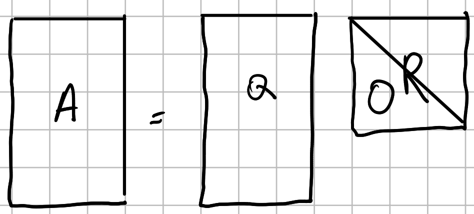
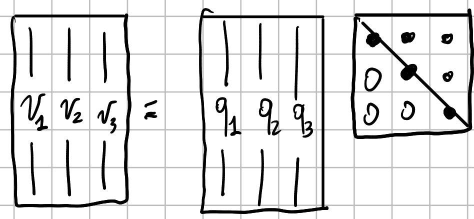
QR factorization of a matrix $A \in \mathbb{R}^{m \times n}$

Q ortonormal $\qquad Q^T Q = Q Q^T = I \qquad Q \in \mathbb{R}^{m\times n}$ or $\mathbb{R}^{m \times m}$

R uppertriangular matrix $\qquad R \in \mathbb{R}^{n \times n}$ or $\mathbb{R}^{m \times n}$

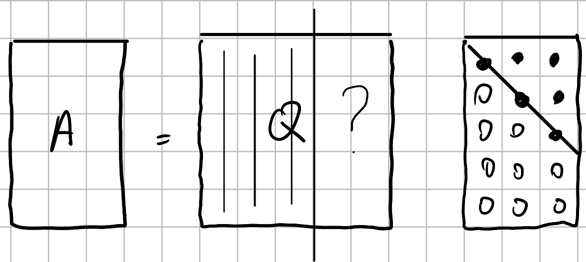
$Q \in \mathbb{R}^{m \times n} \qquad Q^T Q = I \in \mathbb{R}^{n \times n}$





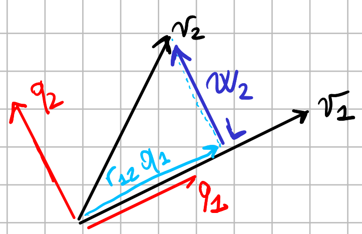
$$q_i^T q_j = q_i \cdot q_j = \delta_{ij}$$

$Q \in \mathbb{R}^{m \times m} \qquad Q^T Q = Q Q^T = I \in \mathbb{R}^{m \times m} \qquad R \in \mathbb{R}^{m \times n}$



How QR?

1) Grahm Schmidt algorithm



$$\begin{bmatrix} | & | \\ v_1 & v_2 \\ | & | \end{bmatrix} = \begin{bmatrix} | & | \\ q_1 & q_2 \\ | & | \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} \\ 0 & r_{22} \end{bmatrix}$$

A $\qquad$ Q $\qquad$ R

$$v_1 = r_{11} q_1$$

$$v_2 = r_{12} q_1 + r_{22} q_2$$

$q_1 = \dfrac{v_1}{\|v_1\|} \qquad q_2 \perp q_1 \qquad r_{11} = \|v_1\|$

v_2 – component of v_1 w.r.t. q_1

$$r_{12} = q_1^T v_2 = q_1 \cdot v_2 = (q_1, v_2)$$

$$= r_{12} q_1$$

$$w_2 = v_2 - r_{12} q_1$$

$q_2 = \dfrac{w_2}{\|w_2\|} \qquad r_{22} = \|w_2\| \qquad w_2 = r_{22} q_2$

$$v_2 = r_{12} q_1 + r_{22} q_2$$

SGSA

input $v_1, \dots, v_n$

output $q_1, \dots, q_n$ and R

$r_{11} = \|v_1\|$

$q_1 = \frac{v_1}{r_{11}}$

for $j = 2, \dots, n$

$w_j = v_j$

for $i = 1, \dots, j-1$

$r_{ij} = q_i^T v_j \longleftrightarrow r_{ij} = q_i^T w_j$

$w_j = w_j - r_{ij} q_i$

end

$r_{jj} = \|w_j\| \leftarrow$ 0 if v_j is l.d. from the previous columns

$q_j = \frac{w_j}{r_{jj}}$

end

MGSA

$q_1 = \frac{v_1}{r_{11}}$

$w_2 = v_2 - r_{12} q_1$ $\qquad r_{12} = q_1^T v_2$ $\qquad q_2 = \frac{w_2}{r_{22}}$ $\qquad r_{22} = \|w_2\|$

$w_3 = v_3 - r_{13} q_1$ $\qquad r_{13} = q_1^T v_3$

$w_3 = w_3 - r_{23} q_2$ $\qquad \boxed{r_{23} = q_2^T w_3}$ $\qquad r_{23} = q_2^T (v_3 - r_{13} q_1) = \underbrace{q_2^T v_3}_{r_{23}^{SGSA}} - r_{13} \underbrace{q_2^T q_1}_{0}$

$q_3 = \frac{w_3}{r_{33}}$ $\qquad r_{33} = \|w_3\| = \|v_3 - r_{13} q_1 - r_{23} q_2\|$

What happens if v_j $j = 1, \dots, n$ are not linearly independent?

