Laczos , Reyleigh quotient $\Longrightarrow$ λ_1, λ_n of A symmetric

Theorem

Let $A \in \mathbb{R}^{n\times n}$ symmetric and $U \in \mathbb{R}^{n \times n-1}$ such that $U^T U = I_{n-1}$

Let $B = U^T A U \in \mathbb{R}^{n-1 \times n-1}$.

Then the eigenvalues $\lambda_1, \dots, \lambda_n$ of A and

the eigenvalues $\mu_1, \dots, \mu_{n-1}$ of B

$$\lambda_n \le \mu_{n-1} \le \lambda_{n-1} \le \dots \le \mu_2 \le \lambda_2 \le \mu_1 \le \lambda_1$$

$U_1 \in \mathbb{R}^{n\times 1}$ $\quad B_1 = U_1^T A U_1 \quad \mu_1^{(1)} \quad \lambda_n \le \mu_1^{(1)} \le \lambda_1$

$U_2 \in \mathbb{R}^{n\times 2}$ $\quad B_2 = U_2^T A U_2 \quad \lambda_n < \mu_2^{(2)} < \mu_1^{(1)} < \mu_1^{(2)} < \lambda_1$

$U_3 \in \mathbb{R}^{n\times 3}$ $\quad B_3 = U_3^T A U_3 \quad \mu_3^{(3)} < \mu_2^{(2)} < \dots \mu_1^{(2)} < \mu_1^{(3)}$

$U_4 \in \mathbb{R}^{n\times 3}$ $\quad B_4 = U_4^T A U_4 \quad \mu_4^{(4)} < \mu_3^{(3)} \quad < \mu_1^{(3)} < \mu_1^{(4)}$

$U_n \in \mathbb{R}^{n\times n-1}$ $\quad \vdots$

$U_n \in \mathbb{R}^{n\times n}$ $\quad B_n = U_n^T A U_n \quad \lambda_n < \mu_{n-1}^{(n-1)} \le \quad \mu_1^{(n-1)} < \lambda_1$

Bn symiler to A $\quad \lambda_n = \mu_n^{(n)} \quad \lambda_1 = \mu_1^{(n)}$

Corollary

Let $A \in \mathbb{R}^{n\times n}$ symmetric

$$\lambda_1 = \max_{\substack{x \neq 0 \\ x \in \mathbb{R}^n}} R_A(x) = \max_{\substack{x \neq 0 \\ x \in \mathbb{R}^n}} \frac{x^T A x}{x^T x}$$

$$\lambda_n = \min_{\substack{x \neq 0 \\ x \in \mathbb{R}^n}} R_A(x) = \min_{\substack{x \neq 0 \\ x \in \mathbb{R}^n}} \frac{x^T A x}{x^T x}$$

Moreover, the functional $R_A(x) : \mathbb{R}^n \to \mathbb{R}$ is stationary in v if and only if $Av = \lambda v$ $\quad \lambda = R_A(v)$

$$R_2 = \frac{x^T A x}{x^T x} \qquad \frac{\partial x}{\partial x_i} = e_i$$

$$(\nabla R_2)_i(x) = \frac{\partial R_2}{\partial x_i}(x) = \left(\frac{\partial}{\partial x_i}(x^T A x) \cdot x^T x - \frac{\partial}{\partial x_i}(x^T x)\; x^T A x\right) \Big/ (x^T x)^2$$

$$= \left((e_i A x + x^T A e_i)\, x^T x - (e_i^T x + x^T e_i)\, x^T A x\right) \Big/ (x^T x)^2$$

$$= 2 e_i^T A x \frac{\cancel{x^T x}}{(x^T x)^{\cancel{2}}} - 2 e_i^T x \frac{x^T A x}{(x^T x)^2}$$

$$= \frac{2 e_i^T}{x^T x}\left(Ax - \frac{x^T A x}{x^T x} x\right) = \frac{2 e_i^T}{x^T x}\left(Ax - R_2(x)\, x\right)$$

$$= \frac{2}{x^T x} e_i^T \left[Ax - R_2(x)\, x\right]$$

$$\nabla R_2(x) = \frac{2}{x^T x}\left[Ax - R_2(x)\, x\right] \qquad \boxed{\nabla R_2(x) \in \text{span}(x, Ax)}$$

$x_1 \qquad R_2(x_1) \nearrow x_2^{\uparrow} = x_1 + \delta_1^{\uparrow} \nabla R_2(x_1) \qquad R_2(x_2^{\uparrow}) > R_2(x_1) \qquad \nabla R_2(x_1) \in \text{span}(x_1, Ax_1)$

