Minimal norm solution

$A \in \mathbb{R}^{m \times n}$ $\qquad m \geq n$

$\operatorname{rank}(A) = r \leq n$

$A = U \Sigma V^T \qquad U \in \mathbb{R}^{m \times m} \qquad \Sigma \in \mathbb{R}^{m \times n} \qquad V \in \mathbb{R}^{n \times n}$

$\ker(A) = \operatorname{span}\{ v_i \quad i = r+1, \dots, n \}$ $\qquad V, U$ orthovormal matrices

$\operatorname{Im}(A) = \operatorname{span}\{ u_i \quad i = 1, \dots, r \}$

$$A = \sum_{i=1}^{r} \sigma_i u_i v_i^T \qquad A_i = u_i v_i^T$$

$Ax = b$ $\qquad b \in$ space generated by the columns of A

$$U \Sigma V^T x = b$$

$$\bar{x} = \boxed{V \Sigma^+ U^T} b \qquad \Sigma^+ \in \mathbb{R}^{n \times m} \qquad \Sigma^+ = \left[\begin{array}{ccc|c} \frac{1}{\sigma_1} & & 0 & \\ & \ddots & & 0 \\ 0 & & \frac{1}{\sigma_r} & \\ \hline & 0 & & 0 \end{array}\right]$$

$$\bar{x} = \left(\sum_{i=1}^{r} \frac{1}{\sigma_i} v_i u_i^T \right) b$$

$$\bar{x} = \sum_{i=1}^{r} \left(\frac{1}{\sigma_i} u_i^T b \right) v_i$$

$x = \bar{x} + \ker(A)$ $\qquad$ affine space

$x = \bar{x} + x_0$ $\qquad x_0 \in \ker(A)$

$$Ax = U \Sigma V^T \Big(\underbrace{\sum_{i=1}^{r} \left(\frac{1}{\sigma_i} u_i^T b \right) v_i}_{\bar{x}} + \underbrace{\sum_{i=r+1}^{n} \gamma_i v_i}_{x_0} \Big)$$

$$= \sum_{i=1}^{r} \left(\frac{1}{\sigma_i} u_i^T b \right) U \Sigma V^T v_i + \sum_{i=r+1}^{n} \gamma_i U \Sigma V^T v_i$$

$$= \sum_{i=1}^{r} \left(\frac{1}{\sigma_i} u_i^T b \right) U \Sigma \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} i + \sum_{i=r+1}^{n} \gamma_i U \Sigma \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} i$$

$$= \sum_{i=1}^{r} \left(\frac{1}{\sigma_i} u_i^T b \right) U \begin{bmatrix} 0 \\ \vdots \\ \sigma_i \\ \vdots \\ 0 \end{bmatrix} + \sum_{i=r+1}^{n} \gamma_i \underbrace{U \quad}_{0}$$

$$Ax = \underbrace{\sum_{i=1}^{r} \frac{1}{\cancel{\sigma_i}} u_i^T b \, \cancel{\sigma_i} u_i}_{b} + 0$$

$$\boxed{A\bar{x} = b}$$

$$\|x\|_2^2 = \sum_{i=1}^{r} \left(\frac{1}{\sigma_i} u_i^T b \right)^2 + \sum_{i=r+1}^{n} \gamma_i^2 = \|\bar{x}\|_2^2 + \|x_0\|_2^2$$

$$P_{inv}(A) = \left(\sum_{i=1}^{r} \frac{1}{\sigma_i} v_i u_i^T \right) \qquad \bar{x} = P_{inv}(A)\, b$$

Eckart-Young theorem

Let $A \in \mathbb{R}^{m \times n}$, for any $k \in [1, \dots, r]$ the truncated SVD decomposition of A

$A_k = \sum_{i=1}^{k} \sigma_i u_i v_i^T$ is the optimal approximation to A

$$\|A - A_k\| = \inf \{ \|A - B\| : B \in \mathbb{R}^{m \times n} \text{ with } \operatorname{rank}(B) = k \}$$

