10. What are the component parts of the mechanical constants of a motor's transfer function?
11. The motor's transfer function relates armature displacement to armature voltage. How can the transfer function that relates load displacement and armature voltage be determined?
12. Summarize the steps taken to linearize a nonlinear system.

# Problems

1. Derive the Laplace transform for the following time functions: [Section: 2.2]
   **a.** $u(t)$
   **b.** $tu(t)$
   **c.** $\sin \omega t\, u(t)$
   **d.** $\cos \omega t\, u(t)$
2. Using the Laplace transform pairs of Table 2.1 and the Laplace transform theorems of Table 2.2, derive the Laplace transforms for the following time functions: [Section: 2.2]
   **a.** $e^{-at}\sin \omega t\, u(t)$
   **b.** $e^{-at}\cos \omega t\, u(t)$
   **c.** $t^3 u(t)$
3. Repeat Problem 19 in Chapter 1, using Laplace transforms. Assume zero initial conditions. [Sections: 2.2; 2.3]
4. Repeat Problem 20 in Chapter 1, using Laplace transforms. Assume that the forcing functions are zero prior to $t = 0-$. [Section: 2.2]
5. Repeat Problem 21 in Chapter 1, using Laplace transforms. Use the following initial conditions for each part as follows: (a) $x(0) = 4$, $x'(0) = -4$; (b) $x(0)=4$, $x'(0)=1$; (c) $x(0)=2, x'(0)=3$, where $x'(0)=\frac{dx}{dt}(0)$. Assume that the forcing functions are zero prior to $t=0-$. [Section: 2.2]
6. Use MATLAB and the Symbolic Math Toolbox to find the Laplace transform of the following time functions: [Section: 2.2]

   Symbolic Math
   SM

   **a.** $f(t) = 8t^2\cos(3t + 45°)$
   **b.** $f(t) = 3te^{-2t}\sin(4t + 60°)$
7. Use MATLAB and the Symbolic Math Toolbox to find the inverse Laplace transform of the following frequency functions: [Section: 2.2]

   Symbolic Math
   SM

   **a.** $G(s) = \dfrac{(s^2+3s+10)(s+5)}{(s+3)(s+4)(s^2+2s+100)}$
   **b.** $G(s) = \dfrac{s^3+4s^2+2s+6}{(s+8)(s^2+8s+3)(s^2+5s+7)}$
8. A system is described by the following differential equation:

$$\frac{d^3y}{dt^3} + 3\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + y = \frac{d^3x}{dt^3} + 4\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 8x$$

   Find the expression for the transfer function of the system, $Y(s)/X(s)$. [Section: 2.3]
9. For each of the following transfer functions, write the corresponding differential equation. [Section: 2.3]
   **a.** $\dfrac{X(s)}{F(s)} = \dfrac{7}{s^2+5s+10}$
   **b.** $\dfrac{X(s)}{F(s)} = \dfrac{15}{(s+10)(s+11)}$
   **c.** $\dfrac{X(s)}{F(s)} = \dfrac{s+3}{s^3+11s^2+12s+18}$
10. Write the differential equation for the system shown in Figure P2.1. [Section: 2.3]

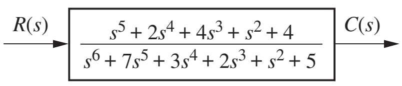


**FIGURE P2.1**

11. Write the differential equation that is mathematically equivalent to the block diagram shown in Figure P2.2. Assume that $r(t) = 3t^3$. [Section: 2.3]

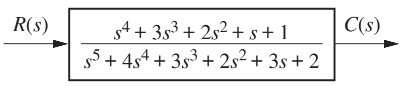


**FIGURE P2.2**

12. A system is described by the following differential equation: [Section 2.3]

$$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 5x = 1$$

   with the initial conditions $x(0) = 1$, $\dot{x}(0) = -1$. Show a block diagram of the system, giving its transfer function and all pertinent inputs and outputs. (Hint: the initial conditions will show up as added inputs to an effective system with zero initial conditions.)

**13.** Use MATLAB to generate the transfer function: [Section: 2.3] MATLAB **ML**

$$G(s) = \frac{5(s+15)(s+26)(s+72)}{s(s+55)(s^2+5s+30)(s+56)(s^2+27s+52)}$$

in the following ways:

**a.** the ratio of factors;

**b.** the ratio of polynomials.

**14.** Repeat Problem 13 for the following transfer function: [Section: 2.3] MATLAB **ML**

$$G(s) = \frac{s^4+25s^3+20s^2+15s+42}{s^5+13s^4+9s^3+37s^2+35s+50}$$

**15.** Use MATLAB to generate the partial-fraction expansion of the following function: [Section: 2.3]

$$F(s) = \frac{10^4(s+5)(s+70)}{s(s+45)(s+55)(s^2+7s+110)(s^2+6s+95)}$$

**16.** Use MATLAB and the Symbolic Math Toolbox to input and form LTI objects in polynomial and factored form for the following frequency functions: [Section: 2.3] Symbolic Math **SM**

a. $G(s) = \dfrac{45(s^2+37s+74)(s^3+28s^2+32s+16)}{(s+39)(s+47)(s^2+2s+100)(s^3+27s^2+18s+15)}$

b. $G(s) = \dfrac{56(s+14)(s^3+49s^2+62s+53)}{(s^3+81s^2+76s+65)(s^2+88s+33)(s^2+56s+77)}$

**17.** Find the transfer function, $G(s) = V_o(s)/V_i(s)$, for each network shown in Figure P2.3. [Section: 2.4]

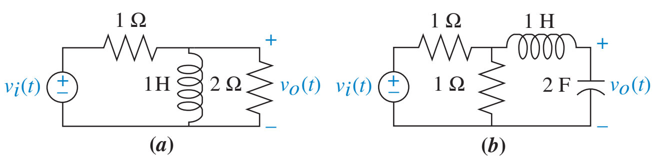


**FIGURE P2.3**

**18.** Find the transfer function, $G(s) = V_L(s)/V(s)$, for each network shown in Figure P2.4. [Section: 2.4]

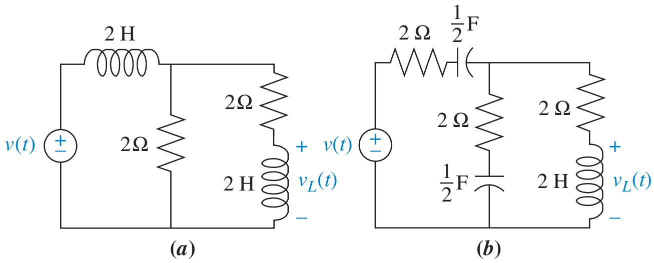


**FIGURE P2.4**

**19.** Find the transfer function, $G(s) = V_o(s)/V_i(s)$, for each network shown in Figure P2.5. Solve the problem using mesh analysis. [Section: 2.4]

