Eigenvalues and eigenvectors approximation

- approximation of some eigenvalues/eigenvectors ←
- approximation of all the eigenvalues/eigenvectors

$$Ax = \lambda x \qquad A \in \mathbb{R}^{n\times n} \qquad \lambda \in \mathbb{C} \qquad x \in \mathbb{C}^n$$

$$x \neq 0$$

$$|\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq \dots \geq |\lambda_n| \qquad \lambda_1 \in \mathbb{R}$$

Power method $\Rightarrow \lambda_1, x_1 \qquad Ax_1 = \lambda_1 x_1$

$$v_0 \in \mathbb{R}^n \qquad v_0 = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 \dots + \alpha_n x_n \qquad \alpha_1 \neq 0$$

$$v_1 = Av_0 = \alpha_1 Ax_1 + \alpha_2 Ax_2 + \dots + \alpha_n Ax_n$$

$x_1, \dots, x_n$ eigenvectors corresponding to eigenvalues $\lambda_1, \dots, \lambda_n$

$$= \alpha_1 \lambda_1 x_1 + \alpha_2 \lambda_2 x_2 + \dots + \alpha_n \lambda_n x_n$$

$$= \lambda_1 \left(\alpha_1 x_1 + \alpha_2 \frac{\lambda_2}{\lambda_1} x_2 + \dots + \alpha_n \frac{\lambda_n}{\lambda_1} x_n\right)$$

$$v_2 = Av_1 = \lambda_1 \left(\alpha_1 \lambda_1 x_1 + \alpha_2 \frac{\lambda_2}{\lambda_1} \lambda_2 x_2 + \dots + \alpha_n \frac{\lambda_n}{\lambda_1} \lambda_n x_n\right)$$

$$= \lambda_1^2 \left(\alpha_1 x_1 + \alpha_2 \left(\frac{\lambda_2}{\lambda_1}\right)^2 x_2 + \dots + \alpha_n \left(\frac{\lambda_n}{\lambda_1}\right)^2 x_n\right)$$

$$v_k = Av_{k-1} = A^k v_0 = \lambda_1^k \left(\alpha_1 x_1 + \alpha_2 \left(\frac{\lambda_2}{\lambda_1}\right)^k x_2 + \dots + \alpha_n \left(\frac{\lambda_n}{\lambda_1}\right)^k x_n\right)$$

$$\frac{1}{\lambda_1^k} v_k = \alpha_1 x_1 + \alpha_2 \left(\frac{\lambda_2}{\lambda_1}\right)^k x_2 + \dots + \alpha_n \left(\frac{\lambda_n}{\lambda_1}\right)^k x_n$$

$$\lim_{k\to\infty} \frac{1}{\lambda_1^k} v_k = \lim_{k\to\infty} \alpha_1 x_1 + \lim_{k\to\infty} \alpha_2 \left(\frac{\lambda_2}{\lambda_1}\right)^k x_2 + \dots + \lim_{k\to\infty} \alpha_n \left(\frac{\lambda_n}{\lambda_1}\right)^k x_n$$

$$= \alpha_1 x_1 \qquad \lim_{k\to\infty} \left(\frac{\lambda_i}{\lambda_1}\right)^k = 0 \qquad |\lambda_i| < |\lambda_1|$$

$$\lim_{k\to\infty} \frac{\frac{1}{\lambda_1^{k+1}} v_{k+1}(i)}{\frac{1}{\lambda_1^k} v_k(i)} = \frac{\lim_{k\to\infty} \frac{1}{\lambda_1^{k+1}} v_{k+1}(i)}{\lim_{k\to\infty} \frac{1}{\lambda_1^k} v_k(i)} = \frac{\alpha_1 x_1(i)}{\alpha_1 x_1(i)} \qquad i = 1, \dots, n$$

$$\lim_{k\to\infty} \frac{v_{k+1}(i)}{v_k(i)} = \lambda_1$$

Algorithm Power method (by components)

