

cart spring pendulum systems

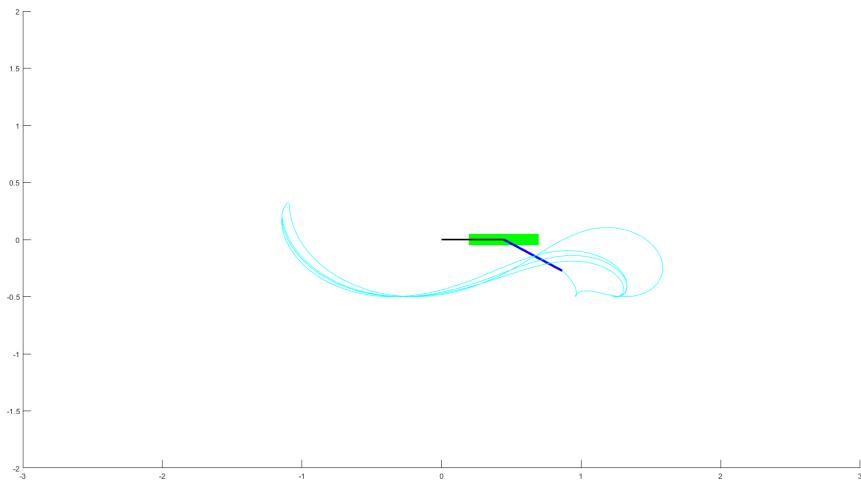
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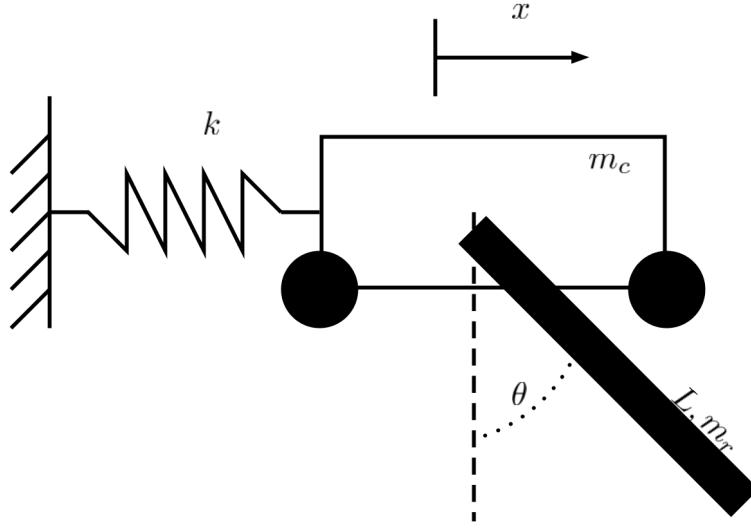
Contents

1	spring-cart-pendulum system	1
1.1	diagram	2
1.2	solve for EOM	2
1.3	MATLAB code	4
1.3.1	dynamics_fxn	4
1.3.2	cart_sim	4
1.4	on paper	5
2	spring-cart-double pendulum system	7
2.1	diagram	7
2.2	solve for EOM	7
2.3	MATLAB code	10
2.3.1	dynamics_fxn2	10
2.3.2	cart_sim2	11
2.4	on paper	13

1 spring-cart-pendulum system



1.1 diagram



Also note that:

$$k = 10 \text{ N/m}$$

$$L = 0.5 \text{ m}$$

$$m_c = 1 \text{ kg}$$

$$m_r = 0.25 \text{ kg}$$

1.2 solve for EOM

\LaTeX typed derivation here, see end for the picture of the actual hand written notes.

$$Y_G = \frac{L}{2}$$

$$I_G = \frac{1}{12}mL^2$$

$$\vec{r}_G = (x + \frac{L}{2}s\theta)\vec{i} + (-\frac{1}{2}Lc\theta)\vec{j}$$

$$\dot{\vec{r}}_G = (\dot{x} + \frac{L}{2}c\theta\dot{\theta})\vec{i} + (\frac{L}{2}s\theta\dot{\theta})\vec{j}$$

note that :

$$T_{total} = T_{cart} + T_{rod}$$

so we first find T_{rod}

$$T_{rod} = \frac{1}{2}m_r\dot{\vec{r}} \cdot \dot{\vec{r}} + \frac{1}{2}I_G\dot{\theta}^2$$

$$\dot{\vec{r}}_G \cdot \dot{\vec{r}}_G = (\dot{x} + \frac{L}{2}c\theta\dot{\theta})(\dot{x} + \frac{L}{2}c\theta\dot{\theta}) + (\frac{L}{2}s\theta\dot{\theta})(\frac{L}{2}s\theta\dot{\theta}) = \dot{x}^2 + Lc\theta\dot{x}\dot{\theta} + \frac{L^2}{4}c^2\theta\dot{\theta}^2 + \frac{L^2}{4}s^2\theta\dot{\theta}^2$$

$$\dot{\vec{r}}_G \cdot \dot{\vec{r}}_G = \dot{x}^2 + Lc\theta\dot{x}\dot{\theta} + \frac{L^2}{4}\dot{\theta}^2$$

$$\frac{1}{2}I_G\dot{\theta}^2 = \frac{1}{2}(\frac{1}{12}m_rL^2)\dot{\theta}^2 = \frac{1}{24}m_rL^2\dot{\theta}^2$$

Let's solve for kinetic energy expression of the rod

$$T_{rod} = \frac{1}{2}m_r\dot{x}^2 + \frac{1}{2}c\theta Lm_r\dot{\theta} + \frac{1}{24}L^2m_r\dot{\theta}^2 + \frac{1}{8}c^2\theta L^2m_r\dot{\theta}^2 + \frac{1}{8}s^2\theta L^2m_r\dot{\theta}^2 = \frac{1}{6}m_r(3\dot{x}^2 + 3Lc\theta\dot{x}\dot{\theta} + L^2\dot{\theta}^2)$$

And note that the kinetic energy expression of the cart is

$$T_{cart} = \frac{1}{2}m_c\dot{x}^2$$

So we can solve for T_{total}

$$T_{total} = T_{cart} + T_{rod} = \frac{1}{2}m_c\dot{x}^2 + \frac{1}{6}m_r(3\dot{x}^2 + 3Lc\theta\dot{x}\dot{\theta} + L^2\dot{\theta}^2)$$

Now we have to solve for V

$$V = \frac{1}{2}kx^2 + \frac{1}{2}m_rgL(1 - c\theta) = \frac{1}{2}kx^2 + \frac{1}{2}m_rgL - \frac{1}{2}m_rgLc\theta$$

by lagrange's eqn, we have:

$$\frac{d}{dt}\left(\frac{dT}{d\dot{q}_k}\right) - \frac{dT}{dq_k} + \frac{dV}{dq_k} = 0$$

So, we need expressions for $\frac{dT}{dx}$, $\frac{dT}{d\dot{x}}$, $\frac{dV}{dx}$, $\frac{dT}{d\dot{\theta}}$, $\frac{dT}{d\theta}$, $\frac{dV}{d\theta}$ to construct our two EOM

$$\frac{dT}{dx} = 0$$

$$\frac{dT}{d\dot{x}} = m_r\dot{x} + m_c\dot{x}\frac{1}{2}c\theta Lm_r\dot{\theta} = (m_c + m_r)\dot{x} + m_c\dot{x}\frac{1}{2}c\theta Lm_r\dot{\theta}$$

$$\frac{dV}{dx} = kx$$

$$\frac{d}{dt}\left(\frac{dT}{d\dot{x}}\right) = \frac{d}{dt}\left((m_c + m_r)\dot{x} + m_c\dot{x}\frac{1}{2}c\theta Lm_r\dot{\theta}\right) = -\frac{1}{2}s\theta Lm_r\dot{\theta}^2 + (m_c + m_r)\ddot{x} + \frac{1}{2}c\theta Lm_r\ddot{\theta}$$

