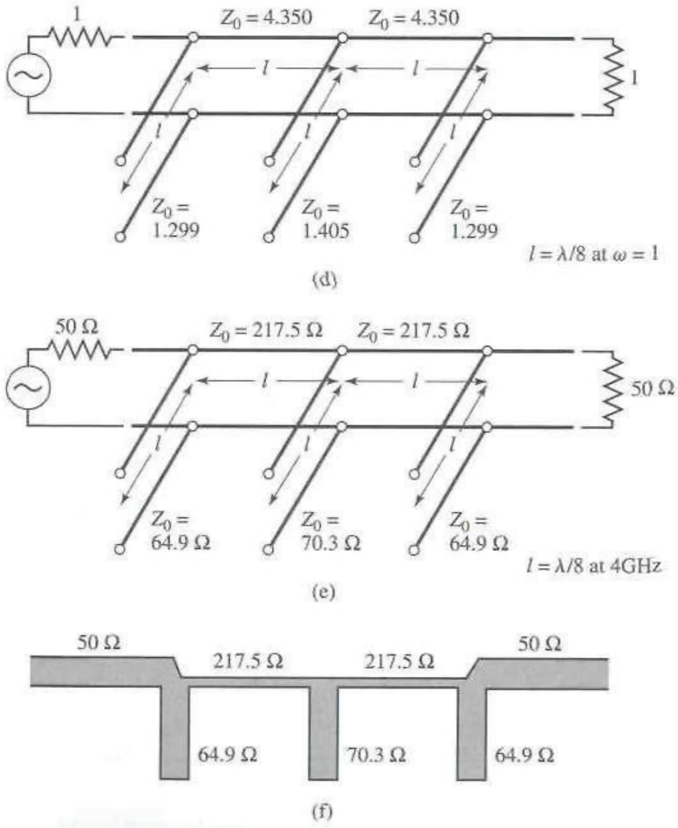



**FIGURE 8.36** Continued. (d) Applying the second Kuroda identity. (e) After impedance and frequency scaling. (f) Microstrip fabrication of final filter.

they are matched to the source and load ($Z_0 = 1$). Then we can apply Kuroda identity (b) from Table 8.7 to both ends of the filter. In both cases we have that

$$n^2 = 1 + \frac{Z_2}{Z_1} = 1 + \frac{1}{3.3487} = 1.299.$$

*The result is shown in Figure 8.36d.*

Finally, we impedance and frequency scale the circuit, which simply involves multiplying the normalized characteristic impedances by 50 Ω and choosing the line and stub lengths to be $\lambda/8$ at 4 GHz. The final circuit is shown in Figure 8.36e, with a microstrip layout in Figure 8.36f.

The calculated amplitude response of this design is plotted in Figure 8.37, along with the response of the lumped-element version. Note that the passband characteristics are very similar up to 4 GHz, but the distributed-element filter has a sharper cutoff. Also notice that the distributed-element filter has a response which repeats every 16 GHz, as a result of the periodic nature of Richard's transformation. ○

### Impedance and Admittance Inverters

As we have seen, it is often desirable to use only series, or only shunt, elements when implementing a filter with a particular type of transmission line. The Kuroda identities can be used for conversions of this form, but another possibility is to use impedance ($K$) or admittance ($J$) inverters [1], [4], [7]. Such inverters are especially useful for bandpass or bandstop filters with narrow ($< 10\%$) bandwidths.

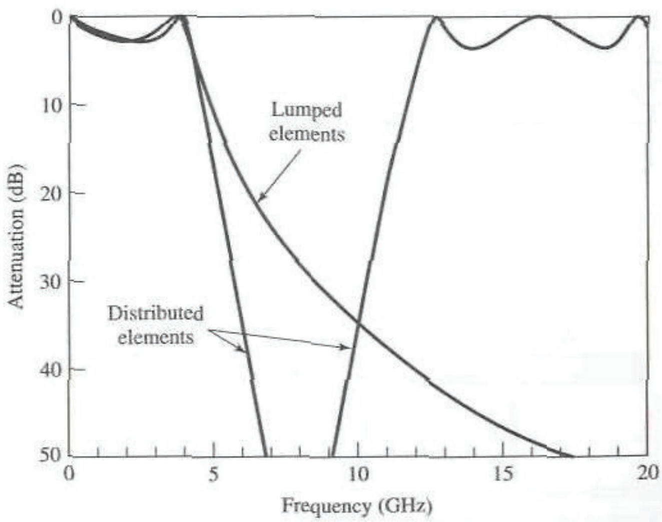



**FIGURE 8.37** Amplitude responses of lumped-element and distributed-element low-pass filter of Example 8.6.

The conceptual operation of impedance and admittance inverters is illustrated in Figure 8.38; since these inverters essentially form the inverse of the load impedance or admittance, they can be used to transform series-connected elements to shunt-connected elements, or vice versa. This procedure will be illustrated in later sections for bandpass and bandstop filters.

In its simplest form, a $J$ or $K$ inverter can be constructed using a quarter-wave transformer of the appropriate characteristic impedance, as shown in Figure 8.38b. This implementation also allows the $ABCD$ matrix of the inverter to be easily found from the $ABCD$ parameters for a length of transmission line given in Table 4.1. Many other types of circuits can also be used as $J$ or $K$ inverters, with one such alternative being shown in Figure 8.38c. Inverters of this form turn out to be useful for modeling the coupled resonator filters of Section 8.8. The lengths, $\theta/2$, of the transmission line sections are generally required to be negative for this type of inverter, but this poses no problem if these lines can be absorbed into connecting transmission lines on either side.

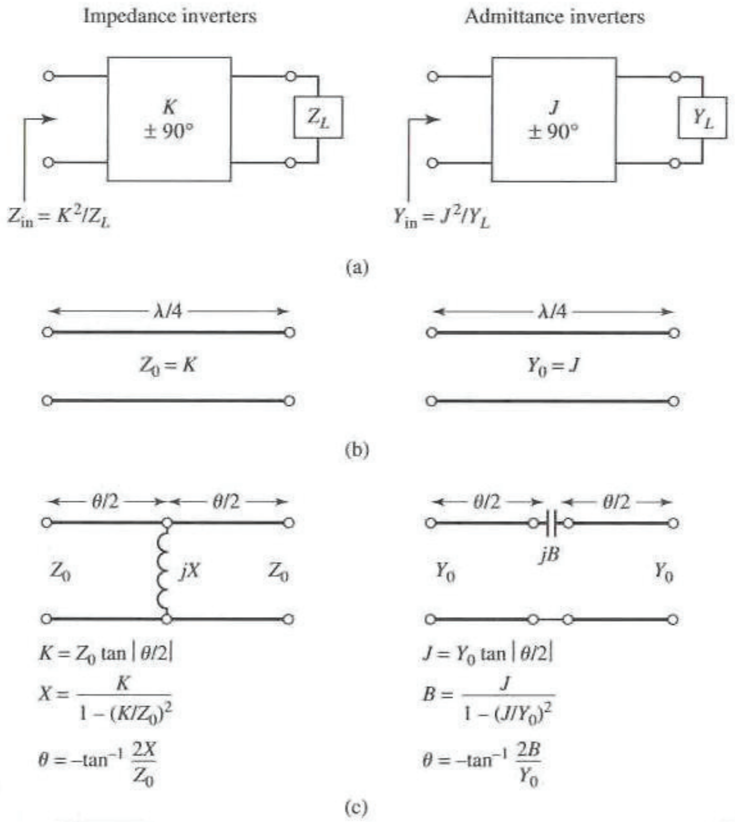



**FIGURE 8.38** Impedance and admittance inverters. (a) Operation of impedance and admittance inverters. (b) Implementation as quarter-wave transformers. (c) An alternative implementation.

