Mathematics
PAPER-I

Analysis-I

Chapter

1

Countable and Uncountable Sets

[April 2016] Define equivalent sets. Prove that open intervals (3, 7) and (5, 9) are equivalent sets. *(6 marks)*

[April 2015] Define equivalent set and prove that set of integers is equivalent to set of natural numbers. *(7 marks)*

[Sept 2014] Prove that set of integers is countable. *(6 marks)*

Chapter

2

Reimann Integration

[April 2016] Let $f(x)$ be defined on [-1, 1] as

$$f(x)=\begin{cases} x^2-1 & \text{if } \quad -1\leq x<0 \\ 2x+1 & \text{if } \quad 0\leq x<1 \end{cases}$$

Prove that f is bounded. Also evaluate L(P, f) where P={-1, -1/3, 2/3, 1}. *(7 marks)*

[April 2016] State and prove necessary and sufficient condition for a bounded function on a closed interval to be Riemann integrable. *(6 marks)*

[April 2015] Evaluate $\underset{n\to\infty}{\text{Lt}}$ L(P,f), where f(x) = x^2 and

$$P=\left\{0,\frac{1}{n},\frac{2}{n},\frac{3}{n},\ldots\ldots,\frac{n}{n}\right\}$$

(5 marks)

[April 2015] Prove that $\int_0^{\pi}\left\{x^2+(\sin x)^3\right\}dx\leq\pi(\pi^2+1)$. *(5 marks)*

[Sept 2014] Prove that $\left|\int_{10}^{19}\frac{\sin x}{1+x^8}dx\right|\leq\frac{1}{10^7}$ *(6 marks)*

[Sept 2014] Let $f\in R[a, b]$ and f^+ and f^- be defined on [a, b] as

$$f^+(x)=\begin{cases} f(x) & \text{if} \quad f(x)\geq 0 \\ 0 & \text{if} \quad f(x)<0 \end{cases} \text{ and } f^-(x)=\begin{cases} 0 & \text{if} \quad f(x)>0 \\ -f(x) & \text{if} \quad f(x)\leq 0 \end{cases}$$

Prove that f^+ and f^- are R-integrable on [a, b]. *(7 marks)*

[Sept 2014] Evaluate L(P, F) and U(P, F) where $f(x) = 2x^2$ and P = {-1, -1/2, ¼, ½, 1}

(5 marks)

[April 2014] Prove that necessary and sufficient condition for a bounded function f to be R-integrable on [a, b] is that to every $\varepsilon>0$, $\exists$ a partition P such that U (P,f)-L (P,f)< ε.

(7 marks)

[April 2014] Prove that $\frac{1}{2} \le \int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-x^{2n}}}dx \le \frac{\pi}{6}, n>1.$ *(5 marks)*

[Sept 2013] Prove that every monotonically increasing function on a closed interval is Rieman integrable.

(6 marks)

[Sept 2013] Prove that $\sqrt{\frac{3}{2}} \le \int_{\pi/4}^{\pi/3} \frac{\sin x}{x} dx \le \frac{\sqrt{2}}{6}$. *(6 marks)*

[April 2013] Prove that a continuous function *f* on [a,b] is Riemann-integrable.

(6 marks)

(Also asked in Apr 2009)

[April 2013] Evaluate:

(i) L (P,f) where $f(x) = x^2$ and $P = \left\{\frac{-3}{2}, \frac{-1}{2}, \frac{1}{4}, 1\right\}$

(ii) U (P,f) where f(x) = sin x and $P = \left\{0, \frac{\pi}{4}, \frac{2\pi}{3}, \pi\right\}$ *(6 marks)*

[Sept 2012] Let f(x)=sin x on $\left[0, \frac{\pi}{2}\right]$. Evaluate $\int_0^{\frac{\pi}{2}} f\,dx$ and $\int_0^{\frac{\pi}{2}} f\,dx$ by dividing

$\left[0, \frac{\pi}{2}\right]$ into *n* equal parts and show that $f \in R(x)$ on $\left[0, \frac{\pi}{2}\right]$.

(6 marks)

[Sept 2012] If *f* is bounded and Integrable in [a, b] then |f| is also bounded and Integrable in[a,b].

Moreover $\left|\int_a^b f\,dx\right| \le \int_a^b f\,|f|\,dx$. But converse is not always true, give suitable example.

(6 marks)

(Also asked in Apr 2012)

[April 2012] Prove that $\frac{1}{\sqrt{2}} \le \int_{0}^{\frac{1}{\sqrt{2}}} \frac{1}{\sqrt{1-x^4}}\,dx \le \sqrt{\frac{2}{3}}$ *(5 marks)*

[Sept 2011] A function f is defined as $f(x) = \begin{cases} K \text{ for } x \neq 0 \\ 0 \text{ for } x = 0 \end{cases}$ $k > 0$

Show f is -R integrable on [-1,1] and find $\int_{-1}^{1} f dx$ *(5 marks)*

[Sept 2011] State first mean value theorem of integral calculus and use it to find ξ if $\int_{-1}^{3} f(x)dx = 2f(\xi)$. *(4 marks)*

(Also asked in Apr 2007)

[Sept 2011] Give an example of a bounded function which is not R-integrable. *(3 marks)*

[Apr 2011] State and prove Fundamental Theorem of integral calculus. *(6 marks)*

[Apr 2011] $f(x)=\frac{1}{x^2}$ defined on [1, 4],

$P_1 = \{1, 2, 3, 4\}$ and $P_3 = \left\{1, \frac{7}{4}, \frac{5}{2}, \frac{13}{4}, 3\right\}$ are two partitions of [1, 4], Verify $L(P^2, f)$ $L(P^1, f)$. *(6 marks)*

[Sept 2010] Define Lower and Upper Riemann Integrals of a bounded function on [a,b] and prove that lower Riemann integral cannot exceed the Upper Riemann integral. *(6 marks)*

[Sept 2010] Show by giving an example of a bounded function f such that f is not R-integrable but [f] is R-integrable. *(6 marks)*

[Apr 2010] Prove that every monotone function defined on a closed interval is Riemann integrable. *(6 marks)*

[Apr 2010] Give an example of a bounded function f such that [f] is R-integrable but f is not. Justify. *(6 marks)*

[Apr 2009] Show that the function *f* defined by :

$f(x) = x^2$, $x \varepsilon [0,a]$, $a > 0$ is R-integrable on $[0,a]$ and $\int_0^a x^2\,dx = \frac{a^3}{3}$

(6 marks)

[Apr 2008] If f is continuous function of [a, b] and let

$F(x) = \int_a^x f(t)dt \ \forall\ x \in [a,b]$, then prove that $F'(x) = f(x) \forall\ x \in [a,b]$.

(6 marks)

[Apr 2008] Let $f(x) = 2 - 3x$ on $[2, 3]$, prove that f is R-integrable and

$\int_2^3 f(x)dx = -\frac{11}{2}$. *(6 marks)*

[Apr 2007] Let *f* be the function defined on $\left[0, \frac{\pi}{4}\right]$ by

$$f(x) = \begin{Bmatrix} \cos x, & \text{if } x \text{ is rational} \\ \sin x, & \text{if } x \text{ is irrational} \end{Bmatrix}$$

Show that *f* is not R-integrable. *(7 marks)*

Chapter

3

Improper Integrals

[April 2016] Show that integral $\int_0^1 \frac{dx}{\sqrt{x-x^2}}$ is convergent and its value is π. *(6 marks)*

[April 2015] Show that $\int_0^\infty \left(\frac{1}{x}-\frac{1}{\sinh x}\right)\frac{dx}{x}$ is convergent. *(7 marks)*

[Sept 2014] State and prove Dirichlet's test for convergence of improper integral. *(6 marks)*

[April 2014] Examine the convergence of the improper integral:

$\int_0^\pi \frac{\sin x}{x}dx$ *(6 marks)*

[April 2014] Discuss the convergence of the integral $\int_0^1 \frac{x^P \log x}{(1+x)^2}dx$ *(6 marks)*

[Sept 2013] State and prove Abel's test for convergence of improper integral. *(6 marks)*

[Sept 2013] Show that $\int_0^\infty \text{Sin}(x^2)\ dx$ is convergent. *(6 marks)*

[April 2013] Prove that improper integral $\int_a^\infty \frac{dx}{x^n}(a>0)$ is convergent iff n>1. Hence examine the convergence of $\int_1^\infty \frac{x^3}{(1+x)^5}dx$. *(6 marks)*

[April 2013] Examine the convergences of improper integral.

$$\int_0^{\infty}\frac{\sin(x^n)}{x^n}dx\,(m>0).$$

(6 marks)

[Sept 2012] Test for convergence of the integral $\int_{-\infty}^{+\infty}\frac{e^x}{1+e^{2x}}dx$. Also find the value of Integral if it is convergent. *(6 marks)*

[April 2012] State Dirichlet's test for convergences of improper integral.

Use it to discuss the convergences of $\int_1^{\infty}\frac{\sin x}{x^p}dx$ *(5 marks)*

[April 2012] Prove that $\int_1^{\infty}\frac{\sin x}{x}dx$ is convergent but not absolutely convergent.

(7 marks)

[Sept 2011] Test the convergence of improper integral

$$\int_a^{\infty}\frac{\sin x}{x^2}dx \text{ where } a>0.$$

(6 marks)

[Apr 2011] Show improper integral $\int_0^{\infty}\frac{\sin^2 x}{x}dx$ is convergent and find it if given $\int_0^{\infty}\frac{\sin x}{x}dx=\frac{\pi}{2}$ *(6 marks)*

[Sept 2010] State Dirichlet's test for improper integrals and hence show $\int_0^{\infty}\sin x^2 dx$ is convergent. *(6 marks)*

[April 2010] Discuss convergence of $\int_0^{\infty}\sin x^2 dx$. *(6 marks)*

[Apr 2010] Discuss convergence of improper integral $\int_0^1\frac{\log x}{\sqrt{x}}dx$.

(6 marks)

[April 2009] Examine the convergence of : $\int_0^2\frac{dx}{x^2-4x+3}$ *(6 marks)*

[April 2008] Show that $\int_0^{\infty}\frac{\sin^2 x}{x}dx$ is convergent.

