Homework 4
Aravind Chandradoss

1) SISO Model

a.

The State Space representation of 4-cylinder SI engine is given by,

$$x_1(k) = \frac{c_{T2}}{\sqrt{\tau_o}}\, p(k-1) + \frac{c_{T3}}{\sqrt{\tau_o}}\, w(k-1) - \frac{c_{T4}}{\sqrt{\tau_o}}\, \alpha(k-1)$$

$$x_2(k) = \frac{c_{T6}}{\sqrt{\tau_o}}\, w(k-1) - \frac{c_{T7}}{\sqrt{\tau_o}}\, T(k-1)$$

$$x_3(k) = p(k-1)$$

$$x_4(k) = x_3(k-1) = p(k-2)$$

$$x_5(k) = x_4(k-1) = p(k-3)$$

State Space Matrices:

$$x(k+1) = \Phi_T x(k) + \Gamma_T\, \alpha(k)$$
$$y(k) = H_T x(k) + D_T\, \alpha(k)$$

$$\Phi_T = \begin{bmatrix} (OM^{-1})_{11} & (OM^{-1})_{12} & 0 & 0 & Q_{12}K_p \\ (OM^{-1})_{21} & (OM^{-1})_{22} & 0 & 0 & Q_{22}K_p \\ (M^{-1})_{11} & (M^{-1})_{12} & 0 & 0 & -(M^{-1}N)_{12}K_p \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \qquad \Gamma_T = \begin{bmatrix} Q_{11} \\ Q_{21} \\ -(M^{-1}N)_{11} \\ 0 \\ 0 \end{bmatrix}$$

$$H_T = \begin{bmatrix} (M^{-1})_{21} & (M^{-1})_{22} & 0 & 0 & -(M^{-1}N)_{22}K_p \end{bmatrix} \qquad D_T = \begin{bmatrix} -(M^{-1}N)_{21} \end{bmatrix}$$

$$M = \begin{bmatrix} \frac{-c_{T_1}}{\sqrt{\tau_0}} & \frac{-c_{T_3}}{\sqrt{\tau_0}} \\ 0 & \frac{-c_{T_5}}{\sqrt{\tau_0}} \end{bmatrix} \qquad N = \begin{bmatrix} \frac{c_{T_4}}{\sqrt{\tau_0}} & 0 \\ 0 & \frac{c_{T_7}}{\sqrt{\tau_0}} \end{bmatrix}$$

$$\mathrm{O} = \begin{bmatrix} \frac{c_{T_2}}{\sqrt{\tau_0}} & \frac{c_{T_3}}{\sqrt{\tau_0}} \\ 0 & \frac{c_{T_6}}{\sqrt{\tau_0}} \end{bmatrix} \qquad P = \begin{bmatrix} \frac{-c_{T_4}}{\sqrt{\tau_0}} & 0 \\ 0 & \frac{-c_{T_7}}{\sqrt{\tau_0}} \end{bmatrix}$$

$$Q = -OM^{-1}N + P$$

$In\ general, X_{ij}\ stands\ for\ the\ element\ in\ i^{th} row\ and\ j^{th} column$

$$\Phi_T = \begin{bmatrix} 0.9724 & -6.1276 & 0 & 0 & 0.0011 \\ 0 & 0.9990 & 0 & 0 & -0.0004 \\ -1.0092 & 3.1353 & 0 & 0 & -0.0006 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \qquad \Gamma_T = \begin{bmatrix} -91.2330 \\ 0 \\ 46.6806 \\ 0 \\ 0 \end{bmatrix}$$

$$H_T = [0 \quad -1.0228 \quad 0 \quad 0 \quad 0.0002] \qquad D_T = [0]$$

$$\begin{aligned}
x_1(k) &= 0.9724\, x_1(k-1) - 6.1276\, x_2(k-1) + 0.0011\, x_5(k-1) \\
x_2(k) &= 0.9990\, x_2(k-1) - 0.0004\, x_5(k-1) \\
x_3(k) &= -1.0092\, x_1(k-1) + 3.1353\, x_2(k-1) - 0.0006\, x_5(k-1) \\
x_4(k) &= x_3(k-1) \\
x_5(k) &= x_4(k-1)
\end{aligned}$$

b. Eigen values

The eigen values of Phi are,

-0.0754 + 0.0000i
0.0354 + 0.0800i
0.0354 - 0.0800i
0.9880 + 0.0461i
0.9880 - 0.0461i

It is **exactly same** as in Homework 2.

c. Transfer function

The continuous transfer function is given by

$$\frac{0.008598\, Z^2 + 0.0172\, \mathrm{Z} + 0.008598}{Z^5 - 1.971\, Z^4 + 0.9714 Z^3 + 0.0005646 Z^2 + 0.001129\, \mathrm{Z} + 0.0005646}$$

The function is **exactly same** as in HW2
(NOTE: The polynomial in the denominator is monic)

d. controllability

Checking the rank of

$$Wc = [\Gamma \quad \Phi\Gamma \quad \Phi^2\Gamma \quad \Phi^3\Gamma \quad \Phi^4\Gamma]$$

The rank is Wc,

Rank(Wc) = 5;

Since **Wc is full rank**, the system is **controllable**.