## 8.6 STEPPED-IMPEDANCE LOW-PASS FILTERS

A relatively easy way to implement low-pass filters in microstrip or stripline is to use alternating sections of very high and very low characteristic impedance lines. Such filters are usually referred to as *stepped-impedance*, or hi-$Z$, low-$Z$ filters, and are popular because they are easier to design and take up less space than a similar low-pass filter using stubs. Because of the approximations involved, however, their electrical performance is not as good, so the use of such filters is usually limited to applications where a sharp cutoff is not required (for instance, in rejecting out-of-band mixer products).

### Approximate Equivalent Circuits for Short Transmission Line Sections

We begin by finding the approximate equivalent circuits for a short length of transmission line having either a very large or very small characteristic impedance. The $ABCD$ parameters of a length, $\ell$, of line having characteristic impedance $Z_0$ are given in Table 4.1; the conversion in Table 4.2 can then be used to find the $Z$-parameters as

$$Z_{11} = Z_{22} = \frac{A}{C} = -jZ_0 \cot \beta\ell, \tag{8.81a}$$

$$Z_{12} = Z_{21} = \frac{1}{C} = -jZ_0 \csc \beta\ell. \tag{8.81b}$$

The series elements of the $T$-equivalent circuit are

$$Z_{11} - Z_{12} = -jZ_0 \left[\frac{\cos \beta\ell - 1}{\sin \beta\ell}\right] = jZ_0 \tan\left(\frac{\beta\ell}{2}\right), \tag{8.82}$$

while the shunt element of the $T$-equivalent is $Z_{12}$. So if $\beta\ell < \pi/2$, the series elements have a positive reactance (inductors), while the shunt element has a negative reactance (capacitor). We thus have the equivalent circuit shown in Figure 8.39a, where

$$\frac{X}{2} = Z_0 \tan\left(\frac{\beta\ell}{2}\right), \tag{8.83a}$$

$$B = \frac{1}{Z_0} \sin \beta\ell. \tag{8.83b}$$

Now assume a short length of line (say $\beta\ell < \pi/4$) and a large characteristic impedance. Then (8.83) approximately reduces to

$$X \simeq Z_0\beta\ell, \tag{8.84a}$$

$$B \simeq 0, \tag{8.84b}$$

which implies the equivalent circuit of Figure 8.39b (a series inductor). For a short length of line and a small characteristic impedance, (8.83) approximately reduces to

$$X \simeq 0, \tag{8.85a}$$

$$B \simeq Y_0\beta\ell, \tag{8.85b}$$

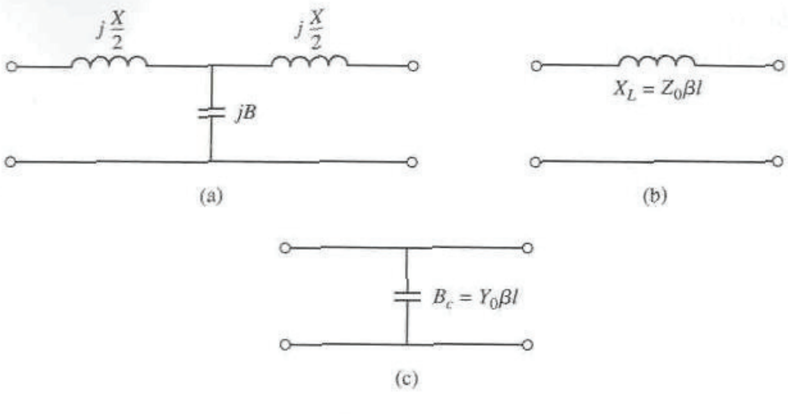



**FIGURE 8.39** Approximate equivalent circuits for short sections of transmission lines. (a) $T$-equivalent circuit for a transmission line section having $\beta\ell << \pi/2$. (b) Equivalent circuit for small $\beta\ell$ and large $Z_0$. (c) Equivalent circuit for small $\beta\ell$ and small $Z_0$.

which implies the equivalent circuit of Figure 8.39c (a shunt capacitor). So the series inductors of a low-pass prototype can be replaced with high-impedance line sections ($Z_0 = Z_h$), and the shunt capacitors can be replaced with low-impedance line sections ($Z_0 = Z_\ell$). The ratio $Z_h/Z_\ell$ should be as high as possible, so the actual values of $Z_h$ and $Z_\ell$ are usually set to the highest and lowest characteristic impedance that can be practically fabricated. The lengths of the lines can then be determined from (8.84) and (8.85); to get the best response near cutoff, these lengths should be evaluated at $\omega = \omega_c$. Combining the results of (8.84) and (8.85) with the scaling equations of (8.67) allows the electrical lengths of the inductor sections to be calculated as

$$\beta\ell = \frac{LR_0}{Z_h} \qquad \text{(inductor)}, \tag{8.86a}$$

and the electrical length of the capacitor sections as

$$\beta\ell = \frac{CZ_\ell}{R_0} \qquad \text{(capacitor)}, \tag{8.86b}$$

where $R_0$ is the filter impedance and $L$ and $C$ are the normalized element values (the $g_k$s) of the low-pass prototype.

**EXAMPLE 8.7 Stepped-Impedance Filter Design**

Design a stepped-impedance low-pass filter having a maximally flat response and a cutoff frequency of 2.5 GHz. It is necessary to have more than 20 dB insertion loss at 4.0 GHz. The filter impedance is 50 Ω; the highest practical line impedance is 150 Ω, and the lowest is 10 Ω.

*Solution*

To use Figure 8.26, we calculate

$$\frac{\omega}{\omega_c} - 1 = \frac{4.0}{2.5} - 1 = 0.6,$$

then the figure indicates $N = 6$ should give the necessary attenuation at 4.0 GHz. Table 8.3 gives the low-pass prototype values as

$$\begin{aligned}
g_1 &= 0.517 = C_1, \\
g_2 &= 1.414 = L_2, \\
g_3 &= 1.932 = C_3, \\
g_4 &= 1.932 = L_4, \\
g_5 &= 1.414 = C_5, \\
g_6 &= 0.517 = L_6.
\end{aligned}$$

The low-pass prototype circuit is shown in Figure 8.40a.