Using $\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}$, show that $\int_0^\infty \frac{\sin^2 x}{x}\,dx = \frac{\pi}{2}$. (6 marks)

[April 2007] For what values of m and n is the integral

$\int_0^1 x^{m-1}(1-x)^{n-1}\log x\,dx$ convergent? (7 marks)

[April 2007] Test for convergence of integral $\int_0^\infty \left(\frac{1}{x}-\frac{1}{\sinh x}\right)\frac{dx}{x}$ (7 marks)

Chapter

4

Beta and Gamma Functions

[April 2016] Prove that $\int_0^p x^m (p^q - x^q)^n \, dx = \frac{p^{qn+m+1}}{q} B\, n+1, \frac{m+1}{q}$ where p>0, q>0; m>-1, n>-1.

(6 marks)

[April 2015] Prove that $\int_0^\infty \frac{1}{\sqrt{1-x^4}} dx = \frac{1}{4\sqrt{2\pi}} \left(\Gamma\left(\frac{1}{4}\right)\right)^3$. *(6 marks)*

[April 2014] Show that $\Gamma(m) \, \Gamma(m+\frac{1}{2}) = \frac{\sqrt{x}}{2^{2m-1}} \Gamma(2m)$ *(6 marks)*

[Sept 2013] Prove that $B\,(m,n) = \frac{\sqrt{m}\sqrt{n}}{\sqrt{(m+n)}}; m>0, n>0.$ *(6 marks)*

[April 2013] Prove that $\int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(a+bx)^{m+n}} dx = \frac{i}{(a+b)^{m,n}} \beta\ (m,n)$ Where m,n>0. *(6 marks)*

[Sept 2012] Show that $\int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(a+x)^{m+n}} dx = \frac{\Gamma(m)\ \Gamma(n)}{a^n(1+a)^m\ (\Gamma(m+n))}$

(6 marks)

[April 2012] Use the relation $\left\{\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(M+n)} \text{ where } m,n>0\right\}$ between Beta and Gamma function. Prove that :

(i) $\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$ (ii) $\int_0^{\infty} e^{-x^2}\,dx=\frac{\sqrt{\pi}}{2}$ *(5 marks)*

[Sept 2011] Define Beta function $\beta(m,n)$ and show

$$B(m,n)=2\int_0^{\pi/2} \sin^{2m-1}\theta\cos^{2n-1}\theta\,d\theta$$

Hence show $\int_0^{\pi/2} \sin^7\theta\,d\theta = 16/35.$ *(6 marks)*

[Apr 2011] State Gamma function and use it to show

$$\int_0^{\infty} e^{-x^3}\,dx=\frac{1}{3}\sqrt{\frac{1}{3}}$$ *(6 marks)*

[Sept 2010] Define Beta and Gamma functions. Using relation between these functions show

$$\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$$ *(6 marks)*

[April 2010] Show $\int_0^{\pi/2} \sqrt{\sin x}\,dx \int_0^{\pi/2} \frac{dx}{\sqrt{\sin x}}=\pi$ *(6 marks)*

[April 2009] Prove that : $\Gamma(m)=\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2m-1}}\Gamma(2m)$ where m is + ve. *(6 marks)*

[April 2008] Prove that $2^n\,\Gamma\left(m+\frac{1}{2}\right) = 1.3.5\ldots\ldots\ldots(2n-1)\sqrt{\pi}$. *(6 marks)*

Integ

[April 2016] Prove

$\int_{\frac{\pi}{2}-\alpha}^{\frac{\pi}{2}} \sin\theta\cos^{-1}$

[April 2015] Prove

[Sept 2014] Prove

[April 2014] Show
$(-\pi, \pi)$.

[April 2013] Find t
hence deduce t

$\int_0^{\pi} \frac{dx}{(a+b\cos x)^2} =$

Chapter

5

Integral as a Function of Parameter

[April 2016] Prove that:

$$\int_{\frac{\pi}{2}-\alpha}^{\frac{\pi}{2}} \sin\theta \cos^{-1}(\cos\alpha \ \mathrm{cosec}\theta)\,d\theta = \frac{\pi}{2}(1-\cos\alpha).$$ *(7 marks)*

[April 2015] Prove that $\displaystyle\int_0^{\pi/2} \frac{dx}{a^2\sin^2 x + b^2\cos^2 x} = \frac{\pi(a^2+b^2)}{4a^3b^3}$. *(6 marks)*

[Sept 2014] Prove that $\displaystyle\int_0^a \frac{\log(1+ax)}{1+x^2}dx = \frac{1}{2}\log(1+a^2)\tan^{-1}a$. *(6 marks)*

[April 2014] Show that $\displaystyle\int_0^\pi \frac{\log(1+\cos\alpha\sin x)}{\sin x}dx = \frac{\pi^2-4\alpha^2}{4}$ for all $\alpha \varepsilon$ $(-\pi, \pi)$. *(6 marks)*

[April 2013] Find the value of $\displaystyle\int_0^\pi \frac{dx}{a+b\cos x}$ (where $a > 0$, $|b| < a$) and hence deduce that

$$\int_0^\pi \frac{dx}{(a+b\cos x)^2} = \frac{\pi a}{(a^2-b^2)^{3/2}} \ \& \ \int_0^\pi \frac{\cos x\,dx}{(a+b\cos x)^2} = \frac{-\pi b}{(a^2-b^2)^{3/2}}.$$

(6 marks)

[Sept 2013] If $|a| < 1$, then evaluate $\int_0^{\pi} \frac{\log(1+a\cos x)}{\cos x} dx$, where "a" is parameter. *(6 marks)*

[Sept 2012] By applying differentiation under the Integral sign, prove that

$$\int_0^{\pi/2} \frac{\log(1+y\sin^2 x)}{\sin^2 x} dx = \pi[\sqrt{1+y}-1] \text{ if } y > -1$$ *(6 marks)*

[April 2012] Prove that $\int_0^{\pi/2} \log(\alpha\cos^2\theta + \beta\sin^2\theta)d\theta = \pi\log\left[\frac{1}{2}(\sqrt{\alpha}+\sqrt{\beta})\right]$ where $\alpha\beta > 0$.

(7 marks)

[Sept 2011] Prove that $\int_a^{\infty} \frac{e^{-ax}\sin bx}{x} = \tan^{-1}\frac{b}{a}$ for a, b > 0 and deduce $\int_a^{\infty} \frac{\sin bx}{x} dx = \pi/2.$ *(6 marks)*

(Also asked in Apr 2010)

[Apr 2011] If f(x, y) and $\frac{\partial}{\partial y} f(x,y)$ are continuous functions of x and y for $a \le x \le b$ and $c \le y \le d$, then show

$$\frac{d}{dy}\left[\int_a^b f(x,y)dx\right] = \left[\int_a^b f(x,y)dx\right] = \int_a^b \frac{\partial}{\partial y} f(x,y)dx$$

(6 marks)

[Apr 2011] By applying differentiation under the integral sign prove

$$\int_0^{\infty} e^{-xy}\frac{\sin x}{x} dx = \cot^{-1} y \text{ for } y > 0$$ *(6 marks)*

[Sept 2010] Given $\int_0^{\frac{\pi}{2}} \frac{dx}{a^2\cos^2 x + b^2\sin^2 x} = \frac{\pi}{2ab}$, by differentiating under the integral sign deduce

$$\int_0^{\frac{\pi}{2}} \frac{dx}{(a^2\cos^2 x + b^2\sin^2 x)^2} = \frac{\pi(a^2+b^2)}{4a^3b^3}$$ *(6 marks)*

[April 2009] Prove that

[April 2008] Show $\in(-\pi, \pi)$.

[April 2007] Assuming sign, prove that

$$\int_0^{\infty} \frac{\tan^{-1} ax}{x(1+x^2)} dx = \frac{\pi}{2}$$

[April 2009] Prove that if $a > b$, $\int_0^{\frac{\pi}{2}} \log\left(\frac{a+b\sin\theta}{a-b\sin\theta}\right)\frac{d\theta}{\sin\theta} = \pi\sin^{-1}\frac{b}{a}$

(6 marks)

[April 2008] Show that $\int_0^{\pi} \frac{\log(1+\cos a\sin x)\,dx}{\sin x} = \pi^2 - 4a^2 \ \forall a \in (-\pi, \pi)$.

(6 marks)

[April 2007] Assuming the validity of differentiation under the integral sign, prove that

$$\int_0^{\infty} \frac{\tan^{-1} ax}{x(1+x^2)}\,dx = \frac{\pi}{2}\log(1+a) \text{ if } a \geq 0.$$

(7 marks)

Mathematics

PAPER-II

Modern Algebra

Chapter

1

Groups

[April 2016] Prove that a semi group in which both the equations ax=b and ya=b have a unique solution is a group. *(6 marks)*

[April 2016] Prove that a group in which every element is its own inverse is abelian. *(6 marks)*

[April 2015] Prove that set of fourth roots of unity forms an abelian group under multiplication of complex numbers. *(6 marks)*

[Sept 2014] Show that the set G = P {1, w, w^2} of cube root of unity forms a finite abelian group of order 3 under multiplication of complex numbers. *(6 marks)*

(Also asked in Apr 2008)

[Sept 2014] If $O(G) = p^2$, where p is prime number then prove that G is an abelian group. *(6 marks)*

(Also asked in Apr 2007)

[April 2014] State and prove Lagrange's Theorem for finite groups. *(6 marks)*

(Also asked in Sept 2013, Apr 2009, 2007)

[Sept 2012] Prove that a finite semigroup in which both the cancellation laws hold is a group. *(6 marks)*

[April 2012] Let G be a set with binary operation which is associative. Assume that for all elements a and b in G, the equations ax = b and ya = b have unique solution in G. Then prove that G is a group. *(6 marks)*

[Sept 2011] Prove that a finite semi-group is a group if and only if it satisfies both the cancellation laws. *(6 marks)*

[Sept 2010] Show that the equation $x^2ax = a^{-1}$ is solvable for x in a group if and only if a is the cube of some element in G. *(6 marks)*

[April 2010] Prove that the order of each sub-group of a finite group is a divisor of the order of the group. *(6 marks)*

[April 2009] Define order of an element of a group G.

