MatGeo Assignment 6.4.7

AI25BTECH11007

Question :
Two motorcycles A and B are running at a speed more than the allowed speed on the road represented by the following lines:

$$\mathbf{r} = \lambda(\mathbf{i} + 2\mathbf{j} - \mathbf{k}), \qquad \mathbf{r} = (3\mathbf{i} + 3\mathbf{j}) + \mu(2\mathbf{i} + \mathbf{j} + \mathbf{k}).$$

Based on the following information, answer the questions:

1) Find the shortest distance between the given lines.
2) Find a point at which the motorcycles may collide.

Solution : Let $\mathbf{x}_1$ and $\mathbf{x}_2$ be the points on the given lines respectively.

$$\mathbf{x}_1 = \lambda \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \qquad \mathbf{x}_2 = \begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

(a) Shortest distance:
Let $\mathbf{A} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix}$

Let

$$\mathbf{M} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \\ -1 & 1 \end{pmatrix}$$

$$(\mathbf{M} \ \mathbf{B} - \mathbf{A}) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ -1 & 1 & 0 \end{pmatrix} \tag{2.1}$$

Row Transformation-1: $R_2 \rightarrow R_2 - 2R_1$

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -3 \\ -1 & 1 & 0 \end{pmatrix} \tag{2.2}$$

Row Transformation-2: $R_3 \rightarrow R_3 + R_1$

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -3 \\ 0 & 3 & 3 \end{pmatrix} \tag{2.3}$$

Row Transformation-3: $R_3 \to R_3 + R_2$

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -3 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.4}$$

Therefore, rank $= 2 < 3 \Rightarrow$ The Lines intersect (not skew).

$$\boxed{\text{The Shortest Distance between the given Lines} = 0 \text{ units}} \tag{2.5}$$

(b) Intersection Point:

At intersection, the points on the lines are equal:

$$\mathbf{x}_1 = \mathbf{x}_2 \quad \Longrightarrow \quad \lambda \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

Equating components gives a system of equations:

$$\begin{cases} \lambda - 2\mu = 3 \\ 2\lambda - \mu = 3 \\ -\lambda - \mu = 0 \end{cases}$$

Write in augmented matrix form:

$$\left[\begin{array}{cc|c} 1 & -2 & 3 \\ 2 & -1 & 3 \\ -1 & -1 & 0 \end{array}\right]$$

Row Reduction-1 : $R_2 \to R_2 - 2R_1$

$$\left[\begin{array}{cc|c} 1 & -2 & 3 \\ 0 & 3 & -3 \\ -1 & -1 & 0 \end{array}\right]$$

Row Reduction-2 : $R_3 \to R_3 + R_1$

$$\left[\begin{array}{cc|c} 1 & -2 & 3 \\ 0 & 3 & -3 \\ 0 & -3 & 3 \end{array}\right]$$

Row Reduction-3 : $R_3 \to R_3 + R_2$

$$\left[\begin{array}{cc|c} 1 & -2 & 3 \\ 0 & 3 & -3 \\ 0 & 0 & 0 \end{array}\right]$$

From R_2: $3\mu = -3 \implies \mu = -1$

From R_1: $\lambda - 2(-1) = 3 \implies \lambda + 2 = 3 \implies \lambda = 1$

$$\therefore \lambda = 1, \quad \mu = -1$$

Intersection point:

$$\mathbf{r} = \lambda \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

$$\mathbf{r} = \begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

$$\boxed{\text{Intersection Point} = (1,\ 2,\ -1)}$$

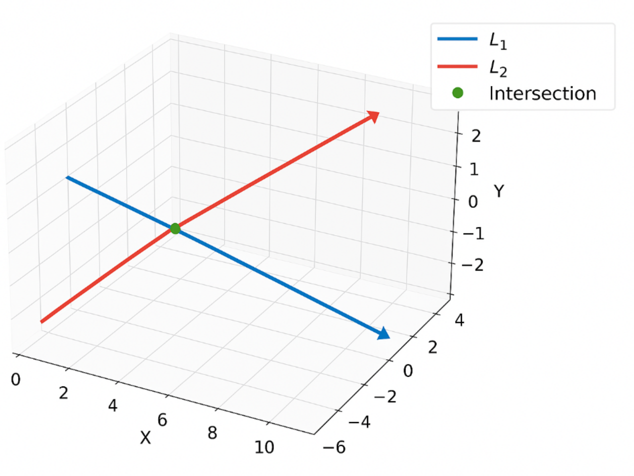

Fig. 2.1: Image