2. SISO – Pole Placement

Design Specification:

- Settling time (radians) = 300 radian
- Overshoot = 250 rpm

Calculation for Desired location of the Dominant pole:

Finding the desire root location (region) for dominant poles,

Overshoot:

$$PO\ \% = 100 * e^{-\frac{\zeta\pi}{\sqrt{(1-\zeta^2)}}}$$

$$\zeta = 0.34 \text{ (Approx.,)}$$

Settling time:

$$T_s = \frac{4}{\zeta\omega_n} = 300\ rad$$

$$\omega_n = 0.039 \text{ (Approx.,)}$$

NOTE: The above parameter are in continuous time domain. One converting the above parameters we get the approximate pole location.

Region of interest:
The region of interest is approximately around (right side of unit circle) z = 0.93+0.1i (approx). The desired poles are chosen in the region of interest by trial and error method.

Desired poles:

The poles were chosen in the desired region, as follows;

0.93+0.1i,
0.93-0.1i,
0.94+0.1i,
0.94+0.1i,
-0.1

The K matrix is as follows,

K= [0.0479 0.0607 0.0579 -0.0097 -0.0017]

The response is as follows,

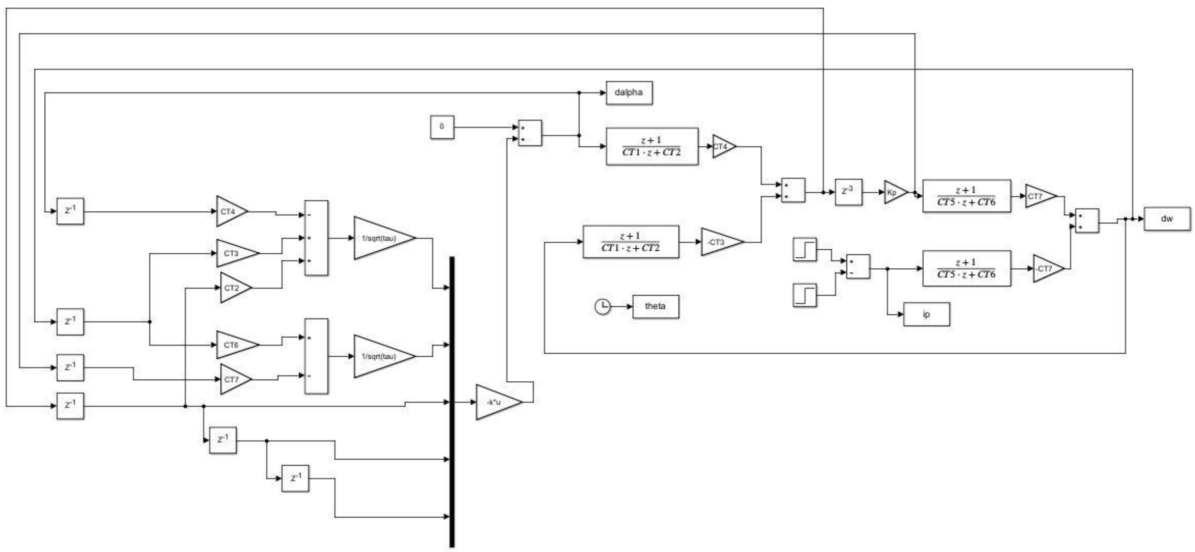

Fig 1: Simulink State Feedback (P alone)