EOM1:

$$\frac{d}{dt}\left(\frac{dT}{d\dot{x}}\right) - \frac{dT}{dx} + \frac{dV}{dx} = 0$$

$$-\frac{1}{2}s\theta Lm_r\dot{\theta}^2 + (m_c + m_r)\ddot{x} + \frac{1}{2}c\theta Lm_r\ddot{\theta} - 0 + kx = 0$$

$$(m_c + m_r)\ddot{x} + \left(\frac{1}{2}c\theta Lm_r\right)\ddot{\theta} - \frac{1}{2}s\theta Lm_r\dot{\theta}^2 + kx = 0$$

$$\frac{dT}{d\theta} = -\frac{1}{2}m_rL\dot{\theta}\dot{x}s\theta$$

$$\frac{dT}{d\dot{\theta}} = \frac{1}{2}m_rLc\theta\dot{x} + \frac{1}{3}m_rL^2\dot{\theta}$$

$$\frac{dV}{d\theta} = \frac{1}{2}m_rgLs\theta$$

$$\frac{d}{dt}\left(\frac{dT}{d\dot{\theta}}\right) = \frac{d}{dt}\left(\frac{1}{2}m_rLc\theta\dot{x} + \frac{1}{3}m_rL^2\dot{\theta}\right) = -\frac{1}{2}s\theta Lm_r\dot{x}\dot{\theta} + \frac{1}{2}c\theta Lm_r\ddot{x} + \frac{1}{3}L^2m_r\ddot{\theta}$$

EOM2:

$$\frac{d}{dt}\left(\frac{dT}{d\dot{\theta}}\right) - \frac{dT}{d\theta} + \frac{dV}{d\theta} = 0$$

$$-\frac{1}{2}Lm_r\dot{x}\dot{\theta}s\theta + \frac{1}{2}c\theta Lm_r\ddot{x} + \frac{1}{3}L^2m_r\ddot{\theta} + \frac{1}{2}m_rL\dot{\theta}\dot{x}s\theta + \frac{1}{2}m_rgLs\theta = 0$$

$$\left(\frac{1}{2}c\theta Lm_r\right)\ddot{x} + \left(\frac{1}{3}L^2m_r\right)\ddot{\theta} + \frac{1}{2}m_rgLs\theta = 0$$

$$\left(\frac{1}{2}c\theta\right)\ddot{x} + \left(\frac{1}{3}L\right)\ddot{\theta} + \frac{1}{2}gs\theta = 0$$

1.3 MATLAB code

1.3.1 dynamics_fxn

```
1 function [ dydt ] = dynamics_fxn( y ,mr,mc,L,k )
2 %spring-cart-pendulum problem EOM; dydt = m\D
3 M = [mr+mc, .5*mr*L*cos(y(3)) ; ...
4 .5*cos(y(3)), (1/3)*L];
5 D = [.5*mr*L*(y(4)^2)*sin(y(3))-k*y(1) ; ...
6 -.5*9.81*sin(y(3)) ];
7 dydt = M\D;
8 end
```

1.3.2 cart_sim

```
1 clear all
2
3 mr = .25; %mass of the rod
4 mc = 1; %mass of the cart
5 L = .5; %length of the rod
6 k = 10; %spring constant
7
8 y1 = 1; %x
9 y2 = 0; %xdot
10 y3 = 0*pi/180; %theta
11 y4 = 0; %thetadot
12
13 deltaTime = .01;
14 tmax = 5;
15
16 y(1,:) = [y1,y2,y3,y4];
17
18 %4th order Runge-Kutta, could also use euler's method (1st order RK)
19 %following example from: http://lpsa.swarthmore.edu/NumInt/NumIntFourth.html
20 for t=1:(tmax/deltaTime+1)
21
22 YDD1(t,:)=dynamics_fxn(y(t,:),mr,mc,L,k);
23 k1(t,:)=[y(t,2),YDD1(t,1),y(t,4),YDD1(t,2)];
24
25 YDD2(t,:)=dynamics_fxn(y(t,:)+k1(t,:)*(deltaTime/2),mr,mc,L,k);
26 k2(t,:)=[y(t,2),YDD2(t,1),y(t,4),YDD2(t,2)];
27
28 YDD3(t,:)=dynamics_fxn(y(t,:)+k2(t,:)*(deltaTime/2),mr,mc,L,k);
29 k3(t,:)=[y(t,2),YDD3(t,1),y(t,4),YDD3(t,2)];
30
31 YDD4(t,:)=dynamics_fxn(y(t,:)+k3(t,:)*(deltaTime),mr,mc,L,k);
32 k4(t,:)=[y(t,2),YDD4(t,1),y(t,4),YDD4(t,2)];
33
34 y(t+1,:)=y(t,:)+(deltaTime/6)*(k1(t,:)+2*k2(t,:)+2*k3(t,:)+k4(t,:));
35
36 end
37 figure('units','normalized','outerposition',[0 0 1 1])
38 for t=1:(tmax/deltaTime+1)
39   clf
40   hold on
41   tic
42   cartStart = [y(t,1),0];
43   cartEnd = cartStart + [.5,0];
44   pendStart = cartStart + [.25,0];
45   pendEnd = pendStart +L*[sin(y(t,3)), -cos(y(t,3))];
46   pendEndStore(:,t) = pendEnd;
```

```

47 line([ cartStart(1) , cartEnd(1) ],[ cartStart(2) , cartEnd(2) ],'Color' , 'green' , 'linewidth'
48 , 15);
49 line([0 , pendStart(1) ],[0 , pendStart(2) ],'Color' , 'black' , 'linewidth' , 2);
50 line([ pendStart(1) , pendEnd(1) ],[ pendStart(2) , pendEnd(2) ],'Color' , 'blue' , 'linewidth'
51 , 3);
52 plot(pendEndStore(1 ,1:end) ,pendEndStore(2 ,1:end) , 'Color' , 'cyan');
53 %legend('cart' , 'rod 1' , 'rod 2' , 'spring' , 'trajectory') %legend is too
54 %resource intensive to plot in real time this way
55 frameTime = toc;
56 axis([-3 ,3 , -2 ,2]);
57 if (deltaTime-frameTime)>0
58 pause((deltaTime-frameTime)) %pause so the sim plays back in real time
59 else
60 pause(10^-10)%in case the deltaTime is too small to play back in real time just
61 print to screen anyway
end

```

1.4 on paper

(1)

$$Y_G = \frac{L}{2}, I_G = \frac{1}{12}mL^2$$

$$\vec{r}_G = (x + \frac{L}{2}\sin\theta)\hat{i} + (-\frac{L}{2}\cos\theta)\hat{j}$$

$$\dot{\vec{r}}_G = (\dot{x} + \frac{L}{2}\cos\theta\dot{\theta})\hat{i} + (\frac{L}{2}\sin\theta\dot{\theta})\hat{j}$$

$$T_{rod} = \frac{1}{2}m_r \dot{\vec{r}} \cdot \dot{\vec{r}} + \frac{1}{2}I_G \dot{\theta}^2$$

$$\dot{\vec{r}}_G \cdot \dot{\vec{r}}_G = (\dot{x} + \frac{L}{2}\cos\theta\dot{\theta})(\dot{x} + \frac{L}{2}\cos\theta\dot{\theta}) + (\frac{L}{2}\sin\theta\dot{\theta})(\frac{L}{2}\sin\theta\dot{\theta})$$

$$= \dot{x}^2 + L\cos\theta\dot{x}\dot{\theta} + \frac{L^2}{4}\cos\theta\dot{\theta}^2 + \frac{L^2}{4}\sin\theta\dot{\theta}^2$$

$$= \dot{x}^2 + L\cos\theta\dot{x}\dot{\theta} + \frac{L^2}{4}\dot{\theta}^2$$

$$T_{rod} = \frac{1}{2}m_r \dot{\vec{r}} \cdot \dot{\vec{r}} + \frac{1}{2}I_G \dot{\theta}^2$$

$$= \frac{1}{2}m_r (\dot{x}^2 + L\cos\theta\dot{x}\dot{\theta} + \frac{L^2}{4}\dot{\theta}^2) + \frac{1}{2}(\frac{1}{12}m_r L^2)\dot{\theta}^2$$

$$= \frac{1}{2}m_r \dot{x}^2 + \frac{1}{2}m_r L\cos\theta\dot{x}\dot{\theta} + \frac{L^2}{8}\dot{\theta}^2 + \frac{1}{24}m_r L^2\dot{\theta}^2$$

$$= \frac{1}{2}m_r \dot{x}^2 + \frac{1}{2}m_r L\cos\theta\dot{x}\dot{\theta} + \frac{1}{6}m_r L^2\dot{\theta}^2$$

$$T_{cart} = \frac{1}{2}m_c \dot{x}^2$$

$$T_{total} = T_{rod} + T_{cart} = \frac{1}{2}m_r \dot{x}^2 + \frac{1}{2}m_r L\cos\theta\dot{x}\dot{\theta} + \frac{1}{6}m_r L^2\dot{\theta}^2 + \frac{1}{2}m_c \dot{x}^2$$

$$= \frac{1}{6}m_r (3\dot{x}^2 + 3L\cos\theta\dot{x}\dot{\theta} + L^2\dot{\theta}^2)$$

$$\frac{\partial}{\partial t} \left(\frac{\partial T}{\partial \dot{x}_k} \right) - \frac{\partial T}{\partial \dot{x}_k} + \frac{\partial V}{\partial x_k} = 0$$