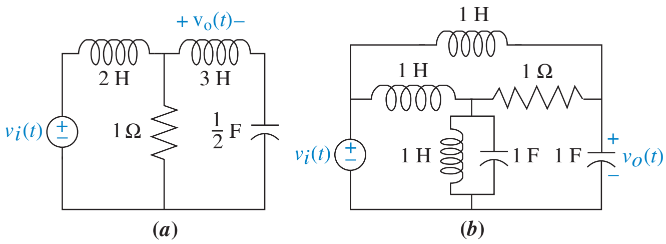


**FIGURE P2.5**

**20.** Repeat Problem 19 using nodal equations. [Section: 2.4]

**21. a.** Write, but do not solve, the mesh and nodal equations for the network of Figure P2.6. [Section: 2.4]

**b.** Use MATLAB, the Symbolic Math Toolbox, and the equations found in part a to solve for the transfer function, $G(s) = V_o(s)/V(s)$. Use both the mesh and nodal equations and show that either set yields the same transfer function. [Section: 2.4] Symbolic Math **SM**

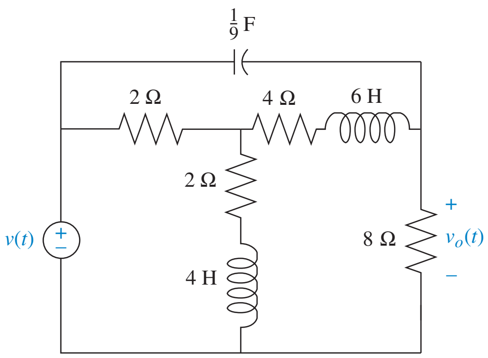


**FIGURE P2.6**

**22.** Find the transfer function, $G(s) = V_o(s)/V_i(s)$, for each operational amplifier circuit shown in Figure P2.7. [Section: 2.4]

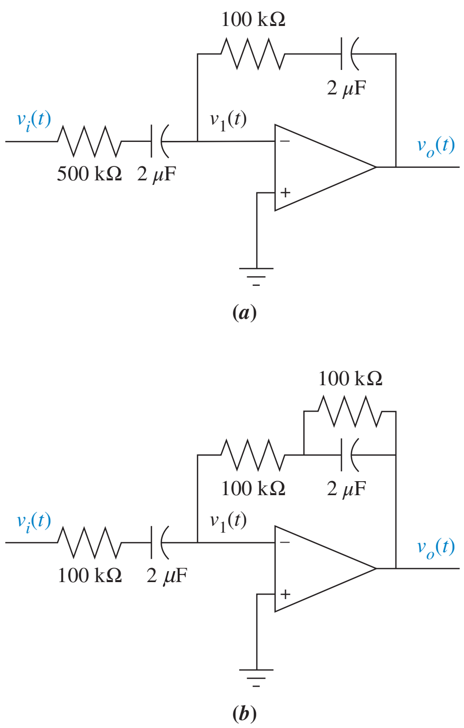


**FIGURE P2.7**

**23.** Find the transfer function, $G(s) = V_o(s)/V_i(s)$, for each operational amplifier circuit shown in Figure P2.8. [Section: 2.4]

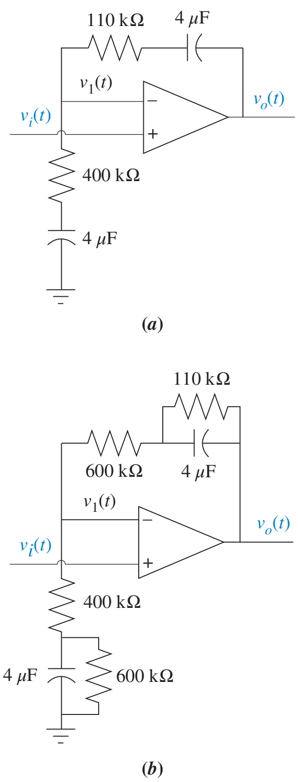


**FIGURE P2.8**

**24.** Find the transfer function, $G(s) = X_1(s)/F(s)$, for the translational mechanical system shown in Figure P2.9. [Section: 2.5]

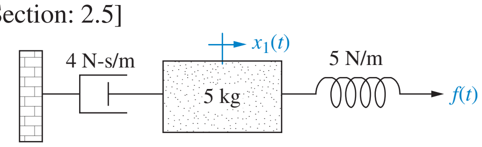


**FIGURE P2.9**

**25.** Find the transfer function, $G(s) = X_2(s)/F(s)$, for the translational mechanical network shown in Figure P2.10. [Section: 2.5]

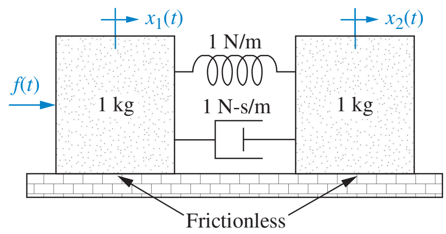


**FIGURE P2.10**

**26.** Find the transfer function, $G(s) = X_2(s)/F(s)$, for the translational mechanical system shown in Figure P2.11. (Hint: place a zero mass at $x_2(t)$.) [Section: 2.5]

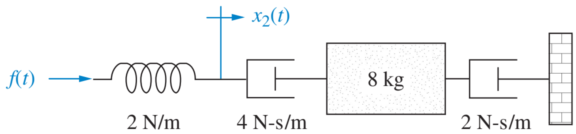


**FIGURE P2.11**

**27.** For the system of Figure P2.12 find the transfer function, $G(s) = X_1(s)/F(s)$. [Section: 2.5]

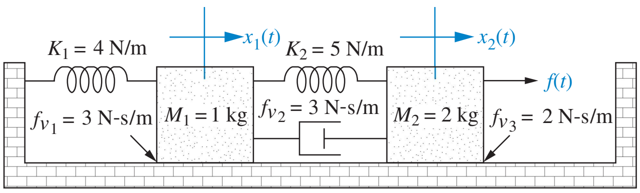


**FIGURE P2.12**

**28.** Find the transfer function, $G(s) = X_3(s)/F(s)$, for the translational mechanical system shown in Figure P2.13. [Section: 2.5]

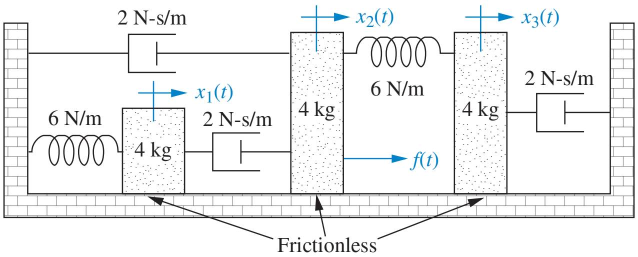


**FIGURE P2.13**

**29.** Find the transfer function, $X_3(s)/F(s)$, for each system shown in Figure P2.14. [Section: 2.5]

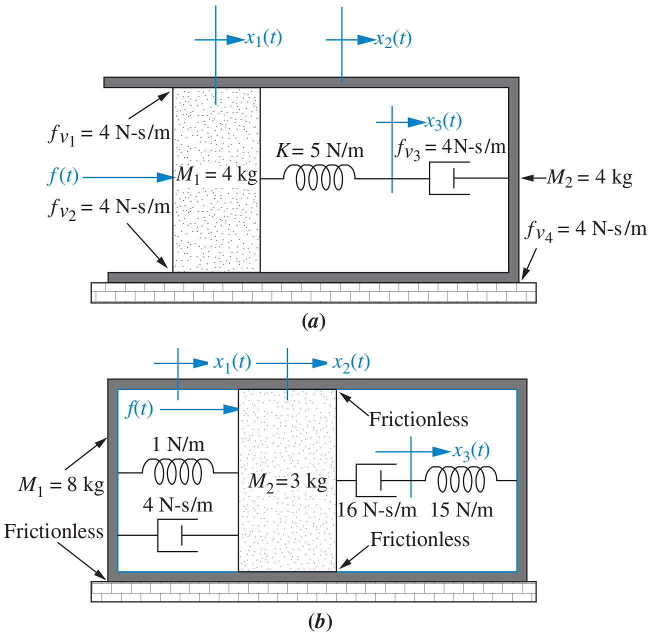


**FIGURE P2.14**

**30.** Write, but do not solve, the equations of motion for the translational mechanical system shown in Figure P2.15. [Section: 2.5]