$\searrow x_2^{\downarrow} = x_1 - \delta_1^{\downarrow} \nabla R_2(x_1) \qquad R_2(x_2^{\downarrow}) < R_2(x_1)$

$x_2^{\uparrow}, x_2^{\downarrow} \in \text{span}(x_1, \nabla R_2(x_1))$

$x_2^{\uparrow} \qquad R_2(x_2^{\uparrow}) \Rightarrow x_3^{\uparrow} = x_2^{\uparrow} + \delta_2^{\uparrow} \nabla R_2(x_2^{\uparrow}) \qquad \nabla R_2(x_2^{\uparrow}) \in \text{span}(x_2^{\uparrow}, A x_2^{\uparrow})$

$\nabla R_2(x_2^{\uparrow}) \in \text{span}(x_1, Ax_1, A^2 x_1)$

$x_3^{\uparrow} \in \text{span}(x_1, Ax_1, A^2 x_1)$

$x_2^{\downarrow} \qquad R_2(x_2^{\downarrow}) \Rightarrow x_3^{\downarrow} = x_2^{\downarrow} - \delta_2^{\downarrow} \nabla R_2(x_2^{\downarrow})$

$\nabla R_2(x_2^{\downarrow}) \in \text{span}(x_1, Ax_1, A^2 x_1)$

$x_3^{\downarrow} \in \text{span}(x_1, Ax_1, A^2 x_1)$

$U_1 \to V_1$

$U_2 \to V_2$

$$V_k^T A V_k = T_k = \begin{bmatrix} \alpha_1 & \beta_2 & & \\ \beta_2 & \alpha_2 & \beta_3 & \\ & \beta_3 & \ddots & \ddots \\ & & \ddots & \alpha_n \end{bmatrix}$$

$$A V_k = V_k T_k$$

$$A v_j = \beta_j v_{j-1} + \alpha_j v_j + \beta_{j+1} v_{j+1}$$

$$v_j^T A v_j = \beta_j \underbrace{\cancel{v_j^T v_{j-1}}}_{0} + \alpha_j \underbrace{v_j^T v_j}_{=1} + \beta_{j+1} \underbrace{\cancel{v_j^T v_{j+1}}}_{0}$$

$$\beta_{j+1} v_{j+1} = -\beta_j v_{j-1} + (A - \alpha_j I) v_j$$

$$w = -\beta_j v_{j-1} + (A - \alpha_j I) v_j \qquad \neq 0$$

$$\beta_{j+1} = \| w \|$$

$$v_{j+1} = w / \beta_{j+1}$$

Symmetric Laczos Algorithm

choose v_1, $\|v_1\| = 1$, $\beta_1 = 0$, $v_0 = 0$

for $j = 1, \dots, k$

$w = A v_j - \beta_j v_{j-1}$

$\alpha_j = (v_j, w)$

$w = w - \alpha_j v_j \qquad (w = -\beta_j v_{j-1} + A v_j - \alpha_j v_j)$

$\beta_{j+1} = \| w \|_2$

If $\beta_{j+1} = 0$ invariant space

else $v_{j+1} = w / \beta_{j+1}$

end

T_n power method $T_n \to \lambda_1$

inverse power method $T_n \to \lambda_n \qquad T_n = L_n U_n$

QR - RQ method

$A \in \mathbb{R}^{n \times n}$

$A = A_1 = Q_1 R_1$

$A_2 = R_1 Q_1 = Q_1^T Q_1 R_1 Q_1 = Q_1^T (Q_1 R_1) Q_1 = Q_1^T A_1 Q_1 =$

A_2 is similar to the matrix A_1

$A_2 = Q_2 R_2$

$A_3 = R_2 Q_2 = Q_2^T Q_2 R_2 Q_2 = Q_2^T A_2 Q_2 = \underbrace{Q_2^T Q_1^T}_{Q^T} A_1 \underbrace{Q_1 Q_2}_{Q}$

$\vdots$

Theorem (Convergence Theorem)

Let $A \in \mathbb{R}^{n \times n}$ such that its eigenvalues have distict modulus

$$0 < |\lambda_n| < |\lambda_{n-1}| < \dots < |\lambda_2| < |\lambda_1|$$

Let X the matrix conteining the corresponding eigenvectors

$$A = X D X^{-1}$$

D: diagonal matrix with λ_j on the diagonal

If X^{-1} admits a LU factorization then the matrices

$$A_{k+1} = Q_k^T A Q_k$$

$(k \to \infty)$ tend to an upper triangular matrix whose diagonal elements are the eigenvalues of the matrix A in decreasing order

