SVD Singular Value decomposition

$A \in \mathbb{R}^{m\times n}$ $\qquad m \geq n$

Definition

A singular value σ of the matrix $A \in \mathbb{R}^{m\times n}$ is a non-negative scalar such that there exist two vectors $v \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ such that

$$Av = \sigma u \qquad\qquad A^T u = \sigma v$$

σ: singular value

v: is the right singular vector corresponding to σ

u: is the left singular vector " " "

Proposition

There exist n singular values for $A \in \mathbb{R}^{m\times n}$ $(m \geq n)$, moreover the corresponding right singular vector $v \in \mathbb{R}^n$ is the eigenvector of the matrix $A^TA \in \mathbb{R}^{n\times n}$

Proof

$$\sigma(A^T u = \sigma v) \qquad Av = \sigma u$$

$$A^T(\sigma u) = \sigma^2 v$$

$$A^TAv = \sigma^2 v$$

σ^2 is an eigenvalue of A^TA and v is the corresponding eigenvalue

A^TA symmetric

$\Downarrow$

n real eigenvalues $\qquad \sigma^2 \Rightarrow \sigma = \sqrt{\sigma^2}$

n real orthonormal eigenvectors $v \in \mathbb{R}^n$

$$A^TA\,V = V\Sigma^2 \qquad V \in \mathbb{R}^{n\times n} \quad \Sigma^2 \in \mathbb{R}^{n\times n} \quad \Sigma^2 = \mathrm{diag}(\sigma_i^2)_{i=1,\dots,n}$$

$$V^TV = VV^T = I$$

σ_i, v_i $\quad i = 1, \dots, n$ $\qquad Av_i = \sigma_i u_i$

$\boxed{\sigma_i \neq 0}$ $\qquad u_i = \frac{1}{\sigma_i} Av_i$

$\boxed{u_i^T u_j = \delta_{ij}}$

$$u_i^T u_j = \left(\frac{1}{\sigma_i} Av_i\right)^T \left(\frac{1}{\sigma_j} Av_j\right) = \frac{1}{\sigma_i\sigma_j} v_i^T A^TAv_j$$

$$= \frac{1}{\sigma_i\sigma_j} v_i^T \sigma_j^2 v_j = \frac{\sigma_j^2}{\sigma_i\sigma_j} v_i^T v_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

$u_i = ?$ $\sigma_i = 0$ $\quad\quad$ $\cancel{u_i = \frac{1}{\sigma_i} A v_i}$

$$\sigma_1 \geq \sigma_2 \geq \sigma_3 \geq \cdots \geq \sigma_r > 0 \qquad \sigma_i = 0 \quad i = r, \dots, n$$

$\sigma_r \neq 0$

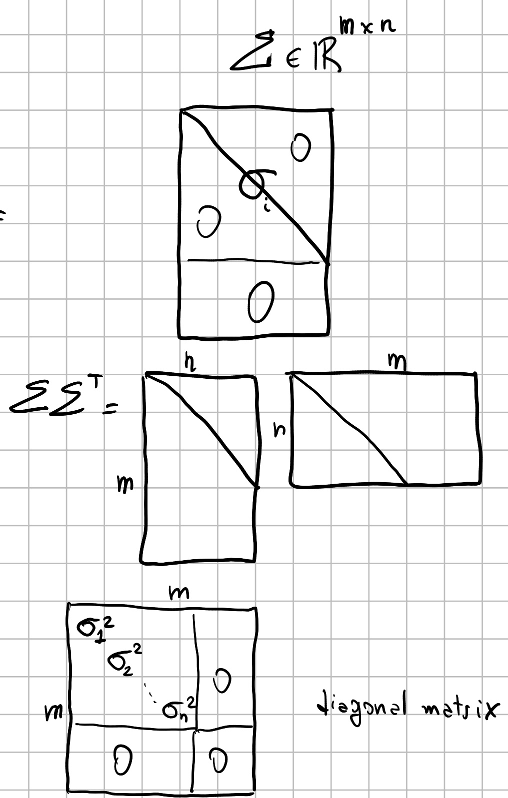
u_i orthonormal $\quad i = 1, \dots, r$ $\qquad$ u_j arbitrary orthonormal vectors orthonormal to u_i $i = 1, \dots, r$

$$U = \left[\begin{array}{c|c} u_i = \frac{1}{\sigma_i} A v_i & U \text{ arbitrary such that } U^T U = U U^T = I \end{array}\right]$$

