The symmetric Lanczos Algorithm $\quad A = A^T$

$$\ulcorner \quad A V_m = V_m H_m + w_m^T e_m = V_{m+1} \bar{H}_m \qquad \text{for any } A \in \mathbb{R}^{n \times n}$$

$$V_m^T A V_m = H_m \quad \lrcorner$$

Theorem

Assume that the Arnoldi method is applied to a real symmetric matrix A then the coefficients generated by the algorithm are such that

a) $h_{ij} = 0 \qquad 1 \le i < j-1$

b) $h_{j,j+1} = h_{j+1,j} \qquad j = 1, \dots, m$

$\Leftrightarrow$ H_m is symmetric and tri-diagonal

Proof

$$H_m = V_m^T A V_m \qquad H_m^T = (V_m^T A V_m)^T = V_m^T A^T V_m = V_m^T A V_m = H_m$$

H_m Hessemberg and symmetric $\Rightarrow$ $h_{ij} = 0 \quad 1 \le i \le j-1$

Hessemberg

$$H_m = \begin{bmatrix} h_{11} & h_{12} & h_{13} & 0 & \dots \\ h_{21} & h_{22} & h_{23} & 0 & \dots \\ 0 & h_{32} & h_{33} & & \dots \\ 0 & 0 & h_{43} & & \dots \\ \vdots & \vdots & \vdots & & \end{bmatrix} \qquad T_m = \begin{bmatrix} \alpha_1 & \beta_2 & & & & \\ \beta_2 & \alpha_2 & \beta_3 & & 0 & \\ & \beta_3 & \alpha_3 & \beta_4 & & \\ & 0 & \beta_4 & \ddots & \ddots & \\ & & & \ddots & \alpha_{m-1} & \beta_m \\ & & & & \beta_m & \alpha_m \end{bmatrix}$$

Arnoldi algorithm

v_1

$w_1 = A v_1$

$$w_1 = A v_1 - \overbrace{(A v_1, v_1)}^{h_{11} = \alpha_1} v_1$$

$$h_{21} = \beta_2 = \| A v_1 - (A v_1, v_1) v_1 \|$$

$$v_2 = \frac{A v_1 - (A v_1, v_1) v_1}{h_{21}} \qquad h_{21} v_2 = A v_1 - h_{11} v_1$$

$$\beta_2 v_2 = A v_1 - \alpha_1 v_1 \qquad A v_1 = \alpha_1 v_1 + \beta_2 v_2$$

$$w_2 = A v_2$$

$$w_2 = A v_2 - \underbrace{(A v_2, v_1)}_{h_{21}=\beta_2} v_1 - \underbrace{(A v_2, v_2)}_{h_{22}=\alpha_2} v_2$$

$$(A v_2, v_1) = v_1^T A v_2 = v_1^T A^T v_2 = (v_2, A v_1) =$$

$$= (v_2, \alpha_1 v_1 + \beta_2 v_2) = \alpha_1 \underbrace{(v_2, v_1)}_{0} + \beta_2 (v_2, v_2)$$

$$w_2 = \beta_3 v_3 = A v_2 - \beta_2 v_1 - \alpha_2 v_2$$

$$A v_2 = \beta_2 v_1 + \alpha_2 v_2 + \beta_3 v_3$$

$$w_3 = A v_3$$

$$w_3 = A v_3 - \underbrace{(A v_3, v_1)}_{h_{31}=0} v_1 - \underbrace{(A v_3, v_2)}_{h_{32}=\beta_3} v_2 - \underbrace{(A v_3, v_3)}_{\alpha_3} v_3 = \beta_4 v_4$$

$$(A v_3, v_1) = (v_3, A v_1) = 0$$

$$A v_3 = \beta_3 v_2 + \alpha_3 v_3 + \beta_4 v_4$$

$$(A v_3, v_2) = (v_3, A v_2) = \beta_3$$

$$A v_1 = \text{"} 0 \, v_0 \text{"} + \alpha_1 v_1 + \beta_2 v_2 \qquad \beta_1 = 0$$