$v_0 \neq 0$ $\qquad v_0 = \alpha_1 x_1 + \alpha_2 x_2 + \qquad + \alpha_n x_n \qquad \alpha_1 \neq 0$

do

$$v_{k+1} = A v_1$$

$$\lambda_1^{(k+1)} = \frac{v_{k+1}(i)}{v_k(i)}$$

$k{+}{+}$

$$\text{while} \left(k \le MaxIter \;\&\&\; \frac{|\lambda_1^{(k+1)} - \lambda_1^{(k)}|}{|\lambda_1^{(k+1)}|} > tol \right)$$

$$v_k = \underbrace{\lambda_1^k}\ \alpha_1 x_1 + \lambda_1^k \left(\cancel{\alpha_2 \left(\frac{\lambda_2}{\lambda_1}\right)^k x_2 + \dots} \right)$$

$|\lambda_1| > 1 \qquad \lambda_1^k \alpha_1 \xrightarrow[k\to\infty]{} \infty$

$|\lambda_1| < 1 \qquad \lambda_1^k \alpha_1 \xrightarrow[k\to\infty]{} 0$

$v_0 \neq 0 \qquad \alpha_1 \neq 0 \qquad v_{old} = v_0 \qquad k = 0 \qquad \lambda_1^{new} = \infty$

$[\|v_{old}\|_\infty, i] = \max(|v_{old}|)$ $\qquad v_{old} = v_{old} / \|v_{old}\|_\infty$

do

$$\lambda_1^{old} = \lambda_1^{new}$$

$$v_{new} = A v_{old}$$

$$\lambda_1^{new} = \frac{v_{new}(i)}{v_{old}(i)} \qquad \left(\pm v_{new}(i) \right)$$

$$v_{old} = v_{new}$$

$$[\|v_{old}\|_\infty, i] = \max(|v_{old}|)$$

$$v_{old} = \frac{v_{old}}{\|v_{old}\|_\infty}$$

$k{+}{+}$

$$\text{while} \left(k < MaxIter \;\&\&\; \frac{|\lambda_1^{new} - \lambda_1^{old}|}{|\lambda_1^{new}|} > tol \right)$$

$$\left.\begin{array}{ll} O(n^2) & \text{matrix vector product} \\ O(n) & \text{max} \\ O(n) & \text{normalization} \end{array}\right\} \sim O(n^2)$$

Reileigh quotient $\quad R_1^{(k)} = \dfrac{v_k^T A v_k}{v_k^T v_k} = \dfrac{v_k^T \boxed{v_{k+1}}}{v_k^T \boxed{v_k}} \longrightarrow \lambda_1$

$$v_k = \lambda_1^k \left(\alpha_1 x_1 + \alpha_2 \left(\frac{\lambda_2}{\lambda_1}\right)^k x_2 + \alpha_3 \left(\frac{\lambda_3}{\lambda_1}\right)^k x_3 \right) \qquad \|x_i\| = 1 \qquad i = 1, \dots, n$$

$$v_{k+1} = \lambda_1^{k+1} \left(\alpha_1 x_1 + \alpha_2 \left(\frac{\lambda_2}{\lambda_1}\right)^{k+1} x_2 + \alpha_3 \left(\frac{\lambda_3}{\lambda_1}\right)^{k+1} x_3 \right)$$

$$\begin{aligned} v_k^T v_k = \lambda_1^{2k} \Bigg(& \alpha_1^2 x_1^T x_1 + \alpha_1 \alpha_2 \left(\frac{\lambda_2}{\lambda_1}\right)^k x_1^T x_2 + \alpha_1 \alpha_3 \left(\frac{\lambda_3}{\lambda_1}\right)^k x_1^T x_2 + \\ & + \alpha_2 \alpha_1 \left(\frac{\lambda_2}{\lambda_1}\right)^k x_2^T x_1 + \alpha_2^2 \left(\frac{\lambda_2}{\lambda_1}\right)^{2k} x_2^T x_2 + \alpha_2 \alpha_3 \left(\frac{\lambda_2}{\lambda_1}\right)^k \left(\frac{\lambda_3}{\lambda_1}\right)^k x_2^T x_3 + \\ & + \alpha_3 \alpha_1 \left(\frac{\lambda_3}{\lambda_1}\right)^k x_3^T x_1 + \alpha_3 \alpha_2 \left(\frac{\lambda_3}{\lambda_1}\right)^k \left(\frac{\lambda_2}{\lambda_1}\right)^k x_3^T x_2 + \alpha_3^2 \left(\frac{\lambda_3}{\lambda_1}\right)^{2k} x_3^T x_3 \Bigg) \end{aligned}$$