Next, we use (8.86) to find the electrical lengths of the hi-$Z$, low-$Z$ transmission line sections to replace the series inductors and shunt capacitors:

$$\beta \ell_1 = g_1 \frac{Z_\ell}{R_0} = 5.9°,$$

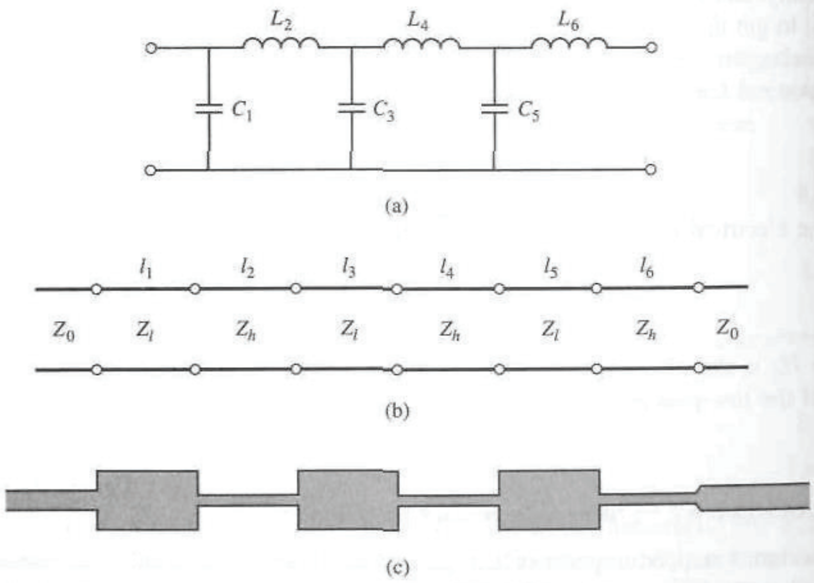



**FIGURE 8.40** Filter design for Example 8.7. (a) Low-pass filter prototype circuit. (b) Stepped-impedance implementation. (c) Microstrip layout of final filter.

$$\beta\ell_2 = g_2\frac{R_0}{Z_h} = 27.0°,$$

$$\beta\ell_3 = g_3\frac{Z_\ell}{R_0} = 22.1°,$$

$$\beta\ell_4 = g_4\frac{R_0}{Z_h} = 36.9°,$$

$$\beta\ell_5 = g_5\frac{Z_\ell}{R_0} = 16.2°,$$

$$\beta\ell_6 = g_6\frac{R_0}{Z_h} = 9.9°.$$

The final filter circuit is shown in Figure 8.40b, where $Z_\ell = 10\,\Omega$ and $Z_h = 150\,\Omega$. Note that $\beta\ell < \pi/4$ in all cases. A layout in microstrip is shown in Figure 8.40c.

Figure 8.41 shows the calculated amplitude response, compared with the response of the corresponding lumped-element filter. The passband characteristics are very similar, but the lumped-element circuit gives more attenuation at higher frequencies. This is because the stepped-impedance filter elements depart significantly from the lumped-element values at the higher frequencies. The stepped-impedance filter may have other passbands at higher frequencies, but the response will not be perfectly periodic because the lines are not commensurate. ○

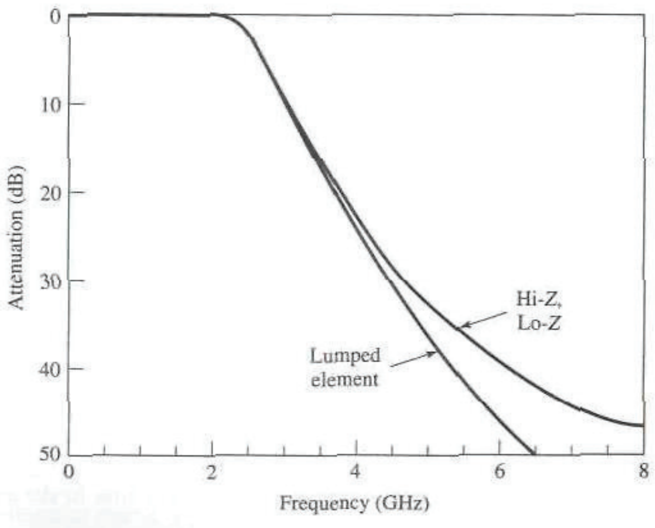



**FIGURE 8.41** Amplitude response of the stepped-impedance low-pass filter of Example 8.7, compared with the corresponding lumped-element design.

## 8.7 COUPLED LINE FILTERS

The parallel coupled transmission lines discussed in Section 7.6 (for directional couplers) can also be used to construct many types of filters. Fabrication of multisection bandpass or bandstop coupled line filters is particularly easy in microstrip or stripline form, for bandwidths less than about 20%. Wider bandwidth filters generally require very tightly coupled lines, which are difficult to fabricate. We will first study the filter characteristics of a single quarter-wave coupled line section, and then show how these sections can be used to design a bandpass filter [7]. Other filter designs using coupled lines can be found in reference [1].

### Filter Properties of a Coupled Line Section

A parallel coupled line section is shown in Figure 8.42a, with port voltage and current definitions. We will derive the open-circuit impedance matrix for this four-port network by considering the superposition of even- and odd-mode excitations [8], which are shown in Figure 8.42b. Thus, the current sources $i_1$ and $i_3$ drive the line in the even mode, while $i_2$ and $i_4$ drive the line in the odd mode. By superposition, we see that the total port currents, $I_i$, can be expressed in terms of the even- and odd-mode currents as

$$I_1 = i_1 + i_2, \tag{8.87a}$$

$$I_2 = i_1 - i_2, \tag{8.87b}$$

$$I_3 = i_3 - i_4, \tag{8.87c}$$

$$I_4 = i_3 + i_4. \tag{8.87d}$$

First consider the line as being driven in the even mode by the $i_1$ current sources. If the other ports are open-circuited, the impedance seen at port 1 or 2 is

$$Z_{\text{in}}^e = -jZ_{0e} \cot \beta \ell. \tag{8.88}$$

The voltage on either conductor can be expressed as

$$\begin{aligned} v_a^1(z) = v_b^1(z) &= V_e^+[e^{-j\beta(z-\ell)} + e^{j\beta(z-\ell)}] \\ &= 2V_e^+ \cos \beta(\ell - z), \end{aligned} \tag{8.89}$$

so the voltage at port 1 or 2 is

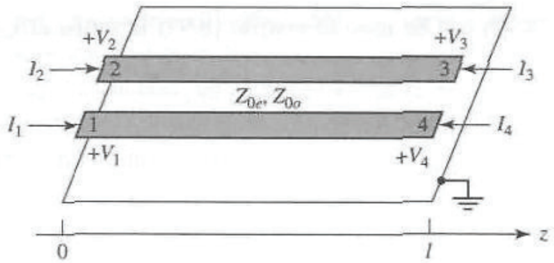
$$v_a^1(0) = v_b^1(0) = 2V_e^+ \cos \beta\ell = i_1 Z_{\text{in}}^e.$$

