cart spring pendulum systems

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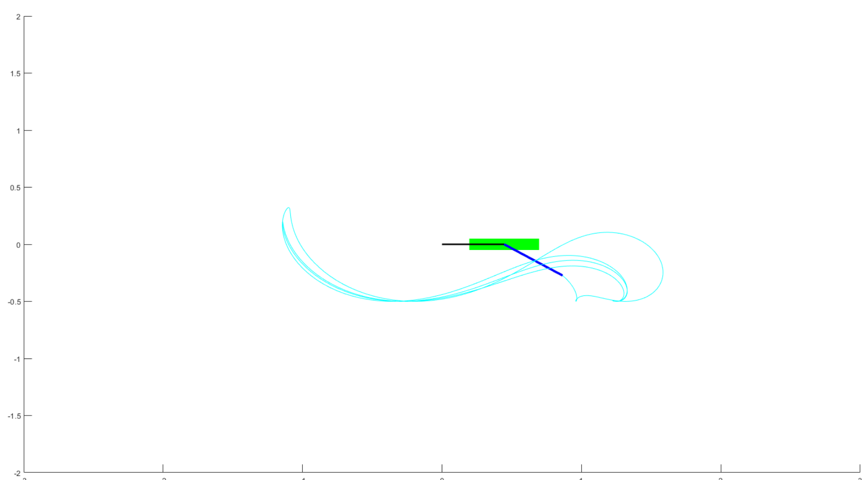
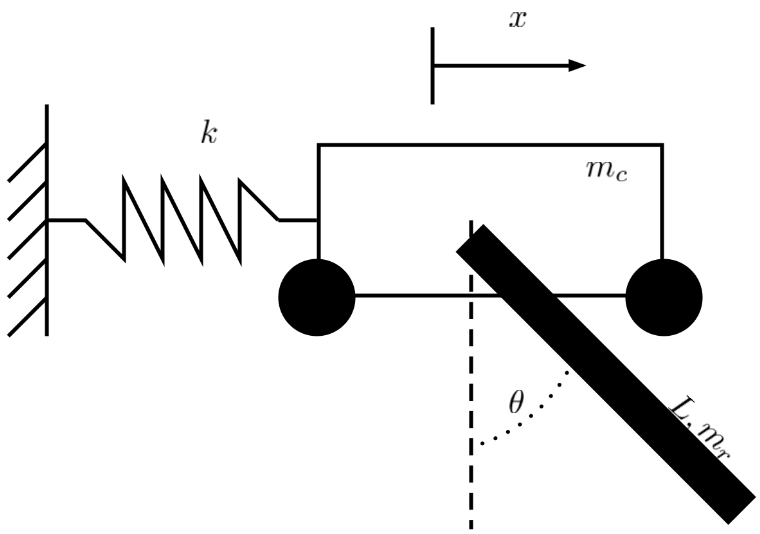
1 spring-cart-pendulum system



1.1 diagram



Also note that:
$k = 10N/m$
$L = 0.5m$
$m_c = 1kg$
$m_r = 0.25kg$

1.2 solve for EOM

LaTeX typed derivation here, see end for the picture of the actual hand written notes.

$$Y_G = \frac{L}{2}$$

$$I_G = \frac{1}{12}mL^2$$

$$r_G = (x + \frac{L}{2}s\theta)\vec{i} + (-\frac{1}{2}Lc\theta)\vec{j}$$

$$\dot{r}_G = (\dot{x} + \frac{L}{2}c\theta\dot{\theta})\vec{i} + (\frac{L}{2}s\theta\dot{\theta})\vec{j}$$

note that :
$T_{total} = T_{cart} + T_{rod}$
so we first find T_{rod}

$$T_{rod} = \frac{1}{2}m_r\dot{r} \cdot \dot{r} + \frac{1}{2}I_G\dot{\theta}^2$$

$$\dot{r}_G \cdot \dot{r}_G = (\dot{x} + \frac{L}{2}c\theta\dot{\theta})(\dot{x} + \frac{L}{2}c\theta\dot{\theta}) + (\frac{L}{2}s\theta\dot{\theta})(\frac{L}{2}s\theta\dot{\theta}) = \dot{x}^2 + Lc\theta\dot{x}\dot{\theta} + \frac{L^2}{4}c^2\theta\dot{\theta}^2 + \frac{L^2}{4}s^2\theta\dot{\theta}^2$$

$$\dot{r}_G \cdot \dot{r}_G = \dot{x}^2 + Lc\theta\dot{x}\dot{\theta} + \frac{L^2}{4}\dot{\theta}^2$$

$$\frac{1}{2}I_G\dot{\theta}^2 = \frac{1}{2}(\frac{1}{12}m_rL^2)\dot{\theta}^2 = \frac{1}{24}m_rL^2\dot{\theta}^2$$

Let's solve for kinetic energy expression of the rod

$$T_{rod} = \frac{1}{2}m_r\dot{x}^2 + \frac{1}{2}c\theta Lm_r\dot{\theta} + \frac{1}{24}L^2m_r\dot{\theta}^2 + \frac{1}{8}c^2\theta L^2m_r\dot{\theta}^2 + \frac{1}{8}s^2\theta L^2m_r\dot{\theta}^2 = \frac{1}{6}m_r(3\dot{x}^2 + 3Lc\theta\dot{x}\dot{\theta} + L^2\dot{\theta}^2)$$

And note that the kinetic energy expression of the cart is

$$T_{cart} = \frac{1}{2}m_c\dot{x}^2$$

So we can solve for T_{total}

$$T_{total} = T_{cart} + T_{rod} = \frac{1}{2}m_c\dot{x}^2 + \frac{1}{6}m_r(3\dot{x}^2 + 3Lc\theta\dot{x}\dot{\theta} + L^2\dot{\theta}^2)$$

Now we have to solve for V

$$V = \frac{1}{2}kx^2 + \frac{1}{2}m_rgL(1 - c\theta) = \frac{1}{2}kx^2 + \frac{1}{2}m_rgL - \frac{1}{2}m_rgLc\theta$$

by lagrange's eqn, we have:

$$\frac{d}{dt}(\frac{dT}{d\dot{q}_k}) - \frac{dT}{dq_k} + \frac{dV}{dq_k} = 0$$

So, we need expressions for $\frac{dT}{dx}$, $\frac{dT}{d\dot{x}}$, $\frac{dV}{dx}$, $\frac{dT}{d\theta}$, $\frac{dT}{d\dot{\theta}}$, $\frac{dV}{d\theta}$ to construct our two EOM

$$\frac{dT}{dx} = 0$$

$$\frac{dT}{d\dot{x}} = m_r\dot{x} + m_c\dot{x}\frac{1}{2}c\theta Lm_r\dot{\theta} = (m_c + m_r)\dot{x} + m_c\dot{x}\frac{1}{2}c\theta Lm_r\dot{\theta}$$

$$\frac{dV}{dx} = kx$$

$$\frac{d}{dt}(\frac{dT}{d\dot{x}}) = \frac{d}{dt}((m_c + m_r)\dot{x} + m_c\dot{x}\frac{1}{2}c\theta Lm_r\dot{\theta}) = -\frac{1}{2}s\theta Lm_r\dot{\theta}^2 + (m_c + m_r)\ddot{x} + \frac{1}{2}c\theta Lm_r\ddot{\theta}$$

EOM1:

$$\frac{d}{dt}(\frac{dT}{d\dot{x}}) - \frac{dT}{dx} + \frac{dV}{dx} = 0$$

$$-\frac{1}{2}s\theta Lm_r\dot{\theta}^2 + (m_c + m_r)\ddot{x} + \frac{1}{2}c\theta Lm_r\ddot{\theta} - 0 + kx = 0$$

$$\boxed{(m_c + m_r)\ddot{x} + (\frac{1}{2}c\theta Lm_r)\ddot{\theta} - \frac{1}{2}s\theta Lm_r\dot{\theta}^2 + kx = 0}$$

$$\frac{dT}{d\theta} = -\frac{1}{2}m_rL\dot{\theta}\dot{x}s\theta$$

$$\frac{dT}{d\dot{\theta}} = \frac{1}{2}m_rLc\theta\dot{x} + \frac{1}{3}m_rL^2\dot{\theta}$$

$$\frac{dV}{d\theta} = \frac{1}{2}m_rgLs\theta$$

$$\frac{d}{dt}(\frac{dT}{d\dot{\theta}}) = \frac{d}{dt}(\frac{1}{2}m_rLc\theta\dot{x} + \frac{1}{3}m_rL^2\dot{\theta}) = -\frac{1}{2}s\theta Lm_r\dot{x}\dot{\theta} + \frac{1}{2}c\theta Lm_r\ddot{x} + \frac{1}{3}L^2m_r\ddot{\theta}$$