②

$$\frac{\partial T}{\partial x} = 0$$

$$\frac{\partial T}{\partial \dot{x}} = \frac{1}{2} m_r \dot{x} + m_c \dot{x} \frac{1}{2} C O L m_r \dot{\theta} = (m_r + m_c) \dot{x} + m_c \dot{x} \frac{1}{2} C O L m_c \dot{\theta}$$

$$\frac{\partial V}{\partial x} = kx$$

$$\frac{\partial}{\partial t} \left(\frac{\partial T}{\partial \dot{x}} \right) = \frac{\partial}{\partial t} \left((m_r + m_c) \dot{x} + \frac{1}{2} C O L m_c \dot{x} \dot{\theta} \right) = (m_r + m_c) \ddot{x} + \frac{1}{2} L m_r m_c \dot{x} \dot{\theta} + \frac{1}{2} C O L m_c \ddot{x} \\ = (m_r + m_c) \ddot{x} - \frac{1}{2} L m_r S O \dot{\theta}^2 + \frac{1}{2} L m_c C O \dot{\theta}$$

$$(m_r + m_c) \ddot{x} - \frac{1}{2} L m_r S O \dot{\theta}^2 + \frac{1}{2} L m_c C O \dot{\theta} - 0 + kx = 0$$

EOM₁

$$(m_r + m_c) \ddot{x} + (\frac{1}{2} C O L m_r) \ddot{\theta} + \frac{1}{2} S O L m_r \dot{\theta}^2 + kx = 0$$

$$\frac{\partial T}{\partial \dot{\theta}} = -\frac{1}{2} n_r L C O \dot{x} S \theta$$

$$\frac{\partial T}{\partial \theta} = \frac{1}{2} m_r L C O \dot{x} \dot{\theta} + \frac{1}{3} m_r L^2 \dot{\theta}$$

$$\frac{\partial V}{\partial \theta} = \frac{1}{2} n_r g L S \theta$$

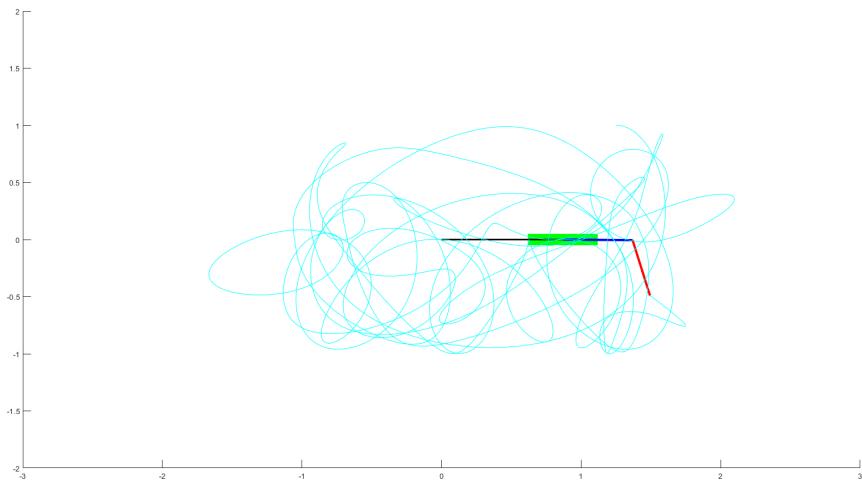
$$\frac{\partial T}{\partial \dot{\theta}} = \frac{1}{2} \left(\frac{1}{2} m_r L C O \dot{x} \dot{\theta} + \frac{1}{3} m_r L^2 \dot{\theta} \right) = \frac{1}{2} m_r L C O \ddot{x} - \frac{1}{2} n_r L S O \dot{x} \dot{\theta} + \frac{1}{3} m_r L^2 \dot{\theta}$$

$$-\frac{1}{2} m_r L S O \dot{x} \dot{\theta} + \frac{1}{3} m_r L^2 \dot{\theta} + \frac{1}{2} n_r L C O \dot{x} \dot{\theta} + \frac{1}{2} n_r L S O \dot{x} \dot{\theta} + \frac{1}{2} m_r g L S \theta = 0$$

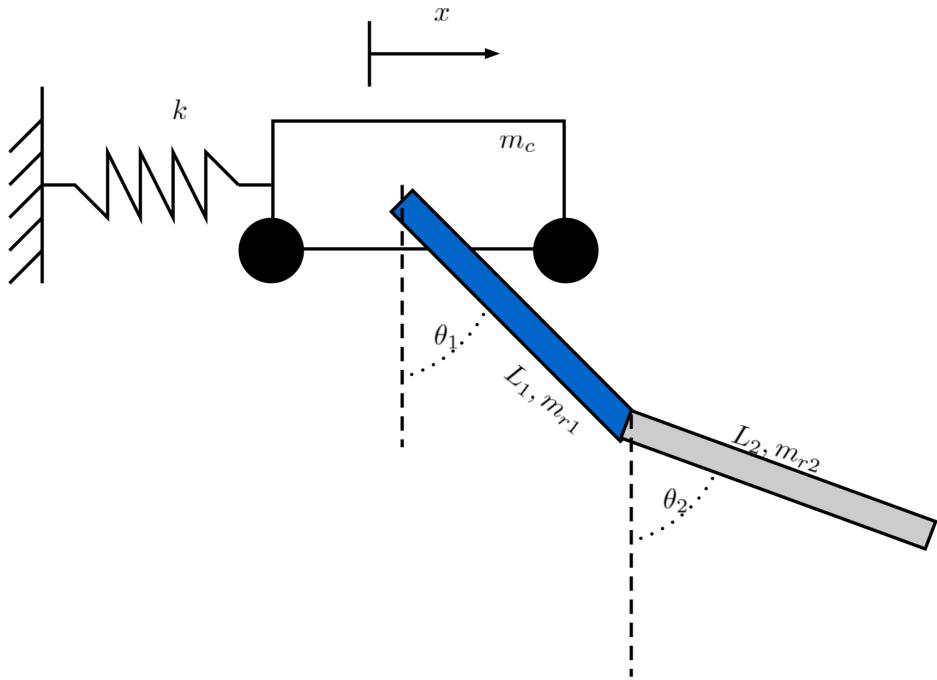
$$\text{EOM}_2 \quad \frac{1}{2} m_r L C O \ddot{x} + \frac{1}{3} m_r L \ddot{\theta} + \frac{1}{2} m_r g L S \theta = 0$$

$$\frac{1}{2} C O \ddot{x} + \frac{1}{3} L \ddot{\theta} + \frac{1}{2} g S \theta = 0$$

2 spring-cart-double pendulum system



2.1 diagram



2.2 solve for EOM

L^AT_EX typed derivation here, see end for the picture of the actual hand written notes.