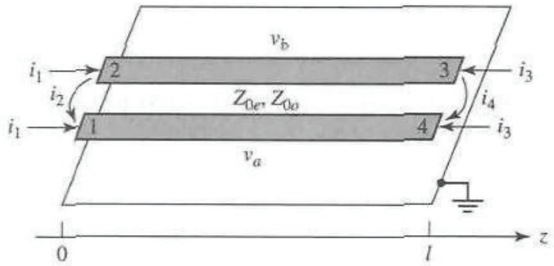
This result and (8.88) can be used to rewrite (8.89) in terms of $i_1$ as

$$v_a^1(z) = v_b^1(z) = -jZ_{0e} \frac{\cos \beta(\ell - z)}{\sin \beta\ell} i_1. \tag{8.90}$$

Similarly, the voltages due to current sources $i_3$ driving the line in the even mode are

$$v_a^3(z) = v_b^3(z) = -jZ_{0e} \frac{\cos \beta z}{\sin \beta\ell} i_3. \tag{8.91}$$



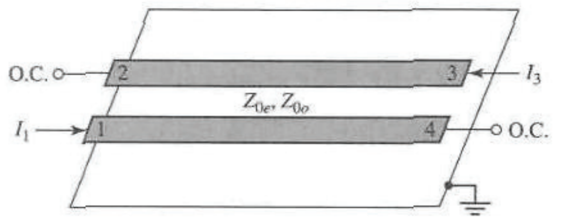




**FIGURE 8.42** Definitions pertaining to a coupled line filter section. (a) A parallel coupled line section with port voltage and current definitions. (b) A parallel coupled line section with even- and odd-mode current sources. (c) A two-port coupled line section having a bandpass response.

Now consider the line as being driven in the odd mode by current $i_2$. If the other ports are open-circuited, the impedance seen at port 1 or 2 is

$$Z_{\text{in}}^{o} = -jZ_{0o}\cot\beta\ell. \qquad 8.92$$

The voltage on either conductor can be expressed as

$$v_a^2(z) = -v_b^2(z) = V_0^+[e^{-j\beta(z-\ell)} + e^{j\beta(z-\ell)}] = 2V_0^+\cos\beta(\ell - z). \qquad 8.93$$

Then the voltage at port 1 or port 2 is

$$v_a^2(0) = -v_b^2(0) = 2V_0^+\cos\beta\ell = i_2 Z_{\text{in}}^{o}.$$

This result and (8.92) can be used to rewrite (8.93) in terms of $i_2$ as

$$v_a^2(z) = -v_b^2(z) = -jZ_{0o}\frac{\cos\beta(\ell - z)}{\sin\beta\ell}i_2. \tag{8.94}$$

Similarly, the voltages due to current $i_4$ driving the line in the odd mode are

$$v_a^4(z) = -v_b^4(z) = -jZ_{0o}\frac{\cos\beta z}{\sin\beta\ell}i_4. \tag{8.95}$$

Now the total voltage at port 1 is

$$\begin{aligned} V_1 &= v_a^1(0) + v_a^2(0) + v_a^3(0) + v_a^4(0) \\ &= -j(Z_{0e}i_1 + Z_{0o}i_2)\cot\theta - j(Z_{0e}i_3 + Z_{0o}i_4)\csc\theta, \end{aligned} \tag{8.96}$$

where the results of (8.90), (8.91), (8.94), and (8.95) were used, and $\theta = \beta\ell$. Next, we solve (8.87) for the $i_j$ in terms of the $I$s:

$$i_1 = \frac{1}{2}(I_1 + I_2), \tag{8.97a}$$

$$i_2 = \frac{1}{2}(I_1 - I_2), \tag{8.97b}$$

$$i_3 = \frac{1}{2}(I_3 + I_4), \tag{8.97c}$$

$$i_4 = \frac{1}{2}(I_4 - I_3), \tag{8.97d}$$

and use these results in (8.96):

$$\begin{aligned} V_1 = &\frac{-j}{2}(Z_{0e}I_1 + Z_{0e}I_2 + Z_{0o}I_1 - Z_{0o}I_2)\cot\theta \\ &\frac{-j}{2}(Z_{0e}I_3 + Z_{0e}I_4 + Z_{0o}I_4 - Z_{0o}I_3)\csc\theta. \end{aligned} \tag{8.98}$$

This result yields the top row of the open-circuit impedance matrix $[Z]$ that describes the coupled line section. From symmetry, all other matrix elements can be found once the first row is known. The matrix elements are then

$$Z_{11} = Z_{22} = Z_{33} = Z_{44} = \frac{-j}{2}(Z_{0e} + Z_{0o})\cot\theta, \tag{8.99a}$$

$$Z_{12} = Z_{21} = Z_{34} = Z_{43} = \frac{-j}{2}(Z_{0e} - Z_{0o})\cot\theta, \tag{8.99b}$$

$$Z_{13} = Z_{31} = Z_{24} = Z_{42} = \frac{-j}{2}(Z_{0e} - Z_{0o})\csc\theta, \tag{8.99c}$$

$$Z_{14} = Z_{41} = Z_{23} = Z_{32} = \frac{-j}{2}(Z_{0e} + Z_{0o})\csc\theta. \tag{8.99d}$$

A two-port network can be formed from the coupled line section by terminating two of the four ports in either open or short circuits; there are ten possible combinations, as illustrated in Table 8.8. As indicated in this table, the various circuits have different frequency responses, including low-pass, bandpass, all pass, and all stop. For bandpass filters, we are most interested in the case shown in Figure 8.42c, as open circuits are easier to fabricate than are short circuits. In this case, $I_2 = I_4 = 0$, so the four-port impedance matrix equations reduce to

$$V_1 = Z_{11}I_1 + Z_{13}I_3, \tag{8.100a}$$

$$V_3 = Z_{31}I_1 + Z_{33}I_3, \tag{8.100b}$$

where $Z_{ij}$ is given in (8.99).