EOM2:

$$\frac{d}{dt}(\frac{dT}{d\dot{\theta}}) - \frac{dT}{d\theta} + \frac{dV}{d\theta} = 0$$

$$\cancel{-\frac{1}{2}Lm_r\dot{x}\dot{\theta}s\theta} + \frac{1}{2}c\theta Lm_r\ddot{x} + \frac{1}{3}L^2m_r\ddot{\theta} + \cancel{\frac{1}{2}m_rL\dot{\theta}\dot{x}s\theta} + \frac{1}{2}m_rgLs\theta = 0$$

$$(\frac{1}{2}c\theta Lm_r)\ddot{x} + (\frac{1}{3}L^2m_r)\ddot{\theta} + \frac{1}{2}m_rgLs\theta = 0$$

$$\boxed{(\frac{1}{2}c\theta)\ddot{x} + (\frac{1}{3}L)\ddot{\theta} + \frac{1}{2}gs\theta = 0}$$

1.3 MATLAB code

1.3.1 dynamics_fxn

```
function [dydt] = dynamics_fxn(y,mr,mc,L,k)
%spring-cart-pendulum problem EOM; dydt = m\D
M = [mr+mc,          .5*mr*L*cos(y(3));...
     .5*cos(y(3)),       (1/3)*L];
D = [.5*mr*L*(y(4)^2)*sin(y(3))-k*y(1);....
     -.5*9.81*sin(y(3))];
dydt = M\D;
end
```

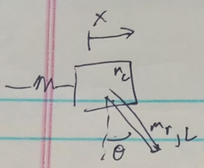
1.3.2 cart_sim

```
clear all

mr = .25;       %mass of the rod
mc = 1;         %mass of the cart
L = .5;         %length of the rod
k = 10;         %spring constant

y1 = 1;         %x
y2 = 0;         %xdot
y3 = 0*pi/180;  %theta
y4 = 0;         %thetadot

deltaTime = .01;
tmax = 5;

y(1,:) = [y1,y2,y3,y4];

%4th order Runge-Kutta, could also use euler's method (1st order RK)
%following example from: http://lpsa.swarthmore.edu/NumInt/NumIntFourth.html
for t=1:(tmax/deltaTime+1)

    YDD1(t,:) = dynamics_fxn(y(t,:),mr,mc,L,k);
    k1(t,:) = [y(t,2),YDD1(t,1),y(t,4),YDD1(t,2)];

    YDD2(t,:) = dynamics_fxn(y(t,:)+k1(t,:)*(deltaTime/2),mr,mc,L,k);
    k2(t,:) = [y(t,2),YDD2(t,1),y(t,4),YDD2(t,2)];

    YDD3(t,:) = dynamics_fxn(y(t,:)+k2(t,:)*(deltaTime/2),mr,mc,L,k);
    k3(t,:) = [y(t,2),YDD3(t,1),y(t,4),YDD3(t,2)];

    YDD4(t,:) = dynamics_fxn(y(t,:)+k3(t,:)*(deltaTime),mr,mc,L,k);
    k4(t,:) = [y(t,2),YDD4(t,1),y(t,4),YDD4(t,2)];

    y(t+1,:) = y(t,:) + (deltaTime/6)*(k1(t,:)+2*k2(t,:)+2*k3(t,:)+k4(t,:));

end
figure( 'units','normalized','outerposition',[0 0 1 1])
for t=1:(tmax/deltaTime+1)
    clf
    hold on
    tic
    cartStart = [y(t,1),0];
    cartEnd = cartStart + [.5,0];
    pendStart = cartStart + [.25,0];
    pendEnd = pendStart +L*[sin(y(t,3)),-cos(y(t,3))];
    pendEndStore(:,t) = pendEnd;
```

```
    line([cartStart(1), cartEnd(1)],[cartStart(2), cartEnd(2)],'Color','green', 'linewidth', 15);
    line([0,pendStart(1)],[0, pendStart(2)],'Color','black', 'linewidth', 2);
    line([pendStart(1), pendEnd(1)],[pendStart(2), pendEnd(2)],'Color','blue', 'linewidth', 3);
    plot(pendEndStore(1,1:end),pendEndStore(2,1:end),'Color','cyan');
    %legend('cart', 'rod 1', 'rod 2', 'spring', 'trajectory') %legend is too
    %resource intensive to plot in real time this way
    frameTime = toc;
    axis([-3,3,-2,2]);
    if (deltaTime-frameTime)>0
        pause((deltaTime-frameTime)) %pause so the sim plays back in real time
    else
        pause(10^-10)%in case the deltaTime is too small to play back in real time just print to screen anyway
    end

end
```

1.4 on paper

1



$Y_G = \frac{L}{2}$, $I_G = \frac{1}{12} m L^2$

$$r_G = (x + \tfrac{L}{2} s\theta)\hat{i} + (-\tfrac{1}{2} L c\theta)\hat{j}$$

$$\dot{r}_G = (\dot{x} + \tfrac{L}{2} c\theta\dot{\theta})\hat{i} + (\tfrac{L}{2} s\theta\dot{\theta})\hat{j}$$

$$T_{rod} = \frac{1}{2} m_r \dot{r}\cdot\dot{r} + \frac{1}{2} I_G \dot{\theta}^2$$

$$\dot{r}_G\cdot\dot{r}_G = (\dot{x} + \tfrac{L}{2} c\theta\dot{\theta})(\dot{x} + \tfrac{L}{2} c\theta\dot{\theta}) + (\tfrac{L}{2} s\theta\dot{\theta})(\tfrac{L}{2} s\theta\dot{\theta})$$

$$= \dot{x}^2 + L c\theta\dot{\theta}\dot{x} + \frac{L^2}{4} c^2\theta\dot{\theta}^2 + \frac{L^2}{4} s^2\theta\dot{\theta}^2$$

$$= \dot{x}^2 + L c\theta\dot{\theta}\dot{x} + \frac{L^2}{4}\dot{\theta}^2$$

$$T_{rod} = \frac{1}{2} m_r \dot{r}\cdot\dot{r} + \frac{1}{2} I_G \dot{\theta}^2$$

$$= \frac{1}{2} m_r\left(\dot{x}^2 + L c\theta\dot{\theta}\dot{x} + \frac{L^2}{4}\dot{\theta}^2\right) + \frac{1}{2}\left(\frac{1}{12} m_r L^2\right)\dot{\theta}^2$$

$$= \frac{1}{2} m_r \dot{x}^2 + \frac{1}{2} m_r L c\theta\dot{\theta}\dot{x} + \frac{L^2}{8} m_r \dot{\theta}^2 + \frac{1}{24} m_r L^2 \dot{\theta}^2$$

$$= \frac{1}{2} m_r \dot{x}^2 + \frac{1}{2} m_r L c\theta\dot{\theta}\dot{x} + \frac{1}{6} m_r L^2 \dot{\theta}^2$$

$$T_{cart} = \frac{1}{2} m_c \dot{x}^2$$

$$T_{total} = T_{rod} + T_{cart} = \frac{1}{2} m_r \dot{x}^2 + \frac{1}{2} m_r L c\theta\dot{\theta}\dot{x} + \frac{1}{6} m_r L^2\dot{\theta} + \frac{1}{2} m_c \dot{x}^2$$

$$= \frac{1}{6} m_r\left(3\dot{x}^2 + 3L c\theta\dot{x}\dot{\theta} + L^2\dot{\theta}^2\right)$$