$$A v_j = \beta_j v_{j-1} + \alpha_j v_j + \beta_{j+1} v_{j+1} \qquad j > 1$$

Lanczos Algorithm

Choose v_1 $\quad \|v_1\| = 1 \quad \beta_1 = 0 \quad v_0 = 0$

for $j = 1, \dots, m$

$\quad w_j = A v_j - \beta_j v_{j-1}$

$\quad \alpha_j = (w_j, v_j)$

$\quad w_j = w_j - \alpha_j v_j$

$\quad \beta_{j+1} = \|w_j\| \leftarrow$ if $\beta_{j+2} == 0$ break

$\quad v_{j+1} = \dfrac{w_j}{\beta_{j+1}}$

end

FOM $\qquad \mathcal{L} = \mathcal{K} = \mathcal{K}_m(A, r_0)$

$$b - A x_m = r_m \perp \mathcal{L} = K_m(A, r_0) \qquad x_m = x_0 + V_m y_m$$

$$V_m^T (b - A x_m) = 0$$

$$V_m^T (b - A x_0 - A V_m y_m) = 0$$

$$V_m^T (r_0 - A V_m y_m) = 0$$

$$\beta e_1 - T_m y_m = 0 \qquad \boxed{T_m y_m = \beta e_1} \leftarrow$$

$$r_m = b - Ax_m = r_0 - AV_m y_m = r_0 - V_m \underline{H_m} y_m - w_m e_m^T y_m$$
$$= r_0 - V_m T_m y_m - \beta_{m+1} v_{m+1} e_m^T y_m$$
$$= r_0 - V_m \beta e_1 - \beta_{m+1} v_{m+1} e_m^T y_m$$
$$= \cancel{r_0} - \cancel{r_0} - \beta_{m+1} v_{m+1} \underline{e_m^T y_m}$$
$$\| r_m \| = \beta_{m+1} | y_m(m) |$$

A_{spd} = A is symmetric positive definite

$$T_m = L_m U_m$$

$$\underset{T_m}{\begin{bmatrix} \alpha_1 & \beta_2 & & \\ \beta_2 & \alpha_2 & \beta_3 & \\ & \beta_3 & \alpha_3 & \ddots \\ & & \ddots & \ddots \end{bmatrix}} = \underset{L_m}{\begin{bmatrix} 1 & 0 & 0 & \\ \lambda_2 & 1 & 0 & \\ & \lambda_3 & 1 & \ddots \\ & & \lambda_4 & \end{bmatrix}} \underset{U_m}{\begin{bmatrix} \eta_1 & \gamma_2 & & & \\ 0 & \eta_2 & \gamma_3 & & \\ 0 & 0 & \eta_3 & \gamma_4 & \\ & & & \ddots & \ddots \end{bmatrix}}$$

$\alpha_1 = \eta_1$

$\beta_2 = \gamma_2$ $\quad \lambda_2 \eta_1 = \beta_2 \quad \lambda_2 = \frac{\beta_2}{\alpha_1}$

$\lambda_2 \gamma_2 + \eta_2 = \alpha_2 \quad \eta_2 = \alpha_2 - \frac{\beta_2}{\alpha_1} \beta_2 \quad \gamma_3 = \beta_3$

$\lambda_3 \eta_2 = \beta_3 \quad \lambda_3 = \frac{\beta_3}{\eta_2}$

$\beta_1 = 0 \quad \lambda_1 = 0 \quad \gamma_1 = 0$ $\quad \lambda_3 \gamma_3 + \eta_3 = \alpha_3 \quad \eta_3 = \alpha_3 - \frac{\beta_3}{\eta_2} \beta_3$