We can analyze the filter characteristics of this circuit by calculating the image impedance (which is the same at ports 1 and 3), and the propagation constant. From Table 8.1, the image impedance in terms of the $Z$-parameters is

$$\begin{aligned} Z_i &= \sqrt{Z_{11}^2 - \frac{Z_{11}Z_{13}^2}{Z_{33}}} \\ &= \frac{1}{2}\sqrt{(Z_{0e} - Z_{0o})^2 \csc^2\theta - (Z_{0e} + Z_{0o})^2 \cot^2\theta}. \end{aligned} \tag{8.101}$$

When the coupled line section is $\lambda/4$ long ($\theta = \pi/2$), the image impedance reduces to

$$Z_i = \frac{1}{2}(Z_{0e} - Z_{0o}), \tag{8.102}$$

which is real and positive, since $Z_{0e} > Z_{0o}$. But when $\theta \rightarrow 0$ or $\pi, Z_i \rightarrow \pm j\infty$, indicating a stopband. The real part of the image impedance is sketched in Figure 8.43, where the cutoff frequencies can be found from (8.101) as

$$\cos\theta_1 = -\cos\theta_2 = \frac{Z_{0e} - Z_{0o}}{Z_{0e} + Z_{0o}}.$$

The propagation constant can also be calculated from the results of Table 8.1 as

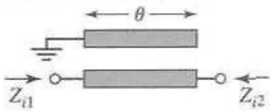
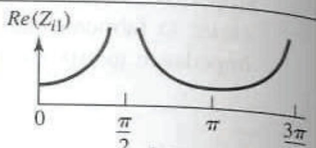
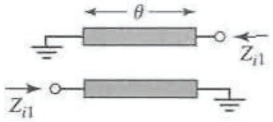
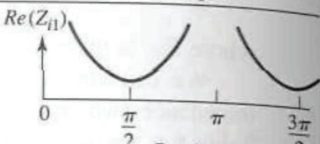
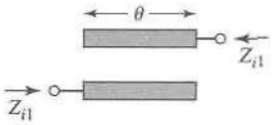
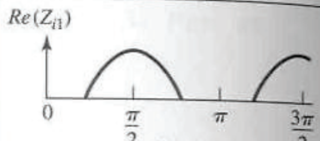
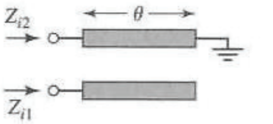
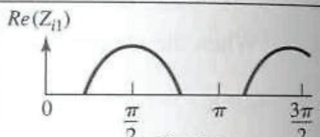
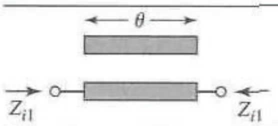
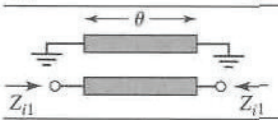
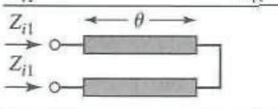
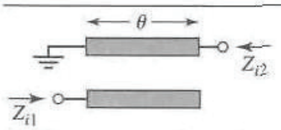
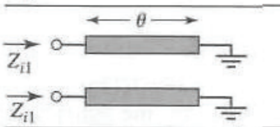
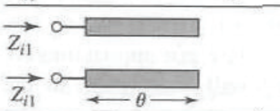
$$\cos\beta = \sqrt{\frac{Z_{11}Z_{33}}{Z_{13}^2}} = \frac{Z_{11}}{Z_{13}} = \frac{Z_{0e} + Z_{0o}}{Z_{0e} - Z_{0o}}\cos\theta, \tag{8.103}$$

which shows $\beta$ is real for $\theta_1 < \theta < \theta_2 = \pi - \theta_1$, where $\cos\theta_1 = (Z_{0e} - Z_{0o})/(Z_{0e} + Z_{0o})$.

### Design of Coupled Line Bandpass Filters

Narrowband bandpass filters can be made with cascaded coupled line sections of the form shown in Figure 8.42c. To derive the design equations for filters of this type, we first show that a single coupled line section can be approximately modeled by the equivalent circuit shown in Figure 8.44. We will do this by calculating the image impedance and propagation constant of the equivalent circuit and showing that they are approximately equal to those of the coupled line section for $\theta = \pi/2$, which will correspond to the center frequency of the bandpass response.

**TABLE 8.8 Ten Canonical Coupled Line Circuits**

| Circuit | Image Impedance | Response |
|---|---|---|
|  | $Z_{i1} = \dfrac{2Z_{0e}Z_{0o}\cos\theta}{\sqrt{(Z_{0e}+Z_{0o})^2\cos^2\theta-(Z_{0e}-Z_{0o})^2}}$ <br> $Z_{i2} = \dfrac{Z_{0e}Z_{0o}}{Z_{i1}}$ |  Low pass |
|  | $Z_{i1} = \dfrac{2Z_{0e}Z_{0o}\sin\theta}{\sqrt{(Z_{0e}-Z_{0o})^2-(Z_{0e}+Z_{0o})^2\cos^2\theta}}$ |  Bandpass |
|  | $Z_{i1} = \dfrac{\sqrt{(Z_{0e}-Z_{0o})^2-(Z_{0e}+Z_{0o})^2\cos^2\theta}}{2\sin\theta}$ |  Bandpass |
|  | $Z_{i1} = \dfrac{\sqrt{Z_{0e}Z_{0o}}\ \sqrt{(Z_{0e}-Z_{0o})^2-(Z_{0e}+Z_{0o})^2\cos^2\theta}}{(Z_{0e}+Z_{0o})\sin\theta}$ <br> $Z_{i2} = \dfrac{Z_{0e}Z_{0o}}{Z_{i1}}$ |  Bandpass |
|  | $Z_{i1} = \dfrac{Z_{0e}+Z_{0o}}{2}$ | All pass |
|  | $Z_{i1} = \dfrac{2Z_{0e}Z_{0o}}{Z_{0e}+Z_{0o}}$ | All pass |
|  | $Z_{i1} = \sqrt{Z_{0e}Z_{0o}}$ | All pass |
|  | $Z_{i1} = -j\dfrac{2Z_{0e}Z_{0o}}{Z_{0e}+Z_{0o}}\cot\theta$ <br> $Z_{i2} = \dfrac{Z_{0e}Z_{0o}}{Z_{i1}}$ | All stop |
|  | $Z_{i1} = j\sqrt{Z_{0e}Z_{0o}}\tan\theta$ | All stop |
|  | $Z_{i1} = -j\sqrt{Z_{0e}Z_{0o}}\cot\theta$ | All stop |