Let a,b ε G be non-identity elements with O(a) = 5 and $aba^{-1} = b^2$, find O(b). *(6 marks)*

[April 2009] Prove that if O (G) = P^2, where p is a prime number, then G is an abelian group. Is group of order |2| abelian? *(6 marks)*

Chapter

2

Subgroups and Quotient Groups

[April 2016] Prove that every subgroup of a cyclic group is cyclic. *(6 marks)*

(Also asked in Apr 2010, 2009, 2008)

[April 2016] Define index of a subgroup and prove that a subgroup of index 2 is normal. *(6 marks)*

[April 2015] Let N be a cyclic normal subgroup of a group G. Show that every subgroup of N is normal in G. *(6 marks)*

[April 2015] If H and K where $H \subseteq K$ are two subgroups of a finite group G then show that [G:H] = [G:K] [K:H]. *(8 marks)*

(Also asked in Sept 2010)

[Sept 2014] Define centre of a group. Prove that the centre Z(G) of a group is a sub-group of G. *(6 marks)*

[Sept 2014] If an abelian group of order 6 contains an element of order 3 then show that it must by cyclic group. *(6 marks)*

[Sept 2014] If H is sub-group of G of index 2 in G, then H is normal sub-group of G. *(6 marks)*

(Also asked in Apr 2010, Sept 2012)

[April 2014] Let H be a subgroup of a group G. If $x^2 \varepsilon H \; \forall \; x \varepsilon G$, then prove that H is a normal subgroup of G. *(6 marks)*

(Also asked in Sept 2013)

[April 2013] Prove that $H \neq \phi \subset G$ is a subgroup of G iff $\forall$ a, b ε H $\Rightarrow$ a $b^{-1} \varepsilon$ H. *(6 marks)*

[April 2013] Prove that order of each subgroup of a finite group is divisor of order of group. (6 marks)

[April 2013] If $G = (a)$ is a cyclic group generated by 'a' Prove that $O(G) = O(a)$. (6 marks)

[April 2012] Let G be a group and H and K subgroups of finite indices in G. Show that $H \cap K$ is also of finite index. If $(G:H) = n$ with $(m,n) = 1$, show that $(G:H \cap K) = mn$. (6 marks)

[April 2012] Define center $Z(G)$ of a group G. Show that if $G/Z(G)$ is cyclic, then G is Abelian. (6 marks)

[Sept 2011] For any subgroup H of a finite group G, prove that $O(G) = O(H)\,[G;H]$. (6 marks)

[Sept 2011] Prove that if H and K are finite subgroups of a group G then $O(HK) = \dfrac{O(H)O(K)}{O(H \cap K)}$. (6 marks)

[Apr 2011] Prove that the order of any subgroup of a finite group G divides the order of the group G. (6 marks)

[Apr 2011] Let H and K be any two subgroups of a group G. Prove that:

i) $H \cap K$ is a subgroup of G.

ii) $H \cap K$ is normal in K if H is normal in G. (6 marks)

[Apr 2011] Let G be a group of order 2n, where n is odd. Show that G has a normal subgroup of order n. (6 marks)

[Sept 2010] Prove that a group of order n is cyclic if and only if it has an element of order n. (6 marks)

[April 2008] Prove that a non-empty sub set H of a group G is a sub group of G if and only if a, b ε H implies that ab^{-1} ε H ∀ a, b ε H. (6 marks)

[April 2008] Let G be a finite group and Z be the centre of G. If $O(G) = P^n$, (p is a prime number and $n \geq 1$) then prove the $Z \neq \{e\}$. (6 marks)

[April 2007] If G is a finite group whose order is a prime number p, then prove that G is cyclic group. (6 marks)

[April 2007] Let H be a subgroup of a group G. Prove that $Ha = Hb$ if and only if ab^{-1} εH. (6 marks)

Hom…
Isomo…

[April 2016] Prove that e…
to $\frac{Z}{n^2}$.

[April 2016] Prove tha…
group.

[Sept 2014] State and p…

[April 2014] Prove that…
to quotient group z…

[April 2013] If ϕ be hon…
$\frac{G}{K} \cong G'$.

[April 2012] Prove tha…
to the Klein 4-grou…

[Sept 2011] If H and …
$H \subseteq K$, then prove …
$\cong \frac{G/H}{K/H}$

[Sept 2011] Prove that…
cyclic and other is i…

Chapter

3

Homomorphism and Isomorphism of Groups

[April 2016] Prove that every finite cyclic group of order n is isomorphic to $\frac{Z}{n^{z}}$.

(6 marks)

[April 2016] Prove that every group is isomorphic to a permutation group.

(6 marks)

[Sept 2014] State and prove Cayley's Theorem. *(6 marks)*

[April 2014] Prove that any finite cyclic group of order n is isomorphic to quotient group z| <n>.

(6 marks)

(Also asked in Sept 2013, 2012)

[April 2013] If ϕ be homorphism of group G into G′ with kernel K then $\frac{G}{K} \cong G'$.

(6 marks)

(Also asked in Apr 2010)

[April 2012] Prove that any non-cyclic group of order 4 is isomorphic to the Klein 4-group.

(6 marks)

[Sept 2011] If H and K are two normal subgroups of G such that $H \subseteq K$, then prove that K/H is a normal subgroup of G/H and G/K $\cong \frac{G/H}{K/H}$

(6 marks)

(Also asked in Apr 2008)

[Sept 2011] Prove that there are only two groups of order six, one is cyclic and other is isomorphic to S_3.

(6 marks)

(Also asked in Apr 2011)

[Sept 2010] If G′ is homomorphic image of G then prove that G′ is isomorphic to some quotient group of G. (6 marks)

[April 2009] Let G and G′ be two groups, If : G→G′ is homomorphism, show that the Kernel of *f* is a normal subgroup of G. (6 marks)

[April 2007] Let *f* be a homomorphism of group G onto group G′ with Kernel K. Then prove that G/K is isomorphic to G′. (7 marks)

Chapter

4

Permutations and Alternating Groups

[April 2015] For any $n > 1$ the subset A_n of S_n of all even permutations is a normal subgroup of S_n of Index 2. *(4 marks)*

[April 2014] Prove that Alternating group A_4, has no subgroup of order 6. *(6 marks)*

(Also asked in Sept 2013, 2012, 2010)

[April 2012] Let S_n be the symmetric group on n symbols and A_n be subset of S_n consisting of all even permutations. Prove that An is a normal subgroup of Sn of index 2. *(6 marks)*

[April 2010] Define class equation of a finite group and prove that if $O(G) = p^n$ (where p is a prime number and $n \geq 1$) then centre $Z=\{e\}$. *(6 marks)*

Chapter

5

Rings and Integral Domains

[April 2015] If R is a commutative ring with unity with characteristic 2 then show that $(a+b)^2 = a^2 + b^2 = (a-b)^2 \ \forall \ a, b \in R$. *(6 marks)*

(Also asked in Apr 2013, 2010)

[April 2014] Let R be a ring with unity such that $(xy)^2 = x^2y^2$ for all $x, y \in R$. Prove that R is commutative. *(6 marks)*

(Also asked in Apr 2011)

[Sept 2013] Find the field of quotients of the integral domain $Z\left[\sqrt{2}\right]$ *(6 marks)*

[April 2009] If R is a ring in which $x^2 = x \ \forall x \in R$, prove that R is a commutative ring of characteristic two. *(6 marks)*

[April 2008] For a positive integer n, prove that ring Zn of all integers modulo n is an integral domain if and only if n is a prime integer. *(6 marks)*

(Also asked in Apr 2007)

Chapter

6

Subrings, Ideals and Quotient Rings

[April 2016] If an ideal M of a commutative ring R with unity is maximal then prove that R/M is a field. Also prove converse part. *(6 marks)*

(Also asked in Sept 2013, 2012, 2011, Apr 2013, 2009, 2008)

[April 2015] If A, B and C are ideal of ring R such that $B \in A$ then show that:

$A \cap (B + C) = B + (A \cap C)$

Give an example to show that in general

$A \cap (B + C) \neq (A \cap B) + (A \cap C)$. *(6 marks)*

[April 2015] An ideal P of a commutative ring R is prime if and only if R/P is an integral domain. *(6 marks)*

(Also asked in Apr 2012, 2011, 2010)

[Sept 2014] If I and J be two ideals of a ring R, then IJ is an ideal of R. Moreover $IJ \subseteq I \cap J$. *(6 marks)*

[April 2014] If U, V are ideals of ring R, let UV be the set of all elements that can be written as finite sum of elements of the form uv where $u \varepsilon U$, $v \varepsilon V$. Prove that UV is an ideal of R and $UV \subset U \cap V$. *(6 marks)*

[Sept 2013] If U, V are ideals of R, let UV be the set of all elements that can be written as finite sum of elements of the form *uv* where $u \in U$ and $v \in V$. Prove that UV is an ideal of R and $UV \subset U \cup V$.

(6 marks)

(Also asked in Sep 2012)

[April 2013] Prove that for any two ideals, I_1 I_2 of a ring R, $I_1 + I_2$ is also an ideal of R containing both I_1 and I_2. *(6 marks)*

[Sept 2012] A communative ring R with unity is simple if and only if R is a field. *(6 marks)*

[April 2012] Show that for every prime p, the ring Z/pZ with the usual modulo operations, is a field. *(6 marks)*

[Sept 2011] For any two ideals A and B of a ring R, prove that $A+B=\langle A \cup B\rangle$ *(6 marks)*

[Apr 2011] Let A and B be any two ideals of a ring R. Prove that $A \cup B$ is an ideal of R if and only if either $A \subseteq B$ or $B \subseteq A$. *(6 marks)*

[Sept 2010] For any two ideals A and B of a ring R prove that

$AB = \{\sum a_1 b_1 | \ a_1 \varepsilon A, b_1 \varepsilon B\}$ is an ideal of R. *(6 marks)*

[April 2010] Define simple ring. Prove that a division ring is a simple ring. *(6 marks)*

[April 2007] If R is a ring and $a \in R$ be any fixed element of R. Let $T = \{x \in R;\ ax = 0\}$. Prove that T is a right ideal of R. *(6 marks)*

[April 2016]

$\cong J(I \cap J)$.

[Sept 2014] L

[April 2014

Homomo

[Sept 2010]

correspo

R/N.

[April 2009]

$A/(A \cap B) \cong$

Chapter

7

Homomorphism and Isomorphism of Rings

[April 2016] Let I and J be two ideals of a ring R then prove that $(I+J)/I \cong J(I \cap J)$.