②

$$\frac{\partial}{\partial t}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k} + \frac{\partial V}{\partial q_k} = 0$$

$$\frac{\partial T}{\partial x} = 0$$

$$\frac{\partial T}{\partial \dot{x}} = m_r \dot{x} + m_c \dot{x} \tfrac{1}{2} c\theta L m_r \dot{\theta} = (m_c + m_r)\dot{x} + m_c \dot{x} \tfrac{1}{2} c\theta L m_r \dot{\theta}$$

$$\frac{\partial V}{\partial x} = kx$$

$$\frac{\partial}{\partial t}\left(\frac{\partial T}{\partial \dot{x}}\right) = \frac{\partial}{\partial t}\left((m_c + m_r)\dot{x} + \tfrac{1}{2} L m_r c\dot{x}\dot{\theta}\right) = (m_c + m_r)\ddot{x} + \tfrac{1}{2} L m_r m_c \dot{x}\dot{\theta} + \tfrac{1}{2} L m_r m_c \dot{x}\ddot{\theta}$$

$$= (m_r + m_c)\ddot{x} - \tfrac{1}{2} L m_r s\theta\, \dot{\theta}^2 + \tfrac{1}{2} L m_r c\theta\, \ddot{\theta}$$

$$(m_r + m_c)\ddot{x} - \tfrac{1}{2} L m_r s\theta\, \dot{\theta}^2 + \tfrac{1}{2} L m_r c\theta\, \ddot{\theta} - 0 + kv = 0$$

EOM$_1$

$$(m_c + m_r)\ddot{x} + \left(\tfrac{1}{2} c\theta L m_r\right)\ddot{\theta} + \tfrac{1}{2} s\theta L m_r \dot{\theta}^2 + kx = 0$$

$$\frac{\partial T}{\partial \theta} = -\tfrac{1}{2} m_r L \dot{\theta} \dot{x} s\theta$$

$$\frac{\partial T}{\partial \dot{\theta}} = \tfrac{1}{2} m_r L c\theta \dot{x} + \tfrac{1}{3} m_r L^2 \dot{\theta}$$

$$\frac{\partial V}{\partial \theta} = \tfrac{1}{2} m_r g L s\theta$$

$$\frac{\partial}{\partial t}\left(\frac{\partial T}{\partial \dot{\theta}}\right) = \frac{\partial}{\partial t}\left(\tfrac{1}{2} m_r L c\theta \dot{x} + \tfrac{1}{3} m_r L^2 \dot{\theta}\right) = \tfrac{1}{2} m_r L c\theta \ddot{x} - \tfrac{1}{2} m_r L s\theta \dot{\theta} \dot{x} + \tfrac{1}{3} m_r L^2 \ddot{\theta}$$

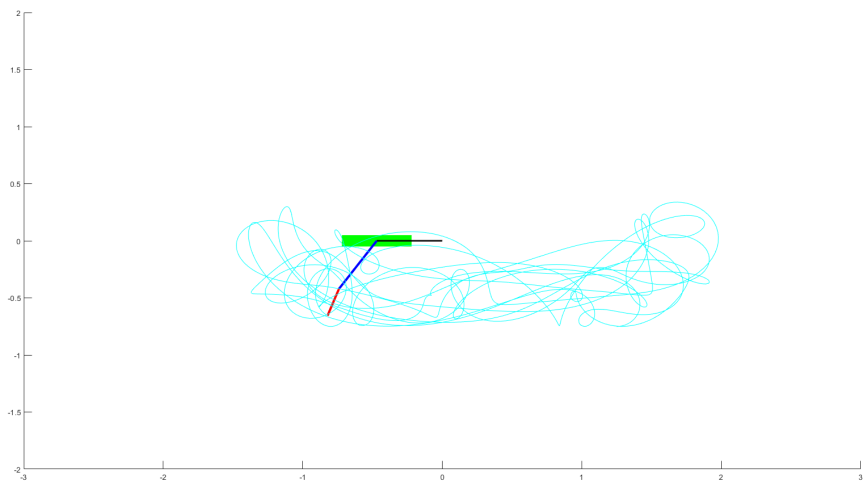
$$\cancel{-\tfrac{1}{2} m_r L s\theta \dot{\theta} \dot{x}} + \tfrac{1}{3} m_r L^2 \ddot{\theta} + \tfrac{1}{2} m_r L c\theta \ddot{x} + \cancel{\tfrac{1}{2} m_r L s\theta \dot{x} \dot{\theta}} + \tfrac{1}{2} m_r g L s\theta = 0$$

$$\tfrac{1}{2} \cancel{m_r L} c\theta \ddot{x} + \tfrac{1}{3} \cancel{m_r L} L \ddot{\theta} + \tfrac{1}{2} \cancel{m_r} g \cancel{L} s\theta = 0$$

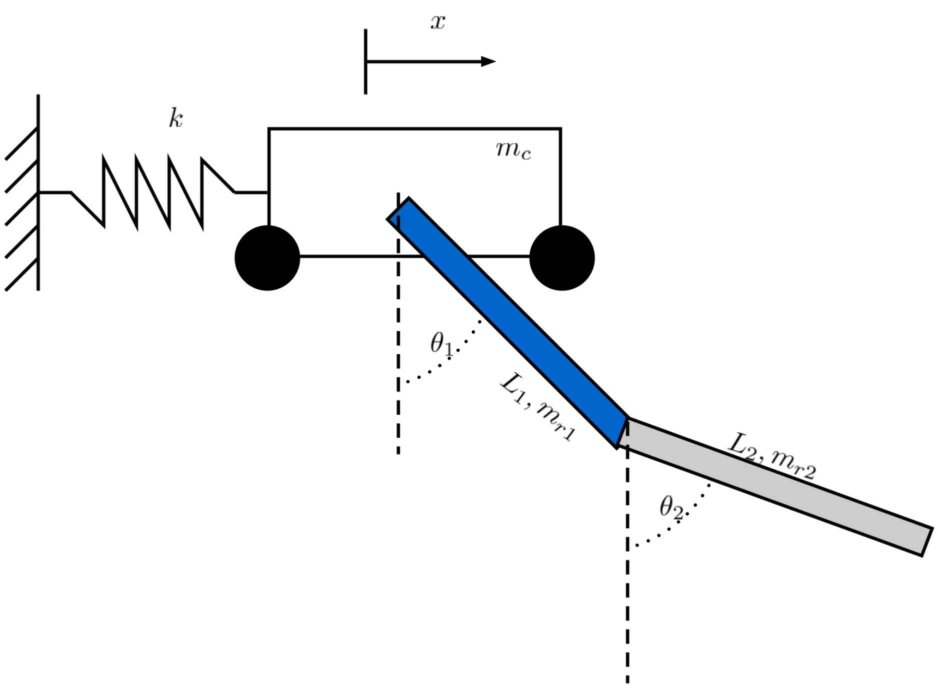
EOM$_2$

$$\tfrac{1}{2} c\theta \ddot{x} + \tfrac{1}{3} L \ddot{\theta} + \tfrac{1}{2} g s\theta = 0$$

2 spring-cart-double pendulum system



2.1 diagram



2.2 solve for EOM

LaTeX typed derivation here, see end for the picture of the actual hand written notes.