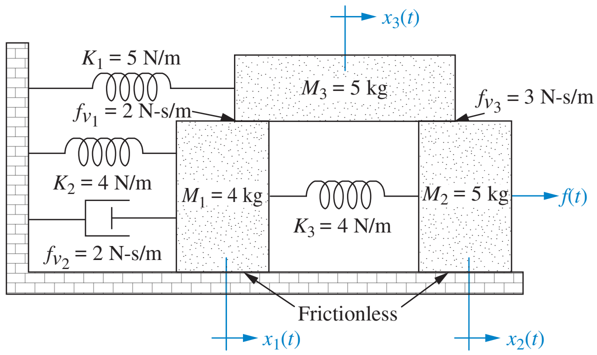


**FIGURE P2.15**

**31.** For the unexcited (no external force applied) system of Figure P2.16, do the following:

**a.** Write the differential equation that describes the system.

**b.** Assuming initial conditions $x(0) = x_0$ and $\dot{x}(0) = x_1$, write a Laplace transform expression for $X(s)$.

**c.** Find $x(t)$ by obtaining the inverse Laplace transform from the result in Part **c**.

**d.** What will be the oscillation frequency in Hz for this system?

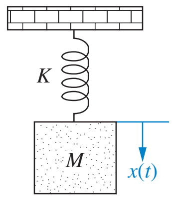


**FIGURE P2.16**

**32.** For each of the rotational mechanical systems shown in Figure P2.17, write, but do not solve, the equations of motion. [Section: 2.6]

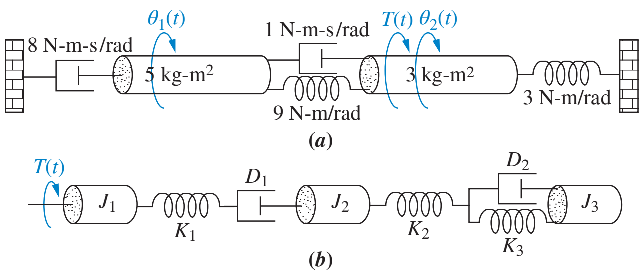


**FIGURE P2.17**

**33.** For the rotational mechanical system shown in Figure P2.18, find the transfer function $G(s) = \theta_2(s)/T(s)$ [Section: 2.6]

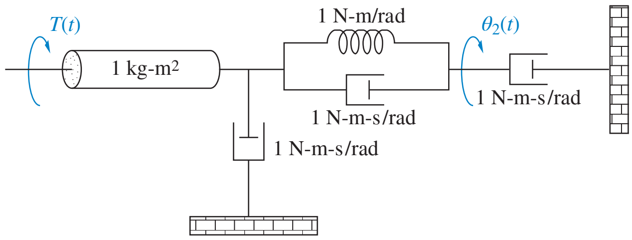


**FIGURE P2.18**

**34.** Find the transfer function, $\dfrac{\theta_1(s)}{T(s)}$, for the system shown in Figure P2.19.

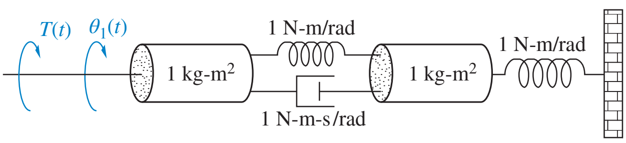


**FIGURE P2.19**

**35.** For the rotational mechanical system with gears shown in Figure P2.20, find the transfer function, $G(s) = \theta_3(s)/T(s)$. The gears have inertia and bearing friction as shown. [Section: 2.7]

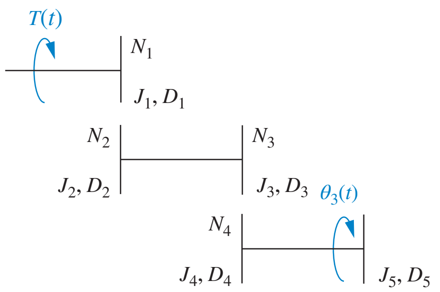


**FIGURE P2.20**

**36.** For the rotational system shown in Figure P2.21, find the transfer function, $G(s) = \theta_2(s)/T(s)$. [Section: 2.7]

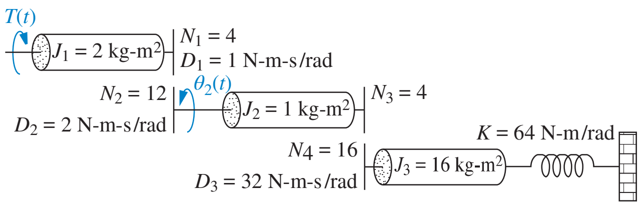


**FIGURE P2.21**

**37.** Find the transfer function, $G(s) = \theta_2(s)/T(s)$, for the rotational mechanical system shown in Figure P2.22. [Section: 2.7]

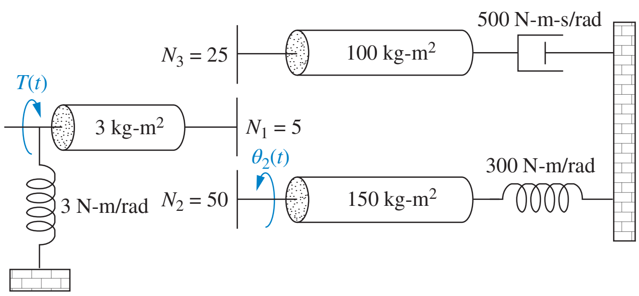


**FIGURE P2.22**

**38.** Find the transfer function, $G(s) = \theta_4(s)/T(s)$, for the rotational system shown in Figure P2.23. [Section: 2.7]

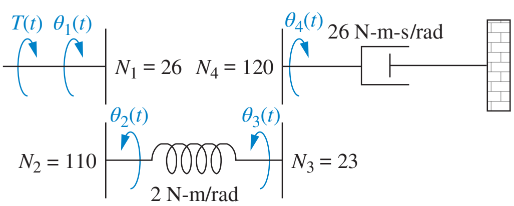


**FIGURE P2.23**

**39.** For the rotational system shown in Figure P2.24, find the transfer function, $G(s) = \theta_L(s)/T(s)$. [Section: 2.7]