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X^{-1} can not be factorized LU we loose the decreasing order of the eigenvalues

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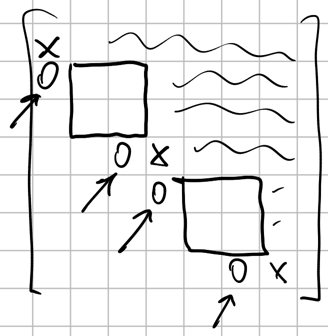
$|\lambda_2| = |\lambda_3|$

$$A_{k+1} \to \begin{bmatrix} X & & & & & \\ 0 & X & X & & & \\ 0 & X & X & & & \\ 0 & 0 & 0 & X & & \\ 0 & 0 & 0 & 0 & X & \\ 0 & 0 & 0 & 0 & 0 & X \end{bmatrix}$$



$A \rightarrow$ Tridiagonal matrix $\qquad$ <u>A symmetric</u>

$$A^{(1)} = A = \begin{bmatrix} a_{11}^{(1)} & a_1^T \\ a_1 & B^{(1)} \end{bmatrix} \qquad a_1 \in \mathbb{R}^{n-1 \times 1} \qquad B^{(1)} = \mathbb{R}^{n-1 \times n-1}$$

$u_1 \in \mathbb{R}^{n-1 \times 1}$ $\qquad H_1 a_1 = \alpha_1 e_1 \qquad e_1 \in \mathbb{R}^{n-1 \times 1}$

$$Q_1 = \begin{bmatrix} 1 & 0 \\ 0 & H_1 \end{bmatrix} \in \mathbb{R}^{n \times n}$$

$$A^{(2)} = Q_1^T A^{(1)} Q_1 = \begin{bmatrix} 1 & 0 \\ 0 & H_1^T \end{bmatrix} \begin{bmatrix} a_{11}^{(1)} & a_1^T \\ a_1 & B^{(1)} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & H_1 \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}^{(1)} & a_1^T \\ H_1^T a_1 & H_1^T B^{(1)} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & H_1 \end{bmatrix} = \begin{bmatrix} a_{11}^{(1)} & a_1^T H_1 \\ H_1^T a_1 & H_1^T B^{(1)} H_1 \end{bmatrix}$$

$H_1^T = H_1$

$a_1^T H_1 = (H_1^T a_1)^T$

$$= \left[\begin{array}{c|ccc} a_{11}^{(1)} & \alpha_1 & \cdots 0 \cdots & 0 \\ \hline \alpha_1 & & & \\ \vdots & & H_1^T B^{(1)} H_1 & \\ 0 & & & \\ \vdots & & & \end{array}\right] = \left[\begin{array}{cc|c} a_{11}^{(2)} & a_{12}^{(2)} & 0^T \\ a_{12}^{(2)} & a_{22}^{(2)} & a_2^T \\ \hline 0 & a_2 & B^{(2)} \end{array}\right] \qquad 0 \in \mathbb{R}^{n-2 \times 1} \quad a_2 \in \mathbb{R}^{n-2 \times 1}$$

$u_2 \in \mathbb{R}^{n-2 \times 1}$ $\qquad H_2 a_2 = \alpha_2 e_1 \qquad e_1 \in \mathbb{R}^{n-2 \times 1}$

$$Q_2 = \left[\begin{array}{c|c} I & 0 \\ \hline 0 & H_2 \end{array}\right] \qquad I \in \mathbb{R}^{2 \times 2} \qquad 0 \in \mathbb{R}^{n-2 \times 2} \qquad H_2 \in \mathbb{R}^{n-2 \times n-2}$$

$$A^{(3)} = Q_2^T A^{(2)} Q_2 = \left[\begin{array}{cc|c} a_{11}^{(2)} & a_{12}^{(2)} & 0^T \\ a_{12}^{(2)} & a_{22}^{(2)} & a_2^T \\ \hline 0 & H_2^T a_2 & H_2^T B^{(2)} \end{array}\right] \begin{bmatrix} I & 0 \\ 0 & H_2 \end{bmatrix}$$

$H_2^T a_2 = H_2 a_2 = \alpha_2 e_1$

$$= \left[\begin{array}{cc|c} a_{11}^{(2)} & a_{12}^{(2)} & 0^T \\ a_{12}^{(2)} & a_{22}^{(2)} & a_2^T H_2 \\ \hline 0 & H_2^T a_2 & H_2^T B^{(2)} H_2 \end{array}\right] = \left[\begin{array}{c|c} T^{(3)} & \begin{matrix} 0^T \\ a_3^T \end{matrix} \\ \hline 0 \quad a_3 & B^{(3)} \end{array}\right] \qquad 0 \in \mathbb{R}^{n-3 \times 2}$$

$$T^{(3)} = \begin{bmatrix} a_{11}^{(2)} & a_{12}^{(2)} & 0 \\ a_{12}^{(2)} & a_{22}^{(2)} & \alpha_2 \\ 0 & \alpha_2 & x \end{bmatrix}$$

