A symmetric matrix $\Rightarrow$ real eigenvalues
orthonormal eigenvectors

$A x = \lambda x$ $\quad$ x eigenvector corresponding to λ $\quad$ $\|x_i\| = 1$

$X : X(:,j) = x_j$ corresponding to λ_j

$AX = X\Lambda$ $\quad$ $\Lambda(i,i) = \lambda_i$ diagonal

$$X^T A X = \Lambda$$

$$A = X \Lambda X^T$$

$$A y_E = X \Lambda X^T y_E$$

$$X^T y_E = \begin{bmatrix} x_1^T y_E \\ x_2^T y_E \\ \vdots \\ x_n^T y_E \end{bmatrix} = y_X$$

$$= X \Lambda y_X$$

$$= X \cdot \begin{bmatrix} \lambda_1 x_1^T y_E \\ \lambda_2 x_2^T y_E \\ \vdots \\ \lambda_n x_n^T y_E \end{bmatrix}$$

$$X_1 = \begin{bmatrix} | \\ x_1 \; - x_1^T - \\ | \end{bmatrix} = x_1 x_1^T$$

$$= (\lambda_1 x_1^T y_E) x_1 + (\lambda_2 x_2^T y_E) x_2 + \dots + (\lambda_n x_n^T y_E) x_n$$

$$= \lambda_1 X_1 y_E + \lambda_2 X_2 y_E + \dots + \lambda_n X_n y$$

$$= \left(\sum_{i=1}^{n} (\lambda_i X_i) \right) y_E$$

$$A = \sum_{i=1}^{n} \lambda_i X_i$$

$$A = \lambda_1 X_1 + \sum_{i=2}^{n} \lambda_i X_i$$

$$A_1 = A - \lambda_1 X_1 = \sum_{i=2}^{n} \lambda_i X_i$$

$$A_1 x_1 = A x_1 - \lambda_1 x_1 \underbrace{x_1^T x_1}_{\|x_1\|_2^2 = 1} = \lambda_1 x_1 - \lambda_1 x_1 = 0\, x_1$$

$$A_1 = \sum_{i=2}^{n} \lambda_i X_i \qquad |\lambda_2| > |\lambda_3| \geq |\lambda_4| \dots \geq |\lambda_n| \geq |0|$$

$$\begin{matrix} \lambda_2, & \lambda_3, & \dots, & \lambda_n, & 0 \\ \downarrow & \downarrow & & \downarrow & \\ x_2 & x_3 & & x_n, & x_1 \end{matrix}$$

$$A_2 = A_1 - \lambda_2 X_2 = \sum_{i=3}^{n} \lambda_i X_i$$

$$\begin{matrix} \lambda_3, & \lambda_4, & \dots, & \lambda_n, & 0, & 0 \\ x_3 & x_4 & & x_n, & x_1, & x_2 \end{matrix}$$

$$A_2 x_2 = A_1 x_2 - \lambda_2 X_2 = \lambda_2 x_2 - \lambda_2 X_2 x_2 = 0\, x_2$$

A not symmetric $\qquad$ X not orthonormal

$$AX = X\Lambda$$

$$A = X\Lambda X^{-1}$$

$$A y_E = X\Lambda (X^{-1} y_E)$$

$$= \underbrace{X \underbrace{\Lambda\, y_X}_{\text{rescaling}}}_{\text{change of basis back to the canonical basis}}$$

$$A = \sum_{i=1}^{n} \lambda_i \underbrace{[X(:,i)\ X^{-1}(i,:)]}_{X_i}$$

Similar matrices

Let S be a non-singular matrix

The matrix $\qquad B = S^{-1} A S$ is said <u>similar</u> to $A = SBS^{-1}$

Theorem

Two similar matrices have the same eigenvalues with the same algebraic and geometric multiciplity

Proof

$$A = SBS^{-1}$$

$$\det(A - \lambda I) = \det(SBS^{-1} - \lambda S I S^{-1}) = \det(S(B-\lambda I)S^{-1})$$

$$= \det(S) \cdot \det(B - \lambda I) \cdot \det(S^{-1})$$

$$= \cancel{\det(S)} \cdot \frac{1}{\cancel{\det(S)}} \cdot \det(B - \lambda I) = \det(B - \lambda I)$$

$$Ax = \lambda x$$

$$SBS^{-1} x = \lambda x$$

$$S^{-1} S B S^{-1} x = S^{-1} \lambda x$$

$$B(S^{-1} x) = \lambda (S^{-1} x)$$

$$By = \lambda y \qquad y = S^{-1} x \qquad Sy = x$$

y unique for each x because S is full rank

x l.i. $\Rightarrow$ y l.i.