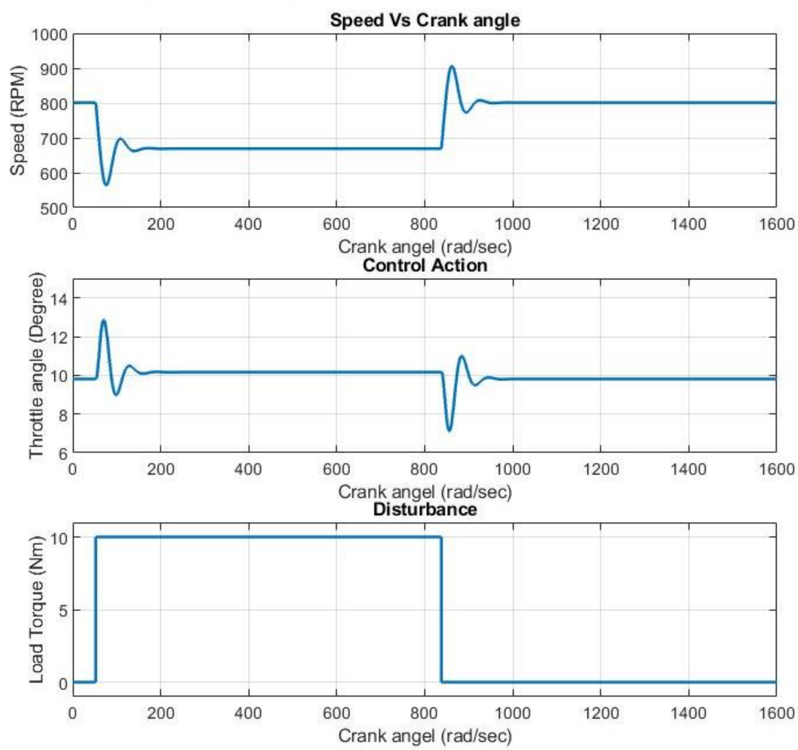

Fig 2: Closed Loop Response of State Feedback (P alone)

Inference,

The throttle angle, overshoot and settling time are within desired region.
The steady state error is non-zero (Because of P-alone control action)

3. SISO Pole Placement – Augmented Systems;

The system is augmented to have integral action. The state space matrix is modifies as follows,

$$\Phi_T = \begin{bmatrix} (OM^{-1})_{11} & (OM^{-1})_{12} & 0 & 0 & Q_{12}K_p & 0 \\ (OM^{-1})_{21} & (OM^{-1})_{22} & 0 & 0 & Q_{22}K_p & 0 \\ (M^{-1})_{11} & (M^{-1})_{11} & 0 & 0 & -(M^{-1}N)_{12}K_p & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ (M^{-1})_{21} & (M^{-1})_{22} & 0 & 0 & -(M^{-1}N)_{22}K_p & 1 \end{bmatrix} \qquad \Gamma_T = \begin{bmatrix} Q_{11} \\ Q_{21} \\ -(M^{-1}N)_{11} \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$H_T = [(M^{-1})_{21} \quad (M^{-1})_{22} \quad 0 \quad 0 \quad -(M^{-1}N)_{22}K_p \quad 0] \qquad D_T = [-(M^{-1}N)_{21}]$$

$$\Phi_T = \begin{bmatrix} 0.9724 & -6.1276 & 0 & 0 & 0.0011 & 0 \\ 0 & 0.9990 & 0 & 0 & -0.0004 & 0 \\ -1.0092 & 3.1353 & 0 & 0 & -0.0006 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1.0228 & 0 & 0 & 0.0002 & 1 \end{bmatrix} \qquad \Gamma_T = \begin{bmatrix} -91.2330 \\ 0 \\ 46.6806 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$H_T = [0 \quad -1.0228 \quad 0 \quad 0 \quad 0.0002 \quad 0\,] \qquad D_T = [0]$$

The desired poles for augmented are chosen in region of interest by trial and error method.
The chosen poles are as follows,

0.9+0.2i,
0.9-0.2i
0.7+0.1i
0.7-0.1i
0.92
0.95

The gain matrix k is as follows

K =[0.0870 0.0428 0.1251 -0.0482 0.0080 0.0006]