$$Y_{G1} = \frac{L_1}{2}$$

$$Y_{G2} = \frac{L_2}{2}$$

$$I_{G1} = \frac{1}{12} m L_1^2$$

$$I_{G2} = \frac{1}{12} m L_2^2$$

$$r_{G1} = (x + \frac{L_1}{2} s\theta_1)\vec{i} + (-\frac{1}{2} L_1 c\theta_1)\vec{j}$$

$$\dot{r}_{G1} = (\dot{x} + \frac{L_1}{2} c\theta_1\dot{\theta}_1)\vec{i} + (\frac{L_1}{2} s\theta_1\dot{\theta}_1)\vec{j}$$

$$r_{G2} = (x + \frac{L_1}{2} s\theta_1 + \frac{L_2}{2} s\theta_2)\vec{i} + (-\frac{1}{2} L_1 c\theta_1 - \frac{1}{2} L_2 c\theta_2)\vec{j}$$

$$\dot{r}_{G2} = (\dot{x} + \frac{L_1}{2} c\theta_1\dot{\theta}_1 + \frac{L_2}{2} c\theta_2\dot{\theta}_2)\vec{i} + (\frac{L_1}{2} s\theta_1\dot{\theta}_1 + \frac{L_2}{2} s\theta_2\dot{\theta}_2)\vec{j}$$

so we first find T_{rod_1}

$$T_{rod1} = \frac{1}{2} m_{r1} \dot{r}_{G1} \cdot \dot{r}_{G1} + \frac{1}{2} I_{G1} \dot{\theta_1}^2$$

$$\dot{r}_{G1} \cdot \dot{r}_{G1} = (\dot{x} + \frac{L}{2} c\theta\dot{\theta})(\dot{x} + \frac{L_1}{2} c\theta_1\dot{\theta}_1) + (\frac{L_1}{2} s\theta_1\dot{\theta}_1)(\frac{L}{2} s\theta_1\dot{\theta}_1) = \dot{x}^2 + L_1 c\theta_1 \dot{x}\dot{\theta}_1 + \frac{L_1^2}{4} c^2\theta_1 \dot{\theta}_1^2 + \frac{L_1^2}{4} s^2\theta_1 \dot{\theta}_1^2$$

$$\dot{r}_{G1} \cdot \dot{r}_{G1} = \dot{x}^2 + L_1 c\theta_1 \dot{x}\dot{\theta}_1 + \frac{L_1^2}{4} \dot{\theta}_1^2$$

$$\frac{1}{2} I_{G1} \dot{\theta_1}^2 = \frac{1}{2} (\frac{1}{12} m_{r1} L_1^2) \dot{\theta_1}^2 = \frac{1}{24} m_{r1} L_1^2 \dot{\theta}_1^2$$

$$T_{rod_1} = \frac{1}{2} m_{r1} \dot{x}^2 + \frac{1}{2} c\theta_1 L_1 m_1 \dot{x}\dot{\theta} + \frac{1}{8} m_{r1} L_1^2 \dot{\theta}^2 + \frac{1}{24} m_{r1} L_1^2 \dot{\theta}_1^2 = \frac{1}{2} m_{r1} \dot{x}^2 + \frac{1}{2} c\theta_1 L_1 m_1 \dot{x}\dot{\theta} + \frac{1}{6} m_{r1} L_1^2 \dot{\theta}_1^2$$

Next we need to find T_{rod_2}

$$T_{rod2} = \frac{1}{2} m_{r2} \dot{r}_{G2} \cdot \dot{r}_{G2} + \frac{1}{2} I_{G2} \dot{\theta_2}^2$$

$$\dot{r}_{G2} \cdot \dot{r}_{G2} = (\dot{x} + \frac{L_1}{2} c\theta_1\dot{\theta}_1 + \frac{L_2}{2} c\theta_2\dot{\theta}_2)(\dot{x} + \frac{L_1}{2} c\theta_1\dot{\theta}_1 + \frac{L_2}{2} c\theta_2\dot{\theta}_2) + (-\frac{1}{2} L_1 c\theta_1 - \frac{1}{2} L_2 c\theta_2)(-\frac{1}{2} L_1 c\theta_1 - \frac{1}{2} L_2 c\theta_2)$$

$$\dot{r}_{G2} \cdot \dot{r}_{G2} = \dot{x}^2 + c\theta_1 L_1 \dot{\theta}_1 \dot{x} + \frac{1}{4} c^2\theta_1 L_1^2 \dot{\theta}_1^2 + \frac{1}{4} s^2\theta_1 L_1^2 \dot{\theta}_1^2 + c\theta_2 L_2 \dot{x}\dot{\theta}_2 + \frac{1}{2} c\theta_1 c\theta_2 L_1 L_2 \dot{\theta}_1\dot{\theta}_2 + \frac{1}{2} s\theta_1 s\theta_2 L_1 L_2 \dot{\theta}_1\dot{\theta}_2 + \frac{1}{4} c^2\theta_2 L_2^2 \dot{\theta}_2^2 + \frac{1}{4} s^2\theta_2 L_2^2 \dot{\theta}_2^2$$

$$\frac{1}{2} m_{r2} \cdot \dot{r}_{G2} \cdot \dot{r}_{G2} =$$

$$(\frac{1}{2} m_{r2} \dot{x}^2 + \frac{1}{4} m_{r2} c\theta_1 L_1 \dot{\theta}_1 \dot{x} + \frac{1}{8} m_{r2} c^2\theta_1 L_1^2 \dot{\theta}_1^2 + \frac{1}{8} m_{r2} s^2\theta_1 L_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_{r2} c\theta_2 L_2 \dot{x}\dot{\theta}_2$$
$$+ \frac{1}{4} m_{r2} c\theta_1 c\theta_2 L_1 L_2 \dot{\theta}_1\dot{\theta}_2 + \frac{1}{4} m_{r2} s\theta_1 s\theta_2 L_1 L_2 \dot{\theta}_1\dot{\theta}_2 + \frac{1}{8} m_{r2} c^2\theta_2 L_2^2 \dot{\theta}_2^2 + \frac{1}{8} m_{r2} s^2\theta_2 L_2^2 \dot{\theta}_2^2)$$

$$I_{G2} = \frac{1}{12} m_{r2} L_2^2$$

$$\frac{1}{2} I_{G2} \dot{\theta_2}^2 = \frac{1}{24} m_{r2} L_2^2 \dot{\theta_2}^2$$

$$T_{rod2} = \frac{1}{2} m_{r2} \dot{r}_{G2} \cdot \dot{r}_{G2} + \frac{1}{2} I_{G2} \dot{\theta_2}^2$$

$$T_{rod2} =$$

$$(\frac{1}{2} m_{r2} \dot{x}^2 + \frac{1}{4} m_{r2} c\theta_1 L_1 \dot{\theta}_1 \dot{x} + \frac{1}{8} m_{r2} c^2\theta_1 L_1^2 \dot{\theta}_1^2 + \frac{1}{8} m_{r2} s^2\theta_1 L_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_{r2} c\theta_2 L_2 \dot{x}\dot{\theta}_2$$
$$+ \frac{1}{4} m_{r2} c\theta_1 c\theta_2 L_1 L_2 \dot{\theta}_1\dot{\theta}_2 + \frac{1}{4} m_{r2} s\theta_1 s\theta_2 L_1 L_2 \dot{\theta}_1\dot{\theta}_2 + \frac{1}{8} m_{r2} c^2\theta_2 L_2^2 \dot{\theta}_2^2 + \frac{1}{8} m_{r2} s^2\theta_2 L_2^2 \dot{\theta}_2^2)$$

$$+\frac{1}{24}m_{r2}L_2^2\dot{\theta_2}^2$$

$$T_{rod2}=\frac{1}{2}m_{r2}\dot{x}^2+\frac{1}{2}c\theta_1L_1m_{r2}\dot{x}\dot{\theta_1}+\frac{1}{8}L_1^2m_{r2}\dot{\theta_1}^2+\frac{1}{2}c\theta_2L_2m_{r2}\dot{x}\dot{\theta_2}+\frac{1}{4}c(\theta_1-\theta_2)L_1L_2m_{r2}\dot{\theta_1}\dot{\theta_2}+\frac{1}{6}L_2^2m_{r2}\dot{\theta_2}^2$$

Also we know T_{cart}

$$T_{cart}=\frac{1}{2}m_c\dot{x}^2$$

note that :
$T_{total}=T_{cart}+T_{rod_1}+T_{rod_2}$
So we can solve for T_{total}

$$T_{total}=\frac{1}{2}m_c\dot{x}^2+\frac{1}{2}m_{r1}\dot{x}^2+\frac{1}{2}c\theta_1L_1m_1\dot{x}\dot{\theta}+\frac{1}{6}m_{r1}L_1^2\dot{\theta}^2+\frac{1}{2}m_{r2}\dot{x}^2+\frac{1}{2}c\theta_1L_1m_{r2}\dot{x}\dot{\theta_1}$$

$$+\frac{1}{8}L_1^2m_{r2}\dot{\theta_1}^2+\frac{1}{2}c\theta_2L_2m_{r2}\dot{x}\dot{\theta_2}+\frac{1}{4}c(\theta_1-\theta_2)L_1L_2m_{r2}\dot{\theta_1}\dot{\theta_2}+\frac{1}{6}L_2^2m_{r2}\dot{\theta_2}^2$$