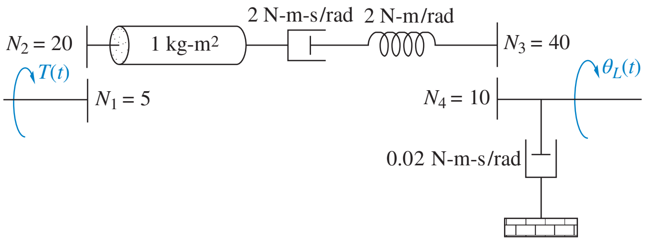


**FIGURE P2.24**

**40.** For the rotational system shown in Figure P2.25, write the equations of motion from which the transfer function, $G(s) = \theta_1(s)/T(s)$, can be found. [Section: 2.7]

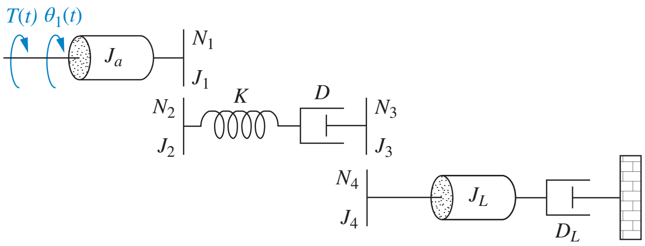


**FIGURE P2.25**

**41.** Given the rotational system shown in Figure P2.26, find the transfer function, $G(s) = \theta_6(s)/\theta_1(s)$. [Section: 2.7]

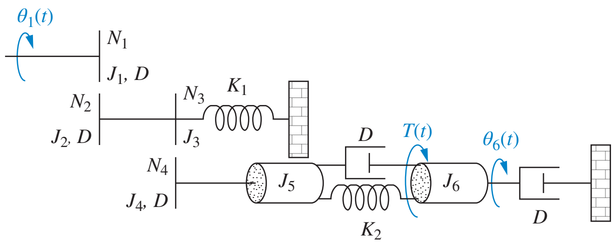


**FIGURE P2.26**

**42.** In the system shown in Figure P2.27, the inertia, $J$, of radius, $r$, is constrained to move only about the stationary axis $A$. A viscous damping force of translational value $f_v$ exists between the bodies $J$ and $M$. If an external force, $f(t)$, is applied to the mass, find the transfer function, $G(s) = \theta(s)/F(s)$. [Sections: 2.5; 2.6]

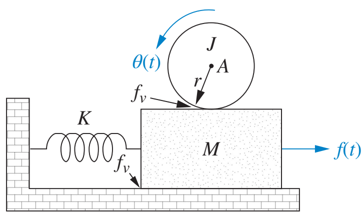


**FIGURE P2.27**

43. For the combined translational and rotational system shown in Figure P2.28, find the transfer function, $G(s) = X(s)/T(s)$. [Sections: 2.5; 2.6; 2.7]

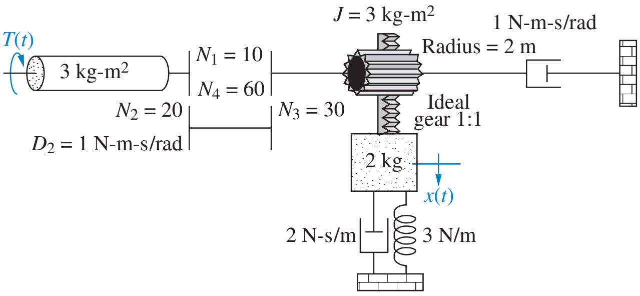


FIGURE P2.28

44. Given the combined translational and rotational system shown in Figure P2.29, find the transfer function, $G(s) = X(s)/T(s)$. [Sections: 2.5; 2.6]

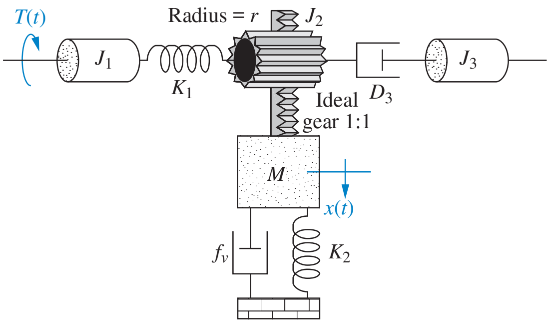


FIGURE P2.29

45. For the motor, load, and torque-speed curve shown in Figure P2.30, find the transfer function, $G(s) = \theta_L(s)/E_a(s)$. [Section: 2.8]

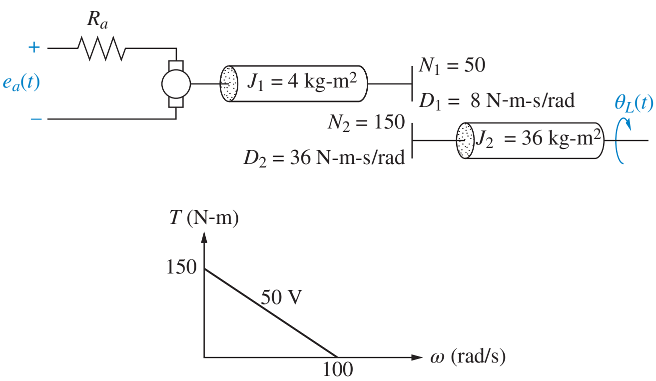


FIGURE P2.30

46. The motor whose torque-speed characteristics are shown in Figure P2.31 drives the load shown in the diagram. Some of the gears have inertia. Find the transfer function, $G(s) = \theta_2(s)/E_a(s)$. [Section: 2.8]

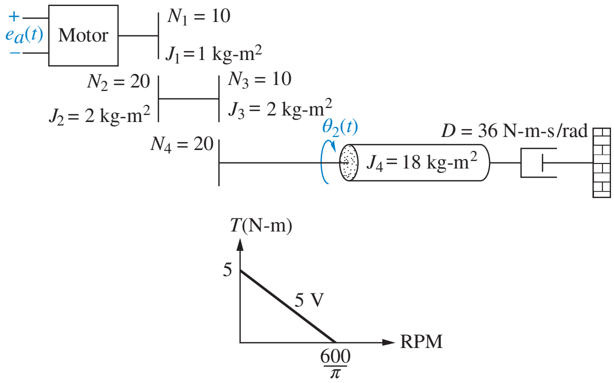


FIGURE P2.31

47. A dc motor develops 55 N-m of torque at a speed of 600 rad/s when 12 volts are applied. It stalls out at this voltage with 100 N-m of torque. If the inertia and damping of the armature are 7 kg-m$^2$ and 3 N-m-s/rad, respectively, find the transfer function, $G(s) = \theta_L(s)/E_a(s)$, of this motor if it drives an inertia load of 105 kg-m$^2$ through a gear train, as shown in Figure P2.32. [Section: 2.8]