$$Y_{G1} = \frac{L_1}{2}$$

$$Y_{G2} = \frac{L_2}{2}$$

$$I_{G1} = \frac{1}{12}mL_1^2$$

$$I_{G2} = \frac{1}{12}mL_2^2$$

$$\begin{aligned} r_{G1} &= (x + \frac{L_1}{2}s\theta_1)\vec{i} + (-\frac{1}{2}L_1c\theta_1)\vec{j} \\ \dot{r}_{G1} &= (\dot{x} + \frac{L_1}{2}c\theta_1\dot{\theta}_1)\vec{i} + (\frac{L_1}{2}s\theta_1\dot{\theta}_1)\vec{j} \\ r_{G2} &= (x + \frac{L_1}{2}s\theta_1 + \frac{L_2}{2}s\theta_2)\vec{i} + (-\frac{1}{2}L_1c\theta_1 - \frac{1}{2}L_2c\theta_2)\vec{j} \\ \dot{r}_{G2} &= (\dot{x} + \frac{L_1}{2}c\theta_1\dot{\theta}_1 + \frac{L_2}{2}c\theta_2\dot{\theta}_2)\vec{i} + (\frac{L_1}{2}s\theta_1\dot{\theta}_1 + \frac{L_2}{2}s\theta_2\dot{\theta}_2)\vec{j} \end{aligned}$$

so we first find T_{rod_1}

$$\begin{aligned} T_{rod1} &= \frac{1}{2}m_{r1}\dot{r}_{G1} \cdot \dot{r}_{G1} + \frac{1}{2}I_{G1}\dot{\theta}_1^2 \\ \dot{r}_{G1} \cdot \dot{r}_{G1} &= (\dot{x} + \frac{L_1}{2}c\theta_1\dot{\theta}_1)(\dot{x} + \frac{L_1}{2}c\theta_1\dot{\theta}_1) + (\frac{L_1}{2}s\theta_1\dot{\theta}_1)(\frac{L_1}{2}s\theta_1\dot{\theta}_1) = \dot{x}^2 + L_1c\theta_1\dot{x}\dot{\theta}_1 + \frac{L_1^2}{4}c^2\theta_1\dot{\theta}_1^2 + \frac{L_1^2}{4}s^2\theta_1\dot{\theta}_1^2 \\ \dot{r}_{G1} \cdot \dot{r}_{G1} &= \dot{x}^2 + L_1c\theta_1\dot{x}\dot{\theta}_1 + \frac{L_1^2}{4}\dot{\theta}_1^2 \\ \frac{1}{2}I_{G1}\dot{\theta}_1^2 &= \frac{1}{2}(\frac{1}{12}m_{r1}L_1^2)\dot{\theta}_1^2 = \frac{1}{24}m_{r1}L_1^2\dot{\theta}_1^2 \\ T_{rod1} &= \frac{1}{2}m_{r1}\dot{x}^2 + \frac{1}{2}c\theta_1L_1m_1\dot{x}\dot{\theta}_1 + \frac{1}{8}m_{r1}L_1^2\dot{\theta}_1^2 + \frac{1}{24}m_{r1}L_1^2\dot{\theta}_1^2 = \frac{1}{2}m_{r1}\dot{x}^2 + \frac{1}{2}c\theta_1L_1m_1\dot{x}\dot{\theta}_1 + \frac{1}{6}m_{r1}L_1^2\dot{\theta}_1^2 \end{aligned}$$

Next we need to find T_{rod_2}

$$\begin{aligned} T_{rod2} &= \frac{1}{2}m_{r2}\dot{r}_{G2} \cdot \dot{r}_{G2} + \frac{1}{2}I_{G2}\dot{\theta}_2^2 \\ \dot{r}_{G2} \cdot \dot{r}_{G2} &= (\dot{x} + \frac{L_1}{2}c\theta_1\dot{\theta}_1 + \frac{L_2}{2}c\theta_2\dot{\theta}_2)(\dot{x} + \frac{L_1}{2}c\theta_1\dot{\theta}_1 + \frac{L_2}{2}c\theta_2\dot{\theta}_2) + (-\frac{1}{2}L_1c\theta_1 - \frac{1}{2}L_2c\theta_2)(-\frac{1}{2}L_1c\theta_1 - \frac{1}{2}L_2c\theta_2) \\ \dot{r}_{G2} \cdot \dot{r}_{G2} &= \dot{x}^2 + c\theta_1L_1\dot{x}\dot{\theta}_1 + \frac{1}{4}c^2\theta_1L_1^2\dot{\theta}_1^2 + \frac{1}{4}s^2\theta_1L_1^2\dot{\theta}_1^2 + c\theta_2L_2\dot{x}\dot{\theta}_2 + \frac{1}{2}c\theta_1c\theta_2L_1L_2\dot{\theta}_1\dot{\theta}_2 + \frac{1}{2}s\theta_1s\theta_2L_1L_2\dot{\theta}_1\dot{\theta}_2 + \frac{1}{4}c^2\theta_2L_2^2\dot{\theta}_2^2 + \frac{1}{4}s^2\theta_2L_2^2\dot{\theta}_2^2 \\ \frac{1}{2}m_{r2} \cdot \dot{r}_{G2} \cdot \dot{r}_{G2} &= \\ (\frac{1}{2}m_{r2}\dot{x}^2 + \frac{1}{4}m_{r2}c\theta_1L_1\dot{x}\dot{\theta}_1 + \frac{1}{8}m_{r2}c^2\theta_1L_1^2\dot{\theta}_1^2 + \frac{1}{8}m_{r2}s^2\theta_1L_1^2\dot{\theta}_1^2 + \frac{1}{2}m_{r2}c\theta_2L_2\dot{x}\dot{\theta}_2 \\ + \frac{1}{4}m_{r2}c\theta_1c\theta_2L_1L_2\dot{\theta}_1\dot{\theta}_2 + \frac{1}{4}m_{r2}s\theta_1s\theta_2L_1L_2\dot{\theta}_1\dot{\theta}_2 + \frac{1}{8}m_{r2}c^2\theta_2L_2^2\dot{\theta}_2^2 + \frac{1}{8}m_{r2}s^2\theta_2L_2^2\dot{\theta}_2^2) \\ I_{G2} &= \frac{1}{12}m_{r2}L_2^2 \\ \frac{1}{2}I_{G2}\dot{\theta}_2^2 &= \frac{1}{24}m_{r2}L_2^2\dot{\theta}_2^2 \\ T_{rod2} &= \frac{1}{2}m_{r2}\dot{r}_{G2} \cdot \dot{r}_{G2} + \frac{1}{2}I_{G2}\dot{\theta}_2^2 \\ T_{rod2} &= \\ (\frac{1}{2}m_{r2}\dot{x}^2 + \frac{1}{4}m_{r2}c\theta_1L_1\dot{x}\dot{\theta}_1 + \frac{1}{8}m_{r2}c^2\theta_1L_1^2\dot{\theta}_1^2 + \frac{1}{8}m_{r2}s^2\theta_1L_1^2\dot{\theta}_1^2 + \frac{1}{2}m_{r2}c\theta_2L_2\dot{x}\dot{\theta}_2 \\ + \frac{1}{4}m_{r2}c\theta_1c\theta_2L_1L_2\dot{\theta}_1\dot{\theta}_2 + \frac{1}{4}m_{r2}s\theta_1s\theta_2L_1L_2\dot{\theta}_1\dot{\theta}_2 + \frac{1}{8}m_{r2}c^2\theta_2L_2^2\dot{\theta}_2^2 + \frac{1}{8}m_{r2}s^2\theta_2L_2^2\dot{\theta}_2^2) \end{aligned}$$

$$+ \frac{1}{24} m_{r2} L_2^2 \dot{\theta}_2^2$$

$$T_{rod2} = \frac{1}{2} m_{r2} \dot{x}^2 + \frac{1}{2} c\theta_1 L_1 m_{r2} \dot{x} \dot{\theta}_1 + \frac{1}{8} L_1^2 m_{r2} \dot{\theta}_1^2 + \frac{1}{2} c\theta_2 L_2 m_{r2} \dot{x} \dot{\theta}_2 + \frac{1}{4} c(\theta_1 - \theta_2) L_1 L_2 m_{r2} \dot{\theta}_1 \dot{\theta}_2 + \frac{1}{6} L_2^2 m_{r2} \dot{\theta}_2^2$$

Also we know T_{cart}

$$T_{cart} = \frac{1}{2} m_c \dot{x}^2$$

note that :

$$T_{total} = T_{cart} + T_{rod_1} + T_{rod_2}$$

So we can solve for T_{total}

$$T_{total} = \frac{1}{2} m_c \dot{x}^2 + \frac{1}{2} m_{r1} \dot{x}^2 + \frac{1}{2} c\theta_1 L_1 m_{r1} \dot{x} \dot{\theta}_1 + \frac{1}{6} m_{r1} L_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_{r2} \dot{x}^2 + \frac{1}{2} c\theta_1 L_1 m_{r2} \dot{x} \dot{\theta}_1 + \frac{1}{8} L_1^2 m_{r2} \dot{\theta}_1^2 + \frac{1}{2} c\theta_2 L_2 m_{r2} \dot{x} \dot{\theta}_2 + \frac{1}{4} c(\theta_1 - \theta_2) L_1 L_2 m_{r2} \dot{\theta}_1 \dot{\theta}_2 + \frac{1}{6} L_2^2 m_{r2} \dot{\theta}_2^2$$