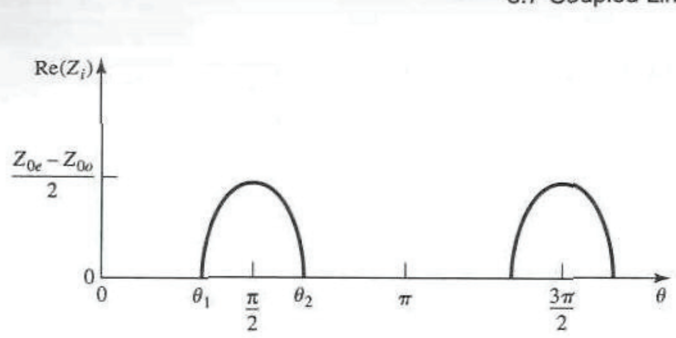



**FIGURE 8.43** The real part of the image impedance of the bandpass network of Figure 8.42c.

The $ABCD$ parameters of the equivalent circuit can be computed using the $ABCD$ matrices for transmission lines from Table 4.1:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} \cos\theta & jZ_0\sin\theta \\ \dfrac{j\sin\theta}{Z_0} & \cos\theta \end{bmatrix} \begin{bmatrix} 0 & -j/J \\ -jJ & 0 \end{bmatrix} \begin{bmatrix} \cos\theta & jZ_0\sin\theta \\ \dfrac{j\sin\theta}{Z_0} & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \left(JZ_0 + \dfrac{1}{JZ_0}\right)\sin\theta\cos\theta & j\left(JZ_0^2\sin^2\theta - \dfrac{\cos^2\theta}{J}\right) \\ j\left(\dfrac{1}{JZ_0^2}\sin^2\theta - J\cos^2\theta\right) & \left(JZ_0 + \dfrac{1}{JZ_0}\right)\sin\theta\cos\theta \end{bmatrix}. \quad 8.104$$

The $ABCD$ parameters of the admittance inverter were obtained by considering it as a quarter-wave length of transmission of characteristic impedance, $1/J$. From (8.27) the image impedance of the equivalent circuit is

$$Z_i = \sqrt{\frac{B}{C}} = \sqrt{\frac{JZ_0^2\sin^2\theta - (1/J)\cos^2\theta}{(1/JZ_0^2)\sin^2\theta - J\cos^2\theta}}, \quad 8.105$$

which reduces to the following value at the center frequency, $\theta = \pi/2$:

$$Z_i = JZ_0^2. \quad 8.106$$

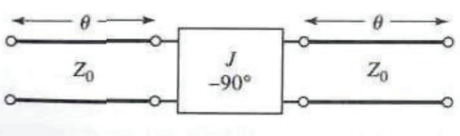



**FIGURE 8.44** Equivalent circuit of the coupled line section of Figure 8.42c.

From (8.31) the propagation constant is

$$\cos \beta = A = \left( JZ_0 + \frac{1}{JZ_0} \right) \sin\theta \cos\theta. \tag{8.107}$$

Equating the image impedances in (8.102) and (8.106), and the propagation constants of (8.103) and (8.107) yields the following equations:

$$\frac{1}{2}(Z_{0e} - Z_{0o}) = JZ_0^2,$$

$$\frac{Z_{0e} + Z_{0o}}{Z_{0e} - Z_{0o}} = JZ_0 + \frac{1}{JZ_0},$$

where we have assumed $\sin\theta \simeq 1$ for $\theta$ near $\pi/2$. These equations can be solved for the even- and odd-mode line impedances to give

$$Z_{0e} = Z_0[1 + JZ_0 + (JZ_0)^2], \tag{8.108a}$$

$$Z_{0o} = Z_0[1 - JZ_0 + (JZ_0)^2]. \tag{8.108b}$$

Now consider a bandpass filter composed of a cascade of $N + 1$ coupled line sections, as shown in Figure 8.45a. The sections are numbered from left to right, with the load on the right, but the filter can be reversed without affecting the response. Since each coupled line section has an equivalent circuit of the form shown in Figure 8.44, the equivalent circuit of the cascade is as shown in Figure 8.45b. Between any two consecutive inverters we have a transmission line section that is effectively $2\theta$ in length. This line is approximately $\lambda/2$ long in the vicinity of the bandpass region of the filter, and has an approximate equivalent circuit that consists of a shunt parallel $LC$ resonator, as in Figure 8.45c.

The first step in establishing this equivalence is to find the parameters for the $T$-equivalent and ideal transformer circuit of Figure 8.45c (an exact equivalent). The $ABCD$ matrix for this circuit can be calculated using the results in Table 4.1 for a $T$-circuit and an ideal transformer:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} \dfrac{Z_{11}}{Z_{12}} & \dfrac{Z_{11}^2 - Z_{12}^2}{Z_{12}} \\ \dfrac{1}{Z_{12}} & \dfrac{Z_{11}}{Z_{12}} \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} \dfrac{-Z_{11}}{Z_{12}} & \dfrac{Z_{12}^2 - Z_{11}^2}{Z_{12}} \\ \dfrac{-1}{Z_{12}} & \dfrac{-Z_{11}}{Z_{12}} \end{bmatrix}. \tag{8.109}$$

Equating this result to the $ABCD$ parameters for a transmission line of length $2\theta$ and characteristic impedance $Z_0$ gives the parameters of the equivalent circuit as

$$Z_{12} = \frac{-1}{C} = \frac{jZ_0}{\sin 2\theta}, \tag{8.110a}$$

$$Z_{11} = Z_{22} = -Z_{12}A = -jZ_0 \cot 2\theta. \tag{8.110b}$$

Then the series arm impedance is

$$Z_{11} - Z_{12} = -jZ_0 \frac{\cos 2\theta + 1}{\sin 2\theta} = -jZ_0 \cot\theta. \tag{8.111}$$

