MatGeo Assignment 5.5.24

AI25BTECH11007

Question:

Using elementary row operations, find the inverse of the matrix $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$

and hence solve the following system of equations:

$$\begin{cases} 3x - 3y + 4z = 21, \\ 2x - 3y + 4z = 20, \\ -y + z = 5. \end{cases}$$

Solution :

$$\text{Let } \mathbf{A} = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$$

Augment $\mathbf{A}$ with the identity: $[\mathbf{A} \,|\, \mathbf{I}] = \left(\begin{array}{ccc|ccc} 3 & -3 & 4 & 1 & 0 & 0 \\ 2 & -3 & 4 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right)$

Row transformation 1: $R_2 \to 3R_2 - 2R_1$

$$\left(\begin{array}{ccc|ccc} 3 & -3 & 4 & 1 & 0 & 0 \\ 0 & -3 & 4 & -2 & 3 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right)$$

Row transformation 2 : $R_3 \to 3R_3 - R_2$

$$\left(\begin{array}{ccc|ccc} 3 & -3 & 4 & 1 & 0 & 0 \\ 0 & -3 & 4 & -2 & 3 & 0 \\ 0 & 0 & -1 & 2 & -3 & 3 \end{array}\right)$$

Row transformation 3 : $R_3 \to -R_3$

$$\left(\begin{array}{ccc|ccc} 3 & -3 & 4 & 1 & 0 & 0 \\ 0 & -3 & 4 & -2 & 3 & 0 \\ 0 & 0 & 1 & -2 & 3 & -3 \end{array}\right)$$

Row transformation 4 and 5 : $R_2 \to R_2 - 4R_3, \quad R_1 \to R_1 - 4R_3$

$$\left(\begin{array}{ccc|ccc} 3 & -3 & 0 & 9 & -12 & 12 \\ 0 & -3 & 0 & 6 & -9 & 12 \\ 0 & 0 & 1 & -2 & 3 & -3 \end{array}\right)$$

Row transformation 6 : Scale rows to get leading 1's:

$$R_2 \to -\tfrac{1}{3}R_2, \qquad R_1 \to \tfrac{1}{3}R_1$$

$$\left(\begin{array}{ccc|ccc} 1 & -1 & 0 & 3 & -4 & 4 \\ 0 & 1 & 0 & -2 & 3 & -4 \\ 0 & 0 & 1 & -2 & 3 & -3 \end{array}\right)$$

Row transformation 7 : $R_1 \to R_1 + R_2$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -2 & 3 & -4 \\ 0 & 0 & 1 & -2 & 3 & -3 \end{array}\right)$$

$$\text{Thus } \boxed{\mathbf{A}^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}}$$

Solving system of equations,

$$\text{We have } \mathbf{A}^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 21 \\ 20 \\ 5 \end{bmatrix}.$$

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{B} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}\begin{bmatrix} 21 \\ 20 \\ 5 \end{bmatrix}$$

$$\mathbf{x} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} \quad \Longrightarrow \quad \boxed{x = 1,\ y = -2,\ z = 3}$$