Now we have to solve for V

$$V=\frac{1}{2}kx^2+\frac{1}{2}m_{r1}gL_1(1-c\theta_1)+\frac{1}{2}m_{r2}g(L_1(1-c\theta_1)+L_2(1-c\theta_2))$$

by lagrange's eqn, we have:

$$\frac{d}{dt}(\frac{dT}{d\dot{q}_k})-\frac{dT}{dq_k}+\frac{dV}{dq_k}=0$$

So, we need expressions for $\frac{dT}{dx}$, $\frac{dT}{d\dot{x}}$, $\frac{dV}{dx}$, $\frac{dT}{d\theta_1}$, $\frac{dT}{d\dot{\theta}_1}$, $\frac{dV}{d\theta_1}$, $\frac{dT}{d\theta_2}$, $\frac{dT}{d\dot{\theta}_2}$, $\frac{dV}{d\theta_2}$ to construct our three EOM

EOM1:

$$\frac{dT}{dx}=0$$

$$\frac{dT}{d\dot{x}}=m_{r1}\dot{x}+m_{r2}\dot{x}+m_c\dot{x}+\frac{1}{2}c\theta_1L_1m_{r1}\dot{\theta_1}+\frac{1}{2}c\theta_1L_1m_{r2}\dot{\theta_1}+\frac{1}{2}c\theta_2L_2m_{r2}\dot{\theta_2}$$

$$\frac{dV}{dx}=kx$$

$$\frac{d}{dt}(\frac{dT}{d\dot{x}})=-\frac{1}{2}s\theta_1L_1m_1\dot{\theta_1}^2-\frac{1}{2}s\theta_1L_1m_2\dot{\theta_1}^2-\frac{1}{2}s\theta_2L_2m_2\dot{\theta_2}^2+m_{r1}\ddot{x}+m_{r2}\ddot{x}+m_c\ddot{x}+\frac{1}{2}c\theta_1L_1m_{r1}\ddot{\theta_1}+\frac{1}{2}c\theta_1L_1m_{r2}\ddot{\theta_1}+\frac{1}{2}c\theta_2L_2m_{r2}\ddot{\theta_2}$$

$$\frac{d}{dt}(\frac{dT}{d\dot{x}})-\frac{dT}{dx}+\frac{dV}{dx}=0$$

$$-\frac{1}{2}s\theta_1L_1m_1\dot{\theta_1}^2-\frac{1}{2}s\theta_1L_1m_2\dot{\theta_1}^2-\frac{1}{2}s\theta_2L_2m_2\dot{\theta_2}^2+m_{r1}\ddot{x}+m_{r2}\ddot{x}+m_c\ddot{x}+\frac{1}{2}c\theta_1L_1m_{r1}\ddot{\theta_1}+\frac{1}{2}c\theta_1L_1m_{r2}\ddot{\theta_1}+\frac{1}{2}c\theta_2L_2m_{r2}\ddot{\theta_2}-0+kx=0$$

Combining like terms to isolate $\ddot{x},\ddot{\theta_1},\ddot{\theta_2}$

$$\boxed{kx-\frac{1}{2}s\theta_1L_1(m_{r1}+m_{r2})\dot{\theta_1}^2-\frac{1}{2}s\theta_2L_2m_{r2}\dot{\theta_2}^2+(m_c+m_{r1}+m_{r2})\ddot{x}+\frac{1}{2}c\theta_1L_1(m_{r1}+m_{r2})\ddot{\theta_1}+\frac{1}{2}c\theta_2L_2m_{r2}\ddot{\theta_2}=0}$$

EOM2:

$$\frac{dT}{d\theta_1}=-\frac{1}{2}s\theta_1L_1m_{r1}\dot{x}\dot{\theta_1}-\frac{1}{2}s\theta_1L_1m_{r2}\dot{x}\dot{\theta_1}-\frac{1}{4}s(\theta_1-\theta_2)L_1L_2m_{r2}\dot{\theta_1}\dot{\theta_2}$$

$$\frac{dT}{d\dot{\theta}_1}=\frac{1}{2}c\theta_1L_1m_{r1}\dot{x}+\frac{1}{2}c\theta_1L_1m_{r2}\dot{x}+\frac{1}{3}L_1^2m_{r1}\dot{\theta_1}+\frac{1}{4}L_1^2m_{r2}\dot{\theta_1}+\frac{1}{4}c(\theta_1-\theta_2)L_1L_2m_{r2}\dot{\theta_2}$$

$$\frac{dV}{d\theta_1}=\frac{1}{2}gs\theta_1L_1(m_{r1}+m_{r2})$$

$$\frac{d}{dt}(\frac{dT}{d\dot{\theta}_1})=-\frac{1}{2}s\theta_1L_1m_{r1}\dot{x}\dot{\theta_1}-\frac{1}{2}s\theta_1L_1m_{r2}\dot{x}\dot{\theta_1}-\frac{1}{4}s(\theta_1-\theta_2)L_1L_2m_{r2}(\dot{\theta_1}-\dot{\theta_2})\dot{\theta_2}$$

$$+\frac{1}{2}c\theta_1 L_1 m_{r1}\ddot{x}+\frac{1}{2}c\theta_1 L_1 m_{r2}\ddot{x}+\frac{1}{3}L_1^2 m_{r1}\ddot{\theta_1}+\frac{1}{4}L_1^2 m_{r2}\ddot{\theta_1}+\frac{1}{4}c(\theta_1-\theta_2)L_1L_2m_{r2}\ddot{\theta_2}$$

$$\frac{d}{dt}(\frac{dT}{d\dot{\theta}_1})-\frac{dT}{d\theta_1}+\frac{dV}{dx\theta_1}=0$$

$$-\frac{1}{2}s\theta_1 L_1 m_{r1}\dot{x}\dot{\theta_1}-\frac{1}{2}s\theta_1 L_1 m_{r2}\dot{x}\dot{\theta_1}-\frac{1}{4}s(\theta_1-\theta_2)L_1L_2m_{r2}(\dot{\theta_1}-\dot{\theta_2})\dot{\theta_2}$$

$$+\frac{1}{2}c\theta_1 L_1 m_{r1}\ddot{x}+\frac{1}{2}c\theta_1 L_1 m_{r2}\ddot{x}+\frac{1}{3}L_1^2 m_{r1}\ddot{\theta_1}+\frac{1}{4}L_1^2 m_{r2}\ddot{\theta_1}+\frac{1}{4}c(\theta_1-\theta_2)L_1L_2m_{r2}\ddot{\theta_2}$$

$$+\frac{1}{2}s\theta_1 L_1 m_{r1}\dot{x}\dot{\theta_1}+\frac{1}{2}s\theta_1 L_1 m_{r2}\dot{x}\dot{\theta_1}+\frac{1}{4}s(\theta_1-\theta_2)L_1L_2m_{r2}\dot{\theta_1}\dot{\theta_2}$$

$$+\frac{1}{2}gs\theta_1 L_1(m_{r1}+m_{r2})=0$$

Combining like terms to isolate $\ddot{x},\ddot{\theta_1},\ddot{\theta_2}$

$$\boxed{\begin{aligned}0=\frac{1}{12}L_1(6gs\theta_1 m_{r1}+6gs\theta_1 m_{r2}+3s(\theta_1-\theta_2)L_2m_{r2}\dot{\theta_2}^2)\\+\frac{1}{12}L_1(6c\theta_1 m_1+6c\theta_1 m_{r2})\ddot{x}\\+\frac{1}{12}L_1(4L_1m_{r1}+3L_1m_{r2})\ddot{\theta_1}\\+\frac{1}{4}c(\theta_1-\theta_2)L_1L_2m_2\ddot{\theta_2}\end{aligned}}$$