$A = QR$ $\qquad Q^T Q = I$ $\qquad Q \in \mathbb{R}^{m \times n} \Rightarrow I \in \mathbb{R}^{n \times n}$

$\|Q^T Q - I\| \sim 10^{-16}$ $\qquad Q \in \mathbb{R}^{m \times m} \Rightarrow I \in \mathbb{R}^{m \times m}$

$\|A - QR\| \sim 10^{-16} \|A\|$

$$\|Q^TQ - I\| \gg 10^{-16}$$

$$\|A - QR\| \gg 10^{-16}\|A\|$$

Re-orthogonalization

if $\|Q^TQ - I\| \gg 10^{-16}$ $\Rightarrow$ the columns of Q are not orthonormal

$A = Q_1 R_1$ $\quad$ Q_1 not good

$Q_1 = Q_2 R_2$

$A = Q_2 R_2 R_1$ $\quad$ Q_2 $\quad \|Q_2^T Q - I\| \sim 10^{-16}$

$R_2 R_1$ upper triangular matrix

$Q_2 = Q_3 R_3$

$A = Q_3 R_3 R_2 R_1$ $\quad$ Q_3

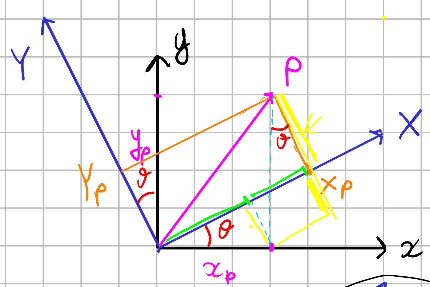
$R_3 R_2 R_1$ Upper triangular matrix

$x \in \mathbb{R}^n$ $\qquad$ $Q \in \mathbb{R}^{n\times n}$ orthonormal

$$\|Qx\|_2 = \|x\|_2$$

$$\|Qx\|_2^2 = (Qx)^T(Qx) = x^T \underbrace{Q^TQ}_{I} x = x^T x = \|x\|_2^2$$

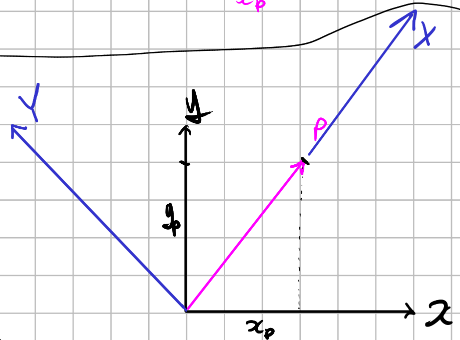
Givens Method



$$\begin{bmatrix} X_p \\ Y_p \end{bmatrix} = \overbrace{\begin{bmatrix} \cos\vartheta & \sin\vartheta \\ -\sin\vartheta & \cos\vartheta \end{bmatrix}}^{G(\vartheta)} \begin{bmatrix} x_p \\ y_p \end{bmatrix}$$

$$X_p = x_p \cos\vartheta + y_p \sin\vartheta$$

$$Y_p = -x_p \sin\vartheta + y_p \cos\vartheta$$



$X_p = \sqrt{x_p^2 + y_p^2} = \|P\|$

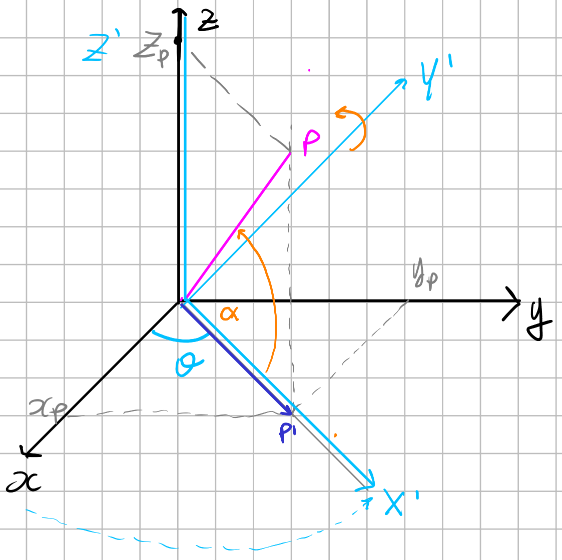
$Y_p = 0 \Rightarrow -x_p \sin\vartheta + y_p \cos\vartheta = 0$

$$\operatorname{tg}\vartheta = \frac{\sin\vartheta}{\cos\vartheta} = \frac{y_p}{x_p}$$

$$\cos\vartheta = \frac{x_p}{\|P\|} = C$$

$$\sin\vartheta = \frac{y_p}{\|P\|} = S$$

$$\begin{bmatrix} X_p \\ 0 \end{bmatrix} = \begin{bmatrix} \|P\| \\ 0 \end{bmatrix} = \begin{bmatrix} C & S \\ -S & C \end{bmatrix} \begin{bmatrix} x_p \\ y_p \end{bmatrix}$$



$$\|P'\| = \sqrt{x_p^2 + y_p^2}$$

$$\begin{bmatrix} \|P\| \\ 0 \\ 0 \end{bmatrix} = \ ? \ \begin{bmatrix} x_p \\ y_p \\ z_p \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} X'_p \\ 0 \\ Z'_p \end{bmatrix} = \begin{bmatrix} x_p \\ y_p \\ z_p \end{bmatrix}$$

$$\begin{bmatrix} X''_p \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} X'_p \\ 0 \\ Y'_p \end{bmatrix}$$

$$\begin{bmatrix} \|P'\| \\ 0 \end{bmatrix} = \begin{bmatrix} C_\theta & S_\theta \\ -S_\theta & C_\theta \end{bmatrix} \begin{bmatrix} x_p \\ y_p \end{bmatrix}$$

$$C_\theta = \frac{x_p}{\|P'\|} \qquad S_\theta = \frac{y_p}{\|P'\|}$$

$$\|P'\| = \sqrt{x_p^2 + y_p^2}$$

$$\begin{bmatrix} \|P'\| \\ 0 \\ z_p \end{bmatrix} = \begin{bmatrix} C_\theta & S_\theta & 0 \\ -S_\theta & C_\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_p \\ y_p \\ z_p \end{bmatrix}$$

