MatGeo Assignment 8.2.43

AI25BTECH11007

Question :
Find the equation of the conic, that satisfies the given conditions.
Focus at (-1, 2), directrix x - 2y + 3 = 0.

Solution :
Let :

$$\mathbf{F} = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \tag{0.1}$$

$$\text{directrix equation is : } \quad x - 2y + 3 = 0 \quad \implies \quad \begin{pmatrix} 1 \\ -2 \end{pmatrix}^T \mathbf{x} = -3 \tag{0.2}$$

The equation of a conic with directrix $\mathbf{n}^T\mathbf{x} = c$, eccentricity e and focus $\mathbf{F}$ is given by:

$$g(\mathbf{x}) = \mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{0.3}$$

where :

$$\mathbf{V} = \|\mathbf{n}\|^2\mathbf{I} - e^2\mathbf{n}\mathbf{n}^T,$$
$$\mathbf{u} = ce^2\mathbf{n} - \|\mathbf{n}\|^2\mathbf{F},$$
$$f = \|\mathbf{n}\|^2\|\mathbf{F}\|^2 - c^2e^2$$

From the question we can say that the conic is a parabola so $e = 1$.
Calculating $\mathbf{V}$, $\mathbf{u}$ and f by using the above equations we get :

$$\mathbf{n} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \qquad \|\mathbf{n}\|^2 = 5, \qquad c = -3, \qquad \|\mathbf{F}\|^2 = 5$$

$$\mathbf{V} = 5\mathbf{I} - \mathbf{n}\mathbf{n}^T = \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix} \tag{0.4}$$

$$\mathbf{u} = c\mathbf{n} - 5\mathbf{F} = \begin{pmatrix} 2 \\ -4 \end{pmatrix} \tag{0.5}$$

$$f = 5 \cdot 5 - (-3)^2 = 16 \tag{0.6}$$

Finding eigen values of $\mathbf{V}$:

$$\det|\mathbf{V} - \lambda\mathbf{I}| = 0 \tag{0.7}$$

$$\det\begin{pmatrix} 4-\lambda & 2 \\ 2 & 1-\lambda \end{pmatrix} = 0 \tag{0.8}$$

$$\lambda = 5 \quad \text{and} \quad 0 \tag{0.9}$$

Eigen vectors $\mathbf{v}$ for any square matrix $\mathbf{A}$ are defined by:

$$\mathbf{A}\mathbf{v} = \lambda\mathbf{v} \tag{0.10}$$

For $\lambda = 0$ and $\lambda = 5$ we get

$$\lambda = 0 \quad \Rightarrow \quad \mathbf{v}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \tag{0.11}$$

$$\lambda = 5 \quad \Rightarrow \quad \mathbf{v}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}. \tag{0.12}$$

Substituting in the equation (0.3) we get the equation of the conic to be :

$$\mathbf{x}^T \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix} \mathbf{x} + 2\begin{pmatrix} 2 & -4 \end{pmatrix} \mathbf{x} + 16 = 0 \tag{0.13}$$

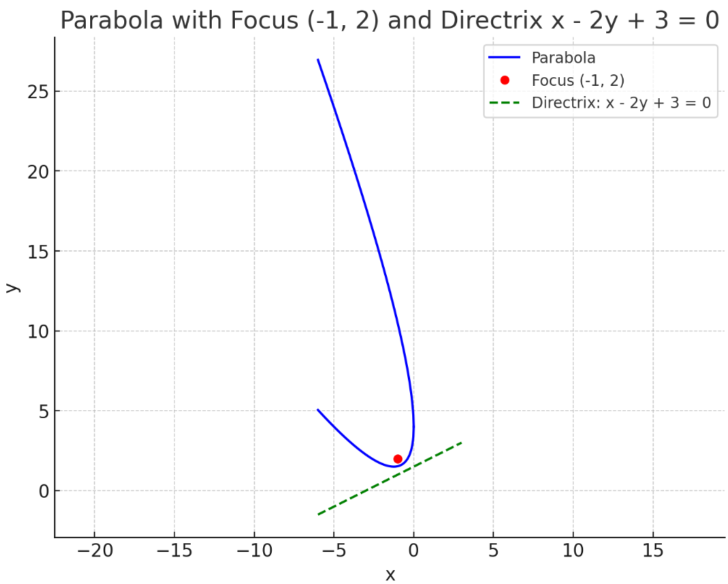

Fig. 0.1: Image