$U \in \mathbb{R}^{m \times m}$

$1 \underbrace{\quad}_{\uparrow} r | r+1 \quad n$

$$AV = U\Sigma$$
$$A = U \Sigma V^T$$
$$AA^T = U \Sigma V^T (U \Sigma V^T)^T =$$
$$= U \Sigma V^T V \Sigma^T U^T$$
$$= U \Sigma \Sigma^T U^T$$



$(AA^T)^T = AA^T$ symmetric

U egenvectors of AA^T corresponding to

σ_i^2 $i = 1, \dots, n$ $\quad$ and $\quad$ 0 $i = n+1, \dots, m$

Proposition

a) The null space of A is the subspace of $\mathbb{R}^n$ spanned by the $n-r$ columns of V corresponding to the singular values $\sigma_{r+1} = \dots = \sigma_n = 0$

$$\ker(A) = \operatorname{span}\{ v_j \;\; j = r+1, \dots, n \}.$$

b) The range (the image) of A is the subspace of $\mathbb{R}^m$ spanned by the columns u_i $i = 1, \dots, r$ of U corresponding to the non-vanishing singular values of A.

c) r is the rank of A.

$$A : \mathbb{R}^n \to \mathbb{R}^m \qquad x \in \mathbb{R}^n$$

$$x = \sum_{i=1}^{n} \alpha_i v_i = \sum_{i=1}^{r} \alpha_i v_i + \sum_{i=r+1}^{n} \alpha_i v_i$$

$$Ax = \sum_{i=1}^{r} \alpha_i A v_i + \sum_{i=r+1}^{n} \alpha_i A v_i =$$

$$= \underbrace{\sum_{i=1}^{r} \alpha_i \sigma_i u_i}_{\in \text{Range}(A)} + \sum_{i=r+1}^{n} \alpha_i \underset{0}{\underset{\|}{\sigma_i}} u_i$$

$$= \qquad \in \text{Range}(A) \qquad \underset{0}{\|}$$

$$\text{span}\{v_i \quad i = r+1, \dots, n\} = \ker(A)$$

$$A \overset{?}{=} \sum_{i=1}^{r} \sigma_i u_i v_i^T \qquad x = \sum_{i=1}^{r} \alpha_i v_i + \sum_{i=r+1}^{n} \alpha_i v_i$$

$$Ax = \sum_{i=1}^{r} \alpha_i \sigma_i u_i$$

$$\sum_{i=1}^{r} \underline{\sigma_i u_i v_i^T} x = \sum_{i=1}^{r} \left(\sum_{j=1}^{r} \sigma_i u_i v_i^T \alpha_j v_j + \sum_{j=r+1}^{n} \sigma_i u_i v_i^T \alpha_j v_j \right)$$

$$= \sum_{i=1}^{r} \left(\sum_{j=1}^{r} \sigma_i \alpha_j u_i v_i^T v_j + \sum_{j=r+1}^{n} \sigma_i \alpha_j u_i v_i^T v_j \right)$$

$$= \sum_{i=1}^{r} \left(\sigma_i \alpha_i u_i + 0 \right) = \sum_{i=1}^{r} \alpha_i \sigma_i u_i$$

$$A = \sum_{i=1}^{r} \sigma_i \underbrace{u_i v_i^T}_{A_i} = \sum_{i=1}^{r} \sigma_i A_i \qquad A_i = u_i v_i^T$$

$$\boxed{n \quad \sigma_i = 0 \quad i = r+1, \dots, n}$$

$$\tilde{A}_k = \sum_{i=1}^{k \le r} \sigma_i A_i \qquad E = A - \tilde{A}_k = \sum_{i=k+1}^{r} \sigma_i A_i$$

$$\|A - \tilde{A}_k\|_2^2 = \|E\|_2^2 = \rho(E^T \cdot E) = \rho\left(\left(\sum_{i=k+1}^{r} \sigma_i u_i v_i^T \right)^T \left(\sum_{i=k+1}^{r} \sigma_i u_i v_i^T \right) \right) =$$

$$= \rho\left(\sum_{i=k+1}^{r} \sum_{j=k+1}^{r} \sigma_i \sigma_j v_i \underbrace{u_i^T u_j}_{\delta_{ij}} v_j^T \right) = \rho\left(\sum_{i=k+1}^{r} \sigma_i^2 v_i v_i^T \right) = \sigma_{k+1}^2$$