$$\begin{bmatrix} \|P\| \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} C_\alpha & 0 & S_\alpha \\ 0 & 1 & 0 \\ -S_\alpha & 0 & C_\alpha \end{bmatrix} \begin{bmatrix} \|P'\| \\ 0 \\ z_p \end{bmatrix}$$

$$C_\alpha = \frac{\|P'\|}{\|P\|} = \frac{\sqrt{x_p^2 + y_p^2}}{\sqrt{x_p^2 + y_p^2 + z_p^2}}$$

$$S_\alpha = \frac{z_p}{\|P\|} = \frac{z_p}{\sqrt{x_p^2 + y_p^2 + z_p^2}}$$

$$\begin{bmatrix} \|P\| \\ 0 \\ 0 \end{bmatrix} = \underbrace{\begin{bmatrix} C_\alpha & 0 & S_\alpha \\ 0 & 1 & 0 \\ -S_\alpha & 0 & C_\alpha \end{bmatrix}}_{G_{13}} \underbrace{\begin{bmatrix} C_\theta & S_\theta & 0 \\ -S_\theta & C_\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{G_{12}} \begin{bmatrix} x_p \\ y_p \\ z_p \end{bmatrix}$$

$$G_\theta = \begin{bmatrix} C & S \\ -S & C \end{bmatrix} \qquad G_\theta^T = \begin{bmatrix} C & -S \\ S & C \end{bmatrix}$$

$$G_\theta^T G_\theta = \begin{bmatrix} C & -S \\ S & C \end{bmatrix} \begin{bmatrix} C & S \\ -S & C \end{bmatrix} = \begin{bmatrix} C^2 + S^2 & \cancel{CS} - \cancel{CS} \\ \cancel{SC} - \cancel{CS} & S^2 + C^2 \end{bmatrix} = I$$

$$C^2 + S^2 = \cos^2\theta + \sin^2\theta = 1$$

$\mathbb{R}^n$

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \to \begin{bmatrix} \sqrt{x_1^2+x_2^2} \\ 0 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} \to \begin{bmatrix} \sqrt{x_1^2+x_2^2+x_3^2} \\ 0 \\ 0 \\ \vdots \\ x_n \end{bmatrix} \to \cdots\cdots \begin{bmatrix} \sqrt{\sum_{i=1}^{n} x_i^2} \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$c_1 = \frac{x_1}{\sqrt{x_1^2+x_2^2}} \qquad s_1 = \frac{x_2}{\sqrt{x_1^2+x_2^2}} \qquad c_2 = \frac{\sqrt{x_1^2+x_2^2}}{\sqrt{x_1^2+x_2^2+x_3^2}} \qquad s_2 = \frac{x_3}{\sqrt{x_1^2+x_2^2+x_3^2}}$$

$$G_{12} = \begin{bmatrix} c_1 & s_1 & 0 & 0 & & \\ -s_1 & c_1 & 0 & 0 & & \\ 0 & 0 & 1 & 0 & & \\ 0 & 0 & 0 & 1 & & \\ 0 & 0 & 0 & & 1 & \\ 0 & 0 & 0 & & & \ddots \end{bmatrix} \qquad G_{13} = \begin{bmatrix} c_2 & 0 & s_2 & 0 & & \\ 0 & 1 & 0 & 0 & & \\ -s_2 & 0 & c_2 & 0 & & \\ 0 & 0 & 0 & 1 & & \\ 0 & 0 & 0 & 0 & 1 & \\ & & & & & 1 \end{bmatrix}$$

$$G = \prod_{j=2}^{n} G_{1j} \qquad G^{-1} = G^T$$

$$G = G_{12} G_{13} G_{14}$$

$$G^T = G_{14}^T G_{13}^T G_{12}^T$$

$$G^T G = \overbrace{G_{14}^T \underbrace{G_{13}^T \overbrace{G_{12}^T G_{12}}^{I} G_{13}}_{I} G_{14}}^{I} = I$$

$A \in \mathbb{R}^{m \times n} \to A = Q\,R$

$$A \xrightarrow{G_1} \begin{bmatrix} x & x & x & x \\ 0 & x & x & x \\ 0 & x & x & x \\ 0 & x & x & x \\ 0 & x & x & x \end{bmatrix} \xrightarrow{G_2} \begin{bmatrix} x & x & x & x \\ 0 & x & x & x \\ 0 & 0 & x & x \\ 0 & 0 & x & x \\ 0 & 0 & x & x \end{bmatrix} \xrightarrow{G_3} \begin{bmatrix} x & x & x & x \\ 0 & x & x & x \\ 0 & 0 & x & x \\ 0 & 0 & 0 & x \\ 0 & 0 & 0 & x \end{bmatrix} \xrightarrow{G_4} \overset{R}{\begin{bmatrix} x & x & x & x \\ 0 & x & x & x \\ 0 & 0 & x & x \\ 0 & 0 & 0 & x \\ 0 & 0 & 0 & 0 \end{bmatrix}}$$

$$\underbrace{G_4 G_3 G_2 G_1}_{Q^T} A = R \qquad Q^T A = R$$

$$A \in \mathbb{R}^{n \times n} \qquad Q Q^T A = Q R \qquad A = Q\,R$$

$$G_1 = G_{15} G_{14} G_{13} G_{12}$$

$$G_2 = \begin{bmatrix} 1 & 0\;0\;0\;0 \\ \begin{matrix}0\\0\\0\\0\end{matrix} & \boxed{G_{25} \cdot G_{24} \cdot G_{23}} \end{bmatrix} \qquad G_{23}, G_{24}, G_{23} \in \mathbb{R}^{4 \times 4}$$