The response of the systems for state feedback (PI) is as follows,

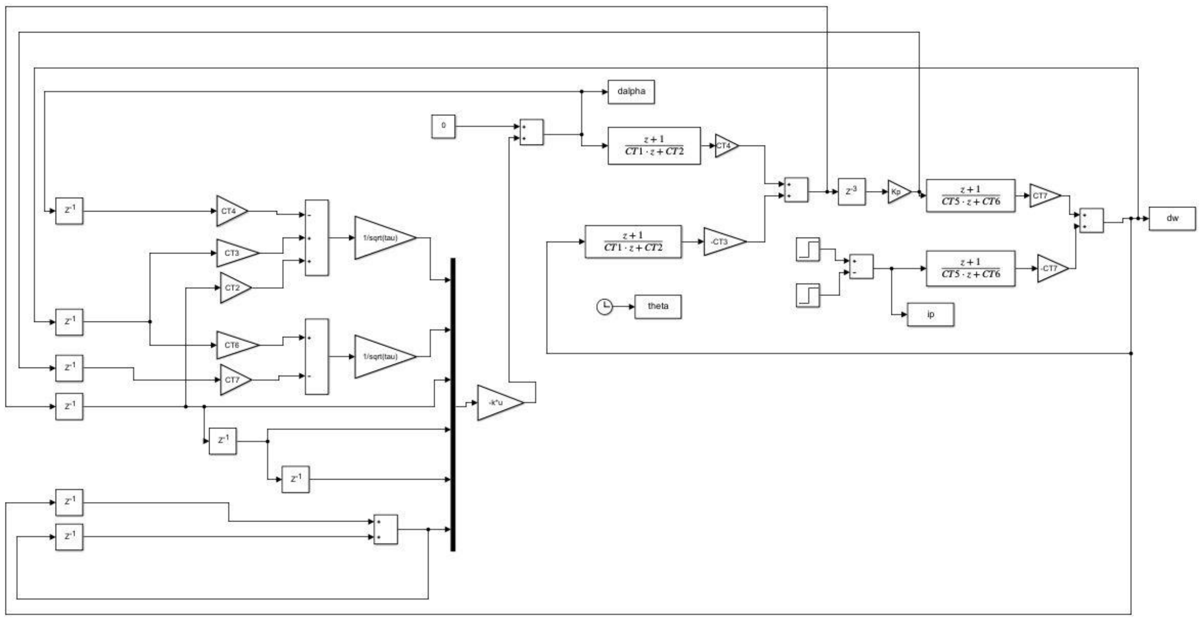

Fig 3: Simulink State Feedback (PI)

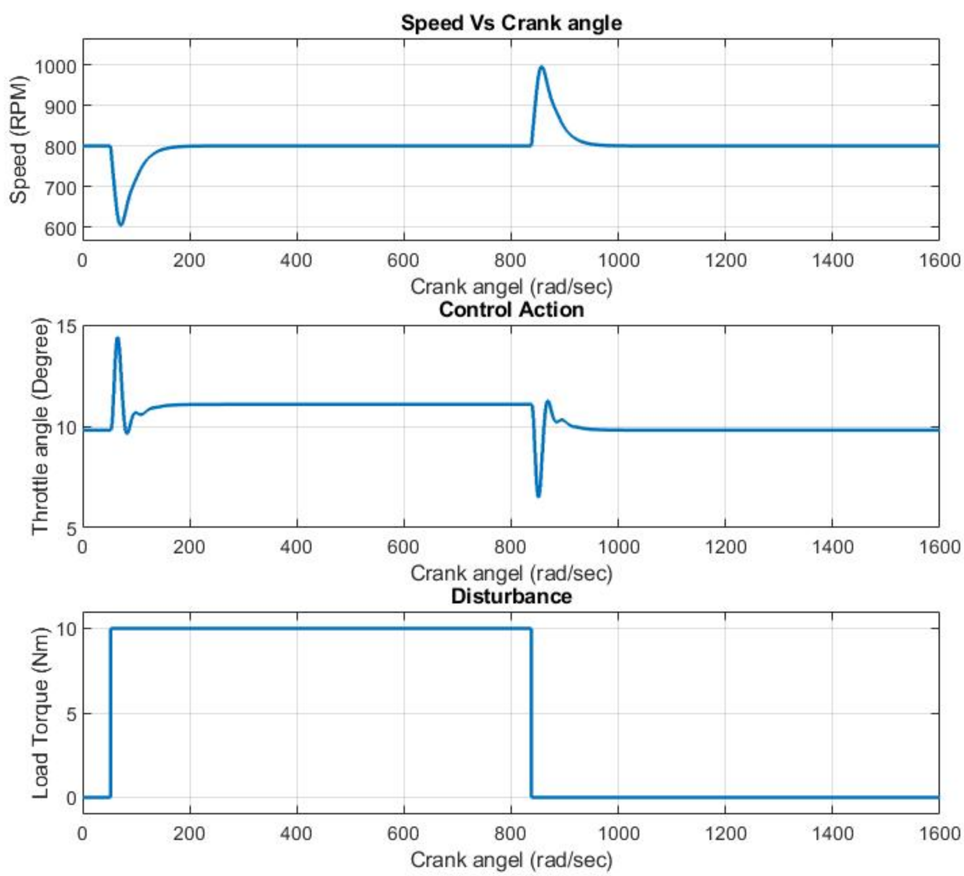

Fig 4: Closed Loop Response of State Feedback (PI)

Inference:

The throttle angel, overshoot, settling time are within desired region.
The steady state error is zero (Because of integral action)