EOM3:

$$\frac{dT}{d\theta_2}=-\frac{1}{2}s\theta_2 L_2 m_{r2}\dot{x}\dot{\theta_2}-\frac{1}{4}s(\theta_1-\theta_2)L_1L_2m_{r2}\dot{x}\dot{\theta_2}$$

$$\frac{dT}{d\dot{\theta}_2}=\frac{1}{2}c\theta_2 L_2 m_{r2}\dot{x}+\frac{1}{4}c(\theta_1-\theta_2)L_1L_2m_{r2}\dot{\theta_1}+\frac{1}{3}L_2^2 m_{r2}\dot{\theta_2}$$

$$\frac{dV}{d\theta_2}=\frac{1}{2}gs\theta_2 L_2 m_{r2}$$

$$\frac{d}{dt}(\frac{dT}{d\dot{\theta}_2})=-\frac{1}{4}s(\theta_1-\theta_2)L_1L_2m_{r2}\dot{\theta_1}(\dot{\theta_1}-\dot{\theta_2})-\frac{1}{2}s\theta_2 L_2 m_{r2}\dot{x}\dot{\theta_2}+\frac{1}{2}c\theta_2 L_2 m_{r2}\ddot{x}+\frac{1}{4}c(\theta_1-\theta_2)L_1L_2m_{r2}\ddot{\theta_2}+\frac{1}{3}L_2^2 m_{r2}\ddot{\theta_2}$$

$$\frac{d}{dt}(\frac{dT}{d\dot{\theta}_2})-\frac{dT}{d\theta_2}+\frac{dV}{dx\theta_2}=0$$

$$-\frac{1}{4}s(\theta_1-\theta_2)L_1L_2m_{r2}\dot{\theta_1}(\dot{\theta_1}-\dot{\theta_2})-\frac{1}{2}s\theta_2 L_2 m_{r2}\dot{x}\dot{\theta_2}+\frac{1}{2}c\theta_2 L_2 m_{r2}\ddot{x}+\frac{1}{4}c(\theta_1-\theta_2)L_1L_2m_{r2}\ddot{\theta_2}+\frac{1}{3}L_2^2 m_{r2}\ddot{\theta_2}$$

$$+\frac{1}{2}s\theta_2 L_2 m_{r2}\dot{x}\dot{\theta_2}+\frac{1}{4}s(\theta_1-\theta_2)L_1L_2m_{r2}\dot{x}\dot{\theta_2}$$

$$+\frac{1}{2}gs\theta_2 L_2 m_{r2}=0$$

Combining like terms to isolate $\ddot{x},\ddot{\theta_1},\ddot{\theta_2}$

$$\boxed{\frac{1}{12}L_2m_{r2}(6gs\theta_2-3s(\theta_1-\theta_2)L_1\dot{\theta_1}^2)+\frac{1}{2}c\theta_2L_2m_{r2}\ddot{x}+\frac{1}{4}c(\theta_1-\theta_2)L_1L_2m_{r2}\ddot{\theta_1}+\frac{1}{3}L_2^2m_{r2}\ddot{\theta_2}=0}$$

See end for actual hand written derivation. note there is a typo in my handwritten notes for the expression for V_{total}

2.3 MATLAB code

2.3.1 dynamics_fxn2

```
function [dydt] = dynamics_fxn2(y,mr1,mr2,mc,L1,L2,k)
%spring-cart-pendulum problem EOM; dydt = m\D
%here our M has to be a 3x3 matrix
M = [mc+mr1+mr2                                   , .5*cos(y(3))*L1*(mr1+mr2)       , .5*cos(y(5))*L2*mr2;...
     (1/12)*L1*(6*cos(y(3))*mr1 + 6*cos(y(3))*mr2), (1/12)*L1*(4*L1*mr1 + 3*L1*mr2), .25*cos(y(3)-y(5))*L1*L2*mr2;...
     .5*cos(y(5))*L2*mr2                          ,.25*cos(y(3)-y(5))*L1*L2*mr2     , (1/3)*(L2^2)*mr2];
D = -[k*y(1) - .5*sin(y(3))*L1*(mr1+mr2)*(y(4)^2) - .5*sin(y(5)) * L2 *mr2 * (y(6)^2);...
      (1/12)*L1*(6*9.81*sin(y(3))*mr1 + 6*9.81*sin(y(3))*mr2 + 3*sin(y(3) - y(5))*L2*mr2*(y(6)^2));...
       (1/12)*L2*mr2*(6*9.81*sin(y(5)) - 3*sin(y(3) - y(5))*L1*(y(4)^2))    ];
dydt = M\D;
end
```

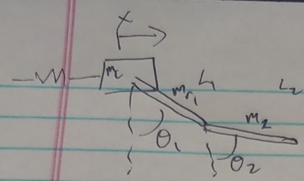
2.3.2 cart_sim2

```
clear all
%this sim is the same as cart_sim.m, but now there is a double pendulum
%attached to the cart.

mr1 = .25; %mass rod 1
mr2 = .1; %mass rod 2
mc = 1;    %mass cart
L1 = .5;   %length rod 1
L2 = .25;   %length rod 2
k = 10;    %spring constant

y1 = 1; %x
y2 = 0; %xdot
y3 = 0*pi/180; %theta 1
y4 = 0; %thetadot
y5 = 0*pi/180; %theta 2
y6 = 0; %thetadot

deltaTime = .006; %delta time
tEnd = 15;

y(1,:) = [y1,y2,y3,y4,y5,y6];


%4th order Runge-Kutta, could also use euler's method (1st order RK)
%following example from: http://lpsa.swarthmore.edu/NumInt/NumIntFourth.html
for t=1:(tEnd/deltaTime+1)

    YDD1(t,:) = dynamics_fxn2(y(t,:),mr1,mr2,mc,L1,L2,k);
    k1(t,:) = [y(t,2),YDD1(t,1),y(t,4),YDD1(t,2), y(t,6), YDD1(t,3)];

    YDD2(t,:) = dynamics_fxn2(y(t,:)+k1(t,:)*(deltaTime/2),mr1,mr2,mc,L1,L2,k);
    k2(t,:) = [y(t,2),YDD2(t,1),y(t,4),YDD2(t,2), y(t,6), YDD2(t,3)];

    YDD3(t,:) = dynamics_fxn2(y(t,:)+k2(t,:)*(deltaTime/2),mr1,mr2,mc,L1,L2,k);
    k3(t,:) = [y(t,2),YDD3(t,1),y(t,4),YDD3(t,2), y(t,6), YDD3(t,3)];

    YDD4(t,:) = dynamics_fxn2(y(t,:)+k3(t,:)*(deltaTime),mr1,mr2,mc,L1,L2,k);
    k4(t,:) = [y(t,2),YDD4(t,1),y(t,4),YDD4(t,2), y(t,6), YDD4(t,3)];
```

```matlab

    y(t+1,:) = y(t,:) + (deltaTime/6)*(k1(t,:)+2*k2(t,:)+2*k3(t,:)+k4(t,:));

end
figure('units','normalized','outerposition',[0 0 1 1])
for t=1:(tEnd/deltaTime+1)
    clf
    hold on
    tic
    %construct the position vectors
    cartStart = [y(t,1),0];
    cartEnd = cartStart + [.5,0];
    pend1Start = cartStart + [.25,0];
    pend1End = pend1Start +L1*[sin(y(t,3)),-cos(y(t,3))];
    pend2Start = pend1End;
    pend2End = pend2Start +L2*[sin(y(t,5)),-cos(y(t,5))];
    pend2EndStore(:,t) = pend2End;

    %plot the objects
    line([cartStart(1), cartEnd(1)],[cartStart(2), cartEnd(2)],'Color','green', 'linewidth', 15);
    line([0,pend1Start(1)],[0, pend1Start(2)],'Color','black', 'linewidth', 2);
    line([pend1Start(1), pend1End(1)],[pend1Start(2), pend1End(2)],'Color','blue', 'linewidth', 3);
    line([pend2Start(1), pend2End(1)],[pend2Start(2), pend2End(2)],'Color','red', 'linewidth', 3);
    plot(pend2EndStore(1,1:end),pend2EndStore(2,1:end),'Color','cyan');
    %legend('cart', 'rod 1', 'rod 2', 'spring', 'trajectory') %legend is too
    %resource intensive to plot in real time this way
    frameTime = toc;
    axis([-3,3,-2,2]);
    if (deltaTime-frameTime)>0
        pause((deltaTime-frameTime)) %pause so the sim plays back in real time
    else
        pause(10^-10)%in case the deltaTime is too small to play back in real time just print to screen anyway
    end
end
```