$$G_3 = \left[\begin{array}{cc|ccc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & & & \\ 0 & 0 & & G_{35}\, G_{34} & \\ 0 & 0 & & & \end{array}\right] \qquad G_{35}, G_{34} \in \mathbb{R}^{3\times 3}$$

$$G_4 = \left[\begin{array}{ccc|cc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & & \\ 0 & 0 & 0 & \multicolumn{2}{c}{G_{45}} \end{array}\right] \qquad G_{45} \in \mathbb{R}^{2\times 2}$$

$$0 \in \mathbb{R}^{m-1}$$

$$A' = G_1 A$$
$$A'' = G_2 A'$$

$$G_2 = \left[\begin{array}{c|cccc} 1 & 0 & 0 & 0 & 0 \\ \hline 0 & & & & \\ 0 & & & & \\ 0 & & & & \\ 0 & & & & \end{array}\right] \rightarrow \left[\begin{array}{c|c} 1 & 0^T \\ \hline 0 & B \end{array}\right] \left[\begin{array}{c|c} a'_{11} & b'_1 \\ \hline 0 & a'_{ij} \end{array}\right] =$$

$$= \boxed{\begin{array}{c|c} a'_{11} & 1 \cdot b'_1 + \cancel{0 \, a'_{ij}} \\ \hline 0 \cdot a'_{11} & 0 \cdot b'_1 + B a'_{ij} \end{array}}$$

$$\left[\begin{array}{c|c} I & 0 \\ \hline 0 & B \end{array}\right] \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline 0 & A_{22} \end{array}\right] = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline 0 & B \cdot A_{22} \end{array}\right]$$

↑

Algorithm : Givens QR factorisation

$A \in \mathbb{R}^{n\times n}$

$W = A \qquad QT = I$

for $j = 1 : n$

    $G = I$

    for $i = j+1 : n$

        $x = W(j,j) \qquad c = \dfrac{x}{\sqrt{x^2+y^2}} \qquad s = \dfrac{y}{\sqrt{x^2+y^2}}$

        $y = W(i,j)$

        $G(j,j) = c \; ; \; G(i,i) = c \; ; \quad G(i,j) = -s \; ; \; G(j,i) = s$

        $QT = G\, QT$

        "$W = G\, W$" G is changing only the rows j and i of W starting from the column j.

            $r_j = c \cdot W(j, j:n) + s\, W(i, j:n)$

            $r_i = -s\, W(j, j:n) + c\, W(i, j:n)$

            $W(j, j:n) = r_j$

            $W(i, j:n) = r_i$

    end

end

$Q = \text{transpose}(QT)$

$R = W$

Householder transformations, Householder reflectors

$$P = I - 2uu^T \qquad u \in \mathbb{R}^n \qquad \|u\|_2 = 1$$

$$uu^T = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \overbrace{\begin{bmatrix} u_1 & \cdots & u_n \end{bmatrix}}^{u^T} = \begin{bmatrix} | & | & & | \\ u_1 u & u_2 u & \cdots & u_n u \\ | & | & & | \end{bmatrix}$$

$\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \to u$

$$(uu^T)_{ij} = u_i u_j \qquad uu^T \text{ is a rank 1 matrix of } \mathbb{R}^{n\times n}$$

Properties

P is symmetric $\qquad P = P^T \qquad (I-2uu^T)^T = I^T - 2(uu^T)^T = I - 2uu^T$

$$(uu^T)_{ij} = u_i u_j = (uu^T)_{ji}$$

P is orthogonal $\qquad P^TP = I \qquad (\|u\|_2 = 1)$

$$P^TP = (I-2uu^T)^T(I-2uu^T) = (I-2uu^T)(I-2uu^T) =$$
$$= I - 2uu^T - 2uu^T + 4(uu^T)(uu^T)$$

$$\begin{pmatrix} | & \\ u & -u- \\ | & \end{pmatrix}\begin{pmatrix} | & \\ u & -u- \\ | & \end{pmatrix}$$

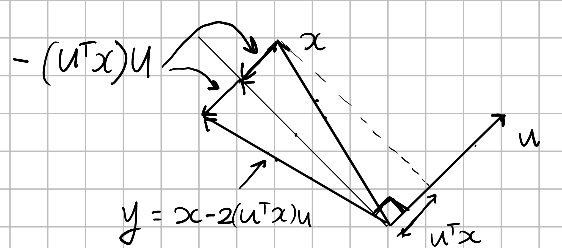
$$u(u^Tu)u^T = uu^T$$

$$= I - 4uu^T + 4uu^T = I$$

P involutive $\qquad P^2 = I$

$$P \cdot P = P^T \cdot P = I$$

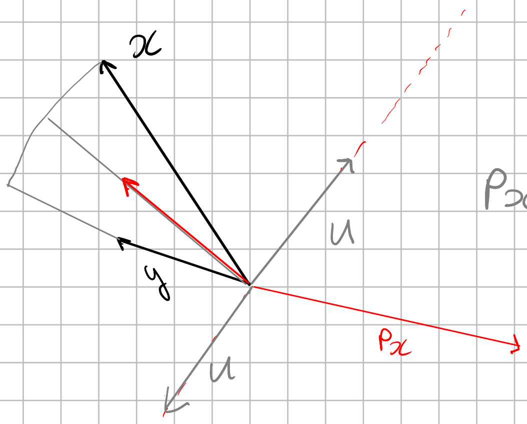
$x \in \mathbb{R}^n \qquad u \in \mathbb{R}^n \qquad P = I - 2uu^T \qquad Px = (I-2uu^T)x = x - 2u(u^Tx)$