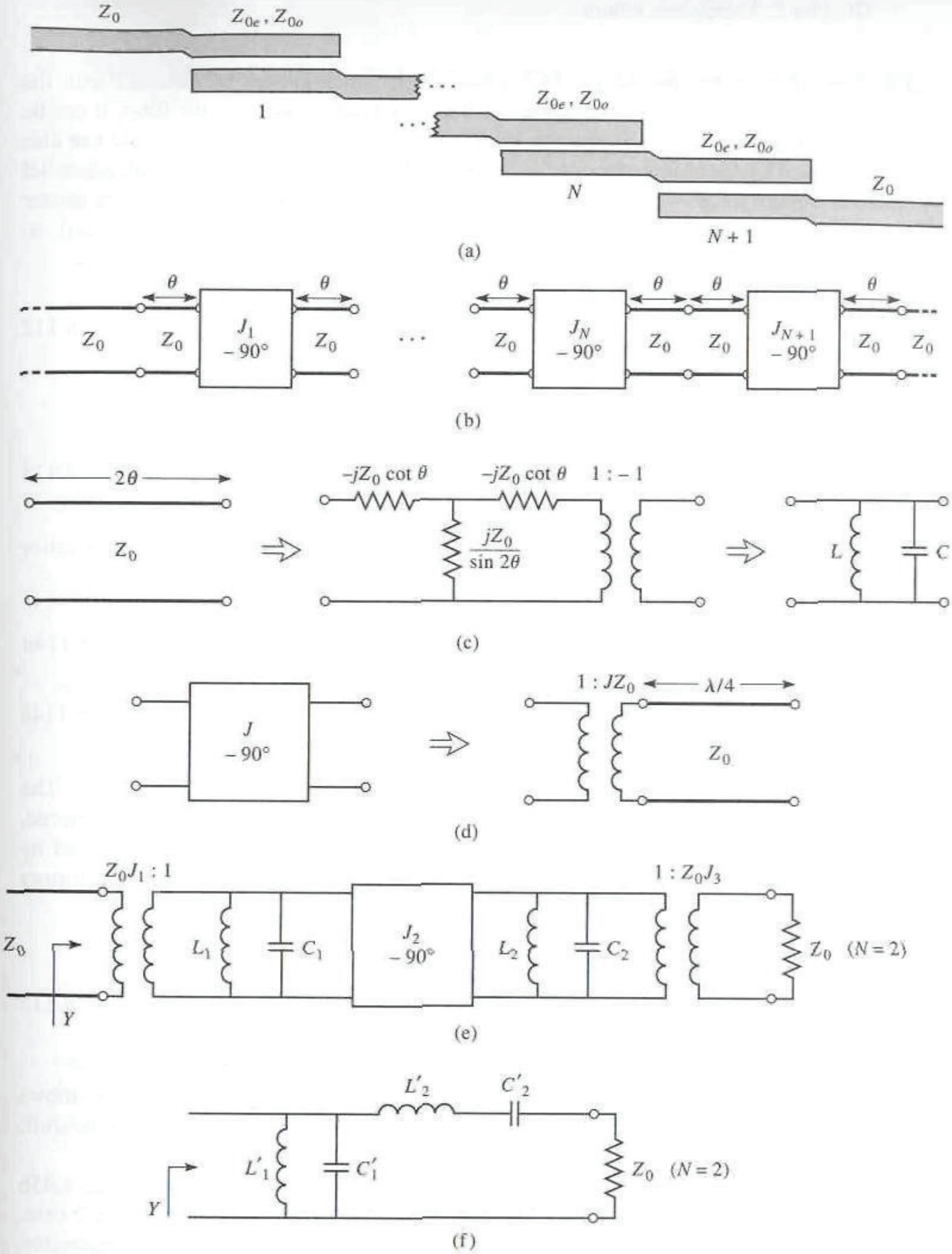



**FIGURE 8.45** Development of an equivalent circuit for derivation of design equations for a coupled line bandpass filter. (a) Layout of an $N+1$ section coupled line bandpass filter. (b) Using equivalent circuit of Figure 8.44 for each coupled line section. (c) Equivalent circuit for transmission lines of length $2\theta$. (d) Equivalent circuit of the admittance inverters. (e) Using results of (c) and (d) for the $N = 2$ case. (f) Lumped-element circuit for a bandpass filter for $N = 2$.

The 1:$-1$ transformer provides a 180° phase shift, which cannot be obtained with the $T$-network alone; since this does not affect the amplitude response of the filter, it can be discarded. For $\theta \sim \pi/2$ the series arm impedances of (8.111) are near zero, and can also be ignored. The shunt impedance $Z_{12}$, however, looks like the impedance of a parallel resonant circuit for $\theta \sim \pi/2$. If we let $\omega = \omega_0 + \Delta\omega$, where $\theta = \pi/2$ at the center frequency $\omega_0$, then we have $2\theta = \beta\ell = \omega\ell/v_p = (\omega_0 + \Delta\omega)\pi/\omega_0 = \pi(1 + \Delta\omega/\omega_0)$, so (8.110a) can be written for small $\Delta\omega$ as

$$Z_{12} = \frac{jZ_0}{\sin \pi(1 + \Delta\omega/\omega_0)} \simeq \frac{-jZ_0\omega_0}{\pi(\omega - \omega_0)}. \qquad 8.112$$

From Section 6.1 the impedance near resonance of a parallel $LC$ circuit is

$$Z = \frac{-jL\omega_0^2}{2(\omega - \omega_0)}, \qquad 8.113$$

with $\omega_0^2 = 1/LC$. Equating this to (8.112) gives the equivalent inductor and capacitor values as

$$L = \frac{2Z_0}{\pi\omega_0}, \qquad 8.114a$$

$$C = \frac{1}{\omega_0^2 L} = \frac{\pi}{2Z_0\omega_0}. \qquad 8.114b$$

The end sections of the circuit of Figure 8.45b require a different treatment. The lines of length $\theta$ on either end of the filter are matched to $Z_0$, and so can be ignored. The end inverters, $J_1$ and $J_{N+1}$, can each be represented as a transformer followed by a $\lambda/4$ section of line, as shown in Figure 8.45d. The $ABCD$ matrix of a transformer with a turns ratio $N$ in cascade with a quarter-wave line is

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} \frac{1}{N} & 0 \\ 0 & N \end{bmatrix} \begin{bmatrix} 0 & -jZ_0 \\ \frac{-j}{Z_0} & 0 \end{bmatrix} = \begin{bmatrix} 0 & \frac{-jZ_0}{N} \\ \frac{-jN}{Z_0} & 0 \end{bmatrix}. \qquad 8.115$$

Comparing this to the $ABCD$ matrix of an admittance inverter (part of (8.104)) shows that the necessary turns ratio is $N = JZ_0$. The $\lambda/4$ line merely produces a phase shift, and so can be ignored.

Using these results for the interior and end sections allows the circuit of Figure 8.45b to be transformed into the circuit of Figure 8.45e, which is specialized to the $N = 2$ case. We see that each pair of coupled line sections leads to an equivalent shunt $LC$ resonator, and an admittance inverter occurs between each pair of $LC$ resonators. Next, we show that the admittance inverters have the effect of transforming a shunt $LC$ resonator into a series $LC$ resonator, leading to the final equivalent circuit of Figure 8.45f (shown for $N = 2$). This will then allow the admittance inverter constants, $J_n$, to be determined from the element values of a low-pass prototype. We will demonstrate this for the $N = 2$ case.

With reference to Figure 8.45e, the admittance just to the right of the $J_2$ inverter is

$$j\omega C_2 + \frac{1}{j\omega L_2} + Z_0 J_3^2 = j\sqrt{\frac{C_2}{L_2}}\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right) + Z_0 J_3^2,$$

since the transformer scales the load admittance by the square of the turns ratio. Then the admittance seen at the input of the filter is

$$Y = \frac{1}{J_1^2 Z_0^2}\left\{j\omega C_1 + \frac{1}{j\omega L_1} + \frac{J_2^2}{j\sqrt{C_2/L_2}\left[(\omega/\omega_0) - (\omega_0/\omega)\right] + Z_0 J_3^2}\right\}$$

$$= \frac{1}{J_1^2 Z_0^2}\left\{j\sqrt{\frac{C_1}{L_1}}\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right) + \frac{J_2^2}{j\sqrt{C_2/L_2}\left[(\omega/\omega_0) - (\omega_0/\omega)\right] + Z_0 J_3^2}\right\}. \quad 8.116$$