[Sept 2014] Let I and J be two ideals of ring R, then $I/(I \cap J) \cong (I+J)/J$. *(6 marks)*

[April 2014] State and prove Fundamental Theorem of Ring Homomorphism. *(6 marks)*

[Sept 2010] If N is an ideal of R then prove that there is 1-1 correspondence between ideals of R containing N and ideals of R/N. *(6 marks)*

[April 2009] Let A and B be two ideals of a ring R, then prove that $A/(A \cap B) \cong (A+B)/B$. *(6 marks)*

Chapter

8

Polynomial Rings

[April 2016] Show that an ideal <x> of Z[x] is a prime ideal but not a maximal ideal.

[April 2015] If R is a commutative ring with unity and f(x), g(x) ∈ R[x] then: deg (f(x)) ≤ deg f(x) + deg g(x)
The equality holds if R is an I.D. *(6 marks)*

(Also asked in Apr 2014)

[Sept 2013] If R is an integral domain, then prove that R [x] is also an integral domain. *(6 marks)*

(Also asked in Sept 2012)

[April 2013] If R is an integral domain, then prove that R [x]; ring of polynomial over R is also an integral domain. *(6 marks)*

[April 2012] What are the units of the polynomial ring $Z_7[x]$? *(6 marks)*

[Sept 2011] Let T be a ring of polynomials over a ring R, Prove that R' = {(a,0,0 ---, 0) \ a ε R} is a subring of T which is isomorphic to R. Further if R has unity 1, then what is the unity of T? *(6 marks)*

[Apr 2011] Let F be a field. Let f, g ∈ F[x], g ≠ 0. Prove that there exist unique polynomials h, r ∈ F[x] such f = hg + r where either r = 0 or degree r < degree g. *(6 marks)*

[Sept 2010] If R is a commutative ring, then prove that R [x] is commutative. Further prove that if R has no proper zero divisors then R[x] also has no proper zero divisors. *(6 marks)*

[April 2008] For any ring R show that R[x]/<x> ≅R. *(6 marks)*

Mathematics
PAPER-III
Probability Theory

Chapter

1

Review of Notion of Probability

[April 2016] If n biscuits are distributed among N beggars, find the probability that a particular beggar receives $r(<n)$ biscuits.

(6 marks)

[April 2016] A and B toss a coin alternately till one of them tossed a head and wins the game. If A starts the game, find their respective probability of winning.

(6 marks)

[April 2016] Bowl I contains 6 red chips and 4 blue chips. Five of these 10 chips are selected at random and without replacement put in bowl II, which was originally empty. One chip is drawn from bowl II. Given that this chip is blue, find the conditional probability that 2 red chips and 3 blue chips are transferred from bowl I to bowl II.

(6 marks)

[April 2015] A girl throws a die. If she gets 5 or 6, she tosses a coin three times and notes the number of heads. If she gets 1,2,3, or 4, she tosses a coin once and notes whether a head or tail is obtained. If she obtained exactly one head, what is the probability that she threw 1,2,3,or 4 with the die?

(6 marks)

(Also asked in Apr 2009)

[April 2015] For any events A_1, A_2 .... , A_n, prove that:

$$P\left(\bigcap_{i=1}^{n} A_i\right) \geq \sum_{i=1}^{n} P(A_i) - (n-1).$$

(6 marks)

(Also asked in Apr 2012, 2008)

[April 2015] The odds in favour of standing first of three students appearing at an examination are 1 :2,2 : 5 and 1: 7 respectively. Find the probability that either of them stands first. *(6 marks)*

[Sept 2014] Bag A contains 6 red and 4 blue balls. Five of these 10 balls are selected at random without replacement and put in bag B, which was originally empty. One ball is then drawn at random from the bag B. Given that this ball is blue, find the conditional probability that 2 red balls and 3 blue balls are transferred from bag A to bag B.

(6 marks)

[Sept 2014] Let $A_1, A_2, \ldots\ldots\ldots\ldots, A_n$ be n events in a sample space. Show that:

$$P\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} P(A_i)$$

When does the equality hold? *(6 marks)*

[April 2014] State and Prove Baye's Theorem on conditional probability. *(6 marks)*

(Also asked in Apr 2011,2007, Sept 2012)

[Sept 2013] A bag contains 5 balls. Two balls are drawn and found to be red. What is the probability of all the balls being red?

(6 marks)

[April 2013] For any n events $E_1, E_2, \ldots\ldots\ldots, E_m$ in a sample space, show that:

$$P\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} P(Ei) - \sum_{\substack{i \neq j \\ i \leq i, j \leq n}} P(E_i \cap E_j) + \sum_{\substack{i \neq j \neq k \\ i \leq i, j, k \leq n}} P(E_i \cap E_j \cap . E_k)$$

$$+ \ldots\ldots + (-1)^n P(E_1 \cap E_2 \cap \ldots\ldots\ldots \cap E_n).$$ *(6 marks)*

[Sept 2011] A and B will throw dice for a prize which is to be won by the player who first throws 6. If A throws the first, what are their respective expectations? *(6 marks)*

[Apr 2011] Define Sample Space. If A and B are two events, then prove that $p(A \cup B) = P(A) + A(B) - P(A \cap B)$. *(6 marks)*

[Apr 2011] A ball is drawn at random from a box containing 6 red balls, 4 white balls and 5 blue balls. Determine the probability that the ball drawn is (i) red (ii) white (iii) blue (iv) red or white. *(6 marks)*

[Sept 2010] A car manufacturing factory has two plants. Plants P manufactures 70% of cars and Plant Q manufactures 30%. At Plant P 80% of cars are rated of standard quality and at Plant Q 90% of cars are rated of standard quality. A car is picked up at random and is found to be of standard quality. What is the probability that it has come from Plant P? *(6 marks)*

[April 2010] A coin is tossed (m+n) times (m>n). Find the probability of atleast m consecutive heads. *(6 marks)*

[April 2007] Give classical definition of probability and its properties. *(3 marks)*

[April 2007] An urn contains a white chips and b blue chips A chip is chosen at random from the urn discarded and replaced by one of the opposite colour and then a second chip is drawn. Find the probability that the second chip drawn is blue. *(3 marks)*

Chapter

2

Random Variables

[Apr 2016] The probability density function of a random variable X follows the probability law.

$$f(x)=\frac{1}{2\theta}\exp\left(-\frac{|x-\theta|}{\theta}\right), -\infty<x<\infty$$

Find the moment generating function of X. *(6 marks)*

[Apr 2016] Find the constant c such that the function:

$$f(x)=\begin{cases} cx^2 & , \quad 0<x<3 \\ 0 & , \quad \text{otherwise} \end{cases}$$

is a probability density function. Hence compute P(1<X<2). *(6 marks)*

[Apr 2016] A bowl contains 10 chips of which 8 are marked \$2 each and 2 are marked \$5 each. Let a person choose at random and without replacement 3 chips from bowl. If the person is to receive the sum of the resulting amount, find the expected amount person will receive. *(6 marks)*

(Also asked in Sept 2011)

[Apr 2015] Let the random variable X takes values $x_i = \frac{(-1)^i 2^i}{i}$ with $P(X = x_i) = 2^{-i}$; I = 1,2,3, ... Check if E (X) exists or not. *(6 marks)*

[Apr 2015] Find relation between cumulants and moments. *(6 marks)*

[Apr 2014] A random variable x has the probability density given by

$$f(x) = \frac{1}{2} e^{-|x|} \text{ for } -\infty < X < \infty.$$

Find its moment generating function. Hence find the variance of the distribution. (6 marks)

[Apr 2014] Define a standardized random variable. What are its mean and variance? (2 marks)

[Sept 2013] If the density function of a random variable x is given by

$$f(x) \begin{cases} x & ; \ 0 < x < 1 \\ 2 - x; & 1 \le x < a \\ 0 & ; \ \text{elsewhere} \end{cases}$$

Find :

(i) The value of a

(ii) The distributive function of x

(iii) $P(0.8 < x < 0 < 0.6a)$. (6 marks)

[Sept 2013] Moment generating function of a random variable. (1 mark)

[Sept 2013] Find the first four moments:

(i) About the origin

(ii) About mean.

for a random variable x having p.d.f.

$$f(x) = \frac{3}{4} x(2 - x)$$

Is the distribution symmetrical about the mean? (6 marks)

[Sept 2013] Find the coefficient of:

(i) Skewness

(ii) Kurtosis

For the distribution having probability density function.

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, -\infty < x < \infty$$

(6 marks)

[April 2013] The probability density function of a random variable is given by $f(x) = \begin{cases} 4x(1-x^2) & 0 \le x \le 1 \\ 0 & otherwise \end{cases}$ find:

(i) Mean (ii) mode (iii) median of x.

(6 marks)

[April 2013] The moment generating function of a random variable x is given by $\left(\frac{2}{3}+\frac{1}{3}e^t\right)^9$. Evaluate p ($\mu - 2\sigma < x < \mu + 2\sigma$).

Where μ is the mean and σ the standard deviation of x. *(6 marks)*

[Sept 2012] If $f(x) = cx^2$, $0<x<1$ is the p.d.f. of a continuous random variable X.

Find (i) P $\left(\frac{1}{3} < X < 2\right)$

(ii) Find 'a' such that $P(X \le a) = P(X>a)$.

(6 marks)

[Sept 2012] Let the random variable X assume the value 'r' with the probability law :

$P(X = r) = q^{r-1}.p, r = 1,2,3,.........$

Find the m.g.f. of X and hence its mean and variance. *(6 marks)*

[Sept 2012] The m.g.f. of binomial variate about origin was found to be $\left(\frac{3}{4}+\frac{1}{4}e^t\right)^8$, find :

(i) Mean, S.D. and Coefficient of Variation

(ii) Mode

(iii) Find P (X = 2).