$x - (u^Tx)u \perp u$

$x - 2(u^Tx)u = Px = y$

$\|Px\| = \|y\| = \|x\|$



$Px = \sigma y \qquad \sigma = \frac{\|x\|}{\|y\|}$

Property

Let x and y be two non-vanishing vectors in $\mathbb{R}^n$ linearly independent.
A unit vector $u \in \mathbb{R}^n$ and a scalar $\sigma \neq 0$ can be found such that

$$Px = \sigma y \qquad P = I - 2uu^T$$

Proof

$$\|Px\|_2 = \|x\|_2 = |\sigma| \, \|y\|_2 \qquad |\sigma| = \frac{\|x\|}{\|y\|} \qquad \sigma = \pm \frac{\|x\|}{\|y\|}$$

$$Px = (I - 2uu^T)x = x - 2(u^Tx)u$$

$$x - \sigma y = x - Px = 2(u^Tx)u$$

$$\|x - \sigma y\| = \|x - Px\| = 2\,|u^Tx|\,\|u\|_2 \qquad 2|u^Tx| = \|x - \sigma y\|$$

$$2 \,\mathrm{sig}(u^Tx)(u^Tx) = \|x - \sigma y\|$$

$$2u^Tx = \mathrm{sign}(u^Tx)\,\|x - \sigma y\|$$

$$u = \frac{x - \sigma y}{2(u^Tx)} = \frac{x - \sigma y}{\mathrm{sign}(u^Tx)\,\|x - \sigma y\|}$$

$$= \mathrm{sign}(u^Tx)\frac{x - \sigma y}{\|x - \sigma y\|} = \mathrm{sign}(u^Tx)\frac{x \mp \frac{\|x\|}{\|y\|}y}{\left\| x \mp \frac{\|x\|}{\|y\|}y \right\|}$$

$y = e_1$

Corollary

For each non vanishing vector $x \in \mathbb{R}^n$ the Householder reflector

$$P = I - 2uu^T$$

with $\quad u = \frac{\tilde{u}}{\|\tilde{u}\|} \qquad \tilde{u} = x + \sigma e_1 \qquad \sigma = \mathrm{sign}(x_1)\|x\|$

is such that $\qquad Px = -\sigma e_1$

$$\sigma^2 = \|x\|_2^2$$

Proof

$$Px = \left(I - 2\frac{\tilde{u}\tilde{u}^T}{\|\tilde{u}\|_2^2}\right)x = x - 2\,\tilde{u}^T x\,\tilde{u}\,\frac{1}{\|\tilde{u}\|_2^2}$$

$$-\sigma e_1 = x - 2\frac{1}{\|\tilde{u}\|_2^2}\,\tilde{u}^T x\,\tilde{u}$$

$$2\frac{1}{\|\tilde{u}\|_2^2}\,\tilde{u}^T x\,\tilde{u} = x + \sigma e_1$$

$$\tilde{u}^T x = (x+\sigma e_1)^T x = x^T x + \sigma e_1^T x = \|x\|_2^2 + \sigma x_1$$
$$= \sigma^2 + \sigma x_1 = \sigma(\sigma + x_1)$$

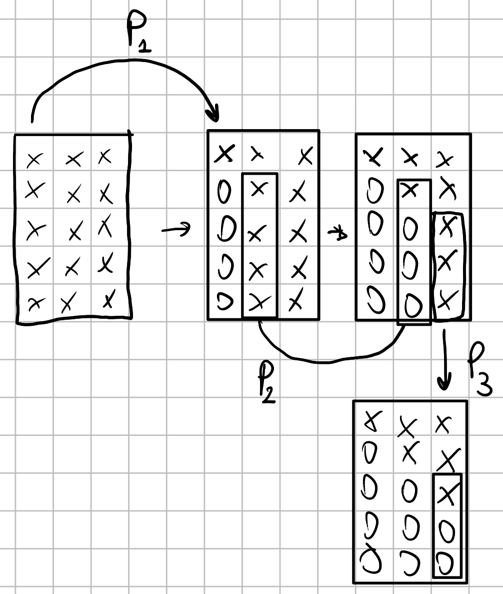
$1 \rightarrow$ $$2\frac{1}{\|\tilde{u}\|_2^2}\,\sigma(\sigma+x_1)\,\underline{\tilde{u}} = x + \sigma e_1$$

$$\|\tilde{u}\|_2^2 = \tilde{u}^T\tilde{u} = (x+\sigma e_1)^T(x+\sigma e_1) = x^T x + 2\sigma e_1^T x + \sigma^2 e_1^T e_1 =$$
$$= \|x\|_2^2 + 2\sigma x_1 + \sigma^2$$
$$= \sigma^2 + 2\sigma x_1 + \sigma^2 = 2\sigma^2 + 2\sigma x_1$$
$$= 2\sigma(\sigma + x_1)$$

$$2\frac{1}{\|\tilde{u}\|_2^2}\,\sigma(\sigma+x_1) = \cancel{2}\frac{1}{\cancel{2\sigma(\sigma+x_1)}}\,\cancel{\sigma(\sigma+x_1)} = 1$$