Now we have to solve for V

$$V = \frac{1}{2} kx^2 + \frac{1}{2} m_{r1} g L_1 (1 - c\theta_1) + \frac{1}{2} m_{r2} g L_2 (1 - c\theta_2)$$

by lagrange's eqn, we have:

$$\frac{d}{dt} \left(\frac{dT}{dq_k} \right) - \frac{dT}{dq_k} + \frac{dV}{dq_k} = 0$$

So, we need expressions for $\frac{dT}{dx}$, $\frac{dT}{d\dot{x}}$, $\frac{dV}{dx}$, $\frac{dT}{d\theta_1}$, $\frac{dT}{d\dot{\theta}_1}$, $\frac{dV}{d\theta_1}$, $\frac{dT}{d\theta_2}$, $\frac{dT}{d\dot{\theta}_2}$, $\frac{dV}{d\theta_2}$ to construct our three EOM

EOM1:

$$\frac{dT}{dx} = 0$$

$$\frac{dT}{d\dot{x}} = m_{r1} \dot{x} + m_{r2} \dot{x} + m_c \dot{x} + \frac{1}{2} c\theta_1 L_1 m_{r1} \dot{\theta}_1 + \frac{1}{2} c\theta_2 L_2 m_{r2} \dot{\theta}_2$$

$$\frac{dV}{dx} = kx$$

$$\frac{d}{dt} \left(\frac{dT}{d\dot{x}} \right) = -\frac{1}{2} s\theta_1 L_1 m_{r1} \dot{\theta}_1^2 - \frac{1}{2} s\theta_1 L_1 m_{r2} \dot{\theta}_1^2 - \frac{1}{2} s\theta_2 L_2 m_{r2} \dot{\theta}_2^2 + m_{r1} \ddot{x} + m_{r2} \ddot{x} + m_c \ddot{x} + \frac{1}{2} c\theta_1 L_1 m_{r1} \ddot{\theta}_1 + \frac{1}{2} c\theta_1 L_1 m_{r2} \ddot{\theta}_1 + \frac{1}{2} c\theta_2 L_2 m_{r2} \ddot{\theta}_2$$

$$\frac{d}{dt} \left(\frac{dT}{d\dot{x}} \right) - \frac{dT}{dx} + \frac{dV}{dx} = 0$$

$$-\frac{1}{2} s\theta_1 L_1 m_{r1} \dot{\theta}_1^2 - \frac{1}{2} s\theta_1 L_1 m_{r2} \dot{\theta}_1^2 - \frac{1}{2} s\theta_2 L_2 m_{r2} \dot{\theta}_2^2 + m_{r1} \ddot{x} + m_{r2} \ddot{x} + m_c \ddot{x} + \frac{1}{2} c\theta_1 L_1 m_{r1} \ddot{\theta}_1 + \frac{1}{2} c\theta_1 L_1 m_{r2} \ddot{\theta}_1 + \frac{1}{2} c\theta_2 L_2 m_{r2} \ddot{\theta}_2 - 0 + kx = 0$$

Combining like terms to isolate $\ddot{x}, \ddot{\theta}_1, \ddot{\theta}_2$

$$kx - \frac{1}{2} s\theta_1 L_1 (m_{r1} + m_{r2}) \dot{\theta}_1^2 - \frac{1}{2} s\theta_2 L_2 m_{r2} \dot{\theta}_2^2 + (m_c + m_{r1} + m_{r2}) \ddot{x} + \frac{1}{2} c\theta_1 L_1 (m_{r1} + m_{r2}) \ddot{\theta}_1 + \frac{1}{2} c\theta_2 L_2 m_{r2} \ddot{\theta}_2 = 0$$

EOM2:

$$\frac{dT}{d\theta_1} = -\frac{1}{2} s\theta_1 L_1 m_{r1} \dot{x} \dot{\theta}_1 - \frac{1}{2} s\theta_1 L_1 m_{r2} \dot{x} \dot{\theta}_1 - \frac{1}{4} s(\theta_1 - \theta_2) L_1 L_2 m_{r2} \dot{\theta}_1 \dot{\theta}_2$$

$$\frac{dT}{d\dot{\theta}_1} = \frac{1}{2} c\theta_1 L_1 m_{r1} \dot{x} + \frac{1}{2} c\theta_1 L_1 m_{r2} \dot{x} + \frac{1}{3} L_1^2 m_{r1} \dot{\theta}_1 + \frac{1}{4} L_1^2 m_{r2} \dot{\theta}_1 + \frac{1}{4} c(\theta_1 - \theta_2) L_1 L_2 m_{r2} \dot{\theta}_1$$

$$\frac{dV}{d\theta_1} = \frac{1}{2} g s\theta_1 L_1 m_{r1}$$

$$\frac{d}{dt} \left(\frac{dT}{d\dot{\theta}_1} \right) = -\frac{1}{2} s\theta_1 L_1 m_{r1} \dot{x} \dot{\theta}_1 - \frac{1}{2} s\theta_1 L_1 m_{r2} \dot{x} \dot{\theta}_1 - \frac{1}{4} s(\theta_1 - \theta_2) L_1 L_2 m_{r2} (\dot{\theta}_1 - \dot{\theta}_2) \dot{\theta}_2$$

$$\begin{aligned}
& + \frac{1}{2}c\theta_1 L_1 m_{r1} \ddot{x} + \frac{1}{2}c\theta_1 L_1 m_{r2} \ddot{x} + \frac{1}{3}L_1^2 m_{r1} \ddot{\theta}_1 + \frac{1}{4}L_1^2 m_{r2} \ddot{\theta}_1 + \frac{1}{4}c(\theta_1 - \theta_2) L_1 L_2 m_{r2} \ddot{\theta}_2 \\
& \frac{d}{dt} \left(\frac{dT}{d\dot{\theta}_1} \right) - \frac{dT}{d\theta_1} + \frac{dV}{dx\theta_1} = 0 \\
& - \frac{1}{2}s\theta_1 L_1 m_{r1} \dot{x}\dot{\theta}_1 - \frac{1}{2}s\theta_1 L_1 m_{r2} \dot{x}\dot{\theta}_1 - \frac{1}{4}s(\theta_1 - \theta_2) L_1 L_2 m_{r2} (\dot{\theta}_1 - \dot{\theta}_2) \dot{\theta}_2 \\
& + \frac{1}{2}c\theta_1 L_1 m_{r1} \ddot{x} + \frac{1}{2}c\theta_1 L_1 m_{r2} \ddot{x} + \frac{1}{3}L_1^2 m_{r1} \ddot{\theta}_1 + \frac{1}{4}L_1^2 m_{r2} \ddot{\theta}_1 + \frac{1}{4}c(\theta_1 - \theta_2) L_1 L_2 m_{r2} \ddot{\theta}_2 \\
& + \frac{1}{2}s\theta_1 L_1 m_{r1} \dot{x}\dot{\theta}_1 + \frac{1}{2}s\theta_1 L_1 m_{r2} \dot{x}\dot{\theta}_1 + \frac{1}{4}s(\theta_1 - \theta_2) L_1 L_2 m_{r2} \dot{\theta}_1 \dot{\theta}_2 \\
& + \frac{1}{2}gs\theta_1 L_1 m_{r1} = 0
\end{aligned}$$

Combining like terms to isolate $\ddot{x}, \ddot{\theta}_1, \ddot{\theta}_2$

$$\boxed{\frac{1}{12}L_1(6gs\theta_1 m_{r1} + 3s(\theta_1 - \theta_2)L_2 m_{r2} \dot{\theta}_2^2) + \frac{1}{12}L_1(6c\theta_1 m_1 + 6c\theta_1 m_{r2})\ddot{x} + \frac{1}{12}L_1(4L_1 m_{r1} + 3L_1 m_{r2})\ddot{\theta}_1 + \frac{1}{4}c(\theta_1 - \theta_2)L_1 L_2 m_2 \ddot{\theta}_2 = 0}$$