(6 marks)

[April 2012] Let X be the number of gallons of ice-cream that is required at a certain store on a hot summer day. Let

$$f(x) = \begin{cases} \frac{12x(100-x)^2}{10^{12}}, & 0 < x < 1000 \\ 0 & \text{elsewhere} \end{cases}$$

be the probability density function of X. How many gallons of ice-cream should the store have on hand each of these days, so that the probability of exhausting its supply on a particular day is 0.05? (6 marks)

[April 2012] The Moment generating function of a random variable X is $\exp[u(e^t-1)]$. Find $P(\mu-2\sigma < X < \mu + 2\sigma)$ where μ is the mean and σ the standard deviation of X. (6 marks)

[Sept 2011] Let first, second and third moments of a random variable about point 7 be 3, 11 and 15 respectively. Find mean and then first, second and third moments about mean. (6 marks)

[Apr 2011] A doctor recommends a patient to go on a particular diet for two weeks and there is equal likelihood for the patient to lose his weight between 2 kg and 4 kg. What is the average amount the patient is expected to loose on this diet? (6 marks)

[Apr 2011] Define Bernoulli and Exponential random variables. (4 marks)

[Sept 2010] Define cumulative distribution of function and its important properties. (6 marks)

[Sept 2010] Let X be a continuous random variable with p.d.f. given by

$$f(x) = \begin{cases} kx & : 0 \le x < 1 \\ k & : 1 \le x < 2 \\ -kx + 3k & : 2 \le x < 3 \\ 0 & : \text{else where} \end{cases}$$

(i) Determine k

(ii) Determine f(x), the cumulative distribution function. (6 marks)

[April 2010] The probability mass function of a discrete random variable X is zero except at the points x = 0, 1,2. Also $P(0) = 3c^3$, $p(1) = 4c - 10c^2$ and $p(2) = 5c - 1$, for some $c>0$.

(i) Determine the value of c.

(ii) Find the largest value of x such that $F(x) < \frac{1}{2}$.

(iii) Find the smallest valu
the distribution funct

[April 2009] A random variab

$f(x) = \frac{a}{x^2+1}, -\infty < x < \infty$

(i) Find the value of the

(ii) Find P

(iii) Find the distribution

[April 2008] Find first four m

(ii) about mean for a

$f(x) = \begin{cases} 4x(9-x^2)/81 \\ 0 \end{cases}$

[April 2008] The spectrum

1,2,....,n and p(X = i) is

(i) p.m.f. of X

(ii) probability distribu

(iii) P (3 < X≤n) and P(X

[April 2007] Let f(x) = Cx e

0

Be the density functio

(i) Determine C

(ii) Find the distributio

(iii) Find P (2≤ x <3)

(iv) Find P (X≥1)

[April 2007] Consider 10
ordered list. At each
these elements indep
element is then mov
position of the elem
operation for a long

(iii) Find the smallest value of x such that $F(x) > \frac{1}{3}$. Here F(x) denotes the distribution function of x.

(6 marks)

[April 2009] A random variable X has the probability density function

$$f(x) = \frac{a}{x^2+1}, \ -\infty < x < \infty$$

(i) Find the value of the constant 'a'

(ii) Find P

(iii) Find the distribution function of X.

(6 marks)

[April 2008] Find first four moments : (i) about origin

(ii) about mean for a random variable X having p.d.f.

$$f(x) = \begin{cases} 4x(9-x^2)/81 & ; 0 \le x \le 3 \\ 0 & ; \text{otherwise} \end{cases}$$

(6 marks)

[April 2008] The spectrum of a random variable X consists of points 1,2,....,n and p(X = i) is proportional to $\frac{1}{i(i+1)}$. Find

(i) p.m.f. of X

(ii) probability distribution function of X

(iii) P (3 < X≤n) and P(X>5).

(7 marks)

[April 2007] Let $f(x) = Cx\, e^{2x}$, $x \ge 0$

0 , otherwise

Be the density function of a random variable X

(i) Determine C

(ii) Find the distribution function F(x) and sketch it.

(iii) Find P (2≤ x <3)

(iv) Find P (X≥1)

(6 marks)

[April 2007] Consider 10 elements that are initially arranged in some ordered list. At each unit of time, a request is made for one of these elements independently of the past. After being moved, the element is then moved to the front of the list. Find the expected position of the element requested after this process has been in operation for a long time.

(4 marks)

Chapter

3

Discrete Random Variables

[April 2016] Prove that the binomial distribution is binomial or unimodal according as $(n+1)\,p$ is an integer or not. Find modal values also. *(6 marks)*

[April 2016] If X is a Poisson random variable with parameter m and μ_r is the rth central moment, then prove that

$$\mu_{r+1}=m\left({}^rc_1\mu_{r-1}+{}^rc_2\mu_{r-2}+\ldots\ldots\ldots\ldots+{}^rc_r\mu_0\right).$$ *(6 marks)*

(Also asked in Apr 2009)

[April 2015] Find the recurrence relation $\mu_{r+1}=pq\left(nr\,\mu_{r-1}+\dfrac{d\mu_r}{dp}\right)$ for the moments of binomial distribution.

(6 marks)

[April 2015] In a lengthy manuscript, it is discovered that only 13.5% of the page contain no typing errors. If we assume the number of errors per page to be a random variable with Poisson distribution, find percentage of pages that have exactly one error. *(6 marks)*

[Sept 2014] A box contains 2^n tickets among which ${}^nC_i$ bear the number $i\,(i=0,1,2,\ldots\ldots,n)$. A group of m tickets is drawn, what is the expectation of the sum of the numbers? *(6 marks)*

[Sept 2014] If X has a Poisson distribution with parameter λ, show that the distribution function of X is given by:

$$F(x)=\frac{1}{\Gamma(x+1)}\int_{\lambda}^{\infty}e^{-t}t^{x}\,dt;\ x=0,1,2,\ldots\ldots\ldots$$ *(6 marks)*

[April 2014] If a random variable x follows binomial distribution with parameters n and p prove that

$$p(x \text{ is even}) = \frac{1}{2}[1+(q-n)^n]$$

(4 marks)

[April 2014] Establish the validity of the Poisson's approximation to the binomial distribution. *(5 marks)*

[April 2014] Define a geometric random variable. *(1 mark)*

[Sept 2013] Define Bernoulli's random variable. *(2 marks)*

[April 2013] Find coefficient (i) Skewness (ii) Kurtosis for the Poisson distribution. *(6 marks)*

[Sept 2012] If X has a Poisson distribution with parameter λ, show that the distribution function of X is given by

$$F(x) = \frac{1}{\Gamma(x+1)}\int_{\lambda}^{\infty} e^{-t}t^{x}\,dt;\ x = 0,1,2,..........$$

(6 marks)

(Also asked in Apr 2009)

[April 2012] A communication system consists of n components, each of which functions independently with probability p. The total system will be able to operate effectively if at least one-half of its components function. For what value of p is a 5-component system more likely to operate effectively than 3- component system?

(6 marks)

[April 2012] Find the mean and variance of a geometric random variable. *(6 marks)*

[Sept 2011] Find the measures of skewness and kurtosis of the Poisson distribution with mean μ. *(6 marks)*

[Apr 2011] Let X is a binomial variable based on n repetitions, then prove that:

$$P(X=K) = \binom{n}{k} p^{k}(1-p^{k})^{n-k},\ \ k = 0,1,.........n.$$ *(8 marks)*

[Sept 2010] Show that in a Poisson distribution with unit mean, mean deviation about mean is $\left(\frac{2}{e}\right)$ times the standard deviation.

(6 marks)

[April 2010] If a random variable X follows binomial distribution with parameters n, p, prove that (X is even) = $\frac{1}{2}$ $[1+(q-p)^n]$. (4 marks)

[April 2010] Let X and Y be two independent Poisson Variates. Show that X + Y is also a Poisson Variate. Does the result hold if X and Y are correlated? Justify your answer.

(5 marks)

[April 2009] Show that mean and variance of the geometric distribution $P(x) = q^x p$; x= 0, 1, 2,....... are respectively $\frac{q}{p}$ and $\frac{q}{p^2}$ (6 marks)

[April 2007] Stating the conditions required, show that a Poisson random variable can be used to approximate a Binomial random variable.

(4 marks)

[April 2007] Define law of large numbers. State and prove Bernoulli's form of law of large numbers.

(6 marks)

Chapter

4

Continuous Random Variables

[April 2016] If families are selected randomly in a certain thickly populated area and their monthly income in excess of Rs. 4000 is treated as exponent random variable with parameter $\lambda = \frac{1}{2000}$ Find the probability of 3 out of 4 families selected have income in excess of Rs. 5000. *(6 marks)*

(Also asked in Sept 2011)

[April 2016] Show that in normal distribution, all odd moments about mean are zero and even moments about mean are given by $\mu_{2n} = 1.3.5 \ldots\ldots (2n-1)\sigma^{2n}$. *(6 marks)*

[April 2016] If X and Y are independent Gamma Varieties with parameters μ and v respectively.

Show that X+Y and $\frac{X}{X+Y}$ are also independent. *(6 marks)*

(Also asked in Apr 2010)

[April 2015] Prove that mean deviation about mean of an exponential distribution is $\frac{2}{\lambda}e^{-1}$. *(6 marks)*

[April 2015] Show that mode of beta distribution is $\frac{a-1}{a+b-2}$. *(6 marks)*

[April 2015] Let X be N (μ, σ^2) such that P (X < 89) = 0.90 and P(X<94) = 0.95. Find and m and σ^2. *(6 marks)*

[Sept 2014] Find the mode of a normal distribution with mean μ. *(5 marks)*

[Sept 2014] Define Gamma distribution. Find Moment generating function of this distribution with parameter λ. *(3 marks)*

[Sept 2014] Find the mean deviation about mean for rectangular distribution. *(4 marks)*

[April 2014] Show that for a normal distribution mean and median are equal. *(6 marks)*

(Also asked in Apr 2008)

[April 2014] Define a uniform distribution. Let x be uniformly distributed in $-2 \leq x \leq 2$.

Find $P\left[|x-1| \geq \frac{1}{2}\right]$ *(6 marks)*

[Sept 2013] Define Rectangular random variable. *(1 mark)*

[April 2013] Assume the mean height of soldiers to be 68.22 inches and a variance of 10.8 inches square. How many soldiers in a regiment of 10,000 would you expect to be over 6 feet tall, given that the area under the standard normal curve between x=0 and x = 0.35 is 0. 1368 and between x = 0 and z=1.5 is 0.3746? *(6 marks)*

[April 2013] Define a rectangular distribution. Find its mean and variance. If x is a rectangular random variable with mean 1 and standard deviation $\frac{2}{\sqrt{3}}$, find $p(x < 0)$. *(6 marks)*

[Sept 2012] Prove that $f(x)=\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$ is p.d.f. *(6 marks)*

[April 2012] Define a Normal random variable. Write the main properties of the normal distribution and normal probability curve. *(6 marks)*

[April 2012] Write a short note on Gamma random variable. *(4 marks)*

(Also asked in Apr 2011)

[Sept 2011] In normal d... below 63. Find mea...