$$\boxed{\tilde{u} = x + \sigma e_1}$$

$$A = QR \qquad Q^T A = R$$



$W = A \qquad Q^T = I$

for $j = 1:n$

$\quad x = W(j:n, j)$

$\quad \sigma = \text{sign}(x_1)\|x\|$

$\quad \tilde{u} = x + \sigma e_1 \qquad e_1 \in \mathbb{R}^{n-j+1}$

$$P' = I - 2\frac{\tilde{u}\tilde{u}^T}{\|\tilde{u}\|_2^2} \qquad I \in \mathbb{R}^{n-j+2 \times n-j+2}$$

$$P_j = \left[\begin{array}{c|c} I^{j-2 \times j-2} & 0 \\ \hline 0 & P' \end{array}\right] \qquad \begin{array}{|c|c|} \hline x & x \\ \hline 0 & P'A' \\ \hline \end{array}$$

$$P_j = \begin{array}{|c|c|} \hline I & 0 \\ \hline 0 & P' \\ \hline \end{array} \qquad \downarrow A \quad \begin{array}{|c|c|} \hline U & M \\ \hline 0 & A' \\ \hline \end{array}$$

$$P_j A = \left[\begin{array}{c|c} IU & IM \\ \hline \underbrace{0U + P'0}_{0} & 0M + P'A' \end{array}\right] = \left[\begin{array}{c|c} U & M \\ \hline 0 & P'A' \end{array}\right]$$

$$QT = P_i\, QT$$
$$W = P_i\, W$$

end

$W = R$ $\qquad Q = (QT)^T \qquad A = QR$

Projectors = Projection operators $\quad P$

$$\boxed{\begin{array}{l} P \text{ is a linear mapping } \mathbb{R}^n \to \mathbb{R}^n \\ P^2 = P \qquad P \text{ is idem potent} \end{array}}$$

Properties

1) P is a projector $\Rightarrow$ $\underline{I - P \text{ is a projector}}$

I is a linear mapping $\qquad P$ is linear $\Rightarrow$ $I-P$ linear

$$(I-P)^2 = (I-P)(I-P) = I - 2IP + P^2 = I - 2P + P^2 \underset{P^2=P}{=} I - 2P + P = \underline{I-P}$$

$$\ker(P) = \{x \in \mathbb{R}^n : Px = 0\}$$
$$\operatorname{Im}(P) = \{y \in \mathbb{R}^n : \exists\, x \in \mathbb{R}^n \;\; Px = y\} = \operatorname{Ran}(P)$$

Property $\quad \boxed{\ker(P) = \operatorname{Im}(I-P)}$

Proof

$x \in \ker(P) \qquad Px = 0 \qquad (I-P)x = x - Px = x \qquad x \in \operatorname{Im}(I-P)$

$x \in \operatorname{Im}(I-P) \qquad \exists\, y \in \mathbb{R}^n \quad (I-P)y = x$

$$P(\; y - Py = x \;)$$
$$Py - P^2 y = Px$$
$$0 = Py - Py = Px \qquad x \in \ker(P)$$

$x \in \ker(P) \Rightarrow x \in \operatorname{Im}(I-P)$

$x \in \operatorname{Im}(I-P) \Rightarrow x \in \ker(P)$ $\qquad \ker(P) = \operatorname{Im}(I-P)$

Corollary

$$\ker(I-P) = \mathrm{Im}(P)$$

$$\ker(I-P) = \mathrm{Im}(I-(I-P)) = \mathrm{Im}(P)$$

Property If P is a projector

$$\ker(P) \cap \mathrm{Im}(P) = \{0_{\mathbb{R}^n}\}$$

Proof

$\exists\, x \in \ker(P) \cap \mathrm{Im}(P)$ $x \neq 0$

$x \in \ker(P)$ $Px = 0$

$x \in \mathrm{Im}(P)$ $\exists y$ $Py = x$ $P(Py = x)$ $P^2 y = Py = Px$

$x = 0$

Corollary

$$\mathbb{R}^n = \ker(P) \oplus \mathrm{Im}(P) = \mathrm{Im}(I-P) \oplus \mathrm{Im}(P)$$

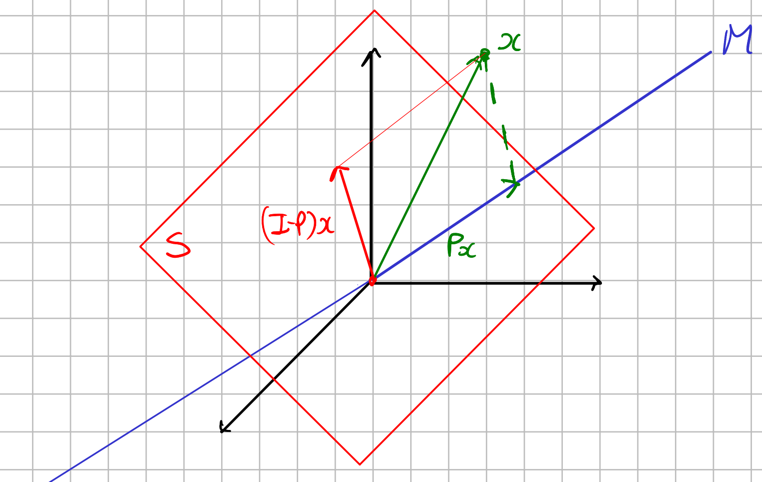
$$x = Px + (I-P)x$$ this splitting is unique

$M, S \subset \mathbb{R}^n$ $\mathbb{R}^n = M \oplus S$

$x \in \mathbb{R}^n$ $x = x_M + x_S$ $x_M \in M$

$x_S \in S$

There exists a unique projector P : $\mathrm{Im}(P) = M$ $\ker(P) = S = \mathrm{Im}(I-P)$