2.4 on paper

1



$$r_{G_1} = \frac{L_1}{2}, \quad I_{G_1} = \frac{1}{12} m_1 L_1^2$$

$$r_{G_2} = \frac{L_2}{2}, \quad I_{G_2} = \frac{1}{12} m_2 L_2^2$$

$$r_{G_1} = \left(x + \frac{L_1}{2} s\theta_1\right)\hat{i} + \left(-\frac{1}{2} L_1 c\theta_1\right)\hat{j}$$

$$\dot{r}_{G_1} = \left(\dot{x} + \frac{L_1}{2} c\theta_1 \dot{\theta}_1\right)\hat{i} + \left(\frac{1}{2} L_1 s\theta_1 \dot{\theta}_1\right)\hat{j}$$

$$\dot{r}_{G_1} \cdot \dot{r}_{G_1} = \left(\dot{x} + \frac{L_1}{2} c\theta_1 \dot{\theta}_1\right)\left(\dot{x} + \frac{L_1}{2} c\theta_1 \dot{\theta}_1\right) + \left(\frac{1}{2} L_1 s\theta_1 \dot{\theta}_1\right)\left(\frac{1}{2} L_1 s\theta_1 \dot{\theta}_1\right)$$

Same as before $= \dot{x}^2 + L_1 c\theta_1 \dot{x}\dot{\theta}_1 + \frac{L_1^2}{4}\dot{\theta}_1^2$

$$T_{rod_1} = \frac{1}{2} m_1 \left(\dot{x}^2 + L_1 c\theta_1 \dot{x}\dot{\theta}_1 + \frac{L_1^2}{4}\dot{\theta}_1^2\right) + \frac{1}{2}\left(\frac{1}{12} m_1 L_1^2\right)\dot{\theta}_1^2$$

$$= \frac{1}{2} m_1 \dot{x}^2 + \frac{1}{2} m_1 L_1 c\theta_1 \dot{x}\dot{\theta}_1 + \frac{1}{8} m_1 L_1^2 \dot{\theta}_1^2 + \frac{1}{24} m_1 L_1^2 \dot{\theta}_1^2$$

$$= \frac{1}{2} m_1 \dot{x}^2 + \frac{1}{2} m_1 L_1 c\theta_1 \dot{x}\dot{\theta}_1 + \frac{1}{6} m_1 L_1^2 \dot{\theta}_1^2$$

$$r_{G_2} = \left(x + \frac{L_1}{2} s\theta_1 + \frac{L_2}{2} s\theta_2\right)\hat{i} + \left(-\frac{1}{2} L_1 c\theta_1 - \frac{1}{2} L_2 c\theta_2\right)\hat{j}$$

$$\dot{r}_{G_2} \cdot \dot{r}_{G_2} = \left(\dot{x} + \frac{L_1}{2} c\theta_1 \dot{\theta}_1 + \frac{L_2}{2} c\theta_2 \dot{\theta}_2\right)\left(\dot{x} + \frac{L_1}{2} c\theta_1 \dot{\theta}_1 + \frac{L_2}{2} c\theta_2 \dot{\theta}_2\right)$$
$$+ \left(\frac{1}{2} L_1 s\theta_1 \dot{\theta}_1 + \frac{1}{2} L_2 s\theta_2 \dot{\theta}_2\right)\left(\frac{1}{2} L_1 s\theta_1 \dot{\theta}_1 + \frac{1}{2} L_2 s\theta_2 \dot{\theta}_2\right)$$

$$= \dot{x}^2 + L_1 c\theta_1 \dot{x}\dot{\theta}_1 + \frac{L_1^2}{4} c^2\theta_1 \dot{\theta}_1^2 + \frac{L_1^2}{4} s^2\theta_1 \dot{\theta}_1^2 + c\theta_2 L_2 \dot{x}\dot{\theta}_2 + \frac{1}{2} c\theta_1 c\theta_2 L_1 L_2 \dot{\theta}$$
$$+ \frac{1}{2} s\theta_1 s\theta_2 L_1 L_2 \dot{\theta}_1 \dot{\theta}_2 + \frac{1}{4} c^2\theta_2 L_2^2 \dot{\theta}_2^2 + \frac{1}{4} s^2\theta_2 L_2^2 \dot{\theta}_2$$

(2)

$$\frac{1}{2}m(\dot{r}_{G_2}\cdot\dot{r}_{G_2}) =$$

$$\frac{1}{2}m_r\dot{x}^2+\frac{1}{2}m_1 c\theta_1 L_1\dot{\theta}\dot{x}+\frac{1}{8}m_1c^2\theta_1L_1^2\dot{\theta}^2+\frac{1}{8}m_rs^2\theta_1L_1^2\dot{\theta}_1^2+\frac{1}{2}c\theta_2m_1L_2\dot{x}\dot{\theta}_2$$

$$+\frac{1}{4}m_2c\theta_1c\theta_2L_1L_2\dot{\theta}_1\dot{\theta}_2+\frac{1}{4}m_2s\theta_1s\theta_2L_1L_2\dot{\theta}_1\dot{\theta}_2+\frac{1}{8}m_2c^2\theta_2L_2^2\dot{\theta}_2^2+\frac{1}{8}m_2s^2\theta_2L_2^2\dot{\theta}_2^2$$

$$I_{G_2}=\frac{1}{12}m_2L_2^2$$

$$\frac{1}{2}I_G\dot{\theta}^2=\frac{1}{24}m_2L_2^2\dot{\theta}_2^2$$

$$T_{rod_2}=\frac{1}{2}m_2\dot{r}_{G_2}\cdot\dot{r}_{G_2}+\frac{1}{2}I_{G_2}\dot{\theta}_2$$

$$=\frac{1}{2}m_1\dot{x}^2+\frac{1}{2}m_1c\theta_1L_1\dot{\theta}_1+\frac{1}{8}L_1^2m_2\dot{\theta}_1^2\ L_2m_2\dot{x}\dot{\theta}_2+\frac{1}{4}c(\theta_1-\theta_2)L_1L_2m_2\dot{\theta}_1\dot{\theta}_2$$
$$+\frac{1}{6}L_2^2m_2\dot{\theta}_2^2$$

$$T_{cart}=\frac{1}{2}m_c\dot{x}^2$$

$$T_{total}=T_{cart}+T_{rod_1}+T_{rod2}$$

$$=\frac{1}{2}m_1x'^2+\frac{1}{2}m_2\dot{x}^2+\frac{1}{2}m_c\dot{x}^2+\frac{1}{2}c\theta_1L_1m_1\dot{x}\ddot{\theta}_1+\frac{1}{2}c\theta_1L_1m_2\dot{x}\dot{\theta}$$
$$+\frac{1}{6}L_1^2m_1\dot{\theta}_1^2+\frac{1}{8}L_1^2m_2\dot{\theta}_1^2+\frac{1}{2}c\theta_2L_2m_2\dot{x}\dot{\theta}_2+\frac{1}{4}c(\theta_1-\theta_2)L_1L_2m_2\dot{\theta}_1\dot{\theta}_2$$
$$+\frac{1}{6}L_2^2m_2\dot{\theta}_2^2$$

$$=\cancel{\frac{1}{24}(12m_c\dot{x}^2+4m_1}$$

$$V_{total}=\frac{1}{2}gL_1m_1-\frac{1}{2}gc\theta_1L_1m_1+\frac{1}{2}gL_2m_2-\frac{1}{2}gc\theta_2L_2m_2+\frac{1}{2}kx^2$$

$$\frac{\partial}{\partial t}\left(\frac{\partial T}{\partial \dot{x}}\right) - \frac{\partial T}{\partial x} + \frac{\partial V}{\partial x} = 0$$

$$\frac{\partial T}{\partial x} = 0$$

$$\frac{\partial T}{\partial \dot{x}} = m_1\dot{x} + m_2\dot{x} + m_c\dot{x} + \frac{1}{2}C\theta_1 L_1 m_1 \dot{\theta}_1 + \frac{1}{2}C\theta_1 L_1 m_2 \dot{\theta}_1 + \frac{1}{2}C\theta_2 L_2 m_2 \dot{\theta}_2$$