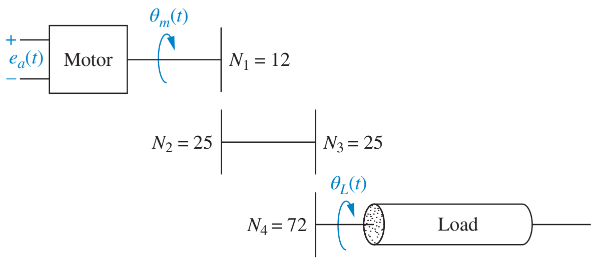


FIGURE P2.32

48. In this chapter, we derived the transfer function of a dc motor relating the angular displacement output to the armature voltage input. Often we want to control the output torque rather than the displacement. Derive the transfer function of the motor that relates output torque to input armature voltage. [Section: 2.8]

49. Find the transfer function, $G(s) = X(s)/E_a(s)$, for the system shown in Figure P2.33. [Sections: 2.5–2.8]

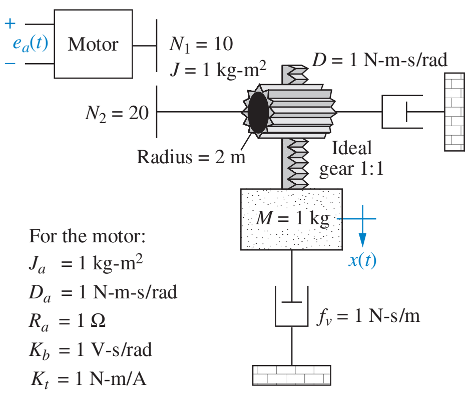


FIGURE P2.33

**50.** Find the series and parallel analogs for the translational mechanical system shown in Figure 2.20 in the text. [Section: 2.9]

**51.** Find the series and parallel analogs for the rotational mechanical systems shown in Figure P2.17(*b*) in the problems. [Section: 2.9]

**52.** A system's output, $c$, is related to the system's input, $r$, by the straight-line relationship, $c = 5r + 7$. Is the system linear? [Section: 2.10]

**53.** Consider the differential equation

$$\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + 2x = f(x)$$

where $f(x)$ is the input and is a function of the output, $x$. If $f(x) = \sin x$, linearize the differential equation for small excursions. [Section: 2.10]

**a.** $x = 0$

**b.** $x = \pi$

**54.** Consider the differential equation

$$\frac{d^3x}{dt^3} + 10\frac{d^2x}{dt^2} + 20\frac{dx}{dt} + 15x = f(x)$$

where $f(x)$ is the input and is a function of the output, $x$. If $f(x) = 3e^{-5x}$, linearize the differential equation for $x$ near 0. [Section: 2.10]

**55.** Many systems are *piecewise* linear. That is, over a *large* range of variable values, the system can be described linearly. A system with amplifier saturation is one such example. Given the differential equation

$$\frac{d^2x}{dt^2} + 17\frac{dx}{dt} + 50x = f(x)$$

assume that $f(x)$ is as shown in Figure P2.34. Write the differential equation for each of the following ranges of $x$: [Section: 2.10]

**a.** $-\infty < x < -3$

**b.** $-3 < x < 3$

**c.** $3 < x < \infty$

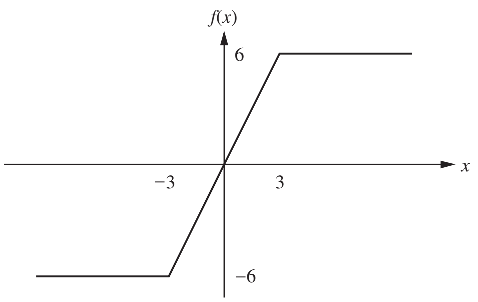


FIGURE P2.34

**56.** For the translational mechanical system with a nonlinear spring shown in Figure P2.35, find the transfer function, $G(s) = X(s)/F(s)$, for small excursions around $f(t) = 1$. The spring is defined by $x_s(t) = 1 - e^{-f_s(t)}$, where $x_s(t)$ is the spring displacement and $f_s(t)$ is the spring force. [Section: 2.10]

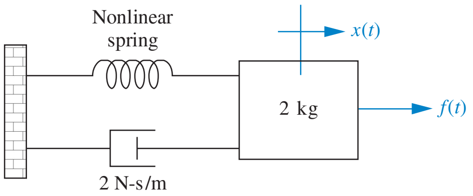


FIGURE P2.35

**57.** Enzymes are large proteins that biological systems use to increase the rate at which reactions occur. For example, food is usually composed of large molecules that are hard to digest; enzymes break down the large molecules into small nutrients as part of the digestive process. One such enzyme is amylase, contained in human saliva. It is commonly known that if you place a piece of uncooked pasta in your mouth its taste will change from paper-like to sweet as amylase breaks down the carbohydrates into sugars. Enzyme breakdown is often expressed by the following relation:

$$S + E \underset{k_1}{\overset{k_s}{\rightleftarrows}} C \xrightarrow{k_1} P$$

In this expression a substrate ($S$) interacts with an enzyme ($E$) to form a combined product ($C$) at a rate $k_1$. The intermediate compound is reversible and gets disassociated at a rate $k_{-1}$. Simultaneously some of the compound is transformed into the final product ($P$) at a rate $k_2$. The kinetics describing this reaction are known as the Michaelis-Menten equations and consist of four nonlinear differential equations. However, under some conditions these equations can be simplified. Let $E_0$ and $S_0$ be the initial concentrations of enzyme and substrate, respectively. It is generally accepted that under some energetic conditions or when the enzyme concentration is very big ($E_0 \gg S_0$), the kinetics for this reaction are given by

$$\frac{dS}{dt} = k_\psi(\tilde{K}_s C - S)$$

$$\frac{dC}{dt} = k_\psi(S - \tilde{K}_M C)$$

$$\frac{dP}{dt} = k_2 C$$

where the following constant terms are used (*Schnell, 2004*):

$$k_\psi = k_1 E_0$$

$$\tilde{K}_s = \frac{k-1}{k_\psi}$$

and

$$\tilde{K}_M = \tilde{K}_s + \frac{k_2}{k_\psi}$$

**a.** Assuming the initial conditions for the reaction are $S(0) = S_0,\ E(0) = E_0,\ C(0) = P(0) = 0,$ find the Laplace transform expressions for $S$, $C$, and $P$: $\mathscr{L}\{S\},\ \mathscr{L}\{C\}$, and $\mathscr{L}\{P\}$, respectively.

**b.** Use the final theorem to find $S(\infty),\ C(\infty)$, and $P(\infty)$.

**58.** Humans are able to stand on two legs through a complex feedback system that includes several sensory inputs—equilibrium and visual along with muscle actuation. In order to gain a better understanding of the workings of the postural feedback mechanism, an individual is asked to stand on a platform to which sensors are attached at the base. Vibration actuators are attached with straps to the individual's calves. As the vibration actuators are stimulated, the individual sways and movements are recorded. It was hypothesized that the human postural dynamics are analogous to those of a cart with a balancing standing pole attached (inverted pendulum). In that case, the dynamics can be described by the following two equations:

$$\begin{aligned} J\frac{d^2\theta}{dt^2} &= mgl\sin\theta(t) + T_{\text{bal}} + T_d(t) \\ T_{\text{bal}}(t) &= -mgl\sin\theta(t) + kJ\theta(t) - \eta J\,\dot{\theta}(t) \\ &\quad -\rho J\int_0^t \theta(t)dt \end{aligned}$$

where $m$ is the individual's mass; $l$ is the height of the individual's center of gravity; $g$ is the gravitational constant; $J$ is the individual's equivalent moment of inertia; $\eta$, $\rho$, and $k$ are constants given by the body's postural control system; $\theta(t)$ is the individual's angle with respect to a vertical line; $T_{\text{bal}}(t)$ is the torque generated by the body muscles to maintain balance; and $T_d(t)$ is the external torque input disturbance. Find the transfer function $\dfrac{\Theta(s)}{T_d(s)}$ (*Johansson, 1988*).