$0 = \alpha_1 x_1 + \dots + \alpha_n x_n \qquad \alpha_i = 0 \quad \forall i$

$y_i = S^{-1} x_i$

$\exists \beta_i$ not all zero such that

$0 = \beta_1 y_1 + \dots + \beta_n y_n$

$0 = \beta_1 S^{-1} x_1 + \dots + \beta_n S^{-1} x_n = S^{-1} (\beta_1 x_1 + \dots + \beta_n x_n)$

S^{-1} full rank $\Rightarrow$ $\ker(S^{-1}) = \{0\}$

$\beta_1 x_1 + \dots + \beta_n x_n = 0$ contraddiction $\Leftrightarrow$ x_i are not l.i.

Theorem (Schur canonical form)

Let $A \in \mathbb{R}^{n \times n}$ and $\lambda_1, \dots, \lambda_n$ its eigenvalues, then there exist an orthogonal matrix U and an upper triangular matrix T whose pricipal elements (diagonal elements) are the eigenvalues λ_i $i = 1, \dots, n$

$$A = U T U^T \qquad T = \begin{bmatrix} \lambda_1 & \cdots & \cdots \\ & \lambda_2 & \cdots \\ 0 & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

$$T = U^T A U$$

Deflation method

A and let us assume that we know λ_1, x_1 $\quad (\|x_1\|_2 = 1)$

$P_1 x_1 = e_1 \qquad P_1$ Householder matrix $\quad P_1 \in \mathbb{R}^{n \times n}$

$P_1 = I - 2 \frac{\tilde{u}\tilde{u}^T}{\|\tilde{u}\|_2^2} \qquad \tilde{u} = x_1 + \sigma e_1$

$P_1 = P_1^T = P_1^{-1}$

$\sigma = \text{sign}(x_1(1)) \|x_1\|_2$
$= \text{sing}(x_1(1))$

$A_1 = A \qquad A_1 x_1 = \lambda_1 x_1$

$P_1 A_1 I x_1 = \lambda_1 P_1 x_1$

$P_1 A_1 P_1^{-1} \underbrace{P_1 x_1}_{e_1} = \lambda_1 \underbrace{P_1 x_1}_{e_1} \qquad P_1^{-1} P_1 = P_1^T P_1 = I \in \mathbb{R}^{n \times n}$

$\underbrace{P_1 A_1 P_1^{-1}}_{B_1} e_1 = \lambda_1 e_1$

$B_1 e_1 = \lambda_1 e_1 = \begin{bmatrix} \lambda_1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ first column of B_1

$B_1 = \begin{bmatrix} \lambda_1 & b_1^T \\ 0 & A_2 \end{bmatrix} \qquad b_1 \in \mathbb{R}^{(n-1)}$

$A_2 \in \mathbb{R}^{(n-1) \times (n-1)}$

$P_1 = S^{-1}$ invertible A and $B_1 = P_1 A P_1^{-1}$ are similar and they have the same eigenvalues

A_2 has $\lambda_2, \dots, \lambda_n$ as eigenvalues

$\exists\, x_2^{(n-1)} \in \mathbb{R}^{n-1} : A_2 x_2^{(n-1)} = \lambda_2 x_2^{(n-1)}$ $\quad \|x_2^{(n-1)}\|_2 = 1 \quad$ $x_2^{(n-1)}$ eigenvector of A_2 corresponding to λ_2

$P_2^{(n-1)} x_2^{(n-1)} = e_1^{(n-1)} \qquad e_1^{(n-1)} \in \mathbb{R}^{n-1} \qquad e_1^{(n-1)} = \begin{array}{c} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \end{array} \begin{array}{l} 1 \\ \\ \\ n-1 \end{array}$

$$P_2 = \begin{bmatrix} 1 & \cdots 0 \cdots \\ \vdots & \\ 0 & P_2^{(n-1)} \\ \vdots & \end{bmatrix} \in \mathbb{R}^{n \times n} \qquad \bar{x}_2 = \begin{bmatrix} \alpha \\ x_2^{(n-1)} \end{bmatrix} \in \mathbb{R}^n$$

$$P_2 \bar{x}_2 = \begin{bmatrix} 1 & 0^T \\ 0 & P_2^{(n-1)} \end{bmatrix} \begin{bmatrix} \alpha \\ x_2^{(n-1)} \end{bmatrix} = \begin{bmatrix} \alpha \\ P_2^{(n-1)} x_2^{(n-1)} \end{bmatrix} = \begin{bmatrix} \alpha \\ e_1^{(n-1)} \end{bmatrix}$$

$$B_2 = P_2 B_1 P_2^{-1} = \begin{bmatrix} 1 & 0^T \\ 0 & P_2^{(n-1)} \end{bmatrix} \begin{bmatrix} \lambda_1 & b_1^T \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} 1 & 0^T \\ 0 & P_2^{(n-1)^{-1}} \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_1 & b_1^T \\ 0 & P_2^{(n-1)} A_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & P_2^{(n-1)^{-1}} \end{bmatrix} = \begin{bmatrix} \lambda_1 & b_1^T P_2^{(n-1)^{-1}} \\ 0 & P_2^{(n-1)} A_2 P_2^{(n-1)^{-1}} \end{bmatrix}$$