EOM3:

$$\begin{aligned}
\frac{dT}{d\theta_2} &= -\frac{1}{2}s\theta_2 L_2 m_{r2} \dot{x}\dot{\theta}_2 - \frac{1}{4}s(\theta_1 - \theta_2) L_1 L_2 m_{r2} \dot{x}\dot{\theta}_2 \\
\frac{dT}{d\dot{\theta}_2} &= \frac{1}{2}c\theta_2 L_2 m_{r2} \dot{x} + \frac{1}{4}c(\theta_1 - \theta_2) L_1 L_2 m_{r2} \dot{\theta}_1 + \frac{1}{3}L_2^2 m_{r2} \dot{\theta}_2 \\
\frac{dV}{d\theta_2} &= \frac{1}{2}gs\theta_2 L_2 m_{r2}
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dt} \left(\frac{dT}{d\dot{\theta}_2} \right) &= -\frac{1}{4}s(\theta_1 - \theta_2) L_1 L_2 m_{r2} \dot{\theta}_1 (\dot{\theta}_1 - \dot{\theta}_2) - \frac{1}{2}s\theta_2 L_2 m_{r2} \dot{x}\dot{\theta}_2 + \frac{1}{2}c\theta_2 L_2 m_{r2} \ddot{x} + \frac{1}{4}c(\theta_1 - \theta_2) L_1 L_2 m_{r2} \ddot{\theta}_2 + \frac{1}{3}L_2^2 m_{r2} \ddot{\theta}_2 \\
& \frac{d}{dt} \left(\frac{dT}{d\theta_2} \right) - \frac{dT}{d\theta_2} + \frac{dV}{dx\theta_2} = 0 \\
& -\frac{1}{4}s(\theta_1 - \theta_2) L_1 L_2 m_{r2} \dot{\theta}_1 (\dot{\theta}_1 - \dot{\theta}_2) - \frac{1}{2}s\theta_2 L_2 m_{r2} \dot{x}\dot{\theta}_2 + \frac{1}{2}c\theta_2 L_2 m_{r2} \ddot{x} + \frac{1}{4}c(\theta_1 - \theta_2) L_1 L_2 m_{r2} \ddot{\theta}_2 + \frac{1}{3}L_2^2 m_{r2} \ddot{\theta}_2 \\
& + \frac{1}{2}s\theta_2 L_2 m_{r2} \dot{x}\dot{\theta}_2 + \frac{1}{4}s(\theta_1 - \theta_2) L_1 L_2 m_{r2} \dot{x}\dot{\theta}_2 \\
& + \frac{1}{2}gs\theta_2 L_2 m_{r2} = 0
\end{aligned}$$

Combining like terms to isolate $\ddot{x}, \ddot{\theta}_1, \ddot{\theta}_2$

$$\boxed{\frac{1}{12}L_2 m_{r2}(6gs\theta_2 - 3s(\theta_1 - \theta_2)L_1 \dot{\theta}_1^2) + \frac{1}{2}c\theta_2 L_2 m_{r2} \ddot{x} + \frac{1}{4}c(\theta_1 - \theta_2) L_1 L_2 m_{r2} \ddot{\theta}_1 + \frac{1}{3}L_2^2 m_{r2} \ddot{\theta}_2 = 0}$$

See end for actual hand written derivation.

2.3 MATLAB code

2.3.1 dynamics_fxn2

```

1 function [ dydt ] = dynamics_fxn2( y ,mr1,mr2,mc,L1,L2,k )
2 %spring-cart-pendulum problem EOM; dydt = m\D
3 %here our M has to be a 3x3 matrix
4 M = [ mc+mr1+mr2 , .5*cos(y(3))*L1*(mr1+mr2) , .5*cos(y(5))*L2*mr2 ; ...
5 (1/12)*L1*(6*cos(y(3))*mr1 + 6*cos(y(3))*mr2) , (1/12)*L1*(4*L1*mr1 + 3*L1*mr2) , .25*cos(y(3)-y(5))*L1*L2*mr2 ; ...

```

```

6      .5*cos(y(5))*L2*mr2           ,.25*cos(y(3)-y(5))*L1*L2*mr2      , (1/3)
7      *(L2^2)*mr2];
8 D = -[k*y(1) - .5*sin(y(3))*L1*(mr1+mr2)*(y(4)^2) - .5*sin(y(5)) * L2 *mr2 * (y(6)^2);...
9      (1/12)*L1*(6*9.81*sin(y(3))*mr1 + 3*sin(y(3) - y(5))*L2*mr2*(y(6)^2));...
10     (1/12)*L2*mr2*(6*9.81*sin(y(5)) - 3*sin(y(3) - y(5))*L1*(y(4)^2))] ;
11 dydt = M\D;
12 end

```

2.3.2 cart_sim2

```

1 clear all
2 %this sim is the same as cart_sim.m, but now there is a double pendulum
3 %attached to the cart.
4
5 mr1 = .25; %mass rod 1
6 mr2 = .05; %mass rod 2
7 mc = 1; %mass cart
8 L1 = .5; %length rod 1
9 L2 = .5; %length rod 2
10 k = 10; %spring constant
11
12 y1 = 1; %x
13 y2 = 0; %xdot
14 y3 = 180*pi/180; %theta 1
15 y4 = 0; %thetadot
16 y5 = 180*pi/180; %theta 2
17 y6 = 0; %thetadot
18
19 deltaTime = .006; %delta time
20 tEnd = 15;
21
22 y(1,:) = [y1,y2,y3,y4,y5,y6];
23
24
25 %4th order Runge-Kutta, could also use euler's method (1st order RK)
26 %following example from: http://lpsa.swarthmore.edu/NumInt/NumIntFourth.html
27 for t=1:(tEnd/deltaTime+1)
28
29     YDD1(t,:) = dynamics_fxn2(y(t,:),mr1,mr2,mc,L1,L2,k);
30     k1(t,:) = [y(t,2),YDD1(t,1),y(t,4),YDD1(t,2),y(t,6),YDD1(t,3)];
31
32     YDD2(t,:) = dynamics_fxn2(y(t,:)+k1(t,:)*(deltaTime/2),mr1,mr2,mc,L1,L2,k);
33     k2(t,:) = [y(t,2),YDD2(t,1),y(t,4),YDD2(t,2),y(t,6),YDD2(t,3)];
34
35     YDD3(t,:) = dynamics_fxn2(y(t,:)+k2(t,:)*(deltaTime/2),mr1,mr2,mc,L1,L2,k);
36     k3(t,:) = [y(t,2),YDD3(t,1),y(t,4),YDD3(t,2),y(t,6),YDD3(t,3)];
37
38     YDD4(t,:) = dynamics_fxn2(y(t,:)+k3(t,:)*(deltaTime),mr1,mr2,mc,L1,L2,k);
39     k4(t,:) = [y(t,2),YDD4(t,1),y(t,4),YDD4(t,2),y(t,6),YDD4(t,3)];
40
41     y(t+1,:) = y(t,:) + (deltaTime/6)*(k1(t,:)+2*k2(t,:)+2*k3(t,:)+k4(t,:));
42
43 end
44 figure('units','normalized','outerposition',[0 0 1 1])
45 for t=1:(tEnd/deltaTime+1)
46     clf
47     hold on
48     tic
49     %construct the position vectors
50     cartStart = [y(t,1),0];
51     cartEnd = cartStart + [.5,0];

```

```

52 pend1Start = cartStart + [.25 ,0];
53 pend1End = pend1Start +L1*[ sin(y(t ,3)),-cos(y(t ,3)) ];
54 pend2Start = pend1End;
55 pend2End = pend2Start +L2*[ sin(y(t ,5)),-cos(y(t ,5)) ];
56 pend2EndStore (:,t) = pend2End;
57
58 %plot the objects
59 line([ cartStart(1), cartEnd(1) ],[ cartStart(2) , cartEnd(2) ],'Color ', 'green ', 'linewidth '
60 , 15);
61 line([0,pend1Start(1)],[0, pend1Start(2)],'Color ', 'black ', 'linewidth ', 2);
62 line([ pend1Start(1), pend1End(1) ],[ pend1Start(2), pend1End(2) ],'Color ', 'blue ', '
63 linewidth ', 3);
64 line([ pend2Start(1), pend2End(1) ],[ pend2Start(2) , pend2End(2) ],'Color ', 'red ', '
65 linewidth ', 3);
66 plot(pend2EndStore(1,1:end),pend2EndStore(2,1:end),'Color ', 'cyan ');
67 %legend('cart', 'rod 1', 'rod 2', 'spring', 'trajectory') %legend is too
68 %resource intensive to plot in real time this way
69 frameTime = toc;
70 axis([-3,3,-2,2]);
71 if (deltaTime-frameTime)>0
72 pause((deltaTime-frameTime)) %pause so the sim plays back in real time
73 else
74 pause(10^-10)%in case the deltaTime is too small to play back in real time just
75 print to screen anyway
76 end
77 end