**59.** Figure P2.36 shows a crane hoisting a load. Although the actual system's model is highly nonlinear, if the rope is considered to be stiff with a fixed length $L$, the system can be modeled using the following equations:

$$\begin{aligned} m_L\ddot{x}_{La} &= m_L g\phi \\ m_T\ddot{x}_T &= f_T - m_L g\phi \\ x_{La} &= x_T - x_L \\ x_L &= L\phi \end{aligned}$$

where $m_L$ is the mass of the load, $m_T$ is the mass of the cart, $x_T$ and $x_L$ are displacements as defined in the figure, $\phi$ is the rope angle with respect to the vertical, and $f_T$ is the force applied to the cart (*Marttinen, 1990*).

**a.** Obtain the transfer function from cart velocity to rope angle $\dfrac{\Phi(s)}{V_T(s)}$.

**b.** Assume that the cart is driven at a constant velocity $V_0$ and obtain an expression for the resulting $\phi(t)$. Show that under this condition, the load will sway with a frequency $\omega_0 = \sqrt{\dfrac{g}{L}}$.

**c.** Find the transfer function from the applied force to the cart's position, $\dfrac{X_T(s)}{F_T(s)}$.

**d.** Show that if a constant force is applied to the cart, its velocity will increase without bound as $t \to \infty$.

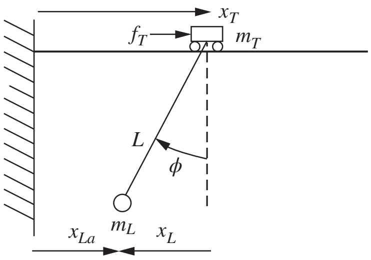


**FIGURE P2.36**[16]

**60.** In 1978, Malthus developed a model for human growth population that is also commonly used to model bacterial growth as follows. Let $N(t)$ be the population density observed at time $t$. Let $K$ be the rate of reproduction per unit time. Neglecting population deaths, the population density at a time $t + \Delta t$ (with small $\Delta t$) is given by

$$N(t+\Delta t) \approx N(t) + KN(t)\Delta t$$

which also can be written as

$$\frac{N(t+\Delta t) - N(t)}{\Delta t} = KN(t)$$

Since $N(t)$ can be considered to be a very large number, letting $\Delta t \to 0$ gives the following differential equation (*Edelstein-Keshet, 2005*):

$$\frac{dN(t)}{dt} = KN(t)$$

[16] Marttinen A., Virkkunen J., Salminen R.T. Control Study with Pilot Crane. *IEEE Transactions on Education*, Vol. 33, No.3, August 1990. Fig. 2. p. 300. IEEE Transactions on Education by Institute of Electrical and Electronics Engineers; IEEE Education Group; IEEE Education Society. Reproduced with permission of Institute of Electrical and Electronics Engineers, in the format Republish in a book via Copyright Clearance Center.

**a.** Assuming an initial population $N(0) = N_0$, solve the differential equation by finding $N(t)$.

**b.** Find the time at which the population is double the initial population.

**61.** In order to design an underwater vehicle that has the characteristics of both a long-range transit vehicle (torpedo-like) and a highly maneuverable low-speed vehicle (boxlike), researchers have developed a thruster that mimics that of squid jet locomotion (*Krieg, 2008*). It has been demonstrated there that the average normalized thrust due to a command step input, $U(s) = \frac{T_{ref}}{s}$ is given by:

$$T(t) = T_{ref}(1 - e^{-\lambda t}) + a\sin(2\pi f t)$$

where $T_{ref}$ is the reference or desired thrust, λ is the system's damping constant, $a$ is the amplitude of the oscillation caused by the pumping action of the actuator, $f$ is the actuator frequency, and $T(t)$ is the average resulting normalized thrust. Find the thruster's transfer function $\frac{T(s)}{U(s)}$. Show all steps.

**62.** The Gompertz growth model is commonly used to model tumor cell growth. Let $v(t)$ be the tumor's volume, then

$$\frac{dv(t)}{dt} = \lambda e^{-\alpha t} v(t)$$

where $\lambda$ and $\alpha$ are two appropriate constants (*Edelstein-Keshet, 2005*).

**a.** Verify that the solution to this equation is given by $v(t) = v_0 e^{\lambda/\alpha(1-e^{-\alpha t})}$, where $v_0$ is the initial tumor volume.

**b.** This model takes into account the fact that when nutrients and oxygen are scarce at the tumor's core, its growth is impaired. Find the final predicted tumor volume (let $t \to \infty$).

**c.** For a specific mouse tumor, it was experimentally found that $\lambda = 2.5$ days, $\alpha = 0.1$ days with $v_0 = 50 \times 10^{-3}$ mm$^3$ (*Chignola, 2005*). Use any method available to make a plot of $v(t)$ vs. $t$.

**d.** Check the result obtained in Part **b** with the results from the graph found in Part **c.**

**63.** A muscle hanging from a beam is shown in Figure P2.37($a$) (*Lessard, 2009*). The $\alpha$-motor neuron can be used to electrically stimulate the muscle to contract and pull the mass, $m$, which under static conditions causes the muscle to stretch. An equivalent mechanical system to this setup is shown in Figure P2.37($b$). The force $F_{iso}$ will be exerted when the muscle contracts. Find an expression for the displacement $X_1(s)$ in terms of $F_1(s)$ and $F_{iso}(s)$.

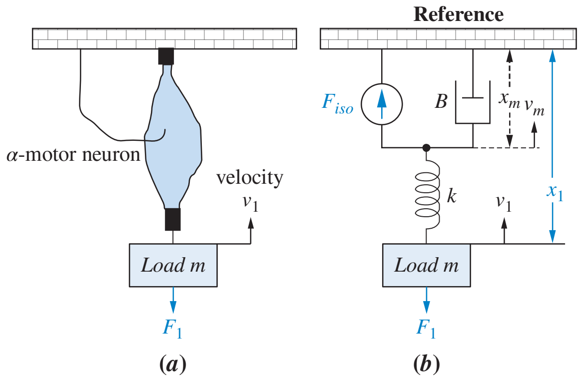


**FIGURE P2.37** **a.** Motor neuron stimulating a muscle;[17] **b.** equivalent circuit[18]

**64.** A three-phase ac/dc converter supplies dc to a battery charging system or dc motor (*Graovac, 2001*). Each phase has an ac filter represented by the equivalent circuit in Figure P2.38.