$\|.\|$ 2-norm or the Frobenious norm

$$\|A - A_k\|_2 = \sigma_{k+1}$$

$$\|A - A_k\|_F = \sum_{i=k+1}^{n} \sigma_i$$

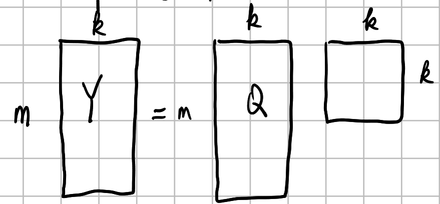
Randomized SVD - RSVD

$A \in \mathbb{R}^{m \times n}$ $\qquad n \leq m$

$X \in \mathbb{R}^{n \times k}$ $\qquad k \leq n \leq m$

$Y = AX \in \mathbb{R}^{m \times k}$

$Y \in \mathbb{R}^{m \times k}$ $\qquad Y = QR$ $\qquad Q \in \mathbb{R}^{m \times k}$ $\quad R \in \mathbb{R}^{k \times k}$



$Im(Y) = Im(Q) \subset Im(A)$

$P = QQ^T \in \mathbb{R}^{m \times m}$

$PA = Q\underbrace{Q^T A}_{B} = QB$ $\qquad B \in \mathbb{R}^{k \times n}$ $\qquad \dim(B) \leq \dim(A)$

$SVD(B)$: $B = U_B \Sigma_B V_B^T$ $\qquad U_B \in \mathbb{R}^{k \times k}$ $\Sigma_B \in \mathbb{R}^{k \times n}$ $V_B \in \mathbb{R}^{n \times n}$

$SVD(PA) = \underbrace{Q U_B}_{U_{PA}} \underbrace{\Sigma_B}_{= \Sigma_{PA}} \underbrace{V_B^T}_{= V_{PA}^T}$ $\qquad PA = A_{small}$

Theorem

Let $A \in \mathbb{R}^{m \times n}$ select $k \geq 2$ and an oversampling peremeter $p \geq 2$

where $k+p \leq \min\{m, n\}$.

Execute the algorithm with a Standard Gaussian matrix $X \in \mathbb{R}^{n \times k+p}$

$$E[\|A - QQ^T A\|] \leq \left(1 + \frac{4\sqrt{k+p}}{p-1} \cdot \sqrt{\min\{m,n\}}\right) \sigma_{k+1}$$

$E[.]$ expetation of ..

$A \in \mathbb{R}^{m \times n}$ $\qquad m \geq n$

$B \in \mathbb{R}^{m \times n}$ $\qquad Q^T Q = I_n$

$$\min_{\substack{Q \in \text{Orthogonal} \\ \text{matrices} \\ \text{in } \mathbb{R}^n}} \|A - BQ\|_F^2$$

$$\|A - BQ\|_F^2 = tr((A - BQ)(A - BQ)^T)$$

$$= tr(AA^T - AQ^T B^T - BQA^T + BQQ^T B^T) \qquad Q \cdot Q^T = Q^T \cdot Q = I_n$$

$$= tr(AA^T) - t(AQ^T B^T) - tr((AQ^T B^T)^T) + tr(BB^T)$$

$$= tr(AA^T) - 2\, tr(AQ^T B^T) + tr(BB^T)$$

$$\max_Q tr(AQ^T B^T)$$

$Q \in \mathbb{R}^{n \times n}$

$A, B \in \mathbb{R}^{m \times n}$

$$tr(ABC) = tr(BCA) = tr(CAB)$$

$Q^T \in \mathbb{R}^{n \times n}$ $\quad B^T \in \mathbb{R}^{n \times m}$ $\quad A \in \mathbb{R}^{m \times n}$

$$tr(AQ^T B^T) = tr(Q^T B^T A) = tr((BQ)^T A)$$
$$= (BQ, A)$$

$Q^T B^T \in \mathbb{R}^{n \times m}$ $\quad A \in \mathbb{R}^{m \times n}$

$Q^T B^T A \in \mathbb{R}^{n \times n}$

$$tr(X^T Y) = tr(YX^T) = tr(XY^T) = tr(Y^T X) = \sum_{ij} X_{ij} Y_{ij} = (X, Y)$$

$x, y \in \mathbb{R}^p$ $\quad x^T y = \sum_{i=1}^{p} x_i y_i \in \mathbb{R}$ $\quad x^T y = (x, y)$

$$tr(Q^T B^T A)$$

$$\boxed{B^T A = U \Sigma V^T} \qquad B^T A \in \mathbb{R}^{n \times n}$$

$$tr(Q^T B^T A) = tr(Q^T U \Sigma V^T) = tr(U \Sigma V^T Q^T) = tr(\Sigma \underbrace{V^T Q^T U}_{Z})$$