After n steps $T^{(n)}$ is a tridiagonal matrix

k: $\quad U_k \in \mathbb{R}^{n-k \times 1} \rightarrow H_k \in \mathbb{R}^{n-k \times n-k}$

$Q_k \in \mathbb{R}^{n \times n} \qquad Q_k = \left[\begin{array}{c|c} I & 0^T \\ \hline 0 & H_k \end{array}\right] \qquad 0 \in \mathbb{R}^{n-k \times k}$

$$A^{(k+1)} = Q_k^T A^{(k)} Q_k = \left[\begin{array}{c|c} T^{(k)} & \begin{array}{c} 0^T \\ a_k^T \end{array} \\ \hline 0 \quad a_k & B^{(k+1)} \end{array}\right]$$

$T^{(k)} \in \mathbb{R}^{k+1 \times k+1}$ tridiagonal

$a_k \in \mathbb{R}^{n-k-1 \times 1}$

$B^{(k+1)} \in \mathbb{R}^{n-k-1 \times n-k-1}$

$A^{(n)} = T^{(n)}$ tridiagonal similar to A

$U_k \quad k = 1, \dots, n-1 \qquad Q$ can be "stored" in a triangular matrix

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A generic matrix (non-symmetric)

$$A^{(1)} = A = \begin{bmatrix} a_{11} & a_1^{*T} \\ a_1 & B^{(1)} \end{bmatrix}$$

$a_1 \in \mathbb{R}^{n-1 \times 1} \qquad a_1^* \in \mathbb{R}^{n-1 \times 1} \quad a_1^* \neq a_1$

$H^{(1)}$: $\quad H_1 a_1 = \alpha_1 e_1 \qquad Q_1 = \begin{bmatrix} 1 & 0 \\ 0 & H_1 \end{bmatrix}$

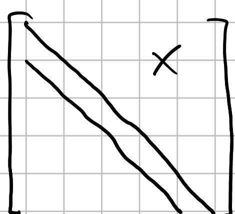
$$A^{(2)} = Q_1^T A^{(1)} Q_1 = \begin{bmatrix} a_{11}^{(1)} & a_1^{*T} H_1 \\ \underset{\uparrow \atop \alpha_1 e_1}{H_1 a_1} & H_1^T B^{(1)} H_1 \end{bmatrix} = \left[\begin{array}{c|c} a_{11}^{(1)} & a_1^{*T} H_1 \\ \hline \begin{array}{c} \alpha_1 \\ 0 \end{array} & H_1^T B^{(1)} H_1 \end{array}\right]$$

$$a_1^{*T} H_1 = (H_1^T a_1^*)^T = (H_1 a_1^*)^T \neq (\alpha e_1)^T$$

$$A^{(2)} = \left[\begin{array}{cc|c} a_{11}^{(2)} & a_{12}^{(2)} & a_1^{(2)*T} \\ a_{21}^{(2)} & a_{22}^{(2)} & a_2^{*T} \\ \hline 0 & a_2 & B^{(2)} \end{array}\right] \qquad [a_{12}^{(2)} \quad a_1^{(2)*T}] = a_1^{*T} H_1$$

$H_2 a_2 = \alpha_2 e_1 \qquad e_1 \in \mathbb{R}^{n-2 \times 1}$

$$Q_2 = \left[\begin{array}{c|c} \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} & 0 \\ \hline 0 & H_2 \end{array}\right]$$

$A^{(n)} = H^{(n)}$ Hessemberg matrix $\qquad H^{(n)} =$ 

$$\boxed{\begin{array}{lll} A & \text{symmetric} \rightarrow & \text{Tridiagonal matrix} \\ A & \text{non-symmetric} \rightarrow & \text{Hessemberg matrix} \end{array}}$$

A strongly sparse (many zeros under the main diagonal)

Givens method instead of the Householder method

$H_{(k)}$ $\qquad G(q,k) \qquad (q,k): \; A^{(k)}(q,k) \neq 0$

QR factorization of $T^{(n)}$ or $H^{(n)}$ is much more convenient then the factorization of A both are converging to a triangular matrix with eigenvalues of A on the diagonal.