$$\|A - \tilde{A}_k\|_2 = \sigma_{k+1}$$

$$\|E\|_2 = \sigma_{k+1}$$

$$\|A-\tilde{A}_k\|_F^2 = \|E\|_F^2 = tr(EE^T) = tr\left(\left(\sum_{i=k+1}^{n} \sigma_i u_i v_i^T\right)\left(\sum_{j=k+1}^{n} \sigma_j u_j v_j^T\right)^T\right)$$

$$= tr\left(\sum_{i=k+1}^{n}\sum_{j=k+1}^{n} \sigma_i\sigma_j\, u_i v_i^T v_j u_j^T\right) =$$

$$= tr\left(\sum_{i=k+1}^{n} \sigma_i^2\, u_i u_i^T\right) = \sum_{i=k+1}^{n} \sigma_i^2$$

Numerical computation of SVD

$A\in \mathbb{R}^{m\times n}$ $\qquad A = U\Sigma V^T$ $\qquad U\in\mathbb{R}^{m\times m}$, $\Sigma\in\mathbb{R}^{m\times n}$, $V\in\mathbb{R}^{n\times n}$

1) Reduce A Bi-diagonal matrix B with orthogonal matrices U_1, V_1

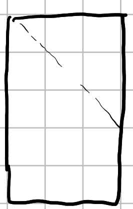
$$A = U_1 B V_1^T \qquad B = U_1^T A V_1$$

2) Compute SVB of B : $B = U_2 \Sigma V_2^T$ $\qquad$ eigenvalues(B^TB)
B^TB tridiagonal

3) $$A = U_1 B V_1^T = \underbrace{U_1 U_2}_{U}\, \Sigma\, \underbrace{V_2^T V_1^T}_{V^T}$$
$$= U \quad \Sigma \quad V^T$$

1)

A



$P_1 A(1,:) = \alpha_1 e_1^{(m)}$

Z W

$$Z_1^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & & \\ 0 & \multicolumn{2}{c}{P_2^T} \end{bmatrix}$$

$$(b_1^T P_2^T)^T = P_2 b_1 = \beta_1 e_1^{(n-1)}$$

$$P_1 A = \begin{bmatrix} \alpha_1 & - & b_1^T - \\ 0 & \times & \times \\ 0 & \times & \times \\ 0 & \times & \times \\ 0 & \times & \times \end{bmatrix} \qquad P_1 A Z_1^T = \begin{bmatrix} \alpha_1 & \beta_1 & 0 \\ 0 & \times & \times \\ 0 & \times & \times \\ 0 & \times & \times \\ 0 & \times & \times \end{bmatrix}$$

$$P_3 \begin{bmatrix} \times \\ \times \\ \times \\ \times \end{bmatrix} = \alpha_2 e_1^{(m-1)} \qquad Q_2 = \begin{bmatrix} 1 & 0^T \\ 0 & P_3 \end{bmatrix} \qquad Q_2 P_1 A Z_1^T = \begin{bmatrix} \alpha_1 & \beta_1 & 0 \\ 0 & \alpha_2 & \times \\ 0 & 0 & \times \\ 0 & 0 & \times \\ 0 & 0 & \times \end{bmatrix}$$

Householder transformation to bidiagonal form

Input $A \in \mathbb{R}^{m \times n}$ $\quad m \geq n$

$B = A$

$U = I_{m \times n}$

$V^T = I_{n \times n}$

for $k = 1, \dots, n$

$P_k \, B(k:m, k) = \alpha_k \, e_1^{(m-k+1)}$

$$Q_k = \begin{array}{cc} & m \\ & \left[\begin{array}{c|c} I_{k-1 \times k-1} & 0 \\ \hline 0 & P_k \end{array}\right] \end{array} m$$

$B = Q_k B$

$U = U \cdot Q_k$

if $k \leq n-2$

$P_k^T \, B(k, k+1:n)^T = \beta_k \, e_1^{(n-k)}$

$$Z_k^T = \begin{array}{cc} & n \\ & \left[\begin{array}{c|c} I & 0 \\ \hline 0 & P_k^T \end{array}\right] \end{array} n$$

$V^T = Z_k V^T$

end

end

$$B = \underbrace{\cdots Q_3 Q_2 Q_1}_{U_1^T} \; \underbrace{A}_{A} \; \underbrace{Z_1 Z_2 \cdots}_{V_1}$$

$A = U_1 B V_1^T$

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2) eigenvalues
eigenvectors $\quad B^T B \quad \underset{V_2}{\Sigma} \Rightarrow U_2 \quad B = U_2 \Sigma V_2^T$