```

2.4 on paper

$\gamma_{G_1} = \frac{L_1}{2}$, $I_{G_1} = \frac{1}{12}m_1L_1^2$
 $\gamma_{G_2} = \frac{L_2}{2}$, $I_{G_2} = \frac{1}{12}m_2L_2^2$
 $\dot{\gamma}_{G_1} = (\ddot{x} + \frac{L_1}{2}\sin\theta_1)\hat{i} + (-\frac{1}{2}L_1\cos\theta_1)\hat{j}$
 $\dot{\gamma}_{G_2} = (\ddot{x} + \frac{L_1}{2}\cos\theta_1\dot{\theta}_1)\hat{i} + (\frac{1}{2}L_1\sin\theta_1\dot{\theta}_1)\hat{j}$
 $\dot{\gamma}_{G_1} \cdot \dot{\gamma}_{G_2} = (\ddot{x} + \frac{L_1}{2}\cos\theta_1\dot{\theta}_1)(\ddot{x} + \frac{L_1}{2}\sin\theta_1\dot{\theta}_1) + (\frac{1}{2}L_1\sin\theta_1\dot{\theta}_1)(\frac{1}{2}L_1\cos\theta_1\dot{\theta}_1)$
 Same as before $= \ddot{x}^2 + L_1\cos\theta_1\dot{x}\theta_1 + \frac{L_1^2}{4}\dot{\theta}_1^2$
 $T_{red} = \frac{1}{2}m_1\ddot{x}^2 + L_1\cos\theta_1\dot{x}\theta_1 + \frac{L_1^2}{4}\dot{\theta}_1^2 + \frac{1}{2}(\frac{1}{12}m_1L_1^2)\dot{\theta}_1^2$
 $= \frac{1}{2}m_1\ddot{x}^2 + \frac{1}{2}m_1L_1\cos\theta_1\dot{x}\theta_1 + \frac{1}{8}m_1L_1^2\dot{\theta}_1^2 + \frac{1}{24}m_1L_1^2\dot{\theta}_1^2$
 $= \frac{1}{2}m_1\ddot{x}^2 + \frac{1}{2}m_1L_1\cos\theta_1\dot{x}\theta_1 + \frac{1}{6}m_1L_1^2\dot{\theta}_1^2$
 $\dot{\gamma}_{G_2} = (\ddot{x} + \frac{L_1}{2}\sin\theta_1 + \frac{L_2}{2}\sin\theta_2)\hat{i} + (-\frac{1}{2}L_1\cos\theta_1\frac{1}{2}L_2\cos\theta_2)\hat{j}$
 $\dot{\gamma}_6 \cdot \dot{\gamma}_{G_2} = (\ddot{x} + \frac{L_1}{2}\cos\theta_1\dot{\theta}_1 + \frac{L_2}{2}\cos\theta_2\dot{\theta}_2)(\ddot{x} + \frac{L_1}{2}\sin\theta_1\dot{\theta}_1 + \frac{L_2}{2}\sin\theta_2\dot{\theta}_2)$
 $+ (\frac{1}{2}L_1\sin\theta_1\dot{\theta}_1 + \frac{1}{2}L_2\sin\theta_2\dot{\theta}_2)(\frac{1}{2}L_1\cos\theta_1\dot{\theta}_1 + \frac{1}{2}L_2\cos\theta_2\dot{\theta}_2)$
 $= \ddot{x}^2 + L_1\cos\theta_1\dot{x}\theta_1 + \frac{L_1^2}{4}\dot{\theta}_1^2 + \frac{L_2^2}{4}\dot{\theta}_2^2 + \cos\theta_1L_1\dot{x}\theta_2 + \frac{1}{2}\cos\theta_1\cos\theta_2L_1L_2\dot{\theta}_1\dot{\theta}_2$
 $+ \frac{1}{2}\sin\theta_1\sin\theta_2L_1L_2\dot{\theta}_1\dot{\theta}_2 + \frac{1}{4}L_2^2\dot{\theta}_2^2 + \frac{1}{4}\sin^2\theta_2L_2^2\dot{\theta}_2^2$

(2)

$$\frac{1}{2}m(\vec{r}_{G_2} - \vec{r}_{G_1}) =$$

$$\frac{1}{2}m\dot{x}^2 + \frac{1}{2}m(c\theta_1\dot{\theta}_1\dot{x} + \frac{1}{8}m(c\theta_1\dot{L}_1^2\dot{\theta}^2 + \frac{1}{8}m_2s^2\theta_1\dot{L}_1^2\dot{\theta}_1^2 + \frac{1}{2}c\theta_1\dot{s}\dot{\theta}_1\dot{\theta}_2)$$

$$+ \frac{1}{4}m_2c\theta_1\dot{c}\theta_2\dot{L}_1\dot{L}_2\dot{\theta}_1\dot{\theta}_2 + \frac{1}{4}m_2s\theta_1\dot{s}\theta_2\dot{L}_1\dot{L}_2\dot{\theta}_1\dot{\theta}_2 + \frac{1}{8}m_2c\theta_2\dot{c}\theta_2\dot{L}_2\dot{\theta}_2^2 + \frac{1}{8}m_2s\theta_2\dot{s}\dot{\theta}_2\dot{\theta}_2^2$$

$$I_{G_2} = \frac{1}{12}m_2L_2^2$$

$$\frac{1}{2}I_{G_2}\dot{\theta}^2 = \frac{1}{12}m_2L_2^2\dot{\theta}_2^2$$

$$T_{rod2} = \frac{1}{2}m_2\vec{r}_{G_2} \cdot \vec{r}_{G_2} + \frac{1}{2}I_{G_2}\dot{\theta}_2$$

$$= \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_1c\theta_1\dot{\theta}_1 + \frac{1}{8}L_1^2m_2s\theta_1\dot{L}_2\dot{n}_2\dot{x}\dot{\theta}_2 + \frac{1}{4}c(\theta_1 - \theta_2)L_1L_2m_2\dot{\theta}_1\dot{\theta}_2$$

$$+ \frac{1}{6}L_2^2m_2\dot{\theta}_2^2$$

$$T_{cart} = \frac{1}{2}m_c\dot{x}^2$$

$$T_{total} = T_{cart} + T_{rod1} + T_{rod2}$$

$$= \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\dot{x}^2 + \frac{1}{2}m_2\dot{x}^2 + \frac{1}{2}c\theta_1L_1m_1\dot{x}\dot{\theta}_1 + \frac{1}{2}c\theta_1L_1m_2\dot{x}\dot{\theta}_1$$

$$+ \frac{1}{6}L_1^2m_1\dot{\theta}_1^2 + \frac{1}{8}L_1^2m_2\dot{\theta}_1^2 + \frac{1}{2}c\theta_2L_2m_2\dot{x}\dot{\theta}_2 + \frac{1}{4}c(\theta_1 - \theta_2)L_1L_2m_2\dot{\theta}_1\dot{\theta}_2$$

$$+ \frac{1}{6}L_2^2m_2\dot{\theta}_2^2$$

$$= \frac{1}{2}m_c\dot{x}^2 + 4m_1\dot{x}^2$$

$$V_{total} = \frac{1}{2}gL_1m_1 - \frac{1}{2}g(c\theta_1L_1m_1 + \frac{1}{2}gL_2m_2) - \frac{1}{2}g(c\theta_2L_2m_2 + \frac{1}{2}gL_2^2)$$