$\eta_1 = \alpha_1$

$\gamma_j = \beta_j \quad \lambda_j = \frac{\beta_j}{\eta_{j-1}} \quad \eta_j = \alpha_j - \frac{\beta_j^2}{\eta_{j-1}}$

$$L_m \underbrace{U_m y_m}_{} = \beta e_1$$
$$L_m \; z_m = \beta e_1$$

$$\underset{L_m}{\begin{bmatrix} 1 & & & \\ \lambda_2 & 1 & & \\ & \lambda_3 & 1 & \\ & & \lambda_4 & 1 \\ & & & \ddots \end{bmatrix}} \begin{bmatrix} z_m(1) \\ \\ \\ \\ z_m(m) \end{bmatrix} = \begin{bmatrix} \beta \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$\boxed{z_m(1) = \beta}$

$z_m(2) = -\lambda_2 \beta \quad \lambda_2 z_m(1) + 1 \, z_m(2) = 0$

$z_m(3) = +\lambda_3 \lambda_2 \beta \quad \lambda_3 z_m(2) + 1 \cdot z_m(3) = 0$

$$\boxed{z_m(j) = (-1)^{j+1} \prod_{i=2}^{j} \lambda_i \; \beta \qquad j > 1}$$

$$U_m \, y_m = z_m$$

$$\begin{bmatrix} \eta_1 & \beta_2 & & & 0 \\ & \eta_2 & \beta_3 & & \\ & & \eta_3 & \beta_4 & \\ & 0 & & \ddots & \beta_m \\ & & & & \eta_m \end{bmatrix} \begin{bmatrix} y_m(1) \\ \\ \\ y_m(m) \end{bmatrix} = \begin{bmatrix} z_m(1) \\ \\ \\ z_m(m) \end{bmatrix}$$

$$\boxed{y_m(m) = \frac{z_m(m)}{\eta_m}}$$

$\rightarrow$ $$y_m(m-1) = \frac{-\beta_m y_m(m) + z_m(m-1)}{\eta_{m-1}}$$

$$y_m = U_m^{-1} z_m$$

$$\|r_m\| = \beta_{m+1} \, |y_m(m)|$$

$$x_m = x_0 + V_m y_m = x_0 + \underbrace{V_m U_m^{-1}}_{P_m} z_m$$

$$P_m = V_m U_m^{-1}$$

$$P_m U_m = V_m$$

$$\begin{bmatrix} | & | & | & & | \\ p_1 & p_2 & p_3 & \cdots & p_m \\ | & | & | & & | \end{bmatrix} \begin{bmatrix} \eta_1 & \beta_2 & & & \\ 0 & \eta_2 & \beta_3 & & \\ 0 & & \ddots & \ddots & \\ 0 & & & & \beta_m \\ 0 & & & & \eta_m \end{bmatrix} = \begin{bmatrix} | & | & & | \\ v_1 & v_2 & & v_m \\ | & | & & | \end{bmatrix}$$

$$\eta_1 p_1 = v_1 \qquad p_1 = \frac{1}{\eta_1} v_1$$

$$\beta_2 p_1 + \eta_2 p_2 = v_2 \qquad p_2 = \frac{1}{\eta_2}(v_2 - \beta_2 p_1)$$

$$\beta_3 p_2 + \eta_3 p_3 = v_3 \qquad p_3 = \frac{1}{\eta_3}(v_3 - \beta_3 p_2)$$

$$x_m = x_0 + P_m z_m$$

D-Lanczos (A s.p.d.)

Given $x_0 \to r_0 = b - Ax_0$ $\quad \beta = \|r_0\|$ $\quad v_1 = \frac{v_0}{\beta}$ $\quad z_m(1) = \beta$

$m=1$ $\quad \eta_1 = \alpha_1 = (Av_1, v_1)$ $\quad \beta_2 = \|Av_1 - \alpha_1 v_1\|$ $\quad v_2 = \frac{Av_1 - \alpha_1 v_1}{\beta_2}$ $\quad p_1 = \frac{v_1}{\eta_1}$

repeat

$m = m+1$

$w = Av_m - \beta_m v_{m-1}$ $\quad \alpha_m = (w, v_m)$ $\quad w = w - \alpha_m v_m$ $\quad \beta_{m+1} = \|w\|$ $\quad v_{m+1} = \frac{w}{\beta_{m+1}}$