$$\begin{aligned} v_k^T v_{k+1} = \lambda_1^{2k+1} \Bigg(& \alpha_1^2 x_1^T x_1 + \alpha_1 \alpha_2 \left(\frac{\lambda_2}{\lambda_1}\right)^{k+1} x_1^T x_2 + \alpha_1 \alpha_3 \left(\frac{\lambda_3}{\lambda_1}\right)^{k+1} x_1^T x_2 + \\ & + \alpha_2 \alpha_1 \left(\frac{\lambda_2}{\lambda_1}\right)^k x_2^T x_1 + \alpha_2^2 \left(\frac{\lambda_2}{\lambda_1}\right)^{2k+1} x_2^T x_1 + \alpha_2 \alpha_3 \left(\frac{\lambda_2}{\lambda_1}\right)^k \left(\frac{\lambda_3}{\lambda_1}\right)^{k+1} x_2^T x_3 + \\ & + \alpha_3 \alpha_1 \left(\frac{\lambda_3}{\lambda_1}\right)^k x_3^T x_1 + \alpha_3 \alpha_2 \left(\frac{\lambda_3}{\lambda_1}\right)^k \left(\frac{\lambda_2}{\lambda_1}\right)^{k+1} x_3^T x_2 + \alpha_3^2 \left(\frac{\lambda_3}{\lambda_1}\right)^{2k+1} x_3^T x_3 \Bigg) \end{aligned}$$

$$\frac{v_k^T v_{k+1}}{v_k^T v_k} = \frac{\lambda_1^{2k+1} \left(\alpha_1^2 x_1^T x_1 + \dots \right)}{\lambda_1^{2k} \left(\alpha_1^2 x_1^T x_1 + \dots \right)}$$

$$\lim_{k \to \infty} \frac{v_k^T v_{k+1}}{v_k^T v_k} = \lambda_1$$

$$\begin{array}{ccccccccc} 1 > \left|\frac{\lambda_2}{\lambda_1}\right| & \geq & \left|\frac{\lambda_3}{\lambda_1}\right| & \geq & \left|\frac{\lambda_4}{\lambda_1}\right| & \geq \dots & \left|\frac{\lambda_n}{\lambda_1}\right| \\ \downarrow & & & & \downarrow & & \downarrow \\ 0 & & & & 0 & & 0 \end{array}$$

$\left|\frac{\lambda_2}{\lambda_1}\right|$ is small the iterative methods converge more fast

$v_0 \neq 0$ $\alpha_1 \neq 0$ $v_{old} = \frac{v_0}{\|v_0\|_2}$ $(v_{old}^T v_{old} = \|v_{old}\|_2^2 = 1)$ $\lambda_1^{new} = \infty$

do

$\lambda_1^{old} = \lambda_1^{new}$

$v_{new} = A v_{old}$

$\lambda_1 = v_{old}^T v_{new}$

$$\frac{v_{old}^T A v_{old}}{v_{old}^T v_{old}} = \frac{v_{old}^T A v_{old}}{1}$$

$v_{old} = \frac{v_{new}}{\|v_{new}\|_2}$ $\longrightarrow$ $x_1^{(k)} = v_{old}$

while ($k{+}{+}$ $k \leq MaxIter$ && $\frac{|\lambda_1^{new} - \lambda_1^{old}|}{|\lambda_1^{new}|} > tol$)

output $\lambda_1^{new} \to \lambda_1$
$v_{old} \to x_1$

$\mathcal{O}(n^2)$ matrix vector
$\mathcal{O}(n)$ λ_1
$\mathcal{O}(n)$ $\|v_{new}\|_2$
$\mathcal{O}(n)$ $v_{new} / \|v_{new}\|_2$

$\Big\} \ \mathcal{O}(n^2)$

Speed of convergence

$$|\lambda_1 - \lambda_1^k| \leq C_\lambda \left(\frac{\lambda_2}{\lambda_1}\right)^k \qquad \|x_1 - x_1^k\| \leq C_x \left(\frac{\lambda_2}{\lambda_1}\right)^k$$