(3)

$$\frac{\partial}{\partial t} \left(\frac{\partial T}{\partial x} \right) - \frac{\partial T}{\partial x} + \frac{\partial V}{\partial x} = 0$$

$$\frac{\partial T}{\partial x} = 0$$

$$\frac{\partial T}{\partial x} = m_1 \dot{x} + m_2 \dot{x} + m_c \dot{x} + \frac{1}{2} C \theta_1 L_1 m_1 \dot{\theta}_1 + \frac{1}{2} C \theta_1 L_1 m_2 \dot{\theta}_1 + \frac{1}{2} C \theta_2 L_2 m_2 \dot{\theta}_2$$

$$\frac{\partial V}{\partial x} = kx$$

$$\begin{aligned} \frac{\partial}{\partial t} \left(\frac{\partial T}{\partial x} \right) &= (m_1 + m_2 + m_c) \ddot{x} + \frac{1}{2} C \theta_1 L_1 (m_1 + m_2) \dot{\theta}_1 + \frac{1}{2} C \theta_2 L_2 (m_2) \dot{\theta}_2 = 0 \\ (m_1 + m_2 + m_c) \ddot{x} &- \frac{1}{2} L_1 (m_1 + m_2) (S \theta_1 \dot{\theta}_1^2 - C \theta_1 \ddot{\theta}_1) \\ &+ \frac{1}{2} L_2 m_2 (-S \theta_2 \dot{\theta}_2^2 + C \theta_2 \ddot{\theta}_2) \end{aligned}$$

$$(m_1 + m_2 + m_c) \ddot{x} - \frac{1}{2} L_1 (m_1 + m_2) (S \theta_1 \dot{\theta}_1^2 - C \theta_1 \ddot{\theta}_1)$$

$$+ \frac{1}{2} L_2 m_2 (-S \theta_2 \dot{\theta}_2^2 + C \theta_2 \ddot{\theta}_2) - 0 + kx = 0$$

EOM,

$$kx - \frac{1}{2} S \theta_1 L_1 (m_1 + m_2) \dot{\theta}_1^2 - \frac{1}{2} S \theta_2 L_2 m_2 \dot{\theta}_2^2$$

$$+ (m_1 + m_2 + m_c) \ddot{x} + \frac{1}{2} C \theta_1 L_1 (m_1 + m_2) \dot{\theta}_1 + \frac{1}{2} C \theta_2 L_2 m_2 \dot{\theta}_2 = 0$$

$$\frac{\partial}{\partial t} \left(\frac{\partial T}{\partial \dot{\theta}_1} \right) - \frac{\partial T}{\partial \theta_1} + \frac{\partial T}{\partial \dot{\theta}_1} = 0$$

(4)

$$\frac{\partial T}{\partial \theta_1} = -\frac{1}{2} s\theta_1 L_1 m_1 \dot{x} \dot{\theta}_1 - \frac{1}{2} s\theta_1 L_1 m_2 \dot{x} \dot{\theta}_1 - \frac{1}{4} s\theta_1 - \theta_2 L_1 L_2 m_2 \dot{\theta}_1 \dot{\theta}_2$$

$$\frac{\partial T}{\partial \dot{\theta}_1} = \frac{1}{2} c\theta_1 L_1 m_1 \dot{x} + \frac{1}{2} c\theta_1 L_1 m_2 \dot{x} + \frac{1}{3} L_1^2 m_1 \dot{\theta}_1 + \frac{1}{4} L_1^2 m_2 \dot{\theta}_1 + \frac{1}{4} (c\theta_1 - \theta_2) L_1 L_2 m_2 \dot{\theta}_2$$

$$\frac{\partial V}{\partial \theta_1} = \frac{1}{2} g s\theta_1 L_1 m_1$$

$$\begin{aligned} \frac{\partial}{\partial t} \left(\frac{\partial T}{\partial \dot{\theta}_1} \right) &= -\frac{1}{2} c\theta_1 L_1 m_1 \dot{x} \dot{\theta}_1^2 - \frac{1}{2} c\theta_1 L_1 m_2 \dot{x} \dot{\theta}_1^2 - \frac{1}{4} c(c\theta_1 - \theta_2) L_1 L_2 m_2 \dot{\theta}_1 (\dot{\theta}_1 - \dot{\theta}_2) \ddot{\theta}_2 \\ &- \frac{1}{2} s\theta_1 L_1 m_1 \ddot{x} \dot{\theta}_1^2 - \frac{1}{2} s\theta_1 L_1 m_2 \ddot{x} \dot{\theta}_1^2 - \frac{1}{2} s\theta_1 L_1 m_2 \dot{\theta}_1 \ddot{x} \\ &- \frac{1}{2} s\theta_1 L_1 m_1 \dot{x} \ddot{\theta}_1 - \frac{1}{2} s\theta_1 L_1 m_2 \dot{x} \ddot{\theta}_1 - \frac{1}{4} s\theta_1 - \theta_2 L_1 L_2 m_2 \dot{\theta}_1 \dot{\theta}_2 \\ &- \frac{1}{4} s(\theta_1 - \theta_2) L_1 L_2 m_2 \ddot{\theta}_1 \ddot{\theta}_2 \end{aligned}$$

FOM2

$$\begin{aligned} &\frac{1}{12} L_2 m_2 (6g s\theta_2 - 3s(\theta_1 - \theta_2) L_1 \dot{\theta}_1^2 + \frac{1}{2} c\theta_2 L_2 m_2 \ddot{x} + \frac{1}{4} c(\theta_1 - \theta_2) L_1 L_2 m_2 \ddot{\theta}_1) \\ &+ \frac{1}{3} L_2^2 m_2 \dot{\theta}_2 = 0 \end{aligned}$$

$$\frac{\partial}{\partial t} \left(\frac{\partial T}{\partial \dot{\theta}_2} \right) - \frac{\partial T}{\partial \theta_2} + \frac{\partial V}{\partial \dot{\theta}_2} = 0$$

(5)

$$\frac{\partial T}{\partial \theta_2} = -\frac{1}{2} S \theta_2 L_2 m_2 \dot{x} \dot{\theta}_2 + \frac{1}{4} (\theta_1 - \theta_2) L_1 L_2 m_2 \ddot{\theta}_1 \dot{\theta}_2$$

$$\frac{\partial T}{\partial \dot{\theta}_2} = \frac{1}{2} C \theta_2 L_2 m_2 \dot{x} - \frac{1}{4} C (\theta_1 - \theta_2) L_1 L_2 m_2 \dot{\theta}_1 + \frac{1}{3} L_2^2 m_2 \ddot{\theta}_2$$

$$\frac{\partial V}{\partial \theta_2} = \frac{1}{2} g S \theta_2 L_2 m_2$$

$$\frac{\partial}{\partial t} \left(\frac{\partial T}{\partial \dot{\theta}_2} \right) =$$

$$-\frac{1}{4} S(\theta_1 - \theta_2) L_1 L_2 m_2 \dot{\theta}_1^2 - \frac{1}{2} S \theta_2 L_2 m_2 \dot{x} \dot{\theta}_2 + \frac{1}{4} S(\theta_1 - \theta_2) L_1 L_2 m_2 \dot{\theta}_1 \dot{\theta}_2$$

$$+ \frac{1}{2} C \theta_2 L_2 m_2 \ddot{x} + \frac{1}{4} C (\theta_1 - \theta_2) L_1 L_2 m_2 \ddot{\theta}_1 + \frac{1}{3} L_2^2 m_2 \ddot{\theta}_2$$

$$-\frac{1}{4} S(\theta_1 - \theta_2) L_1 L_2 m_2 \dot{\theta}_1^2 - \frac{1}{2} S \theta_2 L_2 m_2 \dot{x} \dot{\theta}_2 + \frac{1}{4} S(\theta_1 - \theta_2) L_1 L_2 m_2 \dot{\theta}_1 \dot{\theta}_2$$

$$+ \frac{1}{2} C \theta_2 L_2 m_2 \ddot{x} + \frac{1}{4} C (\theta_1 - \theta_2) L_1 L_2 m_2 \ddot{\theta}_1 + \frac{1}{3} L_2^2 m_2 \ddot{\theta}_2$$

$$+ \frac{1}{2} S \theta_2 L_2 m_2 \dot{x} \dot{\theta}_2 - \frac{1}{4} S(\theta_1 - \theta_2) L_1 L_2 m_2 \dot{\theta}_1 \dot{\theta}_2 + \frac{1}{2} g S \theta_2 L_2 m_2 = 0$$

OM³

$$\frac{1}{12} L_2 m_2 (6 g S \theta_2 - 3 S(\theta_1 - \theta_2) L_1 \dot{\theta}_1^2)$$

$$+ \frac{1}{2} C \theta_2 L_2 m_2 \ddot{x} + \frac{1}{4} C (\theta_1 - \theta_2) L_1 L_2 m_2 \ddot{\theta}_1 + \frac{1}{3} L_2^2 m_2 \ddot{\theta}_2 = 0$$