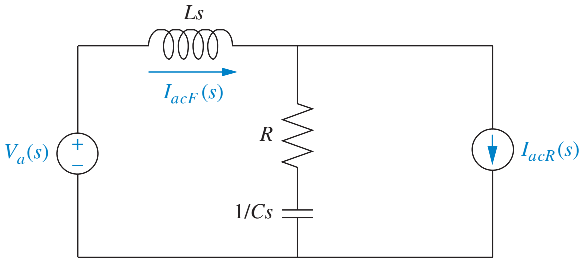


**FIGURE P2.38** AC filter equivalent circuit for a three-phase ac/dc converter

Derive that the inductor current in terms of the two active sources is

$$I_{acF}(s) = \frac{1 + RCs}{LCs^2 + RCs + 1} I_{acR}(s) + \frac{Cs}{LCs^2 + RCs + 1} V_a(s)$$

**65.** A photovoltaic system is used to capture solar energy to be converted to electrical energy. A control system is used to pivot the solar platform to track the sun's movements in order to maximize the captured energy. The system consists of a motor and load similar to that discussed in Section 2.8. A model has been proposed (*Agee, 2012*) that is different from the model developed in the chapter in the following ways: (1) the motor inductance was not neglected and (2) the load, in addition to having inertia and damping, has a spring. Find the transfer function, $\theta_m(s)/E_a(s)$, for this augmented system assuming all load impedances have already been reflected to the motor shaft.

[17] Lessard, C. D. *Basic Feedback Controls in Biomedicine*, Morgan & Claypool, San Rafael, CA, 2009. Figure 2.8, p. 12.
[18] Lessard, C. D. *Basic Feedback Controls in Biomedicine*, Morgan & Claypool, San Rafael, CA, 2009 Figure 2.9, p. 13.

**66.** In a paint mixing plant, two tanks supply fluids to a mixing cistern. The height, $h$, of the fluid in the cistern is dependent upon the difference between the input mass flow rate, $q$, and the output flow rate, $q_e$. A nonlinear differential equation describing this dependency is given by (*Schiop, 2010*)

$$\frac{dh}{dt} + \frac{A_e}{A}\sqrt{2gh} = \frac{q}{\rho A}$$

where $A$ = cross-sectional area of the cistern, $A_e$ = cross-sectional area of the exit pipe, $g$ = acceleration due to gravity, and $\rho$ = liquid density.

**a.** Linearize the nonlinear equation about the equilibrium point $(h_0, q_0)$ and find the transfer function relating the output cistern fluid level, $H(s)$, to the input mass flow rate, $Q(s)$.

**b.** The color of the liquid in the cistern can be kept constant by adjusting the input flow rate, $q$, assuming the input flow's color is specifically controlled. Assuming an average height, $h_{av}$, of the liquid in the cistern, the following equation relates the net flow of color to the cistern to the color in the cistern.

$$e_1 q - e q_e = \frac{d}{dt}(\rho A e h_{av})$$

where $e_1$ = fractional part of flow representing color into the cistern, and $e$ = fractional part of the cistern representing color in the cistern. Assume that the flow out of the cistern is constant and use the relationship, $q_e = \rho A_e \sqrt{2gh_{av}}$, along with the given equation above to find the transfer function, $E(s)/Q(s)$, that relates the color in the cistern to the input flow rate.

## PROGRESSIVE ANALYSIS AND DESIGN PROBLEMS

**67. Control of HIV/AIDS.** HIV inflicts its damage by infecting healthy CD4 + T cells (a type of white blood cell) that are necessary to fight infection. As the virus embeds in a T cell and the immune system produces more of these cells to fight the infection, the virus propagates in an opportunistic fashion. As we now develop a simple HIV model, refer to Figure P2.39. Normally T cells are produced at a rate $s$ and die at a rate $d$. The HIV virus is present in the bloodstream in the infected individual. These viruses in the bloodstream, called *free viruses*, infect healthy T cells at a rate $\beta$. Also, the viruses reproduce through the T cell multiplication process or otherwise at a rate $k$. Free viruses die at a rate $c$. Infected T cells die at a rate $\mu$.

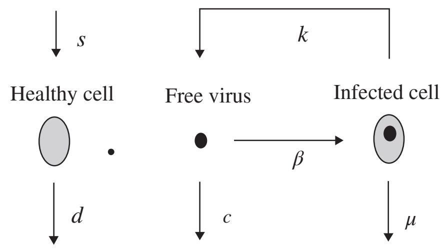


**FIGURE P2.39**[19]

A simple mathematical model that illustrates these interactions is given by the following equations (*Craig, 2004*):

$$\frac{dT}{dt} = s - dT - \beta T v$$

$$\frac{dT^*}{dt} = \beta T v - \mu T^*$$

$$\frac{dv}{dt} = kT^* - cv$$

where

$T$ = number of healthy T cells
$T^*$ = number of infected T cells
$v$ = number of free viruses

**a.** The system is nonlinear; thus linearization is necessary to find transfer functions as you will do in subsequent chapters. The nonlinear nature of this model can be seen from the above equations. Determine which of these equations are linear, which are nonlinear, and explain why.

**b.** The system has two equilibrium points. Show that these are given by

$$(T_0, T_0^*, v_0) = \left(\frac{s}{d}, 0, 0\right)$$

and

$$(T_0, T_0^*, v_0) = \left(\frac{c\mu}{\beta k}, \frac{s}{\mu} - \frac{cd}{\beta k}, \frac{sk}{c\mu} - \frac{d}{\beta}\right)$$

**68. Hybrid vehicle.** Problem 23 in Chapter 1 discusses the cruise control of serial, parallel, and split-power hybrid electric vehicles (HEVs). The functional block diagrams developed for these HEVs indicated that the speed of a

[19] Craig, I. K., Xia, X., and Venter, J. W. Introducing HIV/AIDS Education Into the Electrical Engineering Curriculum at the University of Pretoria. *IEEE Transactions on Education*, vol. 47, no. 1, February 2004, pp. 65–73. Fig. 1, p. 66. IEEE Transactions on Education by Institute of Electrical and Electronics Engineers; IEEE Education Group; IEEE Education Society. Reproduced with permission of Institute of Electrical and Electronics Engineers, in the format Republish in a book via Copyright Clearance Center.

vehicle depends upon the balance between the motive forces (developed by the gasoline engine and/or the electric motor) and running resistive forces. The resistive forces include the aerodynamic drag, rolling resistance, and climbing resistance. Figure P2.40 illustrates the running resistances for a car moving uphill (*Bosch, 2007*).