$B_1 \bar{x}_2 = \lambda_2 \bar{x}_2$

$$\begin{bmatrix} \lambda_1 & b_1^T \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} \alpha \\ x_2^{(n-1)} \end{bmatrix} = \lambda_2 \begin{bmatrix} \alpha \\ x_2^{(n-1)} \end{bmatrix}$$

if $\alpha = \dfrac{b_1^T x_2^{(n-1)}}{\lambda_2 - \lambda_1}$

$\bar{x}_2$ is the eigenvector of B_1 corresponding to λ_2

$\lambda_1 \alpha + b_1^T x_2^{(n-1)} = \lambda_2 \alpha$

$A_2 x_2^{(n-1)} = \lambda_2 x_2^{(n-1)}$ always true by construction

$$\alpha = \frac{b_1^T x_2^{(n-1)}}{\lambda_2 - \lambda_1}$$

$P_2^{-1} = P_2^T = P_2$

$$\underbrace{P_2 B_1 P_2^{-1}}_{B_2} \underbrace{P_2 \bar{x}_2}_{\begin{bmatrix} \alpha \\ e_1^{(n-1)} \end{bmatrix}} = \lambda_2 \underbrace{P_2 \bar{x}_2}_{\begin{bmatrix} \alpha \\ e_1^{(n-1)} \end{bmatrix}}$$

$P_2^{(n-1)} x_2^{(n-1)} = e_1^{(n-1)}$

$P_2 \bar{x}_2 = \begin{bmatrix} \alpha \\ e_1^{(n-1)} \end{bmatrix} =$

$\left(P_2^{(n-1)}\right)^{-1} = P_2^{(n-1)T} = P_2^{(n-1)}$

$$\begin{bmatrix} 1 & 0^T \\ 0 & P_2^{(n-1)} \end{bmatrix} \begin{bmatrix} \lambda_1 & b_1^T \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} 1 & 0^T \\ 0 & P_2^{(n-1)T} \end{bmatrix} = \begin{bmatrix} \lambda_1 & b_1^T \\ 0 & P_2^{(n-1)} A_2 \end{bmatrix} \begin{bmatrix} 1 & 0^T \\ 0 & P_2^{(n-1)T} \end{bmatrix} = \begin{bmatrix} \lambda_1 & b_1^T P_2^{(n-1)T} \\ 0 & P_2^{(n-1)} A_2 P_2^{(n-1)T} \end{bmatrix}$$

$P_2^{(n-1)} A_2 P_2^{(n-1)T}$ is similar to A_2

What is the structure of A_3?

$A_2 x_2^{(n-1)} = \lambda_2 x_2^{(n-1)}$

$$\underbrace{P_2^{(n-2)} A_2 P_2^{(n-2)T}} P_2^{(n-2)} x_2^{(n-2)} = \lambda_2 P_2^{(n-2)} x_2^{(n-2)}$$

$$e_1^{(n-2)} = \lambda_2 e_1^{(n-1)}$$

$$\begin{bmatrix} \lambda_2 & b_2^T \\ 0 & A_3 \end{bmatrix} \begin{bmatrix} 1 \\ \vdots \\ 0 \end{bmatrix} = \lambda_2 \begin{bmatrix} 1 \\ \vdots \\ 0 \end{bmatrix}$$

$$B_2 = \begin{bmatrix} \lambda_1 & b_1^T \\ 0^{(n-2)} & \begin{bmatrix} \lambda_2 & b_2^T \\ 0^{(n-2)} & A_3 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \lambda_1 & - b_1^T - & \\ 0^{n-1} & \lambda_2 & - b_2^T - \\ & 0^{(n-2)} & A_3 \end{bmatrix}$$

Given A_3 λ_3 $x_3^{(n-2)}$ compute $A_3 x_3^{(n-2)} = \lambda_3 x_3^{(n-2)}$ with power method

$$P_3^{(n-2)} x_3^{(n-2)} = e_1^{(n-2)} \qquad \bar{x}_3 = \begin{bmatrix} \alpha \\ \beta \\ x_3^{(n-2)} \end{bmatrix}$$

$$\rightarrow \quad B_2 \bar{x}_3 = \lambda_3 \bar{x}_3$$

$$\begin{cases} \lambda_1 \alpha + b_1^T \begin{bmatrix} \beta \\ x_2^{(n-2)} \end{bmatrix} = \lambda_3 \alpha \\ \lambda_2 \beta + b_2^T x_3^{(n-2)} = \lambda_3 \beta \\ A_3 x_3^{(n-2)} = \lambda_3 x_3^{(n-2)} \end{cases} \qquad \alpha = \frac{b_1^T \begin{bmatrix} \beta \\ x_2^{(n-2)} \end{bmatrix}}{\lambda_3 - \lambda_1} \quad \beta = \frac{b_2^T x_3^{(n-2)}}{\lambda_3 - \lambda_2}$$