A symmetric $x_i \perp x_j$ $x_i^T x_j = \delta_{ij}$

λ_i real

$$v_k^T v_k = \lambda_1^{2k} \left(\alpha_1^2 + \alpha_2^2 \boxed{\left(\frac{\lambda_2}{\lambda_1}\right)^{2k}} + \alpha_3^2 \left(\frac{\lambda_3}{\lambda_1}\right)^{2k} \right)$$

$$v_k^T v_{k+2} = \lambda_1^{2k+2} \left(\alpha_1^2 + \alpha_2^2 \boxed{\left(\frac{\lambda_2}{\lambda_1}\right)^{2k+2}} + \alpha_3^2 \left(\frac{\lambda_3}{\lambda_1}\right)^{2k+1} \right)$$

$$\lim_{k\to\infty} \frac{v_k^T v_{k+2}}{v_k^T v_k} = \frac{\lim_{k\to\infty} \lambda_1^{2k+1} \left(\alpha_1^2 + \alpha_2 \left(\frac{\lambda_2}{\lambda_1}\right)^{2k+1} \cdots \right)}{\lim_{k\to\infty} \lambda_1^{2k} \left(\alpha_1^2 + \alpha_2 \left(\frac{\lambda_2}{\lambda_1}\right)^{2k} + \cdots \right)} = \lambda_1$$

$$|\lambda_1 - \lambda_1^k| \leq C_\lambda \left(\frac{\lambda_2}{\lambda_1}\right)^{2k} \qquad \|x_1 - x_1^k\| \leq C_x \left(\frac{\lambda_2}{\lambda_1}\right)^{2k}$$

Inverse power method

$|\lambda_1| \geq |\lambda_2| \geq \dots \geq |\lambda_{n-1}| > |\lambda_n|$ $\quad x_n$

$\boxed{Ax = \lambda x}$ $\quad$ A invertible $\quad$ $\lambda = 0$ is not an eigenvalue

$A^{-1} A x = \lambda A^{-1} x$

$\frac{1}{\lambda} x = A^{-1} x$ $\quad$ A^{-1} has eigenvalues $\frac{1}{\lambda}$ and eigenvectors x

$A^{-1} x = \sigma x$ $\quad$ $\sigma = \frac{1}{\lambda}$

$v_0 \neq 0 \quad \alpha_n \neq 0 \quad v^{old} = v_0 / \|v_0\| \quad \sigma^{new} = \infty$ $\quad$ $\underline{A = LU}$

do

$\sigma^{old} = \sigma^{new}$

$\cancel{v^{new} = A^{-1} v^{old}}$ $\quad$ $A v^{new} = v^{old}$

$\sigma = v^{old\,T} v^{new}$

$v^{new} = v^{new} / \text{norm}(v^{new})$

$v^{old} = v^{new}$

k++

while $\left(k < \text{maxIter} \;\&\&\; \frac{|\sigma^{new} - \sigma^{old}|}{|\sigma^{new}|} > tol \right)$

$\lambda_n = \frac{1}{\sigma}$

μ find the closest eigenvalue of A to μ

$Ax = \lambda x$

$Ax - \mu x = \lambda x - \mu x$

$(A - \mu I) x = (\lambda - \mu) x$

$|\lambda_1| \geq |\lambda_2| \geq |\lambda_3| \geq \underbrace{|\lambda_4| \geq \mu}$ $\geq |\lambda_5| \geq |\lambda_6|$

$\lambda -$ $\quad$ $> |\lambda_4 - \mu|$ $\quad$ $A - \mu I$

smallest eigenvalue of

$\sigma = \frac{1}{\lambda - \mu}$

Apply inverse power method to $A-\mu I$

$v_0 \neq 0$ $\quad$ $\underline{\alpha_1 \neq 0}$ $\quad$ $v^{new} = v_0 / \|v_0\|_2^2$ $\quad$ $\sigma^{new} = \infty$ $\quad$ $A-\mu I = LU$

do

$\sigma^{old} = \sigma^{new}$

$v^{new} = (A-\mu I)^{-1} v^{old}$ $\quad$ $(A-\mu I) v^{new} = v^{od}$

$\sigma = v^{old} v^{new}$

$v^{new} = v^{new} / \|v^{new}\|$

$v^{old} = v^{new}$

$k{+}{+}$

while $\left\{ k \leq MAXITER \ \&\& \ \frac{|\sigma^{new} - \sigma^{old}|}{\|\sigma^{new}\|} > tol \right)$

$\sigma = \frac{1}{\lambda - \mu}$ $\quad$ $\lambda - \mu = \frac{1}{\sigma}$ $\quad$ $\lambda = \frac{1}{\sigma} + \mu$ $\quad$ v^{new} : approx of corresp. eigenvector