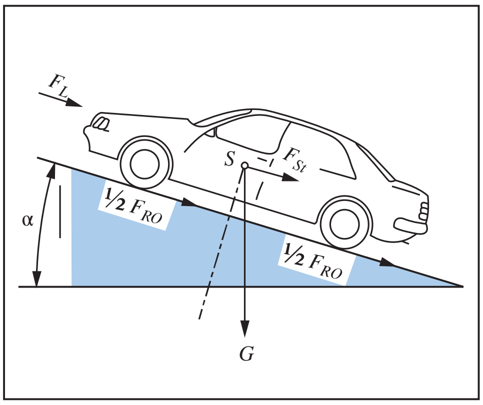


**FIGURE P2.40** Running resistances[20]

The total running resistance, $F_w$, is calculated as $F_w = F_{Ro} + F_L + F_{St}$, where $F_{Ro}$ is the rolling resistance, $F_L$ is the aerodynamic drag, and $F_{St}$ is the climbing resistance. The aerodynamic drag is proportional to the square of the sum of car velocity, $v$, and the head-wind velocity, $v_{hw}$, or $v + v_{hw}$. The other two resistances are functions of car weight, $G$, and the gradient of the road (given by the gradient angle, $\alpha$), as seen from the following equations:

$$F_{Ro} = fG\cos\alpha = fmg\cos\alpha$$

where

$$\begin{aligned} f &= \text{coefficient of rolling resistance} \\ m &= \text{car mass, in kg} \\ g &= \text{gravitational acceleration, in m/s}^2 \\ F_L &= 0.5\rho C_w A(v + v_{\text{hw}})^2. \end{aligned}$$

and

$$\begin{aligned} \rho &= \text{air density, in kg/m}^3 \\ C_w &= \text{coefficient of aerodynamic drag} \\ A &= \text{largest cross-section of the car, in kg/m}^2 \\ F_{St} &= G\sin\alpha = mg\sin\alpha. \end{aligned}$$

The motive force, $F$, available at the drive wheels is:

$$F = \frac{Ti_{tot}}{r}\eta_{tot} = \frac{P\eta_{tot}}{v}$$

where

$$\begin{aligned} T &= \text{motive torque} \\ P &= \text{motive power} \\ i_{tot} &= \text{total transmission ratio} \\ r &= \text{tire radius} \\ \eta_{tot} &= \text{total drive-train efficiency.} \end{aligned}$$

The surplus force, $F - F_w$, accelerates the vehicle (or retards it when $F_w > F$). Letting $a = \dfrac{F - F_w}{k_m \cdot m}$, where $a$ is the acceleration and $k_m$ is a coefficient that compensates for the apparent increase in vehicle mass due to rotating masses (wheels, flywheel, crankshaft, etc.):

**a.** Show that car acceleration,[21] $a$, may be determined from the equation:

$$F = fmg\cos\alpha + mg\sin\alpha + 0.5\rho C_w A(v + v_{hw})^2 + k_m ma$$

**b.** Assuming constant acceleration and using the average value for speed, find the average motive force, $F_{av}$ (in N), and power, $P_{av}$ (in kW) the car needs to accelerate from 40 to 60 km/h in 4 seconds on a level road, ($\alpha = 0°$), under windless conditions, where $v_{hw} = 0$. You are given the following parameters: m = 1590 kg, A = 2 m$^2$, $f = 0.011$, $\rho = 1.2$ kg/m$^3$, $C_w = 0.3$, $\eta_{tot} = 0.9$, $k_m = 1.2$. Furthermore, calculate the additional power, $P_{add}$, the car needs after reaching 60 km/h to maintain its speed while climbing a hill with a gradient $\alpha = 5°$.

**c.** The equation derived in Part **a** describes the non-linear car motion dynamics where $F(t)$ is the input to the system, and $v(t)$ the resulting output. Given that the aerodynamic drag is proportional to $v^2$ under windless conditions, linearize the resulting equation of motion around an average speed, $v_o = 50$ km/h, when the car travels on a level road,[22] where $\alpha = 0°$. (Hint: Expand $v^2 - v_0^2$ in a truncated Taylor series). Write that equation of motion and represent it with a block diagram in which the block $G_v$ represents the vehicle dynamics. The output of that block is the car speed, $v(t)$, and the input is the excess motive force, $F_e(t)$, defined as: $F_e = F - F_{St} - F_{Ro} + F_o$, where $F_o$ is the constant component of the linearized aerodynamic drag.

**d.** Use the equation in Part **c** to find the vehicle transfer function: $G_v(s) = V(s)/F_e(s)$.

**69. Parabolic trough collector**. In a significant number of cases, the open-loop transfer function from fluid flow to

[20] Robert Bosch GmbH, *Bosch Automotive Handbook*, 7th ed. John Wiley & Sons Ltd. UK, 2007. P. 430. Figure at bottom left.

[21] Other quantities, such as top speed, climbing ability, etc., may also be calculated by manipulation from that equation.

[22] Note that on a level road the climbing resistance, $F_{st} = 0$, since $\sin\alpha = \sin 0° = 0$.

fluid temperature in a parabolic trough collector can be approximated (*Camacho*, *2012*) by:

$$P(s) = \frac{K}{1 + \tau s} e^{-sT}$$

**a.** Write an analytic expression for the unit step response of the open-loop system assuming that $h(t)$ represents the output temperature and $q(t)$ the input fluid flow.

**b.** Make a sketch of the unit step response of the open-loop system. Indicate on your figure the time delay, the settling time, the initial and final values of the response, and the value of the response when $t = \tau + T$.

**c.** Call the output temperature $h(t)$ and the input fluid flow $q(t)$. Find the differential equation that represents the open-loop system.

# Cyber Exploration Laboratory

## Experiment 2.1

**Objectives** To learn to use MATLAB to (1) generate polynomials, (2) manipulate polynomials, (3) generate transfer functions, (4) manipulate transfer functions, and (5) perform partial-fraction expansions.

**Minimum Required Software Packages** MATLAB and the Control System Toolbox

### Prelab

1. Calculate the following by hand or with a calculator:
   - **a.** The roots of $P_1 = s^6 + 7s^5 + 2s^4 + 9s^3 + 10s^2 + 12s + 15$
   - **b.** The roots of $P_2 = s^6 + 9s^5 + 8s^4 + 9s^3 + 12s^2 + 15s + 20$
   - **c.** $P_3 = P_1 + P_2;\ P_4 = P_1 - P_2;\ P_5 = P_1P_2$
2. Calculate by hand or with a calculator the polynomial

$$P_6 = (s+7)(s+8)(s+3)(s+5)(s+9)(s+10)$$

3. Calculate by hand or with a calculator the following transfer functions:
   - **a.** $G_1(s) = \dfrac{20(s+2)(s+3)(s+6)(s+8)}{s(s+7)(s+9)(s+10)(s+15)}$,

     represented as a numerator polynomial divided by a denominator polynomial.
   - **b.** $G_2(s) = \dfrac{s^4 + 17s^3 + 99s^2 + 223s + 140}{s^5 + 32s^4 + 363s^3 + 2092s^2 + 5052s + 4320}$,

     expressed as factors in the numerator divided by factors in the denominator, similar to the form of $G_1(s)$ in Prelab 3**a.**
   - **c.** $G_3(s) = G_1(s) + G_2(s);\ G_4(s) = G_1(s) - G_2(s);\ G_5(s) = G_1(s)G_2(s)$

     expressed as factors divided by factors and expressed as polynomials divided by polynomials.
4. Calculate by hand or with a calculator the partial-fraction expansion of the following transfer functions:
   - **a.** $G_6 = \dfrac{5(s+2)}{s(s^2 + 8s + 15)}$
   - **b.** $G_7 = \dfrac{5(s+2)}{s(s^2 + 6s + 9)}$
   - **c.** $G_8 = \dfrac{5(s+2)}{s(s^2 + 6s + 34)}$