always true by construction

$$P_3 = \left[\begin{array}{cc|c} 1 & 0 & 0^T \\ 0 & 1 & 0^T \\ \hline 0 & 0 & P_3^{(n-2)} \end{array}\right] \qquad P_3 \bar{x}_3 = \begin{bmatrix} \alpha \\ \beta \\ P_3^{(n-2)} x_3^{(n-2)} \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \\ e_1^{(n-2)} \end{bmatrix}$$

$$\underbrace{P_3 B_2 P_3^T} P_3 \bar{x}_3 = \lambda_3 P_3 \bar{x}_3$$

$$B_3 \begin{bmatrix} \alpha \\ \beta \\ e_1^{(n-2)} \end{bmatrix} = \lambda_3 \begin{bmatrix} \alpha \\ \beta \\ e_1^{(n-2)} \end{bmatrix} \qquad b_1^T = \begin{bmatrix} b_{11} & b_{12}^T \end{bmatrix}$$

$$B_3 = P_3 B_2 P_3^T = \left[\begin{array}{c|cc} \lambda_1 & b_1^T & \\ \hline 0^{(n-1)} & \lambda_2 & b_2^T \\ \cline{2-3} & 0 & P_3^{(n-2)} A_3 \end{array}\right] P_3^T = \left[\begin{array}{cc|c} \lambda_1 & b_{11} & b_{12}^T P_3^{(n-2)T} \\ 0 & \lambda_2 & b_2^T P_3^{(n-2)T} \\ \hline 0^{(n-2)} & 0^{(n-2)} & P_3^{(n-2)} A_3 P_3^{(n-2)T} \end{array}\right]$$

$$A_3 x_3^{(n-2)} = \lambda_3 x_3^{(n-2)}$$

$$\underbrace{P_3^{(n-2)} A_3 P_3^{(n-2)T}} \underbrace{P_3^{(n-2)} x_3^{(n-2)}} = \lambda_3 \underbrace{P_3^{(n-2)} x_3^{(n-2)}}$$

$$\left[P_3^{(n-2)} A_3 P_3^{(n-2)T}\right] e_1^{(n-2)} = \lambda_3 e_1^{(n-2)}$$

$$\begin{bmatrix} \lambda_3 & b_3^T \\ 0^{(n-3)} & A_4 \end{bmatrix} \begin{bmatrix} 1 \\ 0^{(n-3)} \end{bmatrix} = \lambda_3 \begin{bmatrix} 1 \\ 0^{(n-3)} \end{bmatrix}$$

$A_4 \in \mathbb{R}^{n-3 \times n-3}$ compute $\lambda_4 \; x_4^{(n-3)}$ by the power method

$$P_4^{(n-3)} x_4^{(n-3)} = e_1^{(n-3)}$$

$$\bar{x}_4 = \begin{bmatrix} \alpha \\ \beta \\ \gamma \\ x_4^{(n-3)} \end{bmatrix}$$

$\lambda_1, x_1 \; ; \; \lambda_2, \bar{x}_2 \; ; \; \lambda_3, \bar{x}_3 \; ;$

$$B_1 \bar{x}_2 = \lambda_2 \bar{x}_2 \qquad B_2 \bar{x}_3 = \lambda_3 \bar{x}_3$$

$$B_1 = P_1 A P_1^T \qquad B_2 = P_2 B_1 P_2^T .$$

$$P_1^T \left(P_1 A P_2^T \bar{x}_2 = \lambda_2 \bar{x}_2 \right)$$

$$A P_1^T \bar{x}_2 = \lambda_2 P_1^T \bar{x}_2$$

$$A x_2 = \lambda_2 x_2 \qquad x_2 = P_2^T \bar{x}_2$$

$$B_2 \bar{x}_3 = \lambda_3 \bar{x}_3$$

$$P_2 \underbrace{(P_1 A P_1^T)}_{B_1} P_2^T \bar{x}_3 = \lambda_3 \bar{x}_3$$

$$\cancel{P_1^T} \cancel{P_2^T} \cdot \quad \cancel{P_2} \cancel{P_1} A P_2^T P_2^T \bar{X}_3 = \lambda_3 P_2^T P_2^T \bar{x}_3$$

$$A P_1^T P_2^T \bar{x}_3 = \lambda_3 P_2^T P_2^T \bar{x}_3 \qquad x_3 = P_1^T P_2^T \bar{x}_3$$

$$A x_3 = \lambda_3 x_3$$