$x = Px + (I-P)x$

$x = Px + (I-P)x$ $Px \in M$

$(I-P)x \in S$

$$P\,\boxed{\overset{\ker(P)}{(I-P)x}} = Px - Px = 0 \qquad P(Px) = Px$$

$S, M \qquad \mathbb{R}^n = M \oplus S \qquad\qquad L \perp S \Rightarrow \mathbb{R}^n = L \oplus S$

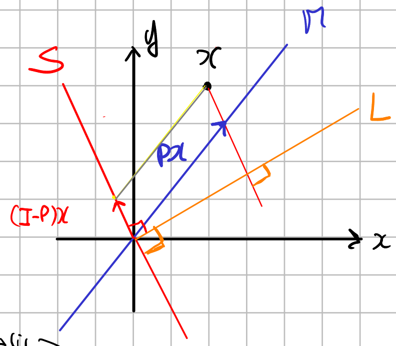
$n = \underset{\overset{\shortparallel}{m}}{\dim(M)} + \dim(S) \Rightarrow \dim(S) = n - m \qquad n = \dim(L) + \underset{\overset{\shortparallel}{n-m}}{\dim(S)}$

$$\Downarrow$$

$\dim(L) = n - (n-m) = m$

$\dim(L) = \dim(M) = m$

$M, S \Rightarrow \underset{\uparrow}{P} \Leftarrow M, L$



Vectors w.r.t the canonical basis

$M \in \mathbb{R}^n \qquad V = [v_1, \ldots, v_m] \qquad M = \text{Span}\{v_1, \ldots, v_m\} \qquad V$ is a basis for M

$L \in \mathbb{R}^n \qquad L \perp S \qquad W = [w_1, \ldots, w_m] \qquad L = \text{Span}\{w_1, \ldots, w_m\} \qquad W$ is a basis for L

$V \in \mathbb{R}^{n \times m} \qquad W \in \mathbb{R}^{n \times m}$

$x \in \mathbb{R}^n \qquad Px \in M \qquad Px = \sum_{i=1}^{m} y_i v_i = Vy \qquad y \in \mathbb{R}^m$

$v_i \in \mathbb{R}^n \ \forall i = 1, \ldots, m$

$(I-P)x = x - Px \perp L \qquad x - Px \perp w_j \quad \forall j = 1, \ldots, m$

$x - Px \in \mathbb{R}^n$

$W^T(x - Px) = 0_{\mathbb{R}^n}$

$W^T(x - Vy) = 0 \qquad x \in \mathbb{R}^n$

$y \in \mathbb{R}^m$

$W^T x = W^T V y \Rightarrow y = (W^T V)^{-1} W^T x \qquad \underset{\overset{\downarrow}{\mathbb{R}^{m \times n}}}{W^T} \underset{\overset{\searrow}{\mathbb{R}^{n \times m}}}{V} \in \mathbb{R}^{m \times m}$

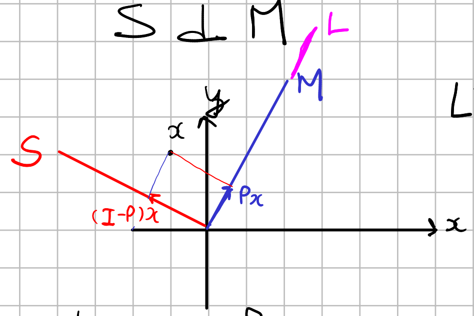
$$Vy = \underbrace{V (W^T V)^{-1} W^T}_{\mathbb{P}} x$$

$\mathbb{P}$ matrix representation of the projector P defined by M and L with respect to the canonical basis

$\mathbb{P} \in \mathbb{R}^{n \times n} \qquad \dim(\ker(\mathbb{P})) = n - m = \dim(S)$

$\text{rank}(\mathbb{P}) = m = \dim(M)$

Special case $S \perp M$ L P is an orthogonal projection



$L \equiv M$

$\mathbb{R}^n = S \oplus M$

Orthogonal projection to $L \equiv M$ P is projectin any vector x onto M orthogonally

V is the matrix containing an orthonormal basis of M columnwise $V = [v_1, \dots, v_m]$

W should be a bassis for $L \equiv M$ $W = V$

$$P = V(W^T V)^{-1} W^T = V(V^T V)^{-1} V^T \qquad V^T V = I \in \mathbb{R}^{m \times m}$$

$$= V I V^T = V V^T$$

$$P = V V^T \qquad Px = V(V^T x)$$

$$V^T x = \underbrace{\begin{bmatrix} v_1^T x \\ v_2^T x \\ \vdots \\ v_m^T x \end{bmatrix}}_{z}$$

$\rightarrow$ the components of x w.r.t. v_1

$\rightarrow$ " " " " " v_2

$$Vz = z_1 v_1 + z_2 v_2 \cdots + z_m v_m$$

"P the matrix of a projector $P \Rightarrow P^T$ is the matrix of the projecto P^T"

1) If P is a projector $\Rightarrow P^T$ is a projector

2) If P is the matrix representing $P \Rightarrow P^T$ is the matrix representing P^T

P: $\quad P = V(W^T V)^{-1} W^T$ $\quad$ L and M can be different

P is the projector tha is projecting any vector $x \in \mathbb{R}^n$ onto M orthogonally to L