[Sept 2010] Find the m... expectation and var...

[Sept 2010] A random va... parameter $\lambda = 3$. Fi...

[April 2010] If X has a unif... function of -2 log X.

[April 2010] Show that fo... $\mu_{2n}=(2n-1)\mu_{2n-2}\sigma^2$ and ... meaning.

[April 2009] Show that fo... the mean is approxima...

[April 2009] If X is a unifor... deviation from the mea...

[April 2008] Show that sum ... gamma variate.

[April 2007] Let X be unifo... distribution function.

[Sept 2011] In normal distribution, 7% items are below 35 and 89% are below 63. Find mean and standard deviation of the distribution. *(6 marks)*

[Sept 2010] Find the m.g.f. of Normal distribution and hence find its expectation and variance. *(6 marks)*

(Also asked in Apr 2007)

[Sept 2010] A random variable X has the exponential distribution with parameter $\lambda = 3$. Find $P(X \geq 4)$. S.D and coefficient of variation. *(6 marks)*

[April 2010] If X has a uniform distribution on [0,1]. Find the distribution function of -2 log X. *(3 marks)*

[April 2010] Show that for a normal distribution $\mu_{2n}=(2n-1)\mu_{2n-2}\sigma^2$ and $\mu_{2n-1} = 0$ where the symbols have their usual meaning. *(6 marks)*

[April 2009] Show that for a normal distribution mean deviation from the mean is approximately equal to $\frac{4}{5}$ th of the standard deviation. *(5 marks)*

[April 2009] If X is a uniformly distributed random variable with mean deviation from the mean is approximately equal to $\frac{4}{3}$, find $P(X<0)$. *(4 marks)*

[April 2008] Show that sum of two independent gamma variates is a gamma variate. *(6 marks)*

[April 2007] Let X be uniform on (0,20). Write down its cumulative distribution function. *(3 marks)*

Chapter

5

Bivariate Random Variables

[April 2016] The joint density function of two continuous random variables X and Y is:

$$f(x,y) = \begin{cases} cxy & , \quad 0<x<4 \,,\, 1<y<5 \\ 0 & , \quad \text{otherwise} \end{cases}$$

(i) Find the value of c.

(ii) Find $P(1 < X < 2, 2 < Y < 3)$.

(iii) Find $P(X \geq 3, Y \leq 2)$.

(6 marks)

(Also asked in Sept 2012)

[April 2016] Show that the coefficient of correlation *r* between two random variables X and Y is given by:

$$r = \frac{\sigma_{X^2} + \sigma_{Y^2} - \sigma^2_{X-Y}}{2\sigma_X \sigma_Y}$$

(6 marks)

[April 2016] If the joint density function of two random variables X and Y is given by:

$$f(x,y) = \begin{cases} \frac{2}{3}(x+2y) & , \quad 0<x<1, 0<y<1. \\ 0 & , \quad \text{elsewhere} \end{cases}$$

Find the conditional mean and the conditional variance of X, given $Y = \frac{1}{2}$.

(6 marks)

(Also asked in Sept 2013)

[April 2016] Two fair dice are tossed simultaneously. Let X denotes the number on the first die and Y denotes the number on the second die. Find $P(X+Y=8)$, $P(X+Y\geq8)$, $P(X=Y)$, $P(X+Y=6|Y=4)$. *(6 marks)*

[April 2015] The joint probability distribution of random variable X and Y is given by:

$P(X = 0, Y = 0) = P(X = 0, Y = 1) = P(X=1, Y = -1)$ and

$P(X = 0, Y = 0) + P(X = 0, Y = 1) + P(X = 1, Y = -1) = 1$

Find:

(i) Marginal distributions of X and Y.

(ii) Conditional probability distribution of X given Y=0, where X and Y take values -1, 0 and 1.

(6 marks)

[April 2015] Two random variables X and Y have the joint p.d.f.:

$$f(x,y)=\begin{cases}2-x-y, 0\leq x\leq 1, 0\leq y\leq 1\\ 0, \quad \text{otherwise}\end{cases}$$

Find: (i) Marginal probability density functions of X and Y.

(ii) Conditional density functions.

(iii) Var (X) and Var (Y)

(iv) Cov(X,Y). *(6 marks)*

[April 2015] Let X and Y have a bivariate normal distribution with parameters $\mu_x = 20$, $\mu_y = 40$, $\sigma_x^2 = 9$, $\sigma_y^2 = 4$ and $\rho = 0.6$.

Find the shortest interval for which 0.90 is the conditional probability that Y is in this interval, given that X =22. *(6 marks)*

[Sept 2014] Let X and Y be continuous random variables having joint density function:

$$f(x,y)=\begin{cases}kxy; & 0<x<4, \quad 1<y<5\\ 0 & \text{Otherwise}\end{cases}$$

(i) The Value of K

(ii) $P(X+Y<3)$ *(5 marks)*

[Sept 2014] Prove that the correlation coefficient is independent of change of origin and scale. *(7 marks)*

(Also asked in Apr 2010,2008)

[April 2014] Show that the random variables X and Y with joint density function

$$f(x,y)=\begin{cases}12xy(1y); & 0<x<1, 0<y<1\\ 0 & ;\ \text{elsewhere}\end{cases}$$

are independent. *(4 marks)*

[April 2014] The joint probability distribution function of two discrete random variables X and Y is given by

$$f(x,y)=\begin{cases}\dfrac{2x+y}{42}; & 0\le x\le 2, 0\le y\le 3\\ 0 & ;\ \text{elsewhere}\end{cases}$$

Find E (X). *(3 marks)*

[April 2014] If r is the coefficient of correlation between two variables x and y, prove that $|r|\le 1$. *(5 marks)*

[Sept 2013] Define Joint probability density function of a pair of continuous random variables. *(2 marks)*

[April 2013] If the random variables x and y have joint density function given by:

$$f(x,y)=\begin{cases}\dfrac{1}{210}(2x+y); & 2<x<6, 0<y<5\\ 0 & ;\text{otherwise}\end{cases}$$

Find (i) The Marginal density function for x (ii) P (x + Y > 4). *(7 marks)*

[April 2013] If coefficient of correlation between two jointly normally distributed variables is zero, then show that they are independent. *(5 marks)*

[Sept 2012] If f(x,y) = 2-x-y, ($0\le x\le 1$, $0\le y\le 1$) and = 0 elsewhere, find :

(i) The marginal probability functions

(ii) The conditional probability functions *(6 marks)*

[April 2012] The random variables X and Y have the joint probability density function.

$$f(x,y) = \begin{cases} x+y & 0<x<1, \quad 0<y<1 \\ 0 & elsewhere \end{cases}$$

Find the coefficient of correlation of X and Y. *(8 marks)*

[Sept 2011] Let $f(x_1/x_2) = \begin{cases} \dfrac{c_1 x_1}{x_2^2}, & 0<x_1<x_2,\ 0<x_2<1 \\ 0 & \text{elsewhere} \end{cases}$ and

$$f_2(x_2) = \begin{cases} c_2 x_2^4, & 0<x_2<1 \\ 0 & \text{elsewhere } x \text{, find} \end{cases}$$

(i) constants c_1 and c_2

(ii) joint p.d.f. of x_1 and x_2

(iii) $P\left(\frac{1}{4}<X_1<\frac{1}{2} \backslash X_2=\frac{5}{8}\right)$

(iv) $P\left(\frac{1}{4}<X_1<\frac{1}{2}\right)$

[Apr 2011] Write short note on Joint distribution. *(4 marks)*

[Apr 2011] Write short note on Correlation coefficient. *(4 marks)*

[Sept 2010] Suppose that the joint p.d.f. of two-dimensional random variable (X,Y) is given by :

$$f(x,y) = \begin{Bmatrix} x^2+\dfrac{xy}{3} & 0<x<1, 0<y<z \\ 0 & elsewhere \end{Bmatrix}$$

Compute (i) $P\left(X>\frac{1}{2}\right)$ (ii) $\left(Y<\frac{1}{2} \mid X<\frac{1}{2}\right)$. *(6 marks)*

[Sept 2010] If X and Y are standardized random variables and r(aX + bY. bX + aY) = $\dfrac{1+2ab}{a^2+b^2}$, find r(x,y), the coefficient of correlation between X and Y.

(6 marks)

[April 2010] The joint density function of the two dimensional random variable (X,Y) is given by : $f(x,y) = \begin{cases} \frac{8}{9}xy; & 1 \le x \le y \le 2 \\ 0 & \text{elsewhere} \end{cases}$

Find the conditional density function of Y given X = x and conditional density function of X given Y = y. (3 marks)

[April 2010] The random variable X and Y have the joint distribution given by the probability density function:

$$f(x,y) = \begin{cases} 6(1-x-y) \text{ for } x>0, y>0, x+y<1 \\ 0 \quad \text{elsewhere} \end{cases}$$

Examine if X and Y are independent. (3 marks)

[April 2009] Let X and Y be two random variables. Prove that the expected value of X is equal to the expectation of the conditional expectation of X given Y. (6 marks)

[April 2009] The joint probability density function of bivariate random variables (X,Y) is given by :

$f(x,y) = 4xye^{-(x^2+y^2)}, x \ge 0, y \ge 0$

(i) Test whether X and Y are independent.