These results also use the fact, from (8.114), that $L_n C_n = 1/\omega_0^2$ for all $LC$ resonators. Now the admittance seen looking into the circuit of Figure 8.45f is

$$Y = j\omega C_1' + \frac{1}{j\omega L_1'} + \frac{1}{j\omega L_2' + (1/j\omega C_2') + Z_0}$$

$$= j\sqrt{\frac{C_1'}{L_1'}}\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right) + \frac{1}{j\sqrt{L_2'/C_2'}\left[(\omega/\omega_0) - (\omega_0/\omega)\right] + Z_0}, \quad 8.117$$

which is identical in form to (8.116). Thus, the two circuits will be equivalent if the following conditions are met:

$$\frac{1}{J_1^2 Z_0^2}\sqrt{\frac{C_1}{L_1}} = \sqrt{\frac{C_1'}{L_1'}}, \quad 8.118a$$

$$\frac{J_1^2 Z_0^2}{J_2^2}\sqrt{\frac{C_2}{L_2}} = \sqrt{\frac{L_2'}{C_2'}}, \quad 8.118b$$

$$\frac{J_1^2 Z_0^3 J_3^2}{J_2^2} = Z_0. \quad 8.118c$$

We know $L_n$ and $C_n$ from (8.114); $L_n'$ and $C_n'$ are determined from the element values of a lumped-element low-pass prototype which has been impedance scaled and frequency transformed to a bandpass filter. Using the results in Table 8.6 and the impedance scaling formulas of (8.64) allows the $L_n'$ and $C_n'$ values to be written as

$$L_1' = \frac{\Delta Z_0}{\omega_0 g_1}, \quad 8.119a$$

$$C_1' = \frac{g_1}{\Delta\omega_0 Z_0}, \quad 8.119b$$

$$L_2' = \frac{g_2 Z_0}{\Delta\omega_0}, \quad 8.119c$$

$$C_2' = \frac{\Delta}{\omega_0 g_2 Z_0}, \quad 8.119d$$

where $\Delta = (\omega_2 - \omega_1)/\omega_0$ is the fractional bandwidth of the filter. Then (8.118) can be solved for the inverter constants with the following results (for $N = 2$):

$$J_1 Z_0 = \left(\frac{C_1 L_1'}{L_1 C_1'}\right)^{1/4} = \sqrt{\frac{\pi\Delta}{2g_1}}, \tag{8.120a}$$

$$J_2 Z_0 = J_1 Z_0^2 \left(\frac{C_2 C_2'}{L_2 L_2'}\right)^{1/4} = \frac{\pi\Delta}{2\sqrt{g_1 g_2}}, \tag{8.120b}$$

$$J_3 Z_0 = \frac{J_2}{J_1} = \sqrt{\frac{\pi\Delta}{2g_2}}. \tag{8.120c}$$

After the $J_n$s are found, $Z_{0e}$ and $Z_{0o}$ for each coupled line section can be calculated from (8.108).

The above results were derived for the special case of $N = 2$ (three coupled line sections), but more general results can be derived for any number of sections, and for the case where $Z_L \neq Z_0$ (or $g_{N+1} \neq 1$, as in the case of an equal-ripple response with $N$ even). Thus, the design equations for a bandpass filter with $N + 1$ coupled line sections are

$$Z_0 J_1 = \sqrt{\frac{\pi\Delta}{2g_1}}, \tag{8.121a}$$

$$Z_0 J_n = \frac{\pi\Delta}{2\sqrt{g_{n-1} g_n}}, \qquad \text{for } n = 2, 3, \ldots, N, \tag{8.121b}$$

$$Z_0 J_{N+1} = \sqrt{\frac{\pi\Delta}{2 g_N g_{N+1}}}. \tag{8.121c}$$

The even and odd mode characteristic impedances for each section are then found from (8.108).

**EXAMPLE 8.8 Coupled Line Bandpass Filter Design**

Design a coupled line bandpass filter with $N = 3$ and a 0.5 dB equal-ripple response. The center frequency is 2.0 GHz, the bandwidth is 10%, and $Z_0 = 50\,\Omega$. What is the attenuation at 1.8 GHz?

*Solution*

The fractional bandwidth is $\Delta = 0.1$. We can use Figure 8.27a to obtain the attenuation at 1.8 GHz, but first we must use (8.71) to convert this frequency to the normalized low-pass form ($\omega_c = 1$):

$$\omega \leftarrow \frac{1}{\Delta}\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right) = \frac{1}{0.1}\left(\frac{1.8}{2.0} - \frac{2.0}{1.8}\right) = -2.11.$$

Then the value on the horizontal scale of Figure 8.27a is

$$\left|\frac{\omega}{\omega_c}\right| - 1 = |-2.11| - 1 = 1.11,$$

which indicates an attenuation of about 20 dB for $N = 3$.

The low-pass prototype values, $g_n$, are given in Table 8.4; then (8.121) can be used to calculate the admittance inverter constants, $J_n$. Finally, the even- and odd-mode characteristic impedances can be found from (8.108). These results are summarized in the following table:

| $n$ | $g_n$ | $Z_0 J_n$ | $Z_{0e}(\Omega)$ | $Z_{0o}(\Omega)$ |
|---|---|---|---|---|
| 1 | 1.5963 | 0.3137 | 70.61 | 39.24 |
| 2 | 1.0967 | 0.1187 | 56.64 | 44.77 |
| 3 | 1.5963 | 0.1187 | 56.64 | 44.77 |
| 4 | 1.0000 | 0.3137 | 70.61 | 39.24 |

Note that the filter sections are symmetric about the midpoint. The calculated response of this filter is shown in Figure 8.46; passbands also occur at 6 GHz, 10 GHz, etc.

Many other types of filters can be constructed using coupled line sections; most of these are of the bandpass or bandstop variety. One particularly compact design is the interdigitated filter, which can be obtained from a coupled line filter by folding the lines at their midpoints; see [1] and [3] for details. ○

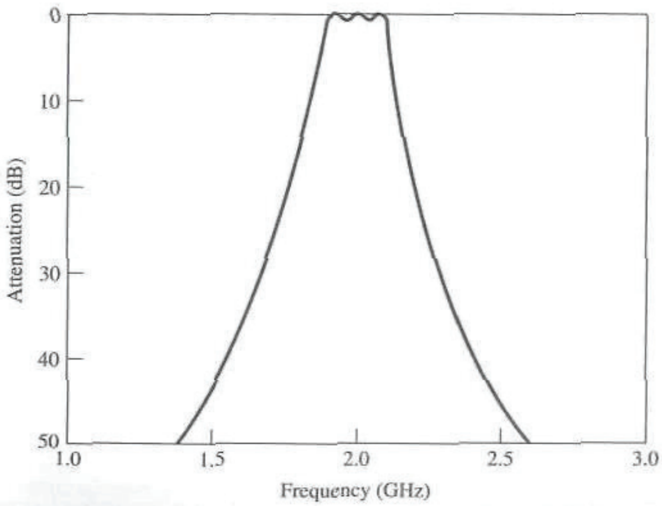



**FIGURE 8.46** Amplitude response of the coupled line bandpass filter of Example 8.8.