Let us consider the projector P^T " " " " " onto L " " M

$L \downarrow W$ $\quad M \downarrow V$

$$P^T = W(V^T W)^{-1} V^T \ ?$$

$$\left(W(V^T W)^{-1} V^T\right)^T = V^{T^T}\left((V^T W)^{-1}\right)^T W^T = \underline{V(W^T V)^{-1} W^T} = P$$

P is an orthogonal projector $\Longleftrightarrow$ P is symmetric

$$P = VV^T$$

$$P^T = (VV^T)^T = VV^T$$

P ~~projector~~ orthogonal projector $\quad \|P\| = 1$

$$\|P\| = \sup_{x \in \mathbb{R}^n} \frac{\|Px\|}{\|x\|}$$

only for orthogonal projectors

$$\boxed{\|Px\| \leq \|x\|} \qquad \|P\| = \sup_{x \in \mathbb{R}^n} \frac{\|Px\|}{\|x\|} \leq 1$$

$$x = \underset{\downarrow\\ Im(P)}{x_M} + \underset{\downarrow\\ ker(P)}{x_S} = Px + (I-P)x$$

$$\|x\|_2^2 = x^T x = (Px + (I-P)x)^T (Px + (I-P)x) = x^T P^T P x + 2x^T P^T (I-P)x +$$
$$+ x^T (I-P)^T (I-P) x$$

$$P = V(W^T V)^{-1} W^T$$

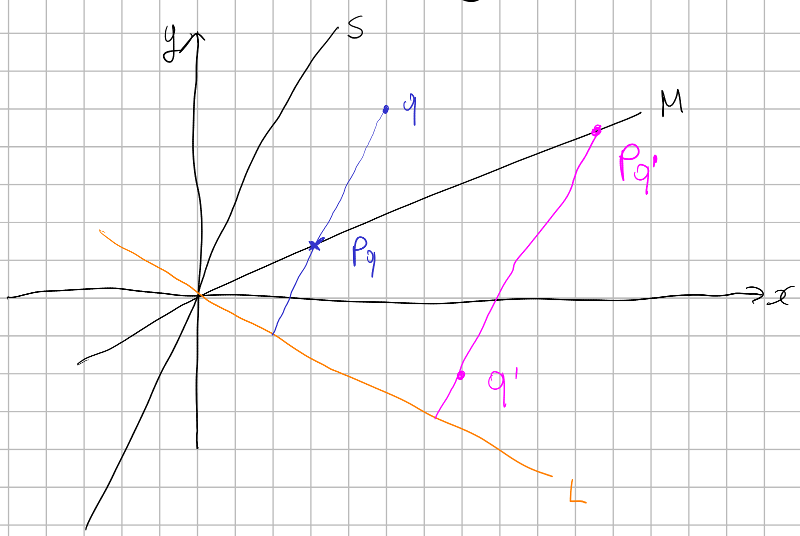
$$P^T = W(V^T W)^{-1} V^T$$

$$P^T P = W(V^T W)^{-1} \underbrace{V^T V}_{I} (W^T V)^{-1} W^T =$$
$$= W (V^T W)^{-1} (W^T V)^{-1} W^T$$
$$((W^T V)^{-1})^T (W^T V)^{-1}$$

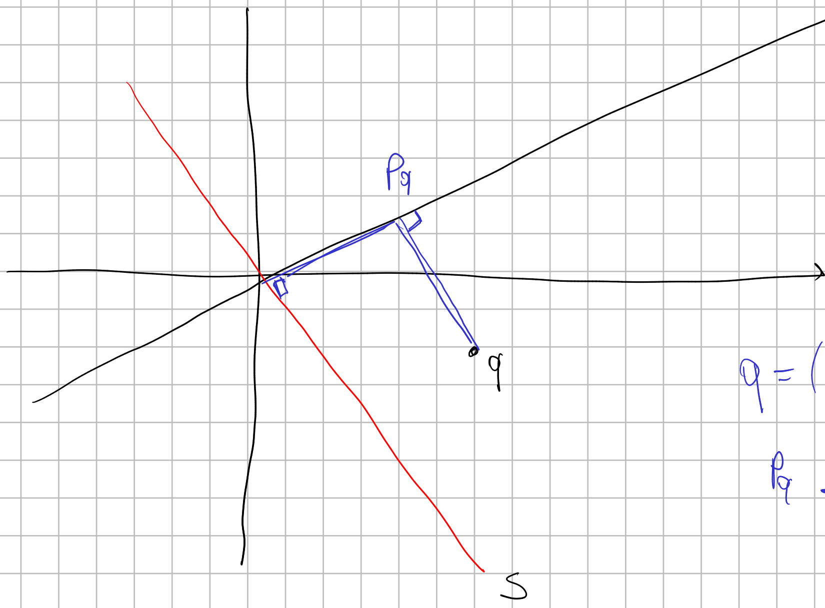
$$x \in M \qquad x = Px \qquad \|x\| = \|Px\| \qquad \|P\| = \sup_{x \in \mathbb{R}^n} \frac{\|Px\|}{\|x\|} = 1$$

$$\boxed{\|P\| = 1}$$

$$P \begin{cases} \lambda = 1 & M \quad x \in M \quad Px = 1 \cdot x \\ \lambda = 0 & \quad x \in S \quad Px = 0 = 0 \cdot x \end{cases}$$



$$\|q\| \geq \|Pq\|$$
$$\|q'\| \leq \|Pq'\|$$



$$\|q\| = \sqrt{\|Pq\|^2 + \|(I-P)q\|^2}$$

$$\|q\| \geq \|Pq\|$$

$$q = (Pq) + (Iq - Pq)$$

$$Pq \perp (Iq - Pq)$$