$$\frac{\partial V}{\partial x} = kx$$

$$\frac{\partial}{\partial t}\left(\frac{\partial T}{\partial \dot{x}}\right) = \cancel{(m_1 + m_2 + m_c)\ddot{x} + \frac{1}{2}C\theta_1 L_1 (m_1 + m_2)\ddot{\theta}_1 + \frac{1}{2}C\theta_2 L_2 m_2 \ddot{\theta}_2 = 0}$$

$$(m_1 + m_2 + m_c)\ddot{x} - \frac{1}{2}L_1(m_1 + m_2)(S\theta_1\dot{\theta}_1^2 - C\theta_1\ddot{\theta}_1)$$

$$+ \frac{1}{2}L_2 m_2(-S\theta_2\dot{\theta}_2^2 + C\theta_2\ddot{\theta}_2)$$

$$(m_1 + m_2 + m_c)\ddot{x} - \frac{1}{2}L_1(m_1 + m_2)(S\theta_1\dot{\theta}_1^2 - C\theta_1\ddot{\theta}_1)$$

$$+ \frac{1}{2}L_2 m_2(-S\theta_2\dot{\theta}_2^2 + C\theta_2\ddot{\theta}_2) - 0 + kx = 0$$

EOM_1

$$kx - \frac{1}{2}S\theta L(m_1 + m_2)\dot{\theta}_1^2 - \frac{1}{2}S\theta_2 L_2 m_2 \dot{\theta}_2^2$$

$$+ (m_1 + m_2 + m_c)\ddot{x}^{3} + \frac{1}{2}C\theta_1 L_1(m_1 + m_2)\ddot{\theta}_1 + \left(\frac{1}{2}C\theta_2 L_2 m_2\right)\ddot{\theta}_2 = 0$$

④

$$\frac{\partial}{\partial t}\left(\frac{\partial T}{\partial \dot{\theta}_1}\right) - \frac{\partial T}{\partial \theta_1} + \frac{\partial V}{\partial \theta_1} = 0$$

$$\frac{\partial T}{\partial \theta_1} = -\frac{1}{2} s\theta_1 L_1 m_1 \dot{x}\dot{\theta}_1 - \frac{1}{2} s\theta_1 L_1 m_2 \dot{x}\dot{\theta}_1 - \frac{1}{4} s(\theta_1 - \theta_2) L_1 L_2 m_2 \dot{\theta}_1 \dot{\theta}_2$$

$$\frac{\partial T}{\partial \dot{\theta}_1} = \frac{1}{2} c\theta_1 L_1 m_1 \dot{x} + \frac{1}{2} c\theta_1 L_1 m_2 \dot{x} + \frac{1}{3} L_1^2 m_1 \dot{\theta}_1 + \frac{1}{4} L_1^2 m_2 \dot{\theta}_1 + \frac{1}{4} c(\theta_1 - \theta_2) L_1 L_2 m_2 \dot{\theta}_2$$

$$\frac{\partial V}{\partial \theta_1} = \frac{1}{2} g s\theta_1 L_1 m_1$$

$$\frac{\partial}{\partial t}\left(\frac{\partial T}{\partial \dot{\theta}_1}\right) = -\frac{1}{2} c\theta L_1 m_1 \dot{x} \dot{\theta}_1^2 - \frac{1}{2} c\theta_1 L_1 m_2 \dot{x} \dot{\theta}_1^2 - \frac{1}{4} c(\theta_1 - \theta_2) L_1 L_2 m_2 \dot{\theta}_1 (\dot{\theta}_1 - \dot{\theta}_2) \dot{\theta}_2$$

$$\cancel{-\frac{1}{2} s\theta_1 L_1 m_2 x \ddot{\theta}_1 \tfrac{1}{4}} - \frac{1}{2} s\theta_1 L_1 m_1 \dot{\theta}_1 \ddot{x} - \frac{1}{2} s\theta_1 L_1 m_2 \dot{\theta}_1 \ddot{x}$$

$$-\frac{1}{2} s\theta_1 L_1 m_1 \dot{x} \ddot{\theta}_1 - \frac{1}{2} s\theta_1 L_1 m_2 \dot{x} \ddot{\theta}_1 - \frac{1}{4} s(\theta_1 - \theta_2) L_1 L_2 m_2 \dot{\theta}_2 \ddot{\theta}_1$$

$$-\frac{1}{4} s(\theta_1 - \theta_2) L_1 L_2 m_2 \ddot{\theta}_1 \ddot{\theta}_2$$

EOM 2

$$\frac{1}{12} L_2 m_2 (6 g s\theta_2 - 3 s(\theta_1 - \theta_2) L_1 \dot{\theta}_1^2 + \frac{1}{2} c\theta_2 L_2 m_2 \ddot{x} + \frac{1}{4} c(\theta_1 - \theta_2) L_1 L_2 m_2 \ddot{\theta}_1$$

$$+ \frac{1}{3} L_2^2 m_2 \ddot{\theta}_2 = 0$$

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$$\frac{\partial}{\partial t}\left(\frac{\partial T}{\partial \dot\theta_2}\right) - \frac{\partial T}{\partial \theta_2} + \frac{\partial V}{\partial \theta_2} = 0$$

$$\frac{\partial T}{\partial \theta_2} = -\frac{1}{2} s\theta_2 L_2 m_2 \dot x \dot\theta_2 + \frac{1}{4} s(\theta_1 - \theta_2) L_1 L_2 m_2 \dot\theta_1 \dot\theta_2$$

$$\frac{\partial T}{\partial \dot\theta_2} = \frac{1}{2} c\theta_2 L_2 m_2 \dot x + \frac{1}{4} c(\theta_1 - \theta_2) L_1 L_2 m_2 \dot\theta_1 + \frac{1}{3} L_2^2 m_2 \dot\theta_2$$

$$\frac{\partial V}{\partial \theta_2} = \frac{1}{2} g s\theta_2 L_2 m_2$$

$$\frac{\partial}{\partial t}\left(\frac{\partial T}{\partial \dot\theta_2}\right) =$$

$$-\frac{1}{4} s(\theta_1 - \theta_2) L_1 L_2 m_2 \dot\theta_1^2 - \frac{1}{2} s\theta_2 L_2 m_2 \dot x \dot\theta_2 + \frac{1}{4} s(\theta_1 - \theta_2) L_1 L_2 m_2 \dot\theta_1 \dot\theta_2$$

$$+ \frac{1}{2} c\theta_2 L_2 m_2 \ddot x + \frac{1}{4} c(\theta_1 - \theta_2) L_1 L_2 m_2 \ddot\theta_1 + \frac{1}{3} L_2^2 m_2 \ddot\theta_2$$

$$-\frac{1}{4} s(\theta_1 - \theta_2) L_1 L_2 m_2 \dot\theta_1^2 - \frac{1}{2} s\theta_2 L_2 m_2 \dot x \dot\theta_2 + \frac{1}{4} s(\theta_1 - \theta_2) L_1 L_2 m_2 \dot\theta_1 \dot\theta_2$$

$$+ \frac{1}{2} c\theta_2 L_2 m_2 \ddot x + \frac{1}{4} c(\theta_1 - \theta_2) L_1 L_2 m_2 \ddot\theta_1 + \frac{1}{3} L_2^2 m_2 \ddot\theta_2$$

$$+ \frac{1}{2} s\theta_2 L_2 m_2 \dot x \dot\theta_2 - \frac{1}{4} s(\theta_1 - \theta_2) L_1 L_2 m_2 \dot\theta_1 \dot\theta_2 + \frac{1}{2} g s\theta_2 L_2 m_2 = 0$$

EOM3

$$\frac{1}{12} L_2 m_2 \left(6 g s\theta_2 - 3 s(\theta_1 - \theta_2) L_1 \dot\theta_1^2\right)$$

$$+ \frac{1}{2} c\theta_2 L_2 m_2 \ddot x + \frac{1}{4} c(\theta_1 - \theta_2) L_1 L_2 m_2 \ddot\theta_1 + \frac{1}{3} L_2^2 m_2 \ddot\theta_2 = 0$$