$\lambda_m = \frac{\beta_m}{\eta_{m-1}}$ $\quad \eta_m = \alpha_m - \frac{\beta_m^2}{\eta_{m-1}} = \alpha_m - \lambda_m \beta_m$ $\quad z_m(m) = -\lambda_m z_m(m-1)$

$$p_m = \frac{1}{\eta_m}(v_m - \beta_m p_{m-1})$$

$$x_m = x_{m-1} + z_m(m) p_m$$

until $\left| \beta_{m+1} \frac{z_m(m)}{\eta_m} \right| < tol$

Given x_0 $\quad r_0 = b - Ax_0$ $\quad \beta = \|r_0\|$ $\quad v^{old} = \frac{r_0}{\beta}$ $\quad z_m^{old} = \beta$ $\quad X_m = X_0$

$\eta^{old} = \alpha^{old} = (Av^{old}, v^{old})$ $\quad w = Av^{old} - \alpha^{old} v^{old}$ $\quad \beta^{new} = \|w\|$ $\quad v^{new} = \frac{w}{\beta^{new}}$

repeat

$w = Av^{new} - \beta^{new} v^{old}$ $\quad \alpha^{new} = (Av^{new}, v^{new})$

$w = w - \alpha^{new} v^{new}$ $\quad \beta^{nn} = \|w\|$ $\quad v^{nn} = \frac{w}{\beta^{nn}}$

$\lambda^{new} = \frac{\beta^{new}}{\eta^{old}}$ $\quad \eta^{new} = \alpha^{new} - \lambda^{new} \beta^{new}$ $\quad z_m^{new} = -\lambda^{new} z_m^{old}$

$p^{new} = \frac{1}{\eta^{new}} (v^{new} - \beta^{new} p^{old})$

$X_m = X_m + z_m^{new} p^{new}$

$\eta^{old} = \eta^{new}$ $\quad \beta^{new} = \beta^{nn}$ $\quad v^{new} = v^{nn}$ $\quad z_m^{old} = z_m^{new}$

until $\left| \beta^{nn} \frac{z_m^{new}}{\eta^{new}} \right| < tol$

X_m is my solution

Proposition $\quad A$ s.p.d. $\quad r_m$ the residual vectors

1) Each residual r_m is orthogonal to all the previous residals

$$(r_m, r_k) = 0 \qquad \forall k < m$$

2) The auxiliary vectors p_m form an A-conjugate system of vectors

$$(Ap_i, p_j) = 0 \quad i \neq j \qquad p_j^T A p_i = 0 \quad i \neq j$$

Proof

1) $\quad r_m = -\beta_{m+1}\, y_m(m)\, v_{m+1}$ $\qquad v_j$ orthonormal $\qquad j = 1, .., m+1$

2) $\quad q_j^T q_i = 0 \qquad q_i \perp q_j$

$q_j^T I q_i = 0$

$q_j^T A p_i = 0 \qquad q_i \perp_A q_j$

$P_m^T A P_m$ is diagonal $\qquad P_m = V_m U_m^{-1}$

$$P_m^T A P_m = (V_m U_m^{-1})^T A V_m U_m^{-1} = U_m^{-T} V_m^T A V_m U_m^{-1} = U_m^{-T} T_m U_m^{-1} = U_m^{-T} L_m \underbrace{U_m U_m^{-1}}_{I}$$

$$= U_m^{-T} L_m$$

lower bi-diagonal matrix

U_m upper bi-diagonal matrix
U_m^{-1} " " "
U_m^{-T} lower " "
L_m " " "

$(P_m^T A P_m)^T = P_m^T A^T P_m = (U_m^{-T} L_m)^T$ $\quad$ upper bi-diagonal matrix

$\qquad \qquad \quad \|$

$P_m^T A P_m = U_m^{-T} L_m$ $\quad$ lowe bi-diagonal "

diagonal matrix