(ii) Find conditional density of X given that Y = y. (5 marks)

[April 2009] The random variable X and Y are jointly normally distributed and U and V are defined by

$U = X\cos\alpha + Y\sin\alpha, V = Y\cos\alpha - X\sin\alpha$

Show that U and V will be uncorrelated if $\tan 2\alpha = \dfrac{2r\sigma_x\sigma_y}{\sigma_x^2 - \sigma_y^2}$

where r is the correlation coefficient between X and Y, σx^2 = Var(X) and σy^2 = Var(Y). (7 marks)

[April 2008] Find the marginal and conditional probability functions if the joint density function is

$$F(x,y) = \begin{cases} 2(2-x-y) & ; 0 \le x \le y \le 1 \\ 0 & ; \text{elsewhere} \end{cases}$$ (6 marks)

[April 2007] Let X and Y be independent random variables each following exponential distribution with mean m. Find the p.d.f of $z = \min(X,Y)$ and $T = \max(X,Y)$. *(4 marks)*

[April 2007] Let X and Y be independent and identically distributed random variables each with p d f

$f(x) = 1000 / x^2$ $\quad x > 1000$

0 $\quad$ otherwise

(3 marks)

Find pdf of $Z = X/Y$

[April 2007] Show that

$E\{E(X / Y)\} = E(X)$

(4 marks)

$E\{E(Y/X)\} = E(Y)$

[April 2007] The joint p d f of X and Y is given by

$F(x,y) = e^{x/y}, 0<x<y, 0<y<\infty$

(3 marks)

Find $E\{X^2/Y = y\}$

Mathematics (Paper-I)
Analysis-I

Section-A

1. (a) Prove that the set $\{\ldots\ldots, 2^{-3}, 2^{-2}, 2^{-1}, 1, 2, 2^2, 2^3, \ldots\ldots\}$ is countable. (3)

(b) If $0 < a < b$, show that:

$$\left|\int_a^b \frac{\sin x}{x}\,dx\right| \leq \frac{2}{a}$$

2. (a) If f is continuous on [a,b], then $f \in R(x)$ on [a, b]. (3)

(b) Give an example of a bounded function f defined on a closed inerval such that $|f|$ is R-integrable but f is not. (3)

3. (a) If f is R-integrable on [a, b] and k is any real number, then k_f is also R-integrable and

$$\int_a^b (k_f)(x)\,dx = k\int_a^b f(x)\,dx$$ (3)

(b) Show that:

$$\int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(a+bx)^{m+n}}\,dx = \frac{1}{(a+b)^m a^n}\beta(m,n)$$ (3)

4. (a) Prove that:

$$\Gamma(m)\Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2m-1}}\Gamma(2m)$$ (3)

(b) Show that:

$$\int_0^\infty \frac{x}{1+x^6}\,dx = \frac{\pi}{3\sqrt{3}}$$ (3)

Section-B

5. (a) Discuss the convergence of $\int_0^\infty \frac{x\tan^{-1}x}{(1+x^4)^{1/3}}\,dx$ (3)

(b) Discuss t

6. (a) If $\phi(x)$

dx is co

converg

(b) Show th

$\int_0^\infty \frac{\sin}{x}$

7. (a) Show th

value is

(b) Show t

where

8. By applyin

following:

(a) $\int_0^{\pi/2}$

(b) $\int_0^{\pi/2} \log$

1. (a) Let G b

Then a

(b) If H is

norm

(b) Discuss the convergence of $\int_a^b \frac{dx}{(x-a)\sqrt{b-x}}$ (3)

6. (a) If $\phi(x)$ is bounded and monotonic in $[a, \infty)$ and $\int_a^\infty f(x)$ dx is convergent at ∞, than prove that $\int_a^\infty f(x)\,\phi(x)\,dx$ is convergent at ∞. (3)

(b) Show that $\int_0^\infty \frac{\sin^2 x}{x^2}dx$ is convergent. Also using

$$\int_0^\infty \frac{\sin x}{x}dx = \frac{\pi}{2}, \text{ show that } \int_0^\infty \frac{\sin^2 x}{x^2}dx = \frac{\pi}{2}$$ (3)

7. (a) Show that $\int_0^{\pi/2} \sin x \,.\, \log \sin x\, dx$ is convergent and its value is $\log \frac{2}{e}$. (3)

(b) Show that $\int_0^\infty \frac{\sin ax \sin bx}{x}dx$ converges to $\frac{1}{2}\log\left(\frac{a+b}{a-b}\right)$ where a + b are positive reals. (3)

8. By applying rule of differentiation under integral sign, prove the following:

(a) $\int_0^{\pi/2} \log\left(\frac{a+b\sin\theta}{a-b\sin\theta}\right) \operatorname{cosec}\theta\, d\theta = \pi \sin^{-1}\frac{b}{a}$ (3)

(b) $\int_0^{\pi/2} \frac{\log(1+y\sin^2 x)}{\sin^2 x}dx = \pi\left[\sqrt{1+y}-1\right]$ (3)

Mathematics (Paper-II)
Modern Algebra

Section-A

1. (a) Let G be a finite group and $a \in G$ be an element of order n. Then $a^m = e$ iff n is a divisor of m. (3)

(b) If H is a sub-group of G of index 2 in G, then prove that H is normal subgroup of G. (3)

2. (a) Let N be a normal sub-group of a group G. Show that G/N is abelian iff for all $x, y \in G$, $xyx^{-1}y^{-1} \in N$. (3)

(b) Let H and K be two subgroups of a group G. Show that:

$(H \cap K)C = H_C \cap K_C$ for all $C \in G$. (3)

3. (a) Let G and G' be two groups and $f : G \to G'$ homomorphism of G onto G'. Then prove that $\frac{G}{H} \cong G'$ where H is the kernel of f.

(b) If $f : G \to G$ defined by $f(x) = x^2 \ \forall \ x \in G$ is an automorphism, show that G is abelian. (3,3)

4. (a) If $O(G) = p^2$, where p is prime number then prove that G is an abelian group. (3)

(b) Prove that A_4 has no subgroup of order six, A_4 is alternative group of order 4. (3)

Section-B

5. (a) If in a ring R, $x^2 = x$ for all $x \in R$, then show that R is commutative.

(b) Let R be a finite ring without zero divisors and $O(R) > 1$. Then show that R is a division ring. (3,3)

6. (a) Prove that intersection of two left (or right) ideals of a ring is a left (or right) ideal. (3)

(b) Prove that a commutative ring with unity is simple iff R is a field. (3)

7. (a) Show that any ideal of z is maximal iff it is generated by some prime element. (3)

(b) Show that the mapping $f : z \to M_2(z)$, defined by:

$$f(r) = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}$$

is a homomorphism of rings, find ker f. Is the mapping an isomorphism? (3)

8. (a) Let R be a commutative ring with identity and $f(x)$. $g(x) \in R[x]$. Then deg $[f(x)\ g(x)] \leq$ deg. $f(x)$ +deg $g(x)$. (3)

(b) For any ring R, show that $R[x] < x > \cong R$. (3)

1. (a) A typical ... of any fo... and the ... probabilit... symbol.

(b) State and...

2. (a) A bowl c... one of th... and with... p.m.f. an... needed t...

(b) Let X be ...

f (x) = ...

Find a n...

3. (a) Let a ran... density ... to x = c.

(b) Let X be...

f (x) = ...

Find me...

4. (a) If a fai... find the... hypoth...

(b) If for a...

Probability Theory

Section-A

1. (a) A typical PIN (Personal Identification Number) is a sequence of any four symbols chosen from 26 letters in the alphabet and the ten digits. If all PINs are equally likely, find the probability that a randomly chosen PIN contains a repeated symbol. (3)

   (b) State and prove Bayes' theorem. (3)

2. (a) A bowl contain 10 balls of same size and shape out of which one of the balls is red. Balls are drawn one by one at random and without replacement untill the red ball is drawn. Find the p.m.f. and c.d.f. of random variable X, the number of trials needed to draw the red chip. (3)

   (b) Let X be a continuous random variable having p.d.f.:

$$f(x) = \begin{cases} 6x(1-x), & 0 \le x \le 1 \\ 0 \quad , & \text{elsewhere} \end{cases}$$

   Find a number *m* such that $P(X < m) = P(X > m)$. (3)

3. (a) Let a random variable X of continuoues types has a probability density function $f(x)$, whose graph is symmetric with respect to $x = c$. If the mean value of X exists, then show that $E(X) = c$.

   (b) Let X be a continuous random variable having p.d.f.:

$$f(x) = \begin{cases} \frac{3}{4}x(2-x) & 0 \le x \le 2 \\ 0 \quad , & \text{elsewhere} \end{cases}$$

   Find measure of skewness and kurtosis of the distribution. (3,3)

4. (a) If a fair coin is tossed at random five independent times, find the conditional probability of five heads relative to the hypothesis that there are at least four heads. (3)

   (b) If for a Poisson random variable X, $E(X^2) = 20$, find $E(X)$. (3)

Section-B

5. (a) Let X be uniformly distributed over $[-\alpha, \alpha]$, where $\alpha > 0$. Find α so that:

(i) $P(X > 1) = \frac{1}{3}$

(ii) $P\left(X < \frac{1}{2}\right) = 0.8$

(iii) $P(|X| < 1) = P(|X| > 1)$ (3)

(b) Find coefficients of skewness and kurtosis of an exponential distribution and describe nature of the distribution. (3)

6. (a) If X has gamma distribution with $\alpha = \frac{r}{2}$, $r \in N$ and $\beta > 0$, then show that $Y = \frac{2X}{\beta}$ is $\chi^2(r)$. (3)

(b) There are six hundred mathematics students in the graduate classes of a university and the probability for any student to need a copy of a paritcular book from the university library on any day is 0.05. How many copies of the book should be kept in the university library so that the probability may be greater than 0.90 that none of the students needing a copy from the library has to come back disappointed? (3)

7. (a) Let two dimensional continuous random variable (X, Y) has joint probability density function given by:

$$f(x, y) = \begin{cases} 6x^2y, & 0<x<1, 0<y<1 \\ 0, & \text{elsewhere} \end{cases}$$

(i) Verify that $\int_0^1 \int_0^1 f(x,y)\,dx\,dy = 1$

(ii) Find $P(0 < X < \frac{3}{4}, \frac{1}{3} < Y < 2)$

(iii) Find $P(X + Y < 1)$

(iv) Find $P(X > Y)$

(v) Find $P(X < 1 \mid Y < 2)$ (3)

(b) Let $f(x, y) = \begin{cases} 8xy, & 0<x<y<1 \\ 0, & \text{elsewhere} \end{cases}$

Find

(i) $E(Y \mid X = x)$

(ii) $E(X \mid Y = y)$

(iii) $Var(Y \mid X = x)$

(iv) $Var(X \mid Y = y)$ (3)

(c) Let the variables X and Y be connected by equation $aX + bY + c = 0$. Prove that the coefficient of corelation between X and Y is -1 if a and b are of same sign and $+1$ if a and b are of opposite signs. (3)

(d) In a certain population of married couples the height X of the husband and the height Y of the wife has a bivariate normal distribution with parameters $\mu_X = 5.8$ feet, $\mu_Y = 5.3$ feet, $\sigma_X = \sigma_Y = 0.2$ feet and $r = 0.6$. If height of the husband is 6.3 feet, find the probability that his wife has a height between 5.28 and 5.92 feet. (3)

Zoology(Paper-A)
Developmental Biology

Explain the following briefly:

Acrosome reaction

Parthenogenesis

Mesolecithal eggs

Sperm Capacitation

Inducers

Radial Cleavage

(b) How do ... prove Wien's displacement ...