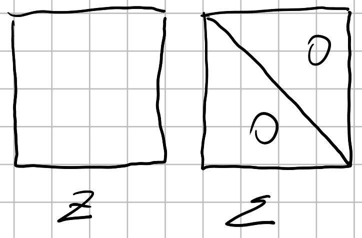
$Z = V^T Q^T U$

Z: orthonormal

$\mathrm{tr}(\Sigma Z) = \mathrm{tr}(Z\Sigma) =$

$$= \sum_{j=1}^{n} z_{jj}\sigma_j \le \sum_{j=1}^{n} |z_{jj}|\sigma_j = \sum_{j=1}^{n} 1\,\sigma_j$$

$Z\Sigma =$



$(Z\Sigma)_{ij} = z_{ij}\sigma_j$

Z is an orthonormal matrix as it is product of orthonormal matrices

The entries of an orthonormal matrix can not exeed 1 because each column $z(:,j)$ has 2-norm equal to one : $\sum_{i=1}^{n} z_{ij}^2 = 1 \quad \forall j \Rightarrow z_{ij}^2 \le 1 \quad \forall ij$

$\forall j = 1, \dots, n \quad |z_{jj}| \le 1$

The condition $\mathrm{tr}(Z\Sigma) = \sum_{j=1}^{n} \sigma_j$ is realized if $Z = I_n$ so we can take

$\boxed{Q = VU^T}$ and $Z = V^T Q U = V^T V U^T U = I_n$ and the maximum is realized

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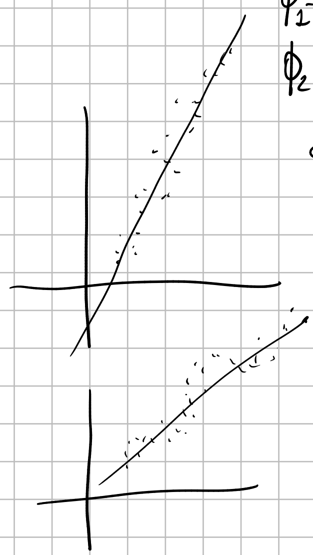
$y = \overset{?}{f}(x) \qquad x_i \to y_i \quad i = 1, \dots, m$

$$f^?(x) \approx \sum_{j=1}^{n} c_j \phi_j(x)$$



$\phi_1 = 1$

$\phi_2 = x$

$c_1 + c_2 x$

$c_1 + c_2 x + c_3 \sin \dots$

$$\sum_{j=1}^{n} c_j \phi_j(x_i) \sim y_i \qquad \forall i = 1, \dots, m$$

$\phi c \qquad Y \qquad \phi_{ij} = \phi_j(x_i)$

$$\min_{c \in \mathbb{R}^n} \|\phi c - Y\|_2^2$$

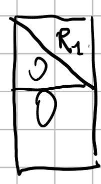
$\boxed{\phi^T \phi c = \phi^T Y \quad \text{system of normal equations}}$ $\qquad m \cdot n^2$

$$\|x\|_2^2 = \|Qx\|_2^2 = (Qx)^T(Qx) = x^T Q^T Q x = x^T x = \|x\|_2^2$$

$\|\phi c - Y\|_2^2 = \|Q^T \phi c - Q^T Y\|_2^2 \qquad \forall Q^T$ orthonormal

$\boxed{\phi = QR} \qquad \phi \in \mathbb{R}^{m\times n} \qquad m \gg n$

$Q \in \mathbb{R}^{m\times m} \quad R \in \mathbb{R}^{m\times n} \qquad Q$ orthonormal $\quad R =$



$R_1 \in \mathbb{R}^{n\times n}$

$$\|\phi c - Y\|_2^2 = \|Q^T Q R c - Q^T Y\|_2^2 = \|Rc - Q^T Y\|_2^2$$

$Q^T Y \in \mathbb{R}^m$ $\qquad$ $Q^T Y = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ $\qquad$ $b_1 \in \mathbb{R}^n$

$b_2 \in \mathbb{R}^{m-n}$

$\in \mathbb{R}^{m-n \times n}$

$$\|\phi c - Y\|_2^2 = \|Rc - Q^T Y\|_2^2 = \|R_1 c - b_1\|_2^2 + \|0 c - b_2\|_2^2$$

$$= \|R_1 c - b_1\|_2^2 + \|b_2\|_2^2 = 0 + \|b_2\|_2^2$$

$$\boxed{R_1 c = b_1}$$