A non-simmetric

Preliminary step $A \rightarrow H^{(n)}$ is $O(n^3)$

each QR fatorization A is $O(n^3)$

" " " $H^{(n)}$ is $O(n^2)$

Stopping criterion $\qquad H^{(n)} = QR$

QR-RQ → Hessemberg matrix

$$H^{(1)} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ 0 & a_{32} & a_{33} & a_{34} \\ 0 & 0 & a_{43} & a_{44} \end{bmatrix} \qquad G_1 H^{(1)} = \begin{bmatrix} a_{11}^{(1)} & a_{12}^{(1)} & a_{13}^{(1)} & a_{14}^{(1)} \\ 0 & a_{22}^{(1)} & a_{23}^{(1)} & a_{24}^{(1)} \\ 0 & a_{32} & a_{33} & a_{34} \\ 0 & 0 & a_{43} & a_{44} \end{bmatrix}$$

$$G_1 = \begin{bmatrix} c_1 & s_1 & 0 & 0 \\ -s_1 & c_1 & 0 & 0 \\ 0 & 0 & & \\ 0 & 0 & & I \end{bmatrix} \qquad G_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c_2 & s_2 & 0 \\ 0 & -s_2 & c_2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad G_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & c_3 & s_3 \\ 0 & 0 & -s_3 & c_3 \end{bmatrix}$$

G_1, G_2, G_3 tridiagonal $\qquad G_3 G_2 G_1$: Hessemberg (lower)

$$G_2 G_1 = \begin{bmatrix} c_1 & s_1 & 0 & 0 \\ -s_1c_2 & c_1c_2 & s_2 & 0 \\ s_1s_2 & -s_2c_2 & c_2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$G_3 G_2 G_1 = \begin{bmatrix} c_1 & s_1 & 0 & 0 \\ -s_1c_2 & c_1c_2 & s_2 & 0 \\ s_1s_2c_3 & -s_2c_2c_3 & c_2c_3 & s_3 \\ -s_1s_2s_3 & +s_2c_2s_3 & -c_2c_3 & c_3 \end{bmatrix}$$

$G = G_3 G_2 G_1$

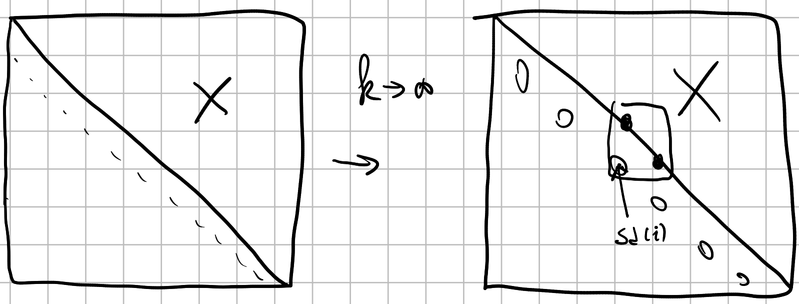
$G H^{(n)} = R$

$H^{(n)} = Q R \qquad Q = G^T = G_1^T G_2^T G_3^T$

$\uparrow$ Hessemberg $\qquad \uparrow$ Hessemberg (upper) triangular

Hessemberg

$RQ =$



$|sd(i)| \sim |d(i)|, |d(i-1)|$

$|sd(i)| << |d(i)| + |d(i-1)|$

$|sd(i)| < (|d(i)| + |d(i-1)|)\,tol$

$k \sim n$

$G = G_3 G_2 G_1 =$ Hessember matrix (lower)

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & c_3 & s_3 \\ 0 & 0 & -s_3 & c_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c_2 & s_2 & 0 \\ 0 & -s_2 & c_2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 & s_1 & 0 & 0 \\ -s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & c_3 & s_3 \\ 0 & 0 & -s_3 & c_3 \end{bmatrix} \begin{bmatrix} c_1 & s_1 & 0 & 0 \\ -s_1c_2 & c_1c_2 & s_2 & 0 \\ s_1s_2 & -c_1s_2 & c_2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} c_1 & s_1 & 0 & 0 \\ -s_1c_2 & c_1c_2 & s_2 & 0 \\ s_1s_2c_3 & -c_1s_2c_3 & c_2s_3 & s_3 \\ -s_1s_2s_3 & c_1s_2s_3 - c_2s_3 & c_3 & \end{bmatrix}$$

G : lower Hessemberg matrix | $H^{(3)}$: upper Hessemberg matrix

$G H^{(3)} = R$

$H^{(3)} = G^T R = Q R \qquad G^T = Q$ upper Hessemberg matrix

RQ is still upper Hessemberg