(c) What is Born-Oppenheimer approximation in Quantum Mechanics?

(d) What are grade and ungrade molecular orlitals? Explain with suitable examples.

(e) What are Photo-inhibitors? How do they work?

(f) What is the physical significance of molar extinction coefficient or molar absorpitivity?

(6x1=6)

MATHEMATICS (Paper-A)
(Analysis-I)

Section - A

1. (a) Prove that the set:

$\{\ldots\ldots\ldots\ldots 2^{-3}, 2^{-2}, 2^{-1}, 1, 2^{1}, 2^{2}, 2^{3}, \ldots\ldots\ldots\ldots\}$ is countable.

(b) If $0 < x < 1$, then show that:

$$\frac{x}{1-x} \geq \log (1-x)^{-1} \geq x$$

2. (a) If $f : [a, b] \rightarrow R$ is a monotonic function, then it is integrable on $[a, b]$.

(b) If f be continuous function defined on $[a, b]$, then show that there exists a real number $\theta \in [0, 1]$ such that:

$$\int_a^b f(x)dx = (b-a)f\{a+\theta(b-a)\}$$

(a) If f is bounded and integrable in [a b], then $|f|$ is also bounded and integrable in [a, b] and:

$$\left|\int_a^b f\,dx\right| \le \int_a^b |f|\,dx$$

(b) Show that

$$\int_0^1 x^{-1/3}(1-x)^{-2/3}\,(1+2x)^{-1}dx$$

$$=\frac{1}{(9)^{1/3}}\beta\left(\frac{2}{3},\frac{1}{3}\right)$$

by substituting $\frac{x}{1-x}=\frac{az}{1-z}$, where a is constant suitably selected.

(a) Prove that:

$$\Gamma(m)\Gamma\left(m+\frac{1}{2}\right)=\frac{\sqrt{\pi}}{2^{2m-1}}(2m)$$

(b) Evaluate:

$$\int_0^{\pi/2}(\sin x)^{2/3}(\cos x)^{-1/2}dx$$

Section - B

5. Examine the convergence of the following:

(a) $\int_0^\infty\left(\frac{1}{1+x}-e^{-x}\right)\frac{dx}{x}$

(b) $\int_a^b\frac{dx}{(x-a)\sqrt{b-x}}$

6. (a) Let $\phi(x)$ be bounded and monotonic in $[a,\infty)$ and tends to 0 as $x\to\infty$. Let $\int_a^t f(x)dx$ be bounded for all $t\ge a$. Then prove that $\int_a^\infty f(x)\phi(x)dx$ is convergent at ∞.

(b) Show that :

$$\int_e^{\infty} \frac{\log x \sin x}{x} dx \text{ is convergent.}$$

7. (a) Discuss the convergence of:

$$\int_0^1 \frac{\log x}{1-x^2} dx$$

(b) Show that $\int_0^{\infty} \frac{\sin ax \sin bx}{x} dx$ converges to $\frac{1}{2}\log\left(\frac{a+b}{a-b}\right)$ where a and b are +ve reals.

8. By applying rule of differentiation under integral sign, prove the following:

(a) $\int_0^{\infty} \frac{e^{-xy} \sin x}{x} dx = \cot^{-1} y, y > 0$

(b) $\int_0^{\pi/2} \frac{\log(1+b\sin^2 x)}{\sin^2 x} dx = \pi[\sqrt{1+b} - 1].$

MATHEMATICS (Paper-B)

(Modern Algebra)

Section-A

1. (a) If a, b $\in$ G such that ab = ba where G is a group, and (O(a), O(b)) = 1 then prove that:
O(ab) =O(a) O(b)

(b) Prove that the centre Z(G) of a group G is a normal subgroup of G.

2. (a) Prove that every quotient group of a cyclic group is cyclic.

(b) Let H and K be two subgroups of a group G. Show that any coset relative to H $\cap$ K is the intersection of a coset relative to H with a coset relative to K.

3. (a) Let ...
show that ...
(b) Prove that i...
G is an auto...
4. (a) If G be a n...
prime num...
G.
(b) Prove tha...
of order ...
5. (a) Prove tha...
domain
(b) If R is
commu...
6. (a) Prove t...
iff one
(b) Show
prime
7. (a) Show
ideal
(b) Show
integ...
8. (a) If R
don...
(b) For
R[x...

(a) Let G and G' be two groups. If $f : G \to G'$ is a homomorphism, show that the kernal of f is a normal subgroup of G.

(b) Prove that if for a group G, $f : G \to G$ is given by $f(x) = x^3, x \in G$ is an automorphism then G is abelian.

(a) If G be a non-abelian group such that $O(G) = p^3$, where p is a prime number, then prove that $O(Z) = p$, where z is centre of G.

(b) Prove that A_4, alternating group of order 4, has no subgroup of order six.

Section - B

5. (a) Prove that a commutative ring R with identity $I \neq 0$ is an integral domain iff the cancellation laws hold for multiplication.

(b) If R is a ring in which $x^2 = x \; \forall \, x \in R$. Prove that R is a commutative ring of characteristics 2.

6. (a) Prove that union of two left (right) ideals of a ring is an ideal iff one is contained in the other.

(b) Show that any ideal of Z is maximal iff it is generated by some prime element.

7. (a) Show that $M \neq \{0\}$ is a maximal ideal of a ring R iff for any ideal I of R, either $I \subseteq M$ or $I + M = R$.

(b) Show that isomorphic images of an integral domain is an integral domain.

8. (a) If R is an integral domain, then prove that R[x] is also integral domain.

(b) For any ring R, show that:

$R[x] | < x > \cong R$

MATHEMATICS (Paper-C)
(Probability theory)

Section – A

1. (a) If p and q are chosen randomly from the set {1, 2, 3, 4, 5, 6, 7, 8, 9, 10} with replacement, determine the probabilities that the roots of the equation $x^2 + px + q = 0$ are real.

   (b) Fatima and John appear in an interview for two vacancies in the same post. The probability of Fatima's selection is $\frac{1}{7}$ and that of John's selection is $\frac{1}{5}$. What is the probability that (i) both of them will be selected? (ii) only one of them is selected? (iii) None of them will be selected? (3,3)

2. (a) Of the students in a college, it is known that 30% has 100% attendance and 70% students are irregular. Previous year results report that 70% of all students who has 100% attendance attain A grade and 10% irregular students attain A grade in their annual examination. At the end of the year, one student is chosen at random from the college and he has A grade. What is the probability that the students have 100% attendance.

   (b) A bowl contains 10 balls of same size and shape, one of the balls is red. Balls are drawn one by one at random and without replacement, until the red ball is drawn. Find the p.m.f. and c.d.f. of random variable X, the number of trials needed to draw the red chip. (3,3)

3. (a) The distribution function for a random variable

$$X \text{ is } F(x) = \begin{cases} 0 & , x < 0. \\ 1 - e^{-2x}, & x \geq 0 \end{cases}$$

   Find (i) the probability density function of X (ii) the probability that $X > 2$ (iii) the probability that $-3 < X \leq 4$.

   (b) Two cards are drawn simultaneously (without replacement) from a well-shuffled pack of 52 cards. Find the mean and variance of the number of red cards. (3,3)

4. (a) A die is thrown 6 times. 'If getting an odd number is a success, what is the probability of (i) 5 successes? (ii) at least 5 successes? (iii) at most 5 successes?

(b) If X has Poisson distribution with parameter m, then show that $P(X = \text{even}) = \frac{1}{2}(1+e^{-2m})$. (3,3)

Section – B

5. (a) Prove that the mean and variance of uniform distribution on [a,b] are $\frac{a+b}{2}$ and $\frac{(b-a)^2}{12}$ respectively.

(b) Find the constant c so that $f(x)=\begin{cases} cx(3-x)^4, 0<x<3 \\ 0 \quad ,\text{elsewhere} \end{cases}$ is a p.d.f. (3,3)

6. (a) If the level of education among adults in a certain region is normally distributed with mean 8 and S.D. 5, what is the probability that in a sample of 100 adults, there is an average level of education (i) between 10 to 14 years (ii) more than 14 years.

(b) Two dice are rolled; Let X denotes the sum on the two faces and Y the absolute value of their difference. Assuming that the dice are fair, find

(i) $P[x = 5) \cap (Y = 1)]$,

(ii) $P[(x = 7) \cap (Y \geq 3)]$,

(iii) $P(X = Y)$. (3,3)

7. (a) The joint probability density function of a bivariate random variable (X, Y) is given by $f(x,y)=\begin{cases} ce^{-(2x+2y)}, & x\geq 0, y\geq 0 \\ 0 \quad , & \text{otherwise} \end{cases}$

Find (i) c (ii) marginal probability density function (iii) conditional probability density function.

(b) If X has gamma distribution with $\alpha = \frac{r}{2}, r \in N$ and $\beta > 0$, then show that $Y = \frac{2X}{\beta}$ is $\chi^2(r)$. (3,3)

8. (a) Let $f(x,y) = \begin{cases} 8xy, & 0 < x < y < 1 \\ 0, & \text{elsewhere} \end{cases}$

Find (i) $E(Y|X = x)$, $E(X|Y = y)$

(ii) $Var(Y|X = x)$.

(b) Let X and Y have a bivariate normal distribution with parameters $\mu_x = 2.8$, $\mu_Y = 110$,

$\sigma_x^2 = 0.16$, $\sigma_Y^2 = 100$ and $r = \frac{3}{5}$. Compute

(i) $P(106 < Y < 124)$

(ii) $P(106 < Y < 124 